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Self-focusing in the complex Ginzburg–Landau limit of the critical nonlinear Schrödinger equation

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Abstract

We analyze self-focusing and singularity formation in the complex Ginzburg–Landau equation (CGL) in the regime where it is close to the critical nonlinear Schrödinger equation. Using modulation theory [Fibich and Papanicolaou, Phys. Lett. A 239 (1998) 167], we derive a reduced system of ordinary differential equations that describes self-focusing in CGL. Analysis of the reduced system shows that in the physical regime of the parameters there is no blowup in CGL. Rather, the solution focuses once and then defocuses. The validity of the analysis is verified by comparison of numerical solutions of CGL with those of the reduced system. © 1998 Elsevier Science B.V.

1. Introduction

The Complex Ginzburg–Landau equation (CGL)

$$\mathrm{i}\psi_t + \Delta\psi + |\psi|^2\psi - \mathrm{i}\epsilon_1\psi - \mathrm{i}\epsilon_2\Delta\psi + \mathrm{i}\epsilon_3|\psi|^2\psi = 0$$

arises in a variety of physical problems – models of chemical turbulence, analysis of Poiseuille flow, Rayleigh–Bérnard convection and Taylor–Couette flow. Its name comes from the field of superconductivity, where it models phase transitions of materials between superconducting and non-superconducting phases (see, e.g., Refs. [1,2] and references therein).

When $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, CGL formally reduces to the nonlinear Schrödinger equation (NLS) which, for spatial dimension $d \geqslant 2$, has solutions that become singular in a finite time. Since CGL solutions are believed to exist globally, one can draw an analogy between NLS as the weak limit of CGL and the Euler equations as the weak limit of the Navier–Stokes equations (see, e.g., Ref. [3]). In Ref. [4] it was suggested that in a periodic d-dimensional domain the CGL equation, considered as a perturbation of the NLS equation, can provide a model for the study of "turbulent" solutions of partial differential equations. This approach was further studied in Ref. [3].

In this work we consider the relation between the two-dimensional CGL on an infinite domain

$$\mathrm{i}\psi_t(t,x,y) + \Delta\psi + |\psi|^2\psi - \mathrm{i}\epsilon_1\psi - \mathrm{i}\epsilon_2\Delta\psi + \mathrm{i}\epsilon_3|\psi|^2\psi = 0 \tag{1}$$

and the 2D critical NLS

$$\mathrm{i}\psi_t(t,x,y) + \Delta\psi + |\psi|^2\psi = 0\,, \tag{2}$$

where from now on

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$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (x, y) \in R^2, \quad t \geqslant 0.$$

Our analysis is restricted to the behavior of solutions of CGL in the regime where it is close to NLS, i.e.

$$|\epsilon_1| \ll 1, \quad |\epsilon_2| \ll 1, \quad |\epsilon_3| \ll 1. \tag{3}$$

We "self-focus" on initial conditions

$$\psi(0, x, y) = \psi_0(x, y)$$

that lead to finite-time singularity in NLS and analyze the behavior of CGL solutions with these initial conditions. While it is difficult to understand the dynamics of these CGL solutions from direct rigorous analysis of CGL, when (3) holds we can formally treat CGL as a perturbation of NLS and analyze it using *modulation theory*, which is a perturbation method that simplifies the analysis of the effect of small perturbations on self-focusing in the critical NLS [5,6].

The Letter is organized as follows. We begin in Section 2 by briefly reviewing self-focusing theory for the critical NLS and state in Proposition 2.1 the main result of modulation theory. In Section 3, modulation theory is applied to the CGL model, resulting in a reduced system of ordinary-differential equations (16) which describes the leading order behavior of self-focusing in CGL. Analysis of this system in Section 4 shows that when $\epsilon := \epsilon_2 + 2\epsilon_3 > 0$ there is no singularity formation in CGL. Instead, there is a single focusing–defocusing event and the maximum growth of the solution has a bound which is exponentially large in ϵ^{-1} (Eq. (26)). In the regime $\epsilon_2 + 2\epsilon_3 < 0$ solutions can blow-up in finite time, which is smaller than the blow-up time for the corresponding NLS equation. In Section 5 we present numerical simulations that demonstrate the agreement between CGL and the reduced system (16).

2. Review of self-focusing theory

We begin with a brief review of self-focusing theory for the critical NLS (2). For more details see Ref. [5].

