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Subgame-Perfection in Quitting Games with Perfect Information and Differential Equations

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We introduce a new approach to studying subgame-perfect equilibrium payoffs in stochastic games: the differential equations approach. We apply our approach to quitting games with perfect information. Those are sequential games in which at every stage one of n players is chosen; each player is chosen with probability $1/n$. The chosen player i decides whether to quit, in which case the game terminates and the terminal payoff is some vector $a_i \in \mathbf{R}^n$, or whether to continue, in which case the game continues to the next stage. If no player ever quits, the payoff is some vector $a_* \in \mathbf{R}^n$. We define a certain differential inclusion, prove that it has at least one solution, and prove that every vector on a solution of this differential inclusion is a subgame-perfect equilibrium payoff.

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1. Introduction. The existence of an equilibrium payoff in multiplayer stochastic games is still an open problem. The classical approach to proving the existence is by using the limit of stationary discounted equilibria. Namely, one takes for every discount factor a stationary discounted equilibrium, and considers the stationary profile, which is the limit of the stationary discounted equilibria, as the discount factor goes to zero. Depending on the exact class of games that is studied, one constructs a nonstationary ϵ-equilibrium in which players play mainly the limit stationary strategy profile, and perturb to other actions with small probability, while monitoring the actions of their opponents to detect deviations.

This approach, which was initiated by Mertens and Neyman [12] to prove the existence of the uniform value in two-player zero-sum stochastic games, was later exploited in numerous studies (see, e.g., Vrieze and Thuijsman [30]; Flesch et al. [5]; Thuijsman and Raghavan [26]; Solan [20]; Vieille [28, 29]; Rosenberg and Vieille [14]; Rosenberg et al. [15, 16]; Solan and Vohra [24]).

The limit of this approach was exhibited by Solan and Vieille [22], who constructed a four-player quitting game in which the simplest equilibrium strategy profile is periodic with period two.[1] Moreover, for ϵ sufficiently small, there is no ϵ-equilibrium in which players play mainly some stationary strategy profile, and perturb to other actions with small probability.

Once the classical approach fails, a need for new approaches arises. Solan and Vieille [21] studied equilibrium payoffs in quitting games. Motivated by dynamical systems, they defined a set-valued function, and proved that every infinite orbit of this function corresponds to an ϵ-equilibrium.

Simon [19] introduced tools from topology to the study of stochastic games. He showed that if a certain topological conjecture holds, then every quitting game admits an equilibrium payoff. However, it is yet not known whether his conjecture holds or not.

[1] *Quitting games* are sequential games in which at every stage each player chooses whether to continue or to quit. The game terminates when at least one player quits, and the terminal payoff vector depends on the subset of players that choose to quit at the terminal stage. If everyone always continues, the payoff is 0 to all players.

Shmaya et al. [18] and Shmaya and Solan [17] used Ramsey's theorem[2] and a stochastic variation of this theorem to prove the existence of an equilibrium payoff in two-player nonzero-sum stopping games.

Here, we present a new approach to study equilibrium payoffs in multiplayer stochastic games: a differential equations approach.

The class of games we study is quitting games with perfect information: at every stage, one of n players is chosen at random, independently of past play; each player i is chosen with probability $1/n$.[3] The chosen player i may decide either (i) to quit, in which case the game terminates, and the terminal payoff is some vector $a_i \in \mathbf{R}^n$, which depends only on the identity of the chosen player or (ii) to continue, in which case the game continues to the next stage. If no player ever quits, the payoff is some vector $a_* \in \mathbf{R}^n$. Observe that this game is a simple multiplayer Dynkin game (see Dynkin [3]).

Because players do not play simultaneously, this game is a game with perfect information. It is well known that games with perfect information admit ϵ-equilibria in pure strategy profiles (see Mertens [11] for a general argument in Borel games or Thuijsman and Raghavan [26], where this argument is adapted to stochastic games). Unfortunately, the ϵ-equilibrium strategy profiles Mertens [11] and Thuijsman and Raghavan [26] constructed use threats of punishment, which might be noncredible.

We study *subgame-perfect* ϵ-equilibria; namely, strategy profiles that are an ϵ-equilibrium after any history.

Roughly speaking, our approach is as follows. Let $W \subset \mathbf{R}^n$ be the compact set that contains all the vectors w in the convex hull of $\{a_1, \ldots, a_n\}$ such that $w^i \le a_i^i$ for at least one player i. We define a certain set-valued function $F\colon W \to \mathbf{R}^n$, characterize the set of equilibrium payoffs that are supported by stationary strategies in terms of F, and prove that the differential inclusion $\dot{w} \in F(w)$ has a solution; namely, there is a continuous function $w\colon \mathbf{R} \to W$ such that $\dot{w}(t) \in F(w(t))$ for almost every t. We then prove that any vector on a solution of the differential inclusion is a subgame-perfect equilibrium payoff. In particular, we deduce that every quitting game with perfect information admits either an equilibrium payoff that is supported by stationary strategies or (a continuum of) subgame-perfect 0-equilibrium payoffs.

The origin of the set-valued function F is in the version of the game in continuous time. That is, consider a version of the game in continuous time, where at every time instance t, each player i is chosen with probability dt. If player i is chosen at time t, he should decide whether to continue or to quit, in which case the game terminates with terminal payoff a_i. Roughly speaking, the set of subgame-perfect 0-equilibria of the game in continuous time coincides with the set of solutions of the differential inclusion $\dot{w} \in F(w)$. Thus, we relate the subgame-perfect equilibrium payoffs of the game in continuous time to those of the game in discrete time.

There are several motivations for our study. First, we try to find new approaches to study equilibrium payoffs in multiplayer stochastic games and multiplayer Dynkin games. Second, subgame-perfect equilibria are more useful than (Nash) equilibria in applications. Third, there are games, like quitting games and stopping games, in which, conditioned on the stage of the game, there is only one possible history, so that deviations from a completely mixed strategy cannot be detected immediately. The study of subgame-perfect equilibria in our model may help us understand (Nash) equilibria in those models.

The use of games in continuous time to study zero-sum stochastic games in discrete time was initiated by Sorin [25] to study zero-sum stochastic games with lack of information on one side. This approach was later exploited by Vieille [27] and Laraki [10] who used

[2] Ramsey [13] proved that for every coloring of a complete infinite graph by finitely many colors there is an infinite complete monochromatic subgraph.

[3] The case where the choice is not uniform is discussed in §5.

differential games to study repeated games with vector payoffs and repeated games with incomplete information on one side, respectively. The dynamics of the differential game Vieille [27] and Laraki [10] used is $\dot{z}(t) = -x(t)Ay(t)$, where $x(t)$ is the control vector at time t of player 1, $y(t)$ is the control vector at time t of player 2, A is a payoff matrix, and $z(t)$ is the parameter at time t. Because in multiplayer games there is multiplicity of equilibria, the dynamics we study is a differential inclusion.

The main difference between our approach and the one used in the literature is the interpretation of the time variable. Sorin [25], Vieille [27], and Laraki [10] considered the n-stage game in discrete time as a game that is played over the time interval $[0, 1]$, where stage k corresponds to the time interval $[(k-1)/n, k/n)$. As n goes to infinity, one obtains a game in continuous time. Here, on the other hand, we study the infinite-stage game, and divide each stage into n substages: substage k of stage m corresponds to the time interval $[m + (k-1)/n, m + k/n)$. We then consider the game in continuous time that is obtained when n goes to infinity.

Differential inclusions were already used in the context of game theory by Kannai and Tennenbaum [7] in the study of dynamical systems that lead to the Nash point in bargaining problems.

This paper is organized as follows. In §2, we present the model, several examples, the differential inclusion that we study, and the main results. In §3, we prove that the differential inclusion we have defined has at least one solution and in §4, we relate solutions of the differential inclusion to subgame-perfect equilibrium payoffs of the game. Extensions and open problems are discussed in §5.

2. The model and the main results.

In this section, we will present the model (§2.1), provide a few examples to illustrate the definitions (§2.2), define the concept of dummy players, who essentially do not participate in the game (§2.3), define the differential inclusion that will play a major role in this paper (§2.4), and finally state our main results (§2.5).

2.1. The model.

A *quitting game with perfect information* Γ is given by

- a finite set $I = \{1, \ldots, n\}$ of players;
- $n+1$ vectors $a_1, \ldots, a_n, a_*$ in $\mathbf{R}^n$.

The game is played as follows. At every stage $k \geq 1$, one of the players is chosen at random; each player is chosen with probability $1/n$, independent of past choices. The chosen player i decides whether to quit, in which case the game terminates and the terminal payoff vector is a_i, or whether to continue, in which case the game continues to the next stage. If no player ever quits, the payoff is a_*.

We assume throughout that $\|a_*\| \leq 1$,[4] and for every $i \in I$, $\|a_i\| \leq 1$ and $a_i^i = 0$.

For technical reasons, it will be more convenient to assume that players choose actions even if the game has already terminated. Setting $B = \{\text{Continue}, \text{Quit}\}$, the set of *histories* of length k is $H_k = (I \times B)^k$, the set of *finite histories* is $H = \bigcup_{k\geq 0} H_k$, and the set of *plays* is $H_\infty = (I \times B)^{\mathbf{N}}$. The space H_∞, equipped with the σ-algebra spanned by the cylinder sets, is a measurable space. We denote by $\mathcal{H}_k$ the sub-σ-algebra induced by the cylinder sets defined by H_k.

A (behavior) *strategy* of player i is a function σ^i: $H \to [0, 1]$; for every $h \in H_k$, $\sigma^i(h)$ is the probability that player i quits if the history h occurs and player i is chosen at stage $k+1$.

A *stationary strategy* is a strategy in which $\sigma^i(h)$ is independent of h; namely, a strategy in which player i quits whenever he is chosen with some fixed probability. We denote by 1^i

[4] The norm we use throughout the paper is the maximum norm: $\|a\| = \max_{i\in I} |a_i|$.

(respectively, 0^i) the strategy of player i in which he quits with probability 1 (respectively, with probability 0) whenever he is chosen.

A *strategy profile*, or simply a *profile*, is a vector $\sigma = (\sigma^i)_{i \in I}$ of strategies; one for each player. A stationary profile, which is a vector of stationary strategies, is identified with a vector $\rho \in [0, 1]^n$; ρ^i is the probability that player i quits whenever he is chosen.

We denote by $\mathbf{i}_k$ and $\mathbf{b}_k$ the player chosen at stage k and the action he chooses, respectively. Those are random variables.

Let $\theta = \min\{k \in \mathbf{N} \mid \mathbf{b}_k = \text{Quit}\}$ be the first stage in which the chosen player decides to quit. If no player ever quits, $\theta = +\infty$. Every profile σ induces a probability distribution $\mathbf{P}_\sigma$ over H_∞. We denote by $\mathbf{E}_\sigma$ the corresponding expectation operator.

A strategy profile σ is *terminating* if $\mathbf{P}_\sigma(\theta < +\infty) = 1$; that is, under σ the game terminates a.s. Observe that a stationary profile ρ is terminating if and only if $\sum_{i \in I} \rho^i > 0$.

