Auction Design	Winter Semester, 2012/2013
Lecture 3: November 13	
Lecturer: Amos Fiat	*Scribe: Ofir Chen*

3.1 A Short Rehearsal

On Lecture 2 we have dealt with complete and incomplete information games. We have defined BNE, DSE, and terms "best response" and "common prior". Today we will use those terms to talk about Bayesian games and Single–Dimensional game, and analyze the agent's strategies and payment in those settings. This lecture is parallel to the course's book chapter 2.4-2.5.

3.2 Bayesian Games

Recall that Bayesian games deal with agent's best responses given other agents' strategies, over distribution F of possible agents' types. In Bayesian games it is important to distinguish between 3 different stages of the games:

- **Ex-ante**: before agents know their types.
- **Interim**: after agents know their types, but not other agents' types. note that at this stage, every agent assumes

$$\underline{v_i} = [v_1, \ldots, v_{i-1}, v_{i+1}, , \ldots, v_n] \sim F_{-i}|v_i.$$

- **Ex-post**: after the game has been played. At this stage, the game's outcome (which we will define later) is already known.

3.2.1 Single–Dimensional Games

A Single–Dimensional game is a Bayesian game with the following settings:

- n Agents. Each agent i has a value v_i for the service, which can be positive or negative.

- F is a prior common distribution of types $\underline{t}$, and it's common knowledge.
- Agent's Actions: for now actions are only bids.
- Vector $\underline{b}$ such that every agent i has a bid b_i. Note that b_i can be different than v_i or t_i.
- Game's outcome is a combination of vectors $\underline{X}$ and $\underline{P}$ such that:
  - $\underline{X}$ outcome vector : Each $x_i \in \{0, 1\}$ indicates if agent i got service. Before results are known, x_i is the probability that agent i got service and $x_i \in \{0, 1\}$.
  - $\underline{P}$ payments vector: Each p_i denotes the payment agent i pays to the mechanism. p_i can be positive or negative. Note that the payment p_i is associated with agent i, whether agent i got service or not.
- Game: a Game G maps agents' actions $\underline{b}$ to an outcome and payment:
  - $x_i^G(\underline{b})$ = outcome to i when actions are $\underline{b}$.
  - $p_i^G(\underline{b})$ = payment from i when actions are $\underline{b}$.
- Strategies Profiles: Vector $\underline{s} = (s_1, s_2, \ldots, s_n)$ such that s_i is agent i's strategy (recall that s_i is a function form types to actions. Specifically here $s_i : v_i \rightarrow b_i$).

  We say that a strategy s_i is **onto** if every action b_i is the outcome of s_i applied to some value v_i.

  Formally: $\forall b_i \; \exists v_i$ s.t. $s_i(v_i) = b_i$. We say that a strategy profile is onto if for every i, s_i is onto.

3.2.2 The Bayesian Single–Dimensional Game

We will now examine the Single–Dimensional Game through the Bayesian game stages:

- **Ex-ante**: from the game's analysis point of view:
  - Allocation rule: $x_i(\underline{v}) = x_i^G(s(\underline{v}))$. x_i is the probability that agent i will get service.

  - Payment rule: $p_i(\underline{v}) = p_i^G(s(\underline{v}))$. p_i is agent i's expected payment. Note that for both cases $s(\underline{v}) = [s_1(v_1), \ldots, s_n(v_n)]$.

  Notations: Game G and strategy vector S are implicit when using the terms $x_i(\underline{v})$ and $p_i(\underline{v})$.

- **Interim**: from agent i's point of view: agent i already knows her type.

  - $x_i(v_i) = Pr[x_i(v_i) = 1|v_i] = E[x_i(v)|v_i]$: x_i is the probability that agent i got service given v_i.
  - $p_i(v_i) = E[p_i(\underline{v})|v_i]$

  We can furthermore calculate agent's utility:

  - $u_i(v_i) = v_i x_i(v_i) - p_i(v_i)$

  Note that at this stage we don't know v, so we assume

$$\underline{v}_{-i} = [v_1, \ldots, v_{i-1}, v_{i+1}, , \ldots, v_n] \sim F_{-i}|v_i,$$

  and calculate the probabilities and expected values.

- **Ex-post**:

  - $x_i(\underline{v}) \in \{0, 1\}$
  - $p_i(\underline{v})$ - the final payment of agent i.

3.3 Analyzing the BNE

3.3.1 BNE for Single–Dimensional game

Lemma 3.1 *For a Single–Dimensional game G and a common prior F, an onto Strategy profile s is in BNE iff for all i, v_i, and z:*

$$v_i \cdot x_i(v_i) - p_i(v_i) \geq v_i \cdot x_i(z) - p_i(z).$$

Informally, that means that if there exists a strategy s^* that gives z, then s^* is not better than our strategy s.