A conserved quantity for solutions of NLS is the power [3]

$$N(z) = \frac{1}{2\pi} \int |\psi|^2 \, dx \, dy \equiv N(0),$$

which is of a particular importance due to its connection with singularity formation in NLS. A necessary condition for singularity formation in (2) is that the initial power exceeds the critical value N_c:

$$N(0) \geqslant N_c \simeq 1.86.$$

To proceed, we construct waveguide solutions of NLS of the form

$$\psi = \exp(it) R(r), \quad r = (x^2 + y^2)^{1/2},$$

where the radial function $R(r)$ satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) R - R + R^3 = 0, \quad R'(0) = 0,$$
$$\lim_{r \to \infty} R(r) = 0. \tag{4}$$

The solution of (4) with the lowest power ("ground state"), the so-called *Townes soliton*, has an important role in self-focusing theory. This positive, monotonically decreasing solution has exactly the critical power for self-focusing and it satisfies

$$\int_0^\infty R^2 \, r \, dr = \int_0^\infty (\nabla R)^2 \, r \, dr = \tfrac{1}{2} \int_0^\infty R^4 \, r \, dr = N_c. \tag{5}$$

Analysis of self-focusing in NLS is based on the assumption (which is supported by numerical and analytical evidence) that in the vicinity of the singularity, the solution is roughly a modulated Townes soliton,

$$\psi \sim \psi_R,$$

where

$$\psi_R := \frac{1}{L(t)} R(\rho) \exp(iS), \quad \rho = \frac{r}{L},$$
$$S = \tau + \frac{L_t}{L}\frac{r^2}{4}, \quad \frac{d\tau}{dt} = \frac{1}{L^2}. \tag{6}$$

More precisely, near the singularity, ψ_s, the inner part of the solution [4] whose power is slightly above critical,

[3] In the context of nonlinear optics, the L^2 norm is related to the power of the laser beam.

[4] A possible definition is $\psi_s = \psi$ for $0 \leqslant \rho \leqslant \rho_c$, with $1 \ll \rho_c$ constant.

collapses towards the singularity in a quasi-self-similar fashion,

$$\psi_s \sim \frac{1}{L} V(\tau, \rho) \exp(\mathrm{i}S) ,$$

where

$$V \to R \quad \text{as} \quad t \to T_c$$

and T_c is the blowup time. Blowup, of course, corresponds to

$$\lim_{t \to T_c} L(t) = 0 .$$

Based on the modulation ansatz ψ_R, it was shown that near the singularity self-focusing can be described by the reduced system [7–9]

$$\beta_t = -\frac{\nu(\beta)}{L^2} , \tag{7}$$

$$L_{tt} = -\frac{\beta}{L^3} , \tag{8}$$

where

$$\nu(\beta) \sim c \exp\left(-\frac{\pi}{\beta^{1/2}}\right) , \quad c \simeq 45.1 .$$

In the context of nonlinear optics, the modulation variable L is the transverse width as well as 1/amplitude of the focusing part of the laser beam ψ_s and β is proportional to the excess power above critical of ψ_s [5,10],

$$\beta(z) \sim \frac{N(\psi_s) - N_c}{M} , \tag{9}$$

where

$$M = \tfrac{1}{4} \int_0^\infty r^2 R(r)\, r \, \mathrm{d}r \simeq 0.55 .$$

The $\nu(\beta)$ term arises from radiation effects (power losses of ψ_s to the background) during self-focusing.

Originally, the reduced system (7), (8) was used to derive the *log–log law* for the blowup rate of critical NLS [7–9]. Later, it turned out that the log–log law is not valid even after L becomes as small as 10^{-10} [11,5]. However, since near the singularity

$$0 \leqslant \beta(t) \ll 1 ,$$

$\nu(\beta)$ is exponentially small and self-focusing is essentially *adiabatic*, following

$$-L^3 L_{tt} \sim \beta(0) . \tag{10}$$

This relation leads to the *adiabatic law* for critical collapse which is valid in the domain of physical interest [11,5].