The *expected payoff* that corresponds to a profile σ is

$$\gamma(\sigma) = \mathbf{E}_\sigma[\mathbf{1}_{\{\theta < +\infty\}} a_{\mathbf{i}_\theta} + \mathbf{1}_{\{\theta = +\infty\}} a_*].$$

Definition 2.1. Let $\epsilon \geq 0$. A profile σ is an *ϵ-equilibrium* if for every player $i \in I$ and every strategy σ'^i of player i,

$$\gamma^i(\sigma) \geq \gamma^i(\sigma^{-i}, \sigma'^i) - \epsilon.$$

The expected payoff that corresponds to an ϵ-equilibrium is an *ϵ-equilibrium payoff*, and any accumulation point of ϵ-equilibrium payoffs, as ϵ goes to 0, is an *equilibrium payoff*. Because payoffs are bounded, an equilibrium payoff exists as soon as an ϵ-equilibrium exists for every $\epsilon > 0$.

By Thuijsman and Raghavan [26], every quitting game with perfect information admits a 0-equilibrium. Unfortunately, the 0-equilibrium strategies that Thuijsman and Raghavan [26] construct use threats of punishment.

Given a strategy σ^i of player i and a finite history $h \in H$, the strategy σ^i_h is given by

$$\sigma^i_h(h') = \sigma^i(h; h')$$

for every finite history h', where $(h; h')$ is the concatenation of h and h'. This is the continuation strategy given the history h occurs.

Given a profile σ and a finite history $h \in H$, we denote $\sigma_h = (\sigma^i_h)_{i \in I}$.

Definition 2.2. Let $\epsilon \geq 0$. A profile σ is a *subgame-perfect ϵ-equilibrium* if for every finite history $h \in H$, the profile σ_h is an ϵ-equilibrium.

Clearly, any ϵ-equilibrium in stationary strategies is a subgame-perfect ϵ-equilibrium.

The payoff that corresponds to a subgame-perfect ϵ-equilibrium is a *subgame-perfect ϵ-equilibrium payoff*, and any accumulation point of subgame-perfect ϵ-equilibrium payoffs, as ϵ goes to 0, is a *subgame-perfect equilibrium payoff*.

2.2. Examples. We provide here a few examples that illustrate some features of the model. In the first two examples, a_* may be arbitrary.

Example 2.1. Take $n = 4$, $a_1 = (0, 3, -1, -1)$, $a_2 = (3, 0, -1, -1)$, $a_3 = (-1, -1, 0, 3)$, and $a_4 = (-1, -1, 3, 0)$. This is an adaptation of the game studied by Solan and Vieille [22].

This game admits a 0-equilibrium in pure stationary strategies: players 1 and 3 quit whenever chosen, and players 2 and 4 continue whenever chosen. The corresponding equilibrium payoff is $(1/2)(0, 3, -1, -1) + (1/2)(-1, -1, 0, 3) = (-1/2, 1, -1/2, 1)$, so indeed only players 1 and 3 have incentive to quit.

Example 2.2. Take $n = 3$, $a_1 = (0, 2, -1)$, $a_2 = (-1, 0, 2)$, and $a_3 = (2, -1, 0)$. This is an adaptation of the game studied by Flesch et al. [6] to our setup. As in the analysis of Flesch et al. [6], it is not clear how one can define a subgame-perfect 0-equilibrium using a limit of discounted stationary equilibria as the discount factor goes to 0.

TABLE 1. Two subgame-perfect 0-equilibria in Markovian strategies.

Stage	Profile	Payoffs (approximate)	Profile	Payoffs
1	1, 0, 1	−0.178, 1.381, −0.202	1, 0, 0	$0, \frac{1}{2}, \frac{1}{2}$
2	1, 0, 0	−0.267, 1.072, 0.196	$\frac{3}{4}, 0, 0$	0, 0, 1
3	1, 0, 0	−0.401, 0.607, 0.794	0, 1, 0	$\frac{1}{2}, 0, \frac{1}{2}$
4	1, 1, 0	−0.202, −0.178, 1.381	$0, \frac{3}{4}, 0$	1, 0, 0
5	0, 1, 0	0.196, −0.267, 1.072	0, 0, 1	$\frac{1}{2}, \frac{1}{2}, 0$
6	0, 1, 0	0.794, −0.401, 0.607	$0, 0, \frac{3}{4}$	0, 1, 0
7	0, 1, 1	1.381, −0.202, −0.178		
8	0, 0, 1	1.072, 0.196, −0.267		
9	0, 0, 1	0.607, 0.794, −0.401		

Table 1 presents two subgame-perfect 0-equilibria in Markovian strategies.[5] The two equilibria are periodic;[6] one is pure with period 9 and the other is mixed with period 6.[7]

In every row appear the probabilities by which the players quit if they are chosen, and the continuation payoff. For example, in the first equilibrium, the equilibrium payoff is (approximately) $(0.607, 0.794, -0.401)$ (the payoff at the end of the period). The continuation payoff (= expected payoff if Stage 2 is reached) is $(-0.178, 1.381, -0.202)$, so that players 1 and 3 want to quit at Stage 1. Indeed,

$$(0.607, 0.794, -0.401) = \frac{1}{3}(0, 2, -1) + \frac{1}{3}(-0.178, 1.381, -0.202) + \frac{1}{3}(2, -1, 0).$$

Similarly, the expected payoff if Stage 3 is reached is $(-0.267, 1.072, 0.196)$, so that at Stage 2, only player 1 wants to quit. And indeed,

$$(-0.178, 1.381, -0.202) = \frac{1}{3}(0, 2, -1) + \frac{2}{3}(-0.267, 1.072, 0.196).$$

The second equilibrium corresponds to the one identified by Flesch et al. [6] for their example. Indeed, in every period, the probability that player 1 quits is $1/3 \times 1 + 2/3 \times 1/3 \times 3/4 = 1/2$. Similarly, the probability that player 2 quits in a given period provided player 1 did not quit in that period is $1/2$, and the probability that player 3 quits in a given period provided players 1 and 2 did not quit is $1/2$.

Observe that one can construct more equilibria. Because $(0, 0, 1)$ (and by symmetry $(0, 1, 0)$ and $(1, 0, 0)$) is an equilibrium payoff (see the second equilibrium), $(1/3, 1/3, 1/3)$ is an equilibrium payoff as well: in the first stage, the chosen player continues, while from the second stage on, the players implement the equilibrium that corresponds to $(0, 0, 1)$ (respectively, $(0, 1, 0)$, $(1, 0, 0)$) if player 1 (respectively, 2, 3) was chosen in the first stage.

In fact, one can show that for this example, every feasible and individually rational payoff vector (that is, every vector in the set $\text{conv}\{a_1, a_2, a_3\} \cap \{x \in \mathbf{R}^3 \mid x^i \geq -1/2 \ \forall i\}$) is a subgame-perfect 0-equilibrium payoff. This observation does not hold in general.

We end this example by describing another subgame-perfect 0-equilibrium in Markovian strategies, which gives the basic idea of the equilibria we construct in the general case. In this equilibrium, the players use a parameter, which is the expected continuation payoff;

[5] A strategy of a player is *Markovian* if the mixed action it prescribes to play at each stage t depends only on t, and not on past play.

[6] A Markovian strategy of a player is *periodic* with period T if the mixed actions prescribed by the strategy at stages t and $t+T$ coincide, for every $t \geq 1$.

[7] A strategy σ^i of player i is *pure* if $\sigma^i(h) \in B$ for every finite history h, and is *mixed* otherwise.

each player's mixed action depends solely on the expected continuation payoff of all players. There are six possible continuation payoffs: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0,1/2,1/2)$, $(1/2,0,1/2)$, and $(1/2,1/2,0)$. We will use the following identities:

$$(0,1,0) = \frac{1}{3}\left(1\times(0,2,-1)+0\times\left(0,\frac{1}{2},\frac{1}{2}\right)\right)+\frac{1}{3}\left(0\times(-1,0,2)+1\times\left(0,\frac{1}{2},\frac{1}{2}\right)\right) + \frac{1}{3}\left(0\times(2,-1,0)+1\times\left(0,\frac{1}{2},\frac{1}{2}\right)\right) \tag{1}$$

and

$$\left(0,\frac{1}{2},\frac{1}{2}\right) = \frac{1}{3}\left(\frac{3}{4}\times(0,2,-1)+\frac{1}{4}\times(0,0,1)\right)+\frac{1}{3}(1\times(-1,0,2)+0\times(0,0,1)) + \frac{1}{3}(0\times(2,-1,0)+1\times(1,0,0)). \tag{2}$$

We describe the subgame-perfect 0-equilibrium when the continuation payoff is $(0,1,0)$ and $(0,1/2,1/2)$. The behavior when the continuation payoff is one of the other four vectors is symmetric.

Assume that the continuation payoff is $(0,1,0)$. If player 1 (respectively, 2, 3) is chosen, he quits with probability 1 (respectively, 0, 0). If the chosen player does not quit, the continuation payoff is $(0,1/2,1/2)$.

Assume that the continuation payoff is $(0,1/2,1/2)$. If player 1 (respectively, 2, 3) is chosen, he quits with probability $3/4$ (respectively, 1, 0). If he does not quit, the continuation payoff is $(0,0,1)$ (respectively, $(0,0,1)$, $(1,0,0)$).

Equations (1) and (2) imply that this is indeed a subgame-perfect 0-equilibrium.

In the following example, there is no subgame-perfect 0-equilibrium.

Example 2.3. Take $n=2$, $a_1=(0,1)$, $a_2=(-1,0)$, and $a_*=(1,-1)$. This is an adaptation of Solan and Vieille [23, Example 3]

Here, there is a subgame-perfect ϵ-equilibrium in mixed stationary strategies: player 1 quits whenever chosen with probability 1, and player 2 quits whenever chosen with probability ϵ. The expected payoff is

$$\frac{1}{1+\epsilon}(0,1)+\frac{\epsilon}{1+\epsilon}(-1,0)=\left(-\frac{\epsilon}{1+\epsilon},\frac{1}{1+\epsilon}\right).$$

One can verify that player 1 cannot profit by deviating, while player 2 cannot profit more than $\epsilon/(1+\epsilon)$ by deviating.

The same analysis that was performed by Solan and Vieille [23, Example 3] shows that this game admits no subgame-perfect 0-equilibrium. The basic idea is the following. Consider the three events: $A=\{$the game is terminated by player 1$\}$, $B=\{$the game is terminated by player 2$\}$, and $C=\{$the game continues indefinitely$\}$. Player 1 prefers C to A and A to B, while player 2 prefers A to B and B to C. To achieve event A, which is controlled by player 1, player 2 must threaten player 1 by event B, which is suboptimal for both players. However, because player 1 prefers event C, without such a threat event C will be realized, but event C is worse than B to player 2, so a suboptimal threat is necessary.

2.3. Dummy players. In this section, we define the notion of dummy players, and we see that dummy players essentially never participate in the game—they never quit. One can then eliminate those players from the game. Recall that $a_i^i=0$ for every $i\in I$.

Definition 2.3. A player i is *dummy* if (i) $a_*^i>0$ and (ii) $a_j^i>0$ for every $j\neq i$.

A dummy player never wants to quit: whether the game is going to continue indefinitely, or whether some other player is going to quit, he himself does not want to quit. It is no surprise then that one can eliminate dummy players.

LEMMA 2.1. *Let $i \in I$ be a dummy player in Γ and let $\epsilon > 0$. Let Γ' be the $(n-1)$-player game in which we eliminate player i. Then, any ϵ-equilibrium in Γ' can be extended to an ϵ-equilibrium in Γ by instructing player i to continue whenever he is chosen. Moreover, in every ϵ-equilibrium in Γ, the overall probability that the game is terminated by player i is at most ϵ/A where $A = \min\{a_*^i, a_j^i, j \neq i\}$.*

In particular, every ϵ-equilibrium in Γ can be turned into a $(1+1/A)$ ϵ-equilibrium in which player i never quits, simply by modifying the profile so that player i never quits.