Proof's scheme: $\Rightarrow$ assume: $\forall \underline{v}, i : s(v_i)$ is the best response to $s_{-i}(\underline{v}_{-i})$ where $\underline{v}_{-i} \sim F_{-i}|v_i$. Therefore:

$$\begin{aligned} v_i \cdot x_i(v_i) - p_i(v_i) &= v_i \cdot x_i(s_i(v_i), s_{-i}(\underline{v}_{-i})) - p_i(s_i(v_i), s_{-i}(\underline{v}_{-i})) \\ &\geq v_i \cdot x_i(s'(v_i), s_{-i}(\underline{v}_{-i})) - p_i(s'_i(v_i), s_{-i}(\underline{v}_{-i})) \\ &= v_i \cdot x_i(s_i(z), s_{-i}(\underline{v}_{-i})) - p_i(s_i(z), s_{-i}(\underline{v}_{-i})). \end{aligned}$$

Where the last transition is because s_i is onto. Recall that when s_i is onto, $\forall v, s'_i \; \exists z \mid s'_i(v) = s_i(z)$. Finally, we got: $v_i \cdot x_i(v_i) = p_i(v_i) \geq v_i \cdot x_i(z) - p_i(z)$.

$\Leftarrow$ Now assume: $\forall v, i \; \exists z \mid v_i \cdot x_i(v_i) = p_i(v_i) \geq v_i \cdot x_i(z) - p_i(z)$ and we will prove that $\forall v, i \; s(v_i)$ is the best response to $s_{-i}(v_{-i})$ where $v_{-i} \sim F_{-i}|v_i$.

$$\begin{aligned} v_i \cdot x_i(s_i(v_i), s_{-i}(\underline{v}_{-i})) - p_i(s_i(v_i), s_{-i}(\underline{v}_{-i})) &\geq \\ v_i \cdot x_i(s_i(z), s_{-i}(\underline{v}_{-i})) - p_i(s_i(z), s_{-i}(\underline{v}_{-i})) &= \\ v_i \cdot x_i(s'_i(v_i), s_{-i}(\underline{v}_{-i})) - p_i(s'_i(v_i), s_{-i}(\underline{v}_{-i})). & \end{aligned}$$

Recall that $s'_i(z)$ can be anything because s'_i is onto, and $s_i(v_i)$ can be anything. Therefore s_i is the best strategy.

3.3.2 Agent's Payment in BNE

Theorem 3.2 *When values are drawn from a continuous joint distribution $F : G$, s, and F are in BNE iff for all i:*

1. *(monotonicity) $x_i(v_i)$ is monotone non-decreasing.*
2. *(payment identity) $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$.*

We will often simply ignore $p_i(0)$, i.e. $p_i(0) = 0$.

Proof Let's examine Figure 3.1 to understand this theorem. We will call our agent Alice.

$\Rightarrow$ Let's assume that x_i is monotonic and that the price is defined as above, and prove a BNE: x_i is given and monotonically increasing. Let's assume Alice deviates from her value and bids v'. The utility difference Alice will suffer is denoted as $u(v, v') = v \cdot x(v') - p(v')$. It's easy to see from Figure 3.1 that $\forall v'$, it holds that $u(v, v) > u(v, v')$: On the left bottom side we can see Alice's utility given $v' = v$. In the middle $v' < v$ and to the right, $v' > v$. Note that R—the red upper part of the rightmost graph—is negative

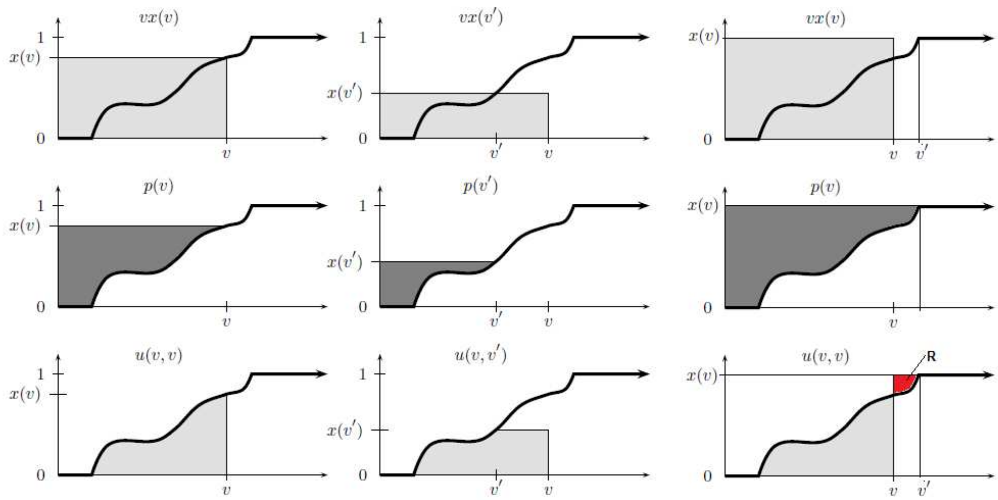

Figure 3.1: The utilities and payments in case Alice deviates from her true value.

utility therefore, the total utility is $u(v,v)-R$. The behavior is assured to be consistent since x_i is monotonic. Since the utility is maximized, the strategy is a BNE over F.