2.1. Modulation theory for the perturbed critical NLS

The NLS equation in spatial dimension D and with nonlinearity $|\psi|^{2\sigma}\psi$ is called *critical* when $\sigma D = 2$. Self-focusing in critical NLS has unique features, being the borderline case between *subcritical* NLS ($\sigma D < 2$) where the focusing nonlinearity cannot dominate over diffraction and all solutions exists globally, and the *supercritical* NLS ($\sigma D > 2$) where the focusing nonlinearity can dominate over diffraction, in which case the singularity formation is relatively insensitive to small perturbations in the equation. As a result, a unique feature of self-focusing in the critical NLS (see, e.g., Eq. (2), where $\sigma = 1$ and $D = 2$) is that addition of small perturbations to the equation can have a large effect on the formation of singularity. Therefore, analysis of self-focusing in equations of the form

$$\mathrm{i}\psi_t + \Delta\psi + |\psi|^2\psi + \epsilon F(\psi, \psi_t, \nabla\psi, \ldots) = 0 , \quad |\epsilon| \ll 1 , \tag{11}$$

requires a delicate perturbation method which is based on the modulation ansatz ψ_R (6). Recently, such a method, called *modulation theory*, was developed in [5,6] and its main result is the following.

Proposition 2.1. If

$$\psi \sim \psi_R , \quad |\beta(t)| \ll 1$$

and the perturbation is small

$$|\epsilon F| \ll |\Delta\psi| \quad \text{and} \quad |\epsilon F| \ll |\psi|^3 ,$$

then self-focusing in the perturbed NLS (11) is given to a leading order by the reduced system

$$\beta_t(t) + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_t - \frac{2\epsilon}{M} f_2 , \quad L_{tt}(t) = -\frac{\beta}{L^3} . \tag{12}$$

The auxiliary functions f_1 and f_2 are given by

$$f_1(t) = 2L(t)\,\mathrm{Re}\left[\int F(\psi_R)\exp(-\mathrm{i}S) \times [R(\rho)+\rho\nabla R(\rho)]\,r\,\mathrm{d}r\right], \qquad (13)$$

$$f_2(t) = \mathrm{Im}\left[\int \psi_R^* F(\psi_R)\,r\,\mathrm{d}r\right]. \qquad (14)$$

3. Derivation of the reduced equations

In order to apply modulation theory to CGL, we rewrite Eq. (1) in the form

$$\mathrm{i}\psi_t + \Delta\psi + |\psi|^2\psi + \epsilon_1 F + \epsilon_2 G + \epsilon_3 H = 0, \qquad (15)$$

where

$$F = -\mathrm{i}\psi, \quad G = -\mathrm{i}\Delta\psi, \quad H = \mathrm{i}|\psi|^2\psi.$$

The auxiliary functions in modulation theory which correspond to F are

$$f_1(t) = 2L(t)\mathrm{Re}\int[-\mathrm{i}\psi_R]\exp(-\mathrm{i}S)[R+\rho\nabla R] = 0,$$

$$f_2(t) = \mathrm{Im}\int[-\mathrm{i}\psi_R]\psi_R^* = -N_c.$$

The derivation of the auxiliary functions which correspond to G is left to Appendix A. This computation yields

$$g_1(t) = 2(E-N_c)LL_t,$$

$$E = \int \rho^3[\nabla R(\rho)]^2\rho\,\mathrm{d}\rho \approx 2.9680,$$

$$g_2(t) = \frac{N_c}{L^2} + ML_t^2.$$

Finally, the auxiliary functions corresponding to H are

$$h_1(t) = 2L(t)\mathrm{Re}\int[\mathrm{i}|\psi_R|^2\psi_R]\exp(-\mathrm{i}S) \times [R+\rho\nabla R] = 0,$$

$$h_2(t) = \mathrm{Im}\int[\mathrm{i}|\psi_R|^2\psi_R]\psi_R^* = \int|\psi_R|^4 = \frac{2N_c}{L^2}.$$

Therefore, to a leading order self-focusing in CGL is given by the reduced system

$$\beta_t(t) = \frac{2N_c}{M}\left(\epsilon_1 - \frac{\epsilon_2+2\epsilon_3}{L^2}\right) + \epsilon_2\left(\frac{E-N_c}{M}-2\right)L_t^2 - \epsilon_2\frac{E-N_c}{M}\frac{\beta}{L^2} - \frac{\nu(\beta)}{L^2},$$

$$L_{tt}(t) = -\frac{\beta}{L^3}. \qquad (16)$$

4. Analysis of the reduced system

Since in this work we are considering the effect of CGL “extra terms” on singularity formation, we analyze (16) when

$$0 < L \ll 1, \quad |\beta| \ll 1. \qquad (17)$$

Under the assumptions (17), to a leading order the first equation in (16) reduces to [5]

$$\beta_t = -\frac{\epsilon}{L^2}, \quad \epsilon := \frac{2N_c}{M}(\epsilon_2+2\epsilon_3). \qquad (18)$$