The proof of Lemma 2.1 is straightforward and omitted.

From now on, we assume that the game contains no dummy players.

Observe that if a dummy player is eliminated from the game, some other player, who was not initially a dummy player, may become a dummy player in the $(n-1)$-player game.

2.4. A differential inclusion. An (autonomous) differential inclusion is an equation of the form $\dot{w} \in G(w)$, where G is a set-valued function. This is a generalization of standard differential equations. A solution of a differential inclusion is an absolutely continuous function w such that $\dot{w}(t) \in G(w(t))$ for almost every t. Differential inclusions have been extensively studied (see, e.g., Aubin and Cellina [1] or Filippov [4]).

In this section, we construct a certain set-valued function F from the data of the game. Our goal is to relate solutions of the differential inclusion $\dot{w} \in F(w)$ to subgame-perfect equilibrium payoffs of the game.

Set

$$W = \{w \in \operatorname{conv}\{a_1, \ldots, a_n\} \mid w^i \leq 0 \text{ for some } i \in I\}.$$

The set W is nonempty as it contains a_i, $i \in I$. It is also compact, but not necessarily convex or even connected (e.g., if $n = 2$, $a_1 = (0, 1)$, and $a_2 = (1, 0)$, then $W = \{(0, 1), (1, 0)\}$).

For every $w \in W$, define

$$\begin{aligned} I_N(w) &= \{i \in I \mid w^i < 0\}, \\ I_Z(w) &= \{i \in I \mid w^i = 0\}, \text{ and} \\ I_P(w) &= \{i \in I \mid w^i > 0\}. \end{aligned}$$

Observe that for every $w \in W$, $I_N(w) \cup I_Z(w) \neq \emptyset$.

Set[8]

$$\Delta(w) = \left\{\rho \in [0,1]^n \;\middle|\; i \in I_P(w) \Rightarrow \rho^i = 0, i \in I_N(w) \Rightarrow \rho^i = 1, \sum_{i \in I} \rho^i \geq 1\right\}.$$

This definition captures the following idea. If w is the continuation payoff and player i is chosen, i will quit with probability 1 if $w^i < 0$ and with probability 0 if $w^i > 0$. Thus, any $\rho \in \Delta(w)$ is a possible description of the behavior of rational players at a given stage, when the continuation payoff is w.

Because $I_N(w) \cup I_Z(w) \neq \emptyset$ for every $w \in W$, Δ has nonempty values.

Define

$$F(w) = \left\{\sum_{i \in I} \rho^i (w - a_i) \;\middle|\; \rho \in \Delta(w)\right\} \subset \mathbf{R}^n.$$

We are interested in solutions of the equation

$$\dot{w} \in F(w). \tag{3}$$

The reader may wonder why the differential inclusion (3) is of interest.

[8] In the definition of $\Delta(w)$, one can take $\sum_{i \in I} \rho^i \geq c$ for any fixed $0 < c \leq 1$.

Consider a version of the game that is played in continuous time. At every time instance t, each player is chosen with probability dt and with probability $1-n\,dt$, no player is chosen. If player i is chosen, he can decide whether to quit, in which case the game terminates and the payoff is a_i or to continue. We allow players to use Markovian strategies, so that a strategy of player i is a measurable function ρ^i: $[0,\infty)\to[0,1]$; $\rho^i(t)$ is the probability that player i quits at time t if he is chosen at that time, and the game was not terminated before.

Every strategy profile $\rho=(\rho^i)_{i\in I}$ induces a payoff function w: $[0,\infty)\to\mathbf{R}^n$; w^i_t is the expected continuation payoff to player i under ρ from time t onward. One can verify that the payoff function w must satisfy $\dot{w}(t)=\sum_{i\in I}\rho^i(t)(w(t)-a_i)$.

For ρ to be a subgame-perfect 0-equilibrium, we need $\rho^i(t)=1$ if $w^i(t)<0$ and $\rho^i(t)=0$ if $w^i(t)>0$. Thus, modulo the condition that $\sum_{i\in I}\rho^i(t)\geq 1$ for every t, a strategy profile ρ is a subgame-perfect 0-equilibrium in the game in continuous time if and only if it defines a solution of the differential inclusion $\dot{w}\in F(w)$ via the equation $\dot{w}(t)=\sum_{i\in I}\rho^i(t)(w(t)-a_i)$.

The condition $\sum_{i\in I}\rho^i\geq 1$ ensures that the game will eventually terminate, as at every time instance, the probability that the game terminates is at least $n\,dt$.

To summarize, we relate subgame-perfect equilibria of the game in continuous time to subgame-perfect equilibria of the game in discrete time.

2.5. The main results. Our results can be divided into three groups. First, we characterize the set of equilibrium payoffs that are supported by stationary strategies (and, in particular, are subgame perfect) by means of fixed points of the set-valued function F. Second, we prove that the differential inclusion $\dot{w}\in F(w)$ always has a solution. Third, we relate solutions of the differential inclusion $\dot{w}\in F(w)$ to subgame-perfect equilibrium payoffs.

2.5.1. Stationary equilibria. Because the payoff that a player receives when he quits alone is $a^i_i=0$, if $a^i_*\geq 0$ for every $i\in I$, the strategy profile in which all players always continue is a stationary equilibrium.

Our first goal is to characterize all equilibrium payoffs that are supported by *terminating* stationary strategies in terms of the set-valued function F.

A *fixed point* of a differential inclusion $\dot{w}\in G(w)$ is any vector w_0 such that $\vec{0}\in G(w_0)$.

Proposition 2.1. *Assume that there are no dummy players and let $w\in W$. If $\vec{0}\in F(w)$, then w is an equilibrium payoff that is supported by terminating stationary profiles.*

The converse of Proposition 2.1 is given by the following proposition.

Proposition 2.2. *Let $w\in W$. If w is an equilibrium payoff that is supported by terminating stationary profiles, then $\vec{0}\in F(w)$.*

Lemma 2.2 gives two cases where we can point at specific vectors w_0 such that $\vec{0}\in F(w_0)$.

Lemma 2.2. (1) *If $\vec{0}\in W$, then $\vec{0}\in F(\vec{0})$.*
(2) *If for some $i\in I$, $a^j_i\geq 0$ for every $j\in I$, then $\vec{0}\in F(a_i)$.*

Proof. We start with the first assertion. Assume that $\vec{0}\in W$. Then, there is $\rho\in[0,1]^n$ that satisfies (i) $\sum_{i\in I}\rho^i=1$ and (ii) $\sum_{i\in I}\rho^i a_i=\vec{0}$. Because $I_Z(\vec{0})=I$, it follows that $\rho\in\Delta(\vec{0})$, and therefore $\vec{0}\in F(\vec{0})$.

We continue with the second assertion. Under the assumptions, the vector ρ that is defined by $\rho^i=1$ and $\rho^j=0$ for every $j\neq i$ is in $\Delta(a_i)$, so that $\vec{0}\in F(a_i)$. □

2.5.2. Existence of a solution to the differential inclusion $\dot{w}\in F(w)$.

Definition 2.4. A function g: $\mathbf{R}\to\mathbf{R}^n$ is *absolutely continuous* if for every $\epsilon>0$, there is $\delta>0$ such that for every $m\in\mathbf{N}$ and every collection $(x_i,y_i)_{i=1}^m$ of real numbers, if $\sum_{i=1}^m|x_i-y_i|<\delta$, then $\sum_{i=1}^m\|g(x_i)-g(y_i)\|<\epsilon$.

Observe that if g is absolutely continuous, it is, in particular, uniformly continuous.[9]

[9] A function g is *uniformly continuous* if the condition in Definition 2.4 holds for $m=1$.

Definition 2.5. A *solution* of a differential inclusion $\dot{w} \in G(w)$ is an absolutely continuous function $w: \mathbf{R} \to W$ that satisfies $\dot{w}(t) \in G(w(t))$ for almost every $t \in \mathbf{R}$.[10]

The following proposition asserts that the differential inclusion $\dot{w} \in F(w)$ has a solution.

Proposition 2.3. *The differential inclusion $\dot{w} \in F(w)$ has at least one solution.*

2.5.3. Solutions of $\dot{w} \in F(w)$ and equilibrium payoffs. For every solution w of $\dot{w} \in F(w)$, denote the range of w by

$$Y_w = \{w(t) \mid t \in \mathbf{R}\} \subseteq W.$$

For every set $Y \subseteq \mathbf{R}^n$, $\overline{Y}$ is the closure of Y.

Definition 2.6. A solution w of $\dot{w} \in F(w)$ has *type* 0 if $\vec{0} \in F(y)$ for some $y \in \overline{Y}_w$. It has *type* 1 otherwise.

By Proposition 2.1, if w is a solution of type 0 and there are no dummy players, then the game admits an equilibrium payoff that is supported by terminating stationary strategies.

The following two propositions relate solutions of the differential inclusion $\dot{w} \in F(w)$ to subgame-perfect equilibrium payoffs.

Proposition 2.4. *Assume that there are no dummy players and let w be a solution of $\dot{w} \in F(w)$ of type 1. Then, every $y \in \overline{Y}_w$ is a subgame-perfect 0-equilibrium payoff.*

More generally, our arguments show that if there are no solutions of type 0, then every vector in the closure of the range of all solutions of type 1 is a subgame-perfect 0-equilibrium payoff.

Proposition 2.5. *Assume that there are no dummy players and let w be a solution of $\dot{w} \in F(w)$ of type 0. Then, every $y \in \overline{Y}_w$ is a subgame-perfect ϵ-equilibrium payoff for every $\epsilon > 0$.*

Remark 2.1. The range of all solutions of $\dot{w} \in F(w)$ does *not* necessarily coincide with the set of subgame-perfect equilibrium payoffs. Indeed, the former set is a subset of W, whereas there are subgame-perfect 0-equilibrium payoffs that are not in W (see, e.g., Example 2.2).

Remark 2.2. By Proposition 2.3, the differential inclusion $\dot{w} \in F(w)$ has at least one solution. If it has a solution of type 0, then by Proposition 2.1, there is a stationary ϵ-equilibrium for every $\epsilon > 0$. If, on the other hand, it has a solution of type 1, then there is continuum of vectors in the range of this solution, and thus by Proposition 2.4, a continuum of subgame-perfect 0-equilibrium payoffs.

3. Existence of a solution to $\dot{w} \in F(w)$. In this section, we prove Proposition 2.3, which states that a solution to the differential inclusion $\dot{w} \in F(w)$ always exists.

Lemma 3.1. *The set-valued functions $w \mapsto \Delta(w)$ and $w \mapsto F(w)$ are upper semicontinuous; that is, their graphs are closed sets in $\mathbf{R}^{2n}$. Moreover, they have nonempty and convex values.*

Proof. The fact that both Δ and F have nonempty and convex values easily follows from the definitions.

We now prove that $w \mapsto \Delta(w)$ is upper semicontinuous. Let $(w_k)_{k\in\mathbf{N}}$ be a sequence of elements in W that converges to w, and let $(\rho_k)_{k\in\mathbf{N}}$ be a sequence of elements in $[0,1]^n$ that converges to ρ such that $\rho_k \in \Delta(w_k)$ for every $k \in \mathbf{N}$. We show that $\rho \in \Delta(w)$.

As $\rho_k \in \Delta(w_k)$ for every $k \in \mathbf{N}$, $\sum_{i\in I} \rho_k^i \geq 1$ for every $k \in \mathbf{N}$. Hence, $\sum_{i\in I} \rho^i \geq 1$.

