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Economic Theory

Equilibrium uniqueness with perfect complements*

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Summary. We study a model in which each of finitely many agent cares about a given subset of finitely many goods. We provide minimal conditions that ensure the existence and uniqueness of the equilibrium price vector – a price vector for which supply meets demand.

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Introduction

It is sometimes the case that some economic goods are perfect complements to some agents, but not to all agents. As an illustration, consider the following highly stylized story. A regulator faces the problem of pricing the use of a given network, e.g., a highway network. For our purposes, a network is simply a graph, with no loop. Each edge of the graph has a capacity: the maximal amount of traffic it can accommodate. Demand for capacity originates from the agents, who are considering using their car to commute between two nodes in the network. Whether an agent is willing to commute on the highway, rather than using some alternative form of transportation, is simply a function of the total price he would have to pay, i.e., of the sum of the prices of the edges he would need to use to reach his destination. An equilibrium price is a vector of prices, one for each edge, such that the capacity of each edge exactly meets the total demand of that edge. The regulator's problem is to set prices at an equilibrium level. Similar issues arise with bandwidth allocation by a monopoly operating a telecommunications network.

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Here we prove that under minimal conditions on the demand functions there exists a unique equilibrium allocation.

Our contribution relates to three different strands of literature. On the one hand, an important theme in general equilibrium theory has been to show that, under appropriate substituability conditions between goods, the equilibrium price vector is unique (see, e.g., Pearce and Wise, 1973; Iritani, 1981, and the references therein). Uniqueness of equilibrium is also obtained when endowments are close to the set of Pareto optima (see, e.g., Balasko, 1987). By contrast, we here adopt a partial equilibrium approach, so that the excess demand does not satisfy Walras' law, and the different goods are complements rather than substitutes. Indeed, the various edges of the path used by a given agent are perfect complements, while the other edges in the graph are deemed irrelevant by this agent.

On the other hand, within a partial equilibrium framework, it has been argued that (i) assuming away wealth effects, (ii) provided the markets for the different goods are competitive, and (iii) under sufficient regularity on the fundamentals of the economy (cost and utility functions), such an existence and uniqueness result holds (see, e.g., Mas-Colell et al., 1995, Sect. 10.G). The absence of wealth effects is generally viewed as a reasonable assumption, once agents spend only a small fraction of their income on the set of goods under consideration (see Marshall, 1920; Vives, 1987). In contrast to (i–iii), our assumptions on the demand function are of a quite different nature, and the supply side of the economy is not competitive. Indeed, in the above stylized story, the supply curve is vertical. Plainly also, our story puts considerable structure on the demand functions.

Finally, on a broader level, the uniqueness question relates to proving that the excess demand function is globally one-to-one, as a function of the prices of the various edges. Very few sets of sufficient conditions for this property to hold are known (see Gale and Nikaido, 1965; Nikaido, 1972). None of these sufficient conditions applies here.

The note is organized as follows. We first define some notation in Section 1, which also contains the existence and uniqueness result. In Section 2 we discuss the existence issue, while Section 3 is devoted to uniqueness.

1 The model

It is convenient to deal with a slightly more abstract model than the one sketched in the introduction. We let X be the finite set of goods, and I be the finite set of economic agents. Item $x \in X$ is in fixed supply $Q_x \geq 0$.

Each agent i cares only about a subset of goods $X^i \subseteq X$, and the different goods in X^i are perfect complements for agent i. As a consequence, the individual demand of agent i, as a function of the price vector $p = (p_x)_{x \in X}$, reduces to a function $D^i(\sum_{x \in X^i} p_x)$. Abusing notations, we sometimes write $D^i(p)$ instead of $D^i(\sum_{x \in X^i} p_x)$.

Throughout we make the following assumptions on the data of the game.

A1) The demand goes to zero as the price increases: $\lim_{q \to \infty} D^i(q) = 0$, for each $i \in I$.

A2) The supply of each item is positive: $Q_x > 0$ for every $x \in X$.
A3) The demand functions D^i are continuous and finite except possibly at 0, in which case $\lim_{q \to 0} D^i(q) = \infty$.
A4) The demand functions D^i are non-increasing and decreasing whenever positive: $D^i(q) > D^i(q')$ as soon as $D^i(q) > 0$ and $q > q'$.