Changing the independent variable in (18) to

$$\tau = \int_0^t \frac{1}{L^2(t')}\,\mathrm{d}t'$$

yields, after one integration,

$$\beta(\tau) = \beta_0 - \epsilon\tau, \quad \beta_0 = \beta(0). \qquad (19)$$

We now define

$$A := \frac{1}{L}.$$

Therefore, $A \geqslant 0$, blowup corresponds to $A \to +\infty$ and complete defocusing amounts to $A \searrow 0$. Using the relation $\beta = A_{\tau\tau}/A$, Eq. (19) can be rewritten as

$$A_{\tau\tau} = (\beta_0 - \epsilon\tau)A, \qquad (20)$$

which is Airy’s equation

$$A_{ss} = sA, \quad s = \epsilon^{-2/3}(\beta_0 - \epsilon\tau). \qquad (21)$$

[5] Here we used $L_t^2 \ll L^{-2}$ since $L^2L_t^2 = O(\beta) \ll 1$.

The initial condition for (21) is given at

$$s_0 = s(\tau = 0) = \epsilon^{-2/3}\beta_0 .$$

Since we are interested in the effect of the perturbation as $\epsilon \to 0$, we assume that ϵ is sufficiently small so that $s_0 \gg 1$. The solution of Eq. (21) is a linear combination of the Airy and Bairy functions [12]

$$A(s) = k_1 \mathrm{Ai}(s) + k_2 \mathrm{Bi}(s) . \qquad (22)$$

In order to determine k_1 and k_2 from the initial conditions, we first note that initially the effect of the perturbation in (20) is small, and self-focusing in the reduced CGL follows,

$$A_{\tau\tau} \sim \beta_0 A , \qquad 0 \leqslant \tau \ll \frac{\beta_0}{|\epsilon|} ,$$

which corresponds to adiabatic self-focusing in the unperturbed NLS (10). Therefore,

$$\begin{aligned} A &\sim \frac{1}{2}\left(A_0 + \frac{A_0'}{\sqrt{\beta_0}}\right)\exp(\beta_0^{1/2}\tau) \\ &\quad + \frac{1}{2}\left(A_0 - \frac{A_0'}{\sqrt{\beta_0}}\right)\exp(-\beta_0^{1/2}\tau) , \qquad (23) \\ &\quad 0 \leqslant \tau \ll \frac{\beta_0}{|\epsilon|} , \end{aligned}$$

where $A_0 := A(0)$ and $A_0' = A'(0)$. We recall that for large s (see, e.g., Ref. [12]),

$$\begin{aligned} \mathrm{Ai}(s) &\sim \frac{1}{2\pi^{1/2}} s^{-1/4} \exp(-\tfrac{2}{3}s^{3/2}) , \quad s \gg 1 , \\ \mathrm{Bi}(s) &\sim \frac{1}{\pi^{1/2}} s^{-1/4} \exp(\tfrac{2}{3}s^{3/2}) , \quad s \gg 1 . \qquad (24) \end{aligned}$$

Combining (22,24) and expanding $s = s_0(1 - \epsilon\tau/\beta_0)$ for $0 \leqslant \tau \ll \beta_0/|\epsilon|$ gives

$$\begin{aligned} A &\sim k_1 \mathrm{Ai}(s_0)\exp(\mathrm{sgn}(\epsilon)\beta_0^{1/2}\tau) \\ &\quad + k_2 \mathrm{Bi}(s_0)\exp(-\mathrm{sgn}(\epsilon)\beta_0^{1/2}\tau) . \end{aligned}$$

Matching this relation with (24) results in

$$\begin{aligned} k_1 &= \frac{1}{2\mathrm{Ai}(s_0)}\left(A_0 + \mathrm{sgn}(\epsilon)\frac{A_0'}{\sqrt{\beta_0}}\right) , \\ k_2 &= \frac{1}{2\mathrm{Bi}(s_0)}\left(A_0 - \mathrm{sgn}(\epsilon)\frac{A_0'}{\sqrt{\beta_0}}\right) . \end{aligned}$$

4.1. CGL as a defocusing perturbation ($\epsilon > 0$)

In the physical regime of parameters $\epsilon_1, \epsilon_2, \epsilon_3 \geqslant 0$, in which case $\epsilon > 0$. We begin by following the evolution of the power of the focusing part of the beam. From (9), (19) we see that $N(\psi_s)$ is monotonically decreasing in τ, going below critical at $s_1 = 0$. Therefore, we can expect focusing to be arrested at $s_2 < s_1$, corresponding to a later time $T_{\max} := t(s_2)$. Since the power continues to decrease after the arrest, we do not expect the solution to refocus.