[10] In the literature on differential equations, a solution w is usually defined over the interval $[0, \infty)$ and not over the whole real line. For our purposes, a solution should be defined over $(-\infty, 0]$, so we find the present definition more convenient.

Fix $i \in I$. If $i \in I_N(w)$, then $w^i < 0$. Hence, $i \in I_N(w_k)$ for every k sufficiently large. In particular, $\rho_k^i = 1$ for every k sufficiently large, so that $\rho^i = 1$. If $i \in I_P(w)$, then $w^i > 0$. Hence, $i \in I_P(w_k)$ for every k sufficiently large. In particular, $\rho_k^i = 0$ for every k sufficiently large, so that $\rho^i = 0$. It follows that $\rho \in \Delta(w)$ as desired.

As F is the composition of a continuous function with an upper semicontinuous set-valued function, it is upper semicontinuous. □

Lemma 3.2. *For every $w \in W$, there is $y \in F(w)$ such that $w - \lambda y \in W$ for every $\lambda > 0$ sufficiently small.*

Geometrically, Lemma 3.2 asserts that for every vector $w \in W$, there is a vector $y \in F(w)$ such that at w the direction $-y$ "points into W."

Proof. Fix $w \in W$ and $y \in F(w)$. Then, $y = (\sum_{i\in I} \rho^i)w - \sum_{i\in I} \rho^i a_i$ for some vector $\rho \in [0,1]^n$. In particular, $w - \lambda y = (1 - \sum_{i\in I} \lambda\rho^i)w + \sum_{i\in I} \lambda\rho^i a_i$. Because w is in the convex hull of $\{a_1, \ldots, a_n\}$, so is $w - \lambda y$, provided $\lambda \le 1/n$.

It remains to show that there is $y \in F(w)$ such that $w^i - \lambda y^i \le 0$ for some $i \in I$.

If $I_N(w) \neq \emptyset$, then $w^i < 0$ for some $i \in I$, and any $y \in F(w)$ satisfies this requirement. If $I_N(w) = \emptyset$, then because $w \in W$, we have $I_Z(w) \neq \emptyset$. Then, for every $i \in I_Z(w)$, we have $w^i = 0$ so that $y = w - a_i \in F(w)$. Moreover, $w^i - \lambda y^i = (1-\lambda)w^i + \lambda a_i^i = 0$. □

Proposition 2.3 follows from the following general result.

Theorem 3.1. *Let $W \subseteq \mathbf{R}^n$ be a compact set and let $F\colon W \to \mathbf{R}^n$ be an upper semicontinuous set-valued function with nonempty and convex values such that for every $w \in W$, there is $y \in F(w)$ satisfying $w - \lambda y \in W$ for every $\lambda > 0$ sufficiently small. Then, the differential inclusion $\dot{w} \in F(w)$ has a solution.*

Proof. The differential inclusion $\dot{w} \in -F(w)$ satisfies the conditions of Deimling [2, Theorem 1] or Kunze [8, Theorem 2.2.1] so that for every $w_0 \in W$, there is an absolutely continuous function $w\colon [0, +\infty) \to W$ that satisfies (i) $w(0) = w_0$ and (ii) $\dot{w}(t) \in -F(w(t))$ for almost every $t \in [0, +\infty)$.

By reversing the direction of time, for every $k \in \mathbf{N}$, there is an absolutely continuous function $w_k\colon (-\infty, k] \to W$ that satisfies $\dot{w}_k(t) \in F(w_k(t))$ for almost every $t \in (-\infty, k]$.

Because W is compact, Ascoli-Arzela's Theorem (see Aubin and Cellina [1, Theorem 0.3.4]) implies that the sequence (w_k) has a convergent subsequence: there is a subsequence $(k_j)_{j\in\mathbf{N}}$ and a function $w\colon \mathbf{R} \to W$ such that $\lim_{j\to\infty} w_{k_j}(t) = w(t)$ for every $t \in \mathbf{R}$. Indeed, the functions w_k are uniformly bounded (as their values are in the compact set W), and their derivatives are also uniformly bounded (as the derivatives are a.e. in the compact set $F(W)$).

By Filippov [4, Lemma 2.7.1] w is absolutely continuous over every open and bounded interval and $\dot{w}(t) \in F(w(t))$ for almost every t in this interval. It follows that w satisfies these two properties over $\mathbf{R}$ as well. □

The following lemma asserts that for every solution of $\dot{w} \in F(w)$, one can find a measurable function ρ such that $\rho \in \Delta(w)$ and $\dot{w} = \sum_i \rho^i(w - a_i)$.

Lemma 3.3. *For every solution w of the differential inclusion $\dot{w} \in F(w)$, there is a measurable function $\rho\colon \mathbf{R} \to [0,1]^n$ such that for almost every $t \in \mathbf{R}$,*

(1) *$\rho(t) \in \Delta(w(t))$ and*
(2) *$\dot{w}(t) = \sum_{i\in I} \rho^i(t)(w(t) - a_i)$.*

Proof. Set $R(t) = \{\rho \in \Delta(w(t))\colon \dot{w}(t) = \sum_{i\in I} \rho^i(w(t) - a_i)\}$. Because $\dot{w}(t) \in F(w(t))$ for almost every t, $R(t)$ has nonempty values for almost every t, and it can be easily verified that it has closed values. Because w and F are measurable, R is a measurable set-valued function. By Kuratowski and Ryll-Nardzewski [9], the set-valued function R has a measurable selector ρ, which plainly satisfies the requirements. □

4. From solutions of $\dot{w} \in F(w)$ to equilibria. In this section, we relate solutions of the differential inclusion $\dot{w} \in F(w)$ to subgame-perfect equilibrium payoffs.

We first provide in §4.1, a sufficient condition for the existence of a subgame-perfect equilibrium payoff. We use this sufficient condition in §4.2 to characterize stationary equilibria in terms of the set-valued function F. In §4.3, we provide two representations of a solution of $\dot{w} \in F(w)$ that are used in §4.4 to show that any point on a solution of this differential inclusion is a subgame-perfect equilibrium payoff.

4.1. Conditions for existence of subgame-perfect equilibria. Recall that $\overline{Y}$ is the closure of Y. Lemma 4.1 provides a condition that ensures that any payoff vector in a given set Y is a subgame-perfect equilibrium payoff.

LEMMA 4.1. *Let $Y \subseteq W$ and $\eta > 0$ be given. Assume that for every $y \in Y$, there exist $\rho \in [0, 1]^n$ and $y_1, \ldots, y_n \in Y$ that satisfy the following conditions:*

(C.1) $y = (1/n) \sum_{i\in I} (\rho^i a_i + (1 - \rho^i) y_i)$,

(C.2) $y_i^i > 0$ *implies that* $\rho^i = 0$,

(C.3) $y_i^i < 0$ *implies that* $\rho^i = 1$, *and*

(C.4) $\max\{\rho^1, \ldots, \rho^n\} \geq \eta$.

Then, the following two assertions hold:

(a) *If there are no dummy players, then every $y \in \overline{Y}$ is a subgame-perfect ϵ-equilibrium payoff for every $\epsilon > 0$.*

(b) *Suppose that for every $i \in I$, if $a_i \in \overline{Y}$, then $a_i^j < 0$ for some $j \in I$. Then, every $y \in Y$ is a subgame-perfect* 0*-equilibrium payoff.*

Note that each of the three subgame-perfect equilibria we presented for Example 2.2 defines a set Y that satisfies the conditions in the lemma.

PROOF. Choose an arbitrary $y \in Y$. We show that y is a subgame-perfect equilibrium payoff.

Step 1. *Definition of a strategy σ.* We simultaneously define a profile σ and a function u: $H \to Y$.

Set $u(\emptyset) = y$, where $\emptyset$ is the history at the beginning of the game. Assume that we have already defined $u(h) \in Y$ for some finite history h. By assumption, there exist $\rho \in [0, 1]^n$ and $y_1, \ldots, y_n \in Y$ that satisfy $u(h) = (1/n) \sum_{i\in I} (\rho^i a_i + (1 - \rho^i) y_i)$ and (C.2)–(C.4).

Thus, if y_i is the continuation payoff if player i is chosen, then by (C.2) and (C.3), ρ^i is an optimal response of player i and $u(h)$ is the expected payoff conditioned that h is realized.

Set $\sigma^i(h) = \rho^i$ and $u(h; i, \text{Quit}) = u(h; i, \text{Continue}) = y_i$ for every $i \in I$.

(C.4) implies that under σ, the probability of termination at every stage is at least η/n. Hence, under σ the game eventually terminates, and (C.1) implies that $u(h)$ is the expected payoff under σ_h,

$$\gamma(\sigma_h) = u(h) \quad \forall\, h \in H.$$

Step 2. *Assertion (b): $(\sigma^{-i}, 0^i)$ is terminating for every player i.* In Steps 2 and 3, we assume that the condition in assertion (b) holds. We argue that for every player $i \in I$, the profile $(\sigma^{-i}, 0^i)$ in which all players but i follow σ and player i never quits, is terminating with probability 1.

Indeed, otherwise, for every $\delta > 0$, there is a finite history h and a player i such that the probability the game terminates under $(\sigma_h^{-i}, 0^i)$ is at most δ. But then for every $j \in I$, the probability the game terminates under $(\sigma_{(h;j,\text{Continue})}^{-i}, 0^i)$ is at most $n\delta$. Because under $\sigma_{(h;j,\text{Continue})}$ termination occurs with probability 1, we deduce that under $\sigma_{(h;j,\text{Continue})}$ the probability that player i terminates is at least $1 - n\delta$ so that $u(h; j, \text{Continue})$, the expected payoff under $\sigma_{(h;j,\text{Continue})}$, is within $n\delta$ of a_i.

If a_i is not in the closure of Y, then there is $\delta' > 0$ such that the distance between a_i and the closure of Y is at least δ'. Because $u(h; j, \text{Continue}) = \gamma(\sigma_{(h;j,\text{Continue})})$ is in Y, this leads to a contradiction when $\delta < \delta'/n$.

Otherwise, $a_i \in \overline{Y}$, so that there is $j \in I$ satisfying $a_i^j < 0$. Suppose that δ is sufficiently small such that $a_i^j < -n\delta$. Because $\|u(h; j, \text{Continue}) - a_i\| < n\delta$, it follows that the expected payoff for player j under $\sigma_{(h;j,\text{Continue})}$ is negative. By (C.3), this implies that $\sigma^j(h) = 1$. Therefore, the probability of termination under $(\sigma_h^{-i}, 0^i)$ is at least $1/n$, which leads to a contradiction if $\delta < 1/n$.

Step 3. *Assertion (b): σ is a subgame-perfect 0-equilibrium.* We now prove that no player $i \in I$ can profit by deviating from σ. The same proof holds for any subgame and, hence, σ is a subgame-perfect 0-equilibrium. Fix a player $i \in I$ and a strategy σ'^i of player i.

For every $k \geq 0$, define a r.v. X_k as follows: $X_k = a^i_{\mathbf{i}_\theta}$ if $\theta < k$ and $X_k = \gamma^i(\sigma_k)$ otherwise, where σ_k is the random strategy induced from stage k on (that is, σ_k is a strategy-valued r.v.).