The *aggregate demand* for item x is

$$\mathcal{D}_x(p) = \sum_{i \in I:\ x \in X^i} D^i\Big(\sum_{y \in X^i} p_y\Big).$$

For every $C \subseteq X$, the demand for bundle C is

$$D_C(p) = \sum_{i \in I:\ X^i = C} D^i\Big(\sum_{y \in X^i} p_y\Big).$$

Definition 1. *An* equilibrium price *is a vector* $p \in [0,\infty)^X$ *such that* $\mathcal{D}_x(p) \le Q_x$ *for each* $x \in X$*, with equality whenever* $p_x > 0$.

The assumption that the demand for item x may fall below the supply when its price is 0 is equivalent to saying that when the price is 0 the demand for bundle $\{x\}$ is at least $\mathcal{D}_x(0)$, and it clears the market.

The *allocation* that corresponds to a price vector p is the vector of demands $(D^i(p))_{i \in I}$.

We can now state our result.

Theorem 1. *The following holds:*

- *Under assumptions (**A1**)–(**A3**) there is at least one equilibrium.*
- *Under assumption (**A4**) there is at most one equilibrium allocation.*

Clearly the equilibrium price need not be unique. Indeed, if two goods are perfect complements for all agents, only the sum of their prices matter.

Example. Assume that $X = \{A, B\}$, with $Q_A = Q_B = 1$, and the demand functions for bundles are given by $D_{\{A\}}(p_A) = (5/4 - p_A)\,\mathbf{1}_{p_A < 5/4}$, $D_{\{B\}}(p_B) = (1/4 - p_B)\,\mathbf{1}_{p_B < 1/4}$, and $D_{\{A,B\}}(p_A + p_B) = (1 - (p_A + p_B)/2)\,\mathbf{1}_{p_A + p_B < 2}$.

We observe that the vector (p_A, p_B), that is defined by $p_A = 5/6, p_B = 0$, is an equilibrium price. Indeed, the corresponding demand is

$$\begin{aligned}
D_{\{A\}}(p) &= \frac{5}{4} - \frac{5}{6} = \frac{5}{12},\\
D_{\{B\}}(p) &= \frac{1}{4},\\
D_{\{A,B\}}(p) &= 1 - \frac{5}{12} = \frac{7}{12},
\end{aligned}$$

so that the aggregate demand of item A is 1, and the aggregate demand of item B is less than 1.

Theorem 1 implies that in this case there is no price vector that clears both goods from the market. This can also be verified analytically.

2 Equilibrium existence

Here we prove the first statement in Theorem 1.

By assumptions (**A1**) and (**A2**), there is q_* sufficiently large such that

$$\mathcal{D}_x(q_*) < Q_x, \quad \forall x \in X.$$

Let $\varDelta = [0, q_*]^X$. Define a set-valued function $\phi : \varDelta \to \varDelta$ as follows:

$$\phi(p) = \begin{cases} 0 & \mathcal{D}_x(p) < Q_x, \\ q_* & \mathcal{D}_x(p) > Q_x, \\ [0, q_*] & \mathcal{D}_x(p) = Q_x. \end{cases}$$

By (**A3**) ϕ is an upper-semi-continuous set-valued function with convex values. As $\varDelta$ is compact, Kakutani's fixed point theorem implies that ϕ has a fixed point. But any fixed point of ϕ is an equilibrium.

3 Equilibrium uniqueness

We here prove the uniqueness claim in Theorem 1. All vectors in the sequel are column-vectors. The proof relies on the following lemma:

Lemma 1. *Let $\mathcal{C}$ be a non-empty collection of non-empty subsets of X, and let $\pi = (\pi_x)_{x \in X} \in \mathbf{R}^X$ be given. Then the system (1) below in the variables $(d_C)_{C \in \mathcal{C}}$ has no solution:*

$$\begin{cases} d_C \sum_{x \in C} \pi_x > 0 & \forall C \in \mathcal{C}, \\ \sum_{C \in \mathcal{C}:\, x \in C} d_C = 0 & \forall x \in X. \end{cases} \tag{1}$$

Proof. For $C \in \mathcal{C}$ and $x \in X$, define

$$a_{C,x} = \begin{cases} 0 & x \notin C, \\ \dfrac{1}{\sum_{y \in C} \pi_y} & x \in C. \end{cases}$$

Denote by $A = (a_{C,x})$ the matrix with $|\mathcal{C}|$ rows and $|X|$ columns that consists of all those $a_{C,x}$.

