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Lecture Notes 1: Vickrey-Clark-Groves Auction

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1 Introduction

The course is based on the book Approximation in Economic Design by Jason Hartline. Under the field of Game Theory we explore the branch of Mechanism Design and specifically the area of Auctions and Single Dimensional Auctions.

2 Social Choice Function

Let us define a game as having a set of agents (aka players or bidders) and let A be a set of alternatives, that is different outcomes for the game.

Let T_i be a set of types for player i.
T_i is basically a set of functions, $t_i \in T_i$ is a mapping $t_i : A \to \mathcal{R}$.
For each agent i, $\tilde{t_i}$ is the reported value function.

An outcome that maximizes everybody's 'happiness', or the social welfare, can be defined as:

$$a^* = \arg\max_{a \in A} \sum_{i=1}^{n} t_i(a)$$

where t_i is the true type for each agent i.

3 VCG Mechanism

In a VCG mechanism the best strategy for each player is to report their true type.
The mechanism works as follows:

1. Ask agents for their types: $\tilde{t_1}, \tilde{t_2}, ..., \tilde{t_n}$
2. Choose $\tilde{a} = \arg\max_{a \in A} \sum \tilde{t_1}(a)$
3. Pay agent i a 'bribe' of $\sum_{j \neq i}^{n} \tilde{t_j}(\tilde{a}) + h_i(\tilde{t_{-i}})$.

Where $t_{-1} = (t_1, t_2, ...t_{i-1}, t_{i+1}, ...t_n)$.
The 3-rd step is the addition that turns the mechanism into a VCG.

Let us examine the case of 2 players:

$$u_1 = t_1(\tilde{a}) + \tilde{t_2}(\tilde{a})$$
$$u_2 = t_2(\tilde{a}) + \tilde{t_1}(\tilde{a})$$

With these settings the dominant strategy for each player is to report their true value. We can examine the perspective of player 1 WLOG:
Case a: Report t_1
Case b: Report $\tilde{t_1} \neq t_1$

For case a we get:

$$\tilde{a^*} = \arg\max_{a \in A} t_1(a) + \tilde{t_2}(a)$$
$$u_1 = t_1(\tilde{a^*}) + \tilde{t_2}(\tilde{a^*})$$

For case b we get:

$$\tilde{a} = \arg\max_{a \in A} \tilde{t_1}(a) + \tilde{t_2}(a)$$
$$u_1 = t_1(\tilde{a}) + \tilde{t_2}(\tilde{a})$$

Case a is a dominant strategy since $\tilde{a}$ maximizes the utility function, h_i depends only on what the other players do and so does not change the dominance of the strategy for player i.

Let

$$OPT(t) = \arg\max_{a \in A} \sum_i t_i(a)$$
$$OPT_{-k}(t) = \sum_{j \neq k} t_j(OPT(t))$$

Note that $\sum_{j \neq i} \tilde{t_j}(\tilde{a}) = OPT_{-i}(\tilde{t})$

The VCG payments to agent i is $OPT_{-i}(\tilde{t}) + [h_i(\tilde{t_{-i}})]$ where $h_i(\tilde{t_{-i}}) \equiv -OPT(\tilde{t_{-i}})$
In other words the payment to agent i is:

$$OPT_{-i}(\tilde{t}) - OPT(\tilde{t_{-i}})$$

That is the welfare of every one in the optimal solution minus the cost that each player incurs to society by simply playing the game, or 'waking up in the morning'.
These payments are called CPP: Clarke Pivot Payments.
CPP is the only type of payments that satisfies:
a. No positive transfers - the mechanism never pays the agents.
b. Individually rational

Example 1: Bandwidth allocation.
Given a weighted graph representing a network with different bandwidths and a list of bids for connections (source, destination, required bandwidth), we can ask how much should we charge?
The VCG determines allocation according to what will be most profitable, but finding this allocation can not be done in polynomial time. One of the Problems with VCG is the inability to modify it for using an approximation algorithm.

Example 2: The Town Bridge.
A situation where the people of a town need to decide wether to spend 10 million for building a bridge or not is an example of a game that has a concept of social cost. The cost of the bridge comes at the expense of something else but is not payed directly by the players. CPP don't work in a case like this.

4 Single Dimensional Auctions

We have n agents and we define a 'service'. Every agent has a value for this service: $v_1 ... v_n$.
$v_i \leftarrow \mathcal{R}$ is the value of the service for agent i.
Let $S \subset \{0,1\}^n$ be the subset of agents that can be served.
We want to design a truthful mechanism to decide which agents get served and for what price. VCG will maximize social welfare (but not revenue) and it might also not be polynomial.

An alternative interpretation of CPP is:
We choose $S^* = \arg\max_{s \in S} \sum_{s_i = 1} v_i$

Let c_i be the critical value for agent i. That is the value where s^* changes from i not getting served to i getting served.

If we fix v_{-i} we get:
if i reports v_i: i gets service
if i reports $c_i + \epsilon$: i gets service
if i reports $c_i - \epsilon$: i does not gets service

i can't be charged more than c_i and c_i is the CPP.