By (C.2)–(C.3) and the definition of σ,

$$\mathbf{E}_{\sigma^{-i},\sigma'^i}[X_{k+1} \mid \mathcal{H}_k] \leq X_k \quad \forall k \geq 0. \tag{4}$$

Because the profile $(\sigma^{-i}, 0^i)$ is terminating, so is the profile (σ^{-i}, σ'^i). Thus, (X_k) is a supermartingale under (σ^{-i}, σ'^i), so that by the monotone convergence theorem,

$$\gamma^i(\sigma^{-i}, \sigma'^i) = \mathbf{E}_{\sigma^{-i},\sigma'^i}[a^i_{\mathbf{i}_\theta}\mathbf{1}_{\{\theta<+\infty\}}] = \lim_{k\to+\infty} \mathbf{E}_{\sigma^{-i},\sigma'^i}[X_k] \leq X_0 = \gamma^i(\sigma),$$

as desired.

We now prove assertion (a). It is no longer true that $(\sigma^{-i}, 0^i)$ is terminating for every player i. To fix this, we will define a proper augmentation τ of σ and prove that it is a 4ϵ-equilibrium.

Step 4. *Definition of an augmentation τ.* For the rest of the proof, we assume that the condition in assertion (a) holds. Because there are no dummy players, for every player $i \in I$ either $a_*^i \leq 0$ or there is $j_i \neq i$ such that $a^i_{j_i} \leq 0$ (j_i is a "punisher" of i), or both. For every $j \in I$, set $I_j = \{i \in I\colon j = j_i\}$, the set of players that j punishes. Because for every player we choose at most one punisher, $(I_j)_{j\in I}$ are disjoint sets (some of these sets may be empty).

Define a profile τ as follows:

$$\tau^j(h) = \min\left\{\sigma^j(h) + \frac{\epsilon}{n}\sum_{i\in I_j}\sigma^i(h), 1\right\}.$$

In words, for every player i that j punishes, the probability that j quits is increased by ϵ times the probability that i quits. This implies, in particular, that for every $k \geq 1$,

$$\mathbf{P}_{\sigma^{-i},0^i}(\theta = k+1 \mid \mathcal{H}_k) \leq \mathbf{P}_{\tau^{-i},0^i}(\theta = k+1 \mid \mathcal{H}_k) \leq (1+\epsilon)\mathbf{P}_{\sigma^{-i},0^i}(\theta = k+1 \mid \mathcal{H}_k). \tag{5}$$

Step 5. $\|\gamma(\sigma) - \gamma(\tau)\| < \epsilon$. We argue here, using a coupling argument, that

$$\|\gamma(\sigma) - \gamma(\tau)\| < \epsilon. \tag{6}$$

Consider the probability space $\Omega = (I \times [0, 1])^{\mathbf{N}}$, equipped with the product topology and the infinite product of the uniform distribution over $I \times [0, 1]$. Each point in Ω is a vector $(i_1, \xi_1, i_2, \xi_2, \ldots)$, where i_k is the identity of the chosen player at stage k and ξ_k is the outcome of the coin toss the chosen player uses at stage k.

Given a point $(i_1, \xi_1, i_2, \xi_2, \ldots) \in \Omega$ and a profile σ', one can easily determine the stage in which the game is terminated: if the game has not terminated by stage k, player i_k stops at stage k if and only if $\xi_k \leq \sigma'^{i_k}(h)$, where $h = (i_1, \text{Continue}, \ldots, i_{k-1}, \text{Continue}, i_k)$ is the history up to stage k.

By the definition of τ, for every $(i_1, \xi_1, i_2, \xi_2, \dots)$, the game will terminate under τ no later than it terminates under σ. Moreover, if after the history h the game terminates under τ, while it continues under σ, then we must have

$$\sigma^{i_k}(h) < \xi_k \leq \tau^{i_k}(h).$$

By the definition of τ, the probability that this happens at stage k is at most $\epsilon \mathbf{P}_\sigma(\theta = k \mid \mathcal{H}_k)$.

Therefore, the overall probability that the play under τ is different from the play under σ is at most $\sum_{k\in\mathbf{N}} \epsilon \mathbf{P}_\sigma(\theta = k \mid \mathcal{H}_k) \leq \epsilon$ and the desired result follows.

Step 6. *τ is a subgame-perfect* 4ϵ*-equilibrium.* Fix a player i and a strategy τ'^i of player i. We will prove that $\gamma^i(\tau^{-i}, \tau'^i) \leq \gamma^i(\tau) + 4\epsilon$. The same proof holds for every subgame and hence the result follows.

Define the sequence $(X_k)_{k\in\mathbf{N}}$ as in Step 3: $X_k = a^i_{\mathbf{i}_\theta}$ if $\theta < k$, and $X_k = \gamma^i(\sigma_k)$ otherwise. By the triangle inequality, (5) and (4), one obtains

$$\begin{aligned}\mathbf{E}_{\tau^{-i},\tau'^i}[X_{k+1} \mid \mathcal{H}_k] - X_k &\leq \mathbf{E}_{\sigma^{-i},\tau'^i}[X_{k+1} \mid \mathcal{H}_k] - X_k + \epsilon \mathbf{P}_{\sigma^{-i},\tau'^i}(\theta = k+1 \mid \mathcal{H}_k)\\ &\leq \epsilon \mathbf{P}_{\tau^{-i},\tau'^i}(\theta = k+1 \mid \mathcal{H}_k).\end{aligned} \tag{7}$$

This implies that the process (Y_k) that is defined by

$$Y_k = X_k - \epsilon \sum_{j=0}^{k-1} \mathbf{P}_{\tau^{-i},\tau'^i}(\theta = j+1 \mid \mathcal{H}_j)$$

is a supermartingale.

By the monotone convergence theorem,

$$\gamma^i(\tau^{-i}, \tau'^i) = \lim_{k\to\infty} \mathbf{E}_{\tau^{-i},\tau'^i}[\mathbf{1}_{\theta<k} X_k + \mathbf{1}_{\theta\geq k} a^i_*]. \tag{8}$$

If (τ^{-i}, τ'^i) is terminating, $\lim_{k\to\infty} \mathbf{P}(\theta < k) = 1$, and the right-hand side in (8) is reduced to $\lim_{k\to\infty} \mathbf{E}_{\tau^{-i},\tau'^i}[X_k]$.

If, on the other hand, (τ^{-i}, τ'^i) is not terminating, then the definition of τ implies that i has no punisher. Because i is not a dummy player, $a^i_* \leq 0$. Moreover, as σ is terminating, termination in the last stages is done by player i, and in this case, player i's payoff is 0. In particular, $\lim_{k\to\infty} \mathbf{1}_{\theta\geq k} X_k = 0$.

In both cases, we get that

$$\gamma^i(\tau^{-i}, \tau'^i) \leq \lim_{k\to\infty} \mathbf{E}_{\tau^{-i},\tau'^i}[X_k].$$

By the definition of (Y_k) and because this process is a supermartingale,

$$\begin{aligned}\gamma^i(\tau^{-i}, \tau'^i) &\leq \lim_{k\to\infty} \mathbf{E}_{\tau^{-i},\tau'^i}[X_k]\\ &\leq \lim_{k\to\infty} \mathbf{E}_{\tau^{-i},\tau'^i}[Y_k] + \epsilon \sum_{k=0}^{\infty} \mathbf{P}_{\tau^{-i},\tau'^i}(\theta = k+1)\\ &\leq \mathbf{E}_{\tau^{-i},\tau'^i}[Y_0] + \epsilon\\ &= \mathbf{E}_{\tau^{-i},\tau'^i}[X_0] + \epsilon\\ &= \gamma^i(\sigma) + \epsilon,\end{aligned}$$

as desired. □

Remark 4.1. Actually, in assertion (b), every y in $\overline{Y}$, the closure of Y, is a subgame-perfect 0-equilibrium payoff. Indeed, if a set Y satisfies the conditions of Lemma 4.1, then by a limiting argument, the set $\overline{Y}$ satisfies these conditions as well.

Remark 4.2. One can weaken condition (C.4) in Lemma 4.1. All that is needed is that for every $h \in H$, the profile σ_h that is defined in the proof of Lemma 4.1 is terminating.

4.2. Characterization of stationary equilibrium payoffs. In this section, we prove Propositions 2.1 and 2.2, which characterize stationary equilibrium payoffs in terms of the differential inclusion F.

Proof of Proposition 2.1. If $\vec{0} \in F(w)$, then $\vec{0} = \sum_{i\in I} \rho^i(w - a_i)$ for some $\rho \in \Delta(w)$. But then, $w = \sum_{i\in I} \rho^i a_i / \sum_{i\in I} \rho^i$, which implies that $w = (1/n)\sum_{i\in I}(\rho^i a_i + (1-\rho^i)w)$. The set $Y = \{w\}$, together with $\eta = \max\{\rho^1, \ldots, \rho^n\} \geq 1/n$, satisfies the conditions of Lemma 4.1. $\square$

Proof of Proposition 2.2. Assume that w is an equilibrium payoff that is supported by terminating stationary profiles. That is, for every $\epsilon > 0$, there is a terminating stationary ϵ-equilibrium ρ_ϵ with expected payoff w_ϵ such that $\lim_{\epsilon\to 0} w_\epsilon = w$. In particular, $\sum_{i\in I} \rho^i_\epsilon a_i / \sum_{i\in I} \rho^i_\epsilon = w_\epsilon$. Because by multiplying all the coordinates of ρ_ϵ by a constant larger than 1 one still obtains an ϵ-equilibrium, we can assume w.l.o.g. that $\max\{\rho^1_\epsilon, \ldots, \rho^n_\epsilon\} = 1$ for every $\epsilon > 0$.

By taking a subsequence, we assume w.l.o.g. that the support of $(\rho_\epsilon)_{\epsilon>0}$, that is, the set of players that quit under ρ_ϵ with positive probability whenever chosen, is independent of ϵ.

We will show that any accumulation point ρ of the sequence $(\rho_\epsilon)_{\epsilon>0}$ as ϵ goes to 0 is in $\Delta(w)$. Observe that any such accumulation point satisfies $\sum_{i\in I} \rho^i \geq 1$ and $\sum_{i\in I} \rho^i a_i / \sum_{i\in I} \rho^i = w$, so that one would have $\vec{0} \in F(w)$, as desired.

Case 1. The support of ρ_ϵ contains a single player i for every $\epsilon > 0$.

For every $\epsilon > 0$, $\max\{\rho^1_\epsilon, \ldots, \rho^n_\epsilon\} = 1$ and, hence, $\rho^i_\epsilon = 1$ and $\rho^j_\epsilon = 0$ for every $j \neq i$. Therefore, $w_\epsilon = a_i$ for every $\epsilon > 0$, so that $w = a_i$. Because ρ_ϵ is an ϵ-equilibrium, if player $k \neq i$ quits with probability 1 whenever he is chosen, he gains at most ϵ. Because $a^k_k = 0$, one obtains

$$\frac{1}{2} a^k_i = \gamma^k(\rho^{-k}_\epsilon, 1^k) \leq \gamma^k(\rho_\epsilon) \leq w^k_\epsilon + \epsilon = a^k_i + \epsilon,$$

which implies that $a^k_i \geq -2\epsilon$. Because ϵ is arbitrary, we get $a^k_i \geq 0$ for every $k \neq i$, and by assumption $a^i_i = 0$. This means that the vector ρ that is defined by $\rho^i = 1$ and $\rho^k = 0$ for every $k \neq i$ is in $\Delta(a_i) = \Delta(w)$, as desired.

Case 2. The support of ρ_ϵ contains at least two players for every $\epsilon > 0$.