The vector $A\pi$ is given by

$$(A\pi)_C = \sum_{x \in X} a_{C,x} \pi_x = \sum_{x \in C} \frac{1}{\sum_{y \in C} \pi_y} \pi_x = 1.$$

In particular, the system (S1) $Az \geq 0$, $Az \neq 0$ has a solution in $\mathbf{R}^X$. By Stiemke's theorem (Stiemke (1915), see also Mangasarian (1969, p. 32)), this implies that the system (S2) $y^T A = 0$, $y > 0$ has no solution.

To prove that the system (1) has no solution, we prove that any solution of (1) defines a solution of the system (S2).

Let $(d_C)_{C\in\mathcal{C}}$ be a solution of the system (1), and set $y_C = d_C \sum_{x\in C} \pi_x$ for $C \in \mathcal{C}$. By (1), $y_C > 0$ for every $C \in \mathcal{C}$, and for every $x \in X$,

$$(y^T A)_x = \sum_{C\in\mathcal{C}} y_C a_{C,x} = \sum_{C\in\mathcal{C}} d_C a_{C,x} \sum_{y\in C} \pi_y = \sum_{C\in\mathcal{C}:\, x\in C} d_C = 0.$$

Hence $y = (y_C)_{C\in\mathcal{C}}$ is a solution to (S2) – a contradiction. □

We now turn to the uniqueness proof. We show that if there are two distinct equilibrium allocations then a solution to the system (1) exists.

Assume that p and $\widehat{p}$ are two different equilibrium price vectors, such that the corresponding equilibrium allocations differ, and set $\pi := \widehat{p} - p$.

We now modify the demands D_C for goods which have price 0. Set

$$\widetilde{D}_C(\cdot) = \begin{cases} D_C(\cdot) & |C| \geq 2, \\ Q_x - \sum_{B\subseteq X:\, x\in B, |B|\geq 2} D_B(\cdot) & C = \{x\}. \end{cases}$$

Since p is an equilibrium, if $p_x > 0$ then $\widetilde{D}_{\{x\}}(p) = D_{\{x\}}(p)$, while if $p_x = 0$ then $\widetilde{D}_{\{x\}}(p) \geq D_{\{x\}}(p)$. A similar statement holds for $\widehat{p}$. By definition,

$$\sum_{C\subseteq X:\, x\in C} \widetilde{D}_C(p) = Q_x = \sum_{C\subseteq X:\, x\in C} \widetilde{D}_C(\widehat{p}), \quad \forall x \in X. \tag{2}$$

We define $\mathcal{C} = \left\{C \subseteq X:\, \widetilde{D}_C(\widehat{p}) \neq \widetilde{D}_C(p)\right\}$ to be the set of bundles for which the demand changes when moving from price p to price $\widehat{p}$.

By assumption, the set $\mathcal{C}$ is non-empty. For $C \in \mathcal{C}$, define $d_C = \widetilde{D}_C(p) - \widetilde{D}_C(\widehat{p}) \neq 0$ to be the change in the demand for bundle C. Since $D_C(p) = D_C(\widehat{p})$ for $C \notin \mathcal{C}$, (2) yields

$$\sum_{C\in\mathcal{C}:\, x\in C} d_C = 0. \tag{3}$$

By the monotonicity assumption (**A4**) on the demand functions, one has for each $C \in \mathcal{C}$

$$d_C \sum_{x\in C} \pi_x > 0. \tag{4}$$

Indeed, $\sum_{x\in C} \pi_x < 0$ if and only if $\sum_{x\in C} p_x > \sum_{x\in C} \widehat{p}_x$ if and only if $D_C(p) < D_C(\widehat{p})$. If $C = \{x\}$ and $p_x = 0 < \widehat{p}_x$ then $\widetilde{D}_{\{x\}}(\widehat{p}) = D_{\{x\}}(\widehat{p}) \leq D_{\{x\}}(p) \leq \widetilde{D}_{\{x\}}(p)$, and (4) holds. An analogous argument holds when $p_x > 0 = \widehat{p}_x$. In all other cases $\widetilde{D}_C(p) = D_C(p)$ and $\widetilde{D}_C(\widehat{p}) = D_C(\widehat{p})$, and (4) holds as well.

By (3) and (4), $(d_C)_{C\in\mathcal{C}}$ is a solution of the system (1) – a contradiction.

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