$\Leftarrow$ Let's assume a BNE, and prove the conditions. Since it's a BNE, we can write: $\forall z_1, z_2$:

$$z_2 \cdot x(z_2) - p(z_2) \geq z_2 \cdot x(z_1) - p(z_1), \tag{3.1}$$

as well as:

$$z_1 \cdot x(z_1) - p(z_1) \geq z_1 \cdot x(z_1) - p(z+2). \tag{3.2}$$

We can now subtract Equations (3.2) from (3.1) and get:

$$(z_2 - z_1)(x(z_2) - x(z_1)) \geq 0,$$

meaning that if $z_2 > z_1$ then $x(z_2) > x(z_1)$ and if $z_1 > z_2$ then $x(z_1) > x(z_2)$, which is the definition of monotonicity. From Equations (3.2), (3.1), we can also get the following:

$$z_2 \cdot (x(z_2) - x(z_1)) \geq p(z_2) - p(z_1) \geq z_1 \cdot (x(z_2) - x(z_1)) \tag{3.3}$$

From Equation 3.3, we can measure the amount of change in payment between z_1 and z_2, and it exactly defines the integral operand. Figure 3.2 describes the delta in the payment for every delta in z.

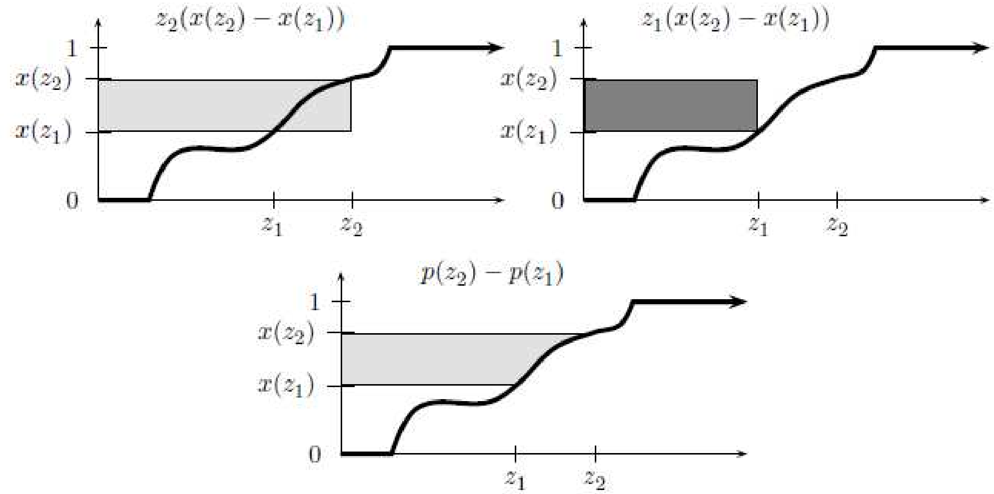

Figure 3.2: The payment's delta given the value's delta.

3.3.3 Examples

Theorem 3.2 shows that We can see the product value equals the sum of the payment and the utility. Let's examine the function x_i for examples we have seen in the past:

Example One object, 2nd price auction, 2 agents who's values distributes $U[0, 1]$

Example One object, 1st price auction, 2 agents who's values distributes $U[0, 1]$.

On both cases calculating x_i is easy since the strategy does not change $x_i(v)$. The probability to win the object is the same as the agent's value when discussing the 2 agents setting (see Figure 3.3).

Example Consider the following auction: agents will pay what the bid, whether they won or not.

At this case, the expected price should be the same as the price from the previous example. The price is the gray area on the above graph, which can be calculated as $v^2/2$.

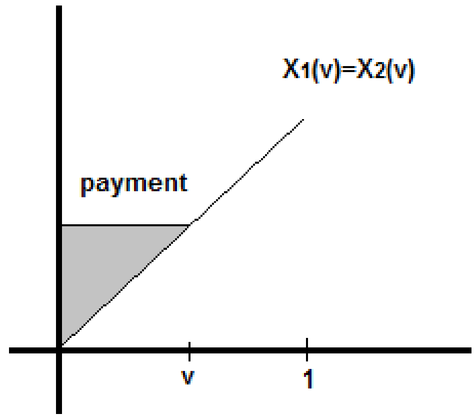

Figure 3.3: Examples 3.3.3 probabilities functions.

Example Let's get back to the 1st price auction. let's calculate the expected price x_i, when we know that the expected payment is $v_i^2/2$:
$x_i = \Pr[i \text{ wins}] \cdot s_i(v_i) + 0 \cdot \Pr[i \text{ loses}] = v_i \cdot s_i(v_i) = v_i^2/2 \rightarrow s_i(v_i) = v_i/2$
we proved that for those settings the best strategy would be "bid $v/2$", like seen in Lecture 2.

Example Consider the 2nd price auction with n agents. let's calculate the expected price :

$$p_i = \Pr[i \text{ wins}] \cdot (\text{second highest price}) = v_i^{n-1} \cdot \frac{n-1}{n} \cdot v.$$

So what is the best strategy here?

$$\Pr[i \text{ wins}] \cdot s_i(v_i) = v_i^{n-1} \cdot s_i(v_i) = v^n \cdot \frac{n-1}{n} \rightarrow s_i(v_i) = v_i \cdot (1 - \frac{1}{n}).$$