Fix $\epsilon > 0$. Because $a^j_j = 0$, one has for every $j \in I$,

$$\sum_{i\in I} \rho^i_\epsilon w^j_\epsilon = \sum_{i\in I} \rho^i_\epsilon a^j_i = \sum_{i\neq j} \rho^i_\epsilon a^j_i. \tag{9}$$

Because ρ_ϵ is an ϵ-equilibrium, if j quits with probability 1 whenever he is chosen, he gains at most ϵ:

$$\frac{\sum_{i\neq j} \rho^i_\epsilon a^j_i}{1 + \sum_{i\neq j} \rho^i_\epsilon} = \frac{a^j_j + \sum_{i\neq j} \rho^i_\epsilon a^j_i}{1 + \sum_{i\neq j} \rho^i_\epsilon} = \gamma^j(\rho^{-j}_\epsilon, 1^j) \leq w^j_\epsilon + \epsilon.$$

Incorporating (9) and because $1 + \sum_{i\neq j} \rho^i_\epsilon \leq n$ for every $i \in I$, this yields $-\epsilon n \leq w^j_\epsilon(1 - \rho^j_\epsilon)$, so that

$$w^j < 0 \Rightarrow w^j_\epsilon < 0 \quad \text{for every } \epsilon \text{ sufficiently small } \Rightarrow \rho^j = 1. \tag{10}$$

Similarly, player j cannot gain more than ϵ if he continues whenever he is chosen:

$$\frac{\sum_{i\neq j} \rho^i_\epsilon a^j_i}{\sum_{i\neq j} \rho^i_\epsilon} = \gamma^j(\rho^{-j}_\epsilon, 0^j) \leq w^j_\epsilon + \epsilon. \tag{11}$$

Because the support of ρ_ϵ contains at least two players, $\sum_{i\neq j} \rho^i_\epsilon > 0$ for every $\epsilon > 0$, so that the denominator in (11) is positive. Incorporating (9) and because $\sum_{i\neq j} \rho^i_\epsilon \leq n-1$ for every $i \in I$, this yields $\rho^j_\epsilon w^j_\epsilon \leq \epsilon(n-1)$, so that

$$w^j > 0 \Rightarrow w^j_\epsilon > 0 \quad \text{for every } \epsilon \text{ sufficiently small} \Rightarrow \rho^j = 0. \tag{12}$$

Because $\max\{\rho^1_\epsilon, \ldots, \rho^n_\epsilon\} = 1$, we have $\max\{\rho^1, \ldots, \rho^n\} = 1$ and $\rho \in \Delta(w)$, as desired. $\square$

4.3. Two representation results. Our goal is to prove that every point on a solution w of $\dot{w} \in F(w)$ is a subgame-perfect equilibrium payoff. To be able to apply Lemma 4.1, it is sufficient to show that for every $t_0 \in \mathbf{R}$, one can find $s_1, \ldots, s_n \in \mathbf{R}$ and $\alpha_1, \ldots, \alpha_n \in [0, 1]$ that satisfy

- $w(t_0) = \frac{1}{n} \sum_{i \in I} (\alpha_i a_i + (1 - \alpha_i) w(s_i))$,
- $\alpha_i = 0$ if $w(s_i) > 0$ and $\alpha_i = 1$ if $w(s_i) < 0$, and
- $\min\{\alpha_1, \ldots, \alpha_n\} > \eta$, where η is independent of t_0.

In this section, we use differential equations to represent $w(t_0)$ as a convex combination of (a_i) and $(w(s_i))$ that satisfies the desired properties. This representation is used in the next section to find the proper (s_i)s for any given t_0.

We will need the following simple observation.

Lemma 4.2. *Let f: $[0, \infty) \to \mathbf{R}$ be a measurable function such that $\int_0^t f(u)\,du$ exists for every $t \geq 0$. Then, the unique solution of the differential equation*

$$\begin{cases} x(0) = 0, \\ \dot{x}(t) = f(t) \times (1 - x(t)), \end{cases}$$

is $x(t) = 1 - \exp(-\int_0^t f(u)\,du)$.

Proof. The fact that $x(t) = 1 - \exp(-\int_0^t f(u)\,du)$ is a solution can be verified by substitution. Uniqueness follows, e.g., by Filippov [4, Theorem 1.1.2]. □

We fix throughout this section a solution w of $\dot{w} \in F(w)$ and $t_0 \in \mathbf{R}$. By Lemma 3.3, there is a measurable function ρ: $\mathbf{R} \to [0, 1]^n$ such that for almost every t,

- $\rho(t) \in \Delta(w(t))$ and
- $\dot{w}(t) = \sum_{i \in I} \rho^i(t)(w(t) - a_i)$.

4.3.1. First representation. Let $(\delta^i)_{i \in I}$ be the unique solution of the following system of differential equations:

$$\begin{cases} \delta^i(t_0) = 0 & \forall\, i \in I, \\ \dot{\delta}^i(t) = \left(1 - \sum_{j \in I} \delta^j(t)\right) \rho^i(t) & \forall\, i \in I, t > 0. \end{cases} \tag{13}$$

The existence of a unique solution follows from Lemma 4.2. Indeed, summing up (13) over $i \in I$, we conclude that $\sum_{i \in I} \dot{\delta}^i(t)$ is a solution of the differential equation

$$\begin{cases} x(0) = 0, \\ \dot{x}(t) = (1 - x(t)) \sum_{i \in I} \rho^i(t), \end{cases} \tag{14}$$

so that

$$\sum_{i \in I} \delta^i(t) = 1 - \exp\left(\int_0^t \sum_{i \in I} \rho^i(u)\,du\right), \tag{15}$$

and therefore

$$\delta^i(t) = \int_0^t \rho^i(v) \exp\left(-\sum_{i \in I} \int_0^v \rho^i(u)\,du\right) dv.$$

The interpretation of the functions (δ^i) is the following. Consider the game in continuous time, in which each player i uses the strategy ρ^i. Then, $\delta^i(t)$ is the overall probability that player i quits in the time interval $[t_0, t]$, provided no player quits before time t_0.

The following lemma lists some properties of the functions (δ^i).

LEMMA 4.3. *For every $i \in I$, the function δ^i: $[t_0, +\infty) \to [0, 1)$ is nondecreasing. Moreover, for every $t \geq t_0$, we have the following assertions*:
(A.1) $1 - \exp(-(t - t_0)) \leq \sum_{i \in I} \delta^i(t) \leq 1 - \exp(-n(t - t_0))$,
(A.2) $w(t_0) = \sum_{i \in I} \delta^i(t) a_i + (1 - \sum_{i \in I} \delta^i(t)) w(t)$,
(A.3) *if* $\rho^i(s) \geq \rho^j(s)$ *for every* $s \in [t_0, t)$, *then* $\delta^i(t) \geq \delta^j(t)$, *and*
(A.4) *if* $\rho^i(s) = 0$ *for every* $s \in [t_0, t)$, *then* $\delta^i(t) = 0$.

(A.1) merely says that one can bound (from below and above) the growth rate of the function $\sum_{i \in I} \delta^i(t)$. (A.2) represents $w(t_0)$ as a convex combination of $(a_i)_{i \in I}$ and $w(t)$ with weights that are given by the functions (δ^i). (A.3) and (A.4) say that if (ρ^i) satisfy certain inequalities, so do (δ^i).

PROOF. Assume w.l.o.g. that $t_0 = 0$.

Because $1 \leq \sum_{i \in I} \rho^i(t) \leq n$ for every $t \geq 0$ and by (15), we have

$$1 - \exp(-t) \leq \sum_{i \in I} \delta^i(t) \leq 1 - \exp(-nt) \quad \forall\, t \geq 0.$$

Therefore, (A.1) holds. This implies by (13) that for every $i \in I$, $\dot{\delta}^i(t) \geq 0$ for every $t \geq 0$ and, hence, δ^i is nondecreasing.

We now prove that (A.2) holds; that is, $w(0) = \sum_{i \in I} \delta^i(t) a_i + (1 - \sum_{i \in I} \delta^i(t)) w(t)$ for every $t \geq 0$. It is enough to show that the derivative of the right-hand side vanishes a.e. This derivative is equal to

$$\sum_{i \in I} \dot{\delta}^i(t) a_i - \left(\sum_{i \in I} \dot{\delta}^i(t)\right) w(t) + \left(1 - \sum_{i \in I} \delta^i(t)\right) \dot{w}(t).$$

Because $\dot{w}(t) = \sum_{i \in I} \rho^i(t)(w(t) - a_i)$, this derivative is equal a.e. to

$$\sum_{i \in I} \dot{\delta}^i(t) a_i - \left(\sum_{i \in I} \dot{\delta}^i_t\right) w(t) + \left(1 - \sum_{i \in I} \delta^i(t)\right)\left(\sum_{i \in I} \rho^i(t)\right) w(t) - \left(1 - \sum_{i \in I} \delta^i(t)\right) \sum_{i \in I} \rho^i(t) a_i.$$

Reordering the terms, the derivative is equal a.e. to

$$\sum_{i \in I} a_i \left(\dot{\delta}^i(t) - \left(1 - \sum_{j \in I} \delta^j(t)\right) \rho^i(t)\right) - w(t) \left(\sum_{i \in I} \dot{\delta}^i(t) - \left(1 - \sum_{j \in I} \delta^j(t)\right)\left(\sum_{i \in I} \rho^i(t)\right)\right).$$

The two terms vanish by (13) and (14).

Finally, we show that (A.3) and (A.4) hold as well. If $\rho^i(s) \geq \rho^j(s)$ for every $s \in [0, t)$, then by (13) $\dot{\delta}^i(s) \geq \dot{\delta}^j(s)$ for every $s \in [0, t)$, so that $\delta^i(t) \geq \delta^j(t)$. If $\rho^i(s) = 0$ for every $s \in [0, t)$, then by (13) $\dot{\delta}^i(s) = 0$ for every $s \in [0, t)$, so that $\delta^i(t) = 0$. □

4.3.2. Second representation. Here, we develop a second representation of $w(t_0)$ as a convex combination of $(a_i)_{i \in I}$ and $(w(s_i))_{i \in I}$ that will be used in the next section to prove that a representation of the form $w(t_0) = (1/n) \sum_{i \in I} (\alpha_i a_i + (1 - \alpha_i) w(s_i))$ is possible.

Fix $s_1, s_2, \ldots, s_n \in [t_0, +\infty]$ (some of the s_is may be equal to $+\infty$). For every $t \geq t_0$, set $J_t = \{i \in I \mid s_i \leq t\}$ and $K_t = \{i \in I \mid s_i > t\}$. Then, $t \mapsto J_t$ and $t \mapsto K_t$ are piecewise constant and $J_t \cup K_t = I$ for every $t \geq 0$.

For every $i \in I$, applying Lemma 4.3 to $t_0 = s_i$, one obtains nondecreasing functions $\delta^{i,j}$: $[s_i, \infty) \to [0, 1)$, $j \in I$, that satisfy for every $t \geq 0$,

$$w(s_i) = \sum_{j \in I} \delta^{i,j}(t) a_j + \left(1 - \sum_{j \in I} \delta^{i,j}(t)\right) w(t) \quad \text{and} \quad \sum_{j \in I} \delta^{i,j}(t) < 1. \tag{16}$$

Let $(\alpha^i)_{i\in I}$ be the unique solution of the following system of differential equations:

$$\begin{cases} \alpha^i(t_0)=0 & \forall\, i\in I,\\ \dot\alpha^i(t)=\rho^i(t)\sum_{k\in K_t}(1-\alpha^k(t))+\sum_{j\in J_t}\dot\alpha^j(t)\delta^{i,j}(t) & \forall\, i\in I,\ t>0. \end{cases} \tag{17}$$

The following lemma establishes the existence of a unique solution.

Lemma 4.4. *The system of differential Equations (17) has a unique solution $(\alpha^i)_{i\in I}$ over $[t_0,\infty)$. Moreover, if $\alpha^i(t)<1$ for every $i\in I$, then $\dot\alpha^i(t)\ge 0$ for every $i\in I$.*

Proof. Assume w.l.o.g. that $0\le s_1\le s_2\le\cdots$. Suppose by induction that the functions (α^i) were already defined on the interval $[0,s_l]$. We here define these functions on the interval (s_l,s_{l+1}).

Observe that the sets J_t and K_t are constant over $t\in(s_l,s_{l+1})$.

For every t in the interval (s_l,s_{l+1}), let A_t be the $n\times n$ matrix

$$A_t=\begin{pmatrix} 1-\delta^{1,1}(t)\mathbf{1}_{1\in J_t} & -\delta^{1,2}(t)\mathbf{1}_{1\in J_t} & \cdots & -\delta^{1,n}(t)\mathbf{1}_{1\in J_t}\\ -\delta^{2,1}(t)\mathbf{1}_{2\in J_t} & 1-\delta^{2,2}(t)\mathbf{1}_{2\in J_t} & \cdots & -\delta^{2,n}(t)\mathbf{1}_{2\in J_t}\\ \cdots & & & \\ -\delta^{n,1}(t)\mathbf{1}_{n\in J_t} & -\delta^{n,2}(t)\mathbf{1}_{n\in J_t} & \cdots & 1-\delta^{n,n}(t)\mathbf{1}_{n\in J_t} \end{pmatrix}.$$

Because $\delta^{i,j}$ is nondecreasing and by (A.1), $0\le\delta^{i,j}(t)<1$, and therefore the matrix A_t is regular. Because A_t has the form $I-B$, where all the entries of B are nonnegative and because $(I-B)^{-1}=\sum_{k=0}^{\infty}B^k$, all the entries of the inverse matrix A_t^{-1} are nonnegative.

The system of differential Equations (17) can be written as

$$A_t\dot\alpha(t)=\rho(t)\sum_{k\in K_t}(1-\alpha^k(t)),$$

or equivalently,

$$\dot\alpha(t)=A_t^{-1}\rho(t)\sum_{k\in K_t}(1-\alpha^k(t)).$$

Because all the entries of the matrix A_t^{-1} are nonnegative, $\dot\alpha^i(t)\ge 0$ whenever $\alpha^k(t)\le 1$ for every $k\in K_t$.

Also, $\sum_{k\in K_t}\alpha^k(t)$ is the unique solution of the differential equation

$$x(0)=\sum_{k\in K_t}\alpha^k(s_l),$$
$$\dot x(t)=f(t)(1-x(t)),$$

where f is some measurable function that satisfies $\int_{s_l}^{t}f(u)\,du<+\infty$. The result follows from Lemma 4.2. $\square$

The following lemma lists several properties of the function α. We set

$$T=\min\big\{\max\{s_1,\ldots,s_n\},\min\{t\ge t_0\colon \alpha^1(T)\ge 1,\ldots,\alpha^n(T)\ge 1\}\big\}.$$

Lemma 4.5. *Over the interval $[t_0,T]$, the functions $(\alpha^i)_{i\in I}$ are nondecreasing and satisfy the following properties*:

(B.1) $\alpha^i(t_0)=0$ *and* $\alpha^i(T)\le 1$ *for every* $i\in I$.

(B.2) $w(t_0)=\frac{1}{n}(\sum_{i\in I}\alpha^i(t)a_i+(1-\alpha^i(t))(\mathbf{1}_{s_i\le t}w(s_i)+\mathbf{1}_{s_i>t}w(t)))$ *for every* $t\in[t_0,T]$, $t<+\infty$.

(B.3) $\sum_{i\in I}\alpha^i(t)\geq 1-\exp(-(t-t_0))$ *for every* $t\geq t_0$, *as well as* $\sum_{i\in I}\alpha^i(t)\leq n-\exp(-2n(t-t_0))$ *provided* $0\leq t-t_0\leq -n/\ln(1-1/2n)$.
Moreover, for every $i,j\in I$ *and every* $t\in[t_0,T]$,
(B.4) *If* $\rho^i(s)\geq\rho^j(s)$ *for every* $s\in[t_0,t)$, *then* $\alpha^i(t)\geq\alpha^j(t)$.
(B.5) *If* $\rho^i(s)=0$ *for every* $s\in[t_0,t)$, *then* $\alpha^i(t)=0$.
(B.6) *If* $w^i(t)>0$ *for every* $t\in[t_0,T)$, *then* $\alpha^i_T=0$.
(B.7) *If* $w^i(t)<0$ *for every* $t\in[t_0,T)$ *and* $s_i>T$, *then* $\alpha^i_T=1$.
(B.8) *If* $w^i(t)<0$ *for every* $t\in[t_0,T)$ *and* $s_i=T=+\infty$, *then* $\lim_{t\to\infty}\alpha^i(t)=1$.

The lemma will be used to show that the conditions of Lemma 4.1 hold for Y_w, the range of w. (B.2) will imply that (C.1) in Lemma 4.1 holds, (B.3) will imply that (C.4) holds, while (B.4)–(B.8) will imply that (C.2) and (C.3) hold.

PROOF. Summing (17) over $i\in I$ gives us

$$\sum_{i\in I}\dot\alpha^i(t)=\left(\sum_{i\in I}\rho^i(t)\right)\left(\sum_{i\in K_t}(1-\alpha^i(t))\right)+\sum_{i\in I}\sum_{j\in J_t}\dot\alpha^j(t)\delta^{i,j}(t). \tag{18}$$

Because $\sum_{i\in I}\rho^i(t)\geq 1$ for almost every t, Equation (18) implies that $\sum_{i\in I}\dot\alpha^i(t)\geq 1-\sum_{i\in I}\alpha^i(t)$. Because the solution of the equation $\dot x=1-x$ with initial condition $x(0)=0$ is $x=1-\exp(-t)$, the first claim in (B.3) follows.

Fix $0\leq t\leq -n/\ln(1-1/2n)$. Then, $n(1-\exp(-nt))\leq 1/2$. By Lemma 4.3(A.1), $\sum_{j\in I}\delta^{j,i}(t)\leq 1/2$. Moreover, $\sum_{i\in I}\rho^i(t)\leq n$. Therefore, by (18),

$$\begin{aligned}\frac{1}{2}\sum_{i\in I}\dot\alpha^i(t)&\leq\sum_{i\in J_t}\dot\alpha^i(t)\left(1-\sum_{j\in I}\delta^{i,j}(t)\right)+\sum_{i\notin J_t}\dot\alpha^i(t)\\&=\left(\sum_{i\in I}\rho^i(t)\right)\left(\sum_{i\in K_t}(1-\alpha^i(t))\right)\\&\leq n\sum_{i\in K_t}(1-\alpha^i(t))\\&\leq n\sum_{i\in I}(1-\alpha^i(t)).\end{aligned}$$

Because the solution of the equation $\dot x=2n^2-2nx$ with the initial condition $x_0=0$ is $x=n-\exp(-2nt)$, the second claim in (B.3) follows.

For (B.2) to be satisfied, we need the derivative of the right-hand side in (B.2) to vanish for almost every $t\geq 0$. We show that the derivative vanishes for every t such that $t\notin\{s_1,\ldots,s_n\}$. The derivative of the right-hand side in (B.2), multiplied by n, is

$$\sum_{i\in I}\dot\alpha^i(t)a_i-\sum_{i\in J_t}\dot\alpha^i(t)w(s_i)-\sum_{i\in K_t}\dot\alpha^i(t)w(t)+\left(\sum_{i\in K_t}(1-\alpha^i(t))\right)\dot w(t).$$

Because $\dot w_t=\sum_{i\in I}\rho^i_t(w_t-a_i)$, by (16), and because $J_t\cup K_t=I$, reordering the terms yields

$$\begin{aligned}&\sum_{i\in I}a_i\left(\dot\alpha^i(t)-\sum_{j\in I}\dot\alpha^i(t)\delta^{i,j}(t)-\sum_{j\in K_t}(1-\alpha^j_t)\rho^i(t)\right)\\&\quad-w(t)\left(\sum_{i\in I}\dot\alpha^i(t)-\sum_{j\in I}\sum_{i\in J_t}\dot\alpha^i(t)\delta^{i,j}(t)-\left(\sum_{i\in K_t}(1-\alpha^i(t))\right)\left(\sum_{i\in I}\rho^i(t)\right)\right).\end{aligned}$$

This sum is zero by (17) and (18).

We now show that (B.4) and (B.5) hold as well. If $\rho^i(s)\geq\rho^j(s)$ for every $s\in[0,t)$, then by Lemma 4.3, $\delta^{k,i}(s)\geq\delta^{k,j}(s)$ for every $j\in I$ and every $s\in[s_j,t]$. By (17), $\dot\alpha^i(s)\geq\dot\alpha^j(s)$ for every $s\in[0,t]$, so that $\alpha^i(t)\geq\alpha^j(t)$.

If $\rho^i(s) = 0$ for every $s \in [0, t)$, then by Lemma 4.3, $\delta^{j,i}(s) = 0$ for every $j \in I$ and every $s \in [s_j, t]$. By (17), $\dot{\alpha}^i(s) = 0$ for every $s \in [0, t]$, so that $\alpha^i(t) = 0$.

Finally, we show that (B.6)–(B.8) hold. If $w^i(t) > 0$ for every $t \in [0, T)$, then $\rho^i(t) = 0$ in this range, and (B.6) follows from (B.5).

Fix $i \in I$. If $w^i(t) < 0$ for every $t \in [0, T)$, then $i \in K_t$ and $\rho^i(t) = 1$ in this range. If $s_i > T$, then by the definition of T, there is $j \in I$ such that $\alpha^j(T) = 1$. Because $\rho^i(t) = 1 \geq \rho^j(t)$ for every $t \in [0, T]$, (B.4) implies that $\alpha^i(T) = 1$, so that (B.7) holds. If $s_i = T = +\infty$, then by (17), $\dot{\alpha}^i(t) \geq 1 - \alpha^i(t)$, so that $\alpha^i(t) \geq 1 - \exp(-t)$, and (B.8) holds. $\square$

4.4. From solutions of $\dot{w} \in F(w)$ to subgame-perfect equilibria. Here, we prove Propositions 2.4 and 2.5. Our goal is to apply Lemma 4.1 to the set $Y = Y_w = \{w(t)\colon t \in \mathbf{R}\}$. To this end, we should choose for every $t \in \mathbf{R}$, real numbers $s^1(t), \ldots, s^n(t) > t$ and $\alpha^1(t), \ldots, \alpha^n(t) \in [0, 1]$ such that $\alpha^i(t) = 1$ if $w^i(s^i(t)) < 0$, $\alpha^i(t) = 0$ if $w^i(s^i(t)) > 0$, and

$$w(t) = \frac{1}{n} \sum_{i \in I} (\alpha_i(t) a_i + (1 - \alpha_i(t)) w(s^i(t))).$$

Moreover, the quantity $\max\{\alpha^1(t), \ldots, \alpha^n(t)\}$ should be bounded away from 0, uniformly over $t \in \mathbf{R}$.

The difficult issue is to ensure that $\max\{\alpha^1(t), \ldots, \alpha^n(t)\}$ is uniformly bounded away from 0. Indeed, by Lemma 4.5, for any choice of $s^1(t), \ldots, s^n(t)$, there are $(\alpha^i(t))_{i \in I}$ such that the other conditions are satisfied (with the additional caveat that we will set $s^i(t)$ to T if it happens to be larger than T).

By (B.3), to ensure that $\max\{\alpha^1(t), \ldots, \alpha^n(t)\}$ is uniformly bounded away from 0, it is sufficient to ensure that $\max\{s^1(t) - t, \ldots, s^n(t) - t\}$ is uniformly bounded away from 0.[11] One can show that if w is a solution of type 1, with the naïve definition

$$s^i(t) = \min\{t' > t\colon w^i(t') = 0\},$$

we indeed get a uniform boundedness away from 0. Because this is not the case for solutions of type 0, our definition is more complicated.

Choose once and for all, two constants $0 < \delta_1 < \delta_2 < -n/\ln(1 - 1/2n)$ that satisfy the following:

(D.1) $n - n\exp(1 - \exp(n\delta_2)) < 1$, and

(D.2) $2\delta_1 < \delta_2$.

Fix for a moment, a solution w of $\dot{w} \in F(w)$. For every player $i \in I$, define

$$U_w^i = \{t \in \mathbf{R} \mid w^i(t) = 0\} \subseteq \mathbf{R}.$$

Because $t \mapsto w(t)$ is continuous, U_w^i is closed. Because $a_i^i = 0$, if $t \in U_w^i$, then player i is indifferent between quitting and continuing when the continuation payoff is w_t.

Define for every $i \in I$, a function $s_w^i\colon \mathbf{R} \to (-\infty, \infty]$ by[12]

$$s_w^i(t) = \begin{cases} \min(U_w^i \cap (t, +\infty)), & w^i(t) \neq 0, \\ \min(U_w^i \cap [t + \delta_1, t + \delta_2]), & w^i(t) = 0,\ U_w^i \cap [t + \delta_1, t + \delta_2] \neq \varnothing, \\ \max(U_w^i \cap (t, t + \delta_1]), & w^i(t) = 0,\ U_w^i \cap [t + \delta_1, t + \delta_2] = \varnothing, \\ & \quad \text{and } U_w^i \cap (t, t + \delta_1] \neq \varnothing, \\ \min(U_w^i \cap (t, +\infty)), & w^i(t) = 0,\ U_w^i \cap (t, t + \delta_2] = \varnothing. \end{cases} \tag{19}$$

[11] Actually, by Remark 4.2, a weaker condition will suffice.

[12] By convention, the minimum of an empty set is $+\infty$.

Observe that $s_w^i(t) > t$ and $s_w^i(t) < +\infty$ as soon as there is $u > t$ such that $w^i(u) = 0$. Moreover, if $s_w^i(t) < +\infty$, then $w^i(s_w^i(t)) = 0$. Set

$$M_w(t) = \max_{i\in I} s_w^i(t) - t > 0.$$

LEMMA 4.6. *Let w be a solution of $\dot{w} \in F(w)$ of type* 1. *Then,* $\inf_{t\in\mathbf{R}} M_w(t) > 0$.

PROOF. Assume, to the contrary, that $\inf_{t\in\mathbf{R}} M_w(t) = 0$. Then, there is a sequence $(t(k))_{k\in\mathbf{N}}$ such that $\lim_{k\to\infty} M_w(t(k)) = 0$. We will prove that $w(t(k)) \to \vec{0}$, which implies that $\vec{0} \in \overline{Y}_w \subseteq W$. By Lemma 2.2, $\vec{0} \in F(\vec{0})$, so that w has type 0, a contradiction.

Fix $\epsilon > 0$. Because w is uniformly continuous, there is $\delta < \delta_1$ such that $|u - t| < \delta$ implies $\|w(u) - w(t)\| < \epsilon$. Let k be sufficiently large such that $M_w(t(k)) < \delta$. For every $i \in I$, $w^i(u) = 0$ for some $u \in (t(k), t(k) + M_w(t(k))]$. This implies that $\|w(t(k))\| < \epsilon$, and the claim follows. □

LEMMA 4.7. *Let w be a solution of $\dot{w} \in F(w)$ of type* 0. *For every $\eta \in (0, \delta_1)$ and every $t \in \mathbf{R}$, at least one of the following statements holds*:

(1) $M_w(t) \geq \eta$,
(2) $M_w(s_w^i(t)) \geq \eta$ *for some $i \in I$, or*
(3) $M_w(s_w^i(s_w^i(t))) \geq \eta$ *for every $i \in I$.*

PROOF. Assume that the first statement does not hold; that is, $M_w(t) < \eta < \delta_1$.

We first assume that $w^i(t) = 0$ for some $i \in I$. Because $s_w^i(t) \leq t + M_w(t) < t + \delta_1$, it follows by the definition of $s_w^i(t)$ that $U_w^i \cap [t + \delta_1, t + \delta_2] = \varnothing$ and $U_w^i \cap (t, t + \delta_1] \neq \varnothing$. Because $2\delta_1 < \delta_2$, $U_w^i \cap (t + s_w^i(t), t + s_w^i(t) + \delta_1] = \varnothing$, so that $s_w^i(s_w^i(t)) \geq s_w^i(t) + \delta_1$, and the second statement holds.

Assume now that $w^i(t) \neq 0$ for every $i \in I$. If the second statement does not hold, then for every $i \in I$, $s_w^i(s_w^i(t)) < t + \eta$, so that $w^i(s_w^i(t)) = 0$. Applying the second paragraph to $s_w^i(t)$ rather than to t, one deduces that the third statement holds. □

PROOF OF PROPOSITION 2.4. Let w be a solution of $\dot{w} \in F(w)$ of type 1. We show that the set $\overline{Y}_w$ satisfies the conditions of assertion (b) in Lemma 4.1.

Because w has type 1, for every i with $a_i \in \overline{Y}_w$, there is $j \in I$ such that $a_i^j < 0$. Indeed, otherwise, there is $i \in I$ with $a_i \in \overline{Y}_w$ such that $a_i^j \geq 0$ for every $j \in I$. By Lemma 2.2, $\vec{0} \in F(a_i)$, so that w has type 0, a contradiction.

Let $y \in Y_w$. Assume w.l.o.g. that $y = w(0)$. By Lemma 4.5, there are $T > t$ and weakly increasing functions $\alpha_t^1, \ldots, \alpha_t^n$ such that for every $t \in [0, T]$, $t < +\infty$,

$$y = w(0) = \frac{1}{n}\sum_{i\in I}\big(\alpha^i(t)a_i + (1 - \alpha_t^i)\big(\mathbf{1}_{s_w^i(0)\leq t}\, w(s_w^i(0)) + \mathbf{1}_{s_w^i(0)>t}\, w(t)\big)\big).$$

If $T < +\infty$, set $y_i = w(\min\{s_w^i(t), T\})$ and $\rho^i = \alpha^i(T)$ for every $i \in I$. If $T = +\infty$, take a sequence $(t_k)_{k\in\mathbf{N}}$ that converges to infinity such that the sequence $(w(t_k))$ converges to w_*. Then, $w_* \in \overline{Y}_w$. Denote $\rho^i = \lim_{k\to\infty} \alpha^i(t_k)$ for $i \in I$. The limits exist because each α^i is weakly increasing and bounded by 1. (C.1) then holds.

We now show that (C.2) and (C.3) hold as well. Fix $i \in I$. Assume first that $s_w^i(t) \geq T$. By the definition of T, $\alpha^j = 1$ for some $j \in I$. By (D.1), $\delta_2 < T$, so that $\delta_2 < s_w^i(t)$. By the definition of $s_w^i(t)$, $w_u^i \neq 0$ for every $u \in (t, s_w^i(t))$. By Lemma 4.5(B.6, B.7, B.8), one of the following statements holds: (i) $w^i(u) > 0$ for every $u \in (t, T]$, in which case $y_i^i = w^i(T) > 0$ and $\alpha^i = 0$; (ii) $w^i(u) < 0$ for every $u \in (t, T]$, in which case $y_i^i = w^i(T) < 0$ and $\alpha^i = 1$; (iii) $T = +\infty$ and $w^i(u) < 0$ for every $u \in (t, +\infty)$, in which case $y_i^i = w_*^i \leq 0$ and $\alpha^i = 1$; or (iv) $T = +\infty$ and $w^i(u) > 0$ for every $u \in (t, +\infty)$, in which case $y_i^i = w_*^i \geq 0$ and $\alpha^i = 0$. Assume now that $s_w^i(t) \leq T$. Then, $y_i^i = w^i(s_w^i(t)) = 0$, and (C.2) and (C.3) trivially hold.

By Lemma 4.6, $\max\{\alpha^1, \ldots, \alpha^n\} \geq \inf_{t\in\mathbf{R}} M_w(t) > 0$, which is independent of t, and (C.4) follows from Lemma 4.5(B.3). □

Proof of Proposition 2.5. The proof is similar to the proof of Proposition 2.4, but instead of applying assertion (b) in Lemma 4.1, we apply assertion (a) in that lemma. Recall that in the proof of Lemma 4.1, (C.4) followed from Lemma 4.6. Unfortunately, when w has type 0, Lemma 4.7 does not give us (C.4). However, by Remark 4.2, to apply Lemma 4.1, it is sufficient to prove that the profile σ we constructed in the proof of Lemma 4.1 is terminating. This fact follows from Lemma 4.7 and Lemma 4.5(B.3). □

Example 2.2 (Continued). A graphic representation of the differential inclusion $\dot{w} \in F(w)$ shows that it has a unique periodic solution (up to time shifts), and the range of this solution coincides with the edges of the triangle that is defined by $\{(1,0,0), (0,1,0), (0,0,1)\}$. Observe that this set is the analog of the set of equilibrium payoffs in the game studied by Flesch et al. [6].

One subgame-perfect 0-equilibrium that is generated by the procedure we used in the proof is the last one we described in §2.2. In fact, all subgame-perfect 0-equilibria that are generated by the procedure we used in the proof coincide with that equilibrium from Stage 2 and on.

If one modifies the definition of $s^i_w(t)$ in the last case to t (rather than $\min(U^i_w \cap (t, +\infty))$), Lemma 4.6, and therefore Proposition 2.4, are still valid, and the generated subgame-perfect 0-equilibrium is the periodic profile with period 6 that was presented in §2.2.

5. Extensions and open problems. The proof we provided here is valid with minor modifications when the probability distribution over I, according to which players are chosen at every stage, is not the uniform distribution but any distribution $p = (p^i)_{i\in I}$. Indeed, the definition of F becomes

$$F(w) = \left\{ \sum_{i\in I} p^i \rho^i (w - a_i) \;\middle|\; \rho \in \Delta(w) \right\},$$

and from that point on, every appearance of ρ^i is changed to $p^i\rho^i$.

We have proven here the existence of a stationary ϵ-equilibrium or a subgame-perfect 0-equilibrium. However, in all the examples the author analyzed in which there is a subgame-perfect 0-equilibrium, there is one that is supported by a *pure Markovian* profile. If this observation is true in general, this might have significant implications on the study of stochastic games and Dynkin games.

It is also not clear whether there is a *periodic* solution w of the differential inclusion $\dot{w} \in F(w)$, and whether for every solution w, the corresponding function ρ is piecewise continuous or not.

The model we have studied is stationary, in the sense that the probability by which a player is chosen and the terminal payoffs are fixed throughout the game. What happens when this is not the case is not known. The simplest case, which we do not know how to analyze is the following. The players are partitioned into two subsets. At odd stages, a player from one subset is chosen according to the uniform distribution, while at even stages, a player from the other subset is chosen according to the uniform distribution.

Another generalization of the model we studied is to allow players to quit simultaneously. This class of games, termed *quitting games*, was studied by Solan and Vieille [21], where partial results were reported.

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