In order to see that this is indeed what happens, we note that as τ increases, s decreases, $\mathrm{Bi}(s)$ becomes exponentially small compared with $\mathrm{Ai}(s)$ and

$$A(s) \sim k_1 \mathrm{Ai}(s) .$$

Since $\mathrm{Ai}(s)$ is bounded for $s \leqslant s_0$, $A(s)$ does not become infinite, i.e., there is no singularity formation. In fact, focusing is arrested when the Airy function attains its absolute maximum at $s_2 \simeq -1.0$. At that point $A(s)$ attains its maximum, which is

$$A(s_2) \sim k_1 \mathrm{Ai}(s_2) \simeq 0.54 k_1 .$$

Therefore, the maximum *amplification factor* (AF) is

$$\begin{aligned} AF &:= \max_{s \leqslant s_0} \frac{A(s)}{A_0} = \frac{A(s_2)}{A_0} \\ &\simeq \frac{0.54}{2\mathrm{Ai}(s_0)}\left(1 + \frac{A_0'}{A_0\sqrt{\beta_0}}\right) . \qquad (25) \end{aligned}$$

Hence, AF increases exponentially with decreasing ϵ. In particular, when ψ_0 is real, $L_t(0) = 0$ and AF can be estimated by

$$AF \simeq 0.54\sqrt{\pi}\beta_0^{1/4}\epsilon^{-1/6}\exp\left(\frac{2}{3}\frac{\beta_0^{3/2}}{\epsilon}\right) . \qquad (26)$$

Note that although $\mathrm{Ai}(s)$ oscillates as $s \to -\infty$, the reduced system is physically relevant only until $A(s)$ vanishes, corresponding to complete defocusing ($L = \infty$). This occurs at $s_3 \simeq -2.3$ where $\mathrm{Ai}(s)$ attains its first zero. Therefore, there is a single focusing–defocusing event. We also note that indeed $s_2 < s_1$, i.e., the arrest occurs after the power has already gone below critical.

It is interesting to compare the effect of CGL extra terms with that of the many perturbations of NLS analyzed in Ref. [5]. Using modulation theory, it was shown there that a small defocusing perturbation typically results in decaying focusing–defocusing oscillations, where the decay between oscillations becomes stronger as ϵ goes to zero. Therefore, unlike CGL (where there is always a single focusing–defocusing event), in those cases the number of oscillations depends on ϵ and only when ϵ is *extremely small* there is a single focusing–defocusing event.

The time of the maximal focusing can be estimated using

$$T_{\max} \sim \epsilon^{-1/3} k_1^{-2} \int_{s_2}^{s_0} \mathrm{Ai}^{-2}(s)\, \mathrm{d}s .$$

From (19), it is clear that

$$\lim_{\epsilon \searrow 0} T_{\max}(\epsilon) = T_c ,$$

where T_c is the blowup time for the solution of NLS with the same initial condition.

4.2. *CGL as a focusing perturbation ($\epsilon < 0$)*

In this case, as τ increases, s also increases, $\mathrm{Bi}(s)$ becomes exponentially large compared with $\mathrm{Ai}(s)$ and

$$A(s) \sim k_2 \mathrm{Bi}(s) .$$

Consequently, the solution blows up at a finite time T^*. From (19), it is clear that T^* is smaller than T_c and that

$$\lim_{\epsilon \nearrow 0} T^*(\epsilon) = T_c .$$

Note that our analysis suggests that the *domain of singularity formation* for CGL is

$$\epsilon_2 + 2\epsilon_3 < 0 .$$

5. Numerical results

In this section we present the results of numerical simulations obtained for both the CGL model, Eq. (1), and for the corresponding reduced system (16). Eq. (1) was solved in the radially symmetric case using an explicit finite-differences method. Spatial derivatives were approximated by fourth-order accurate centered differences. Special attention was given to the computation of the Laplacian near the zero, where it was approximated by a Taylor expansion which amounts to using $\Delta_\perp u \simeq 2u_{rr}$ for $r = 0$. The infinite domain in the radial direction was approximated by a finite domain which was taken large enough as not to introduce any boundary effects on the practically compactly supported solution. An explicit third-order Adams–Bashford method was used for marching in time. The initialization required for the Adams–Bashford time marching was performed using a Runge–Kutta method. The reduced system (16) was solved directly using Matlab's Runge–Kutta ODE45 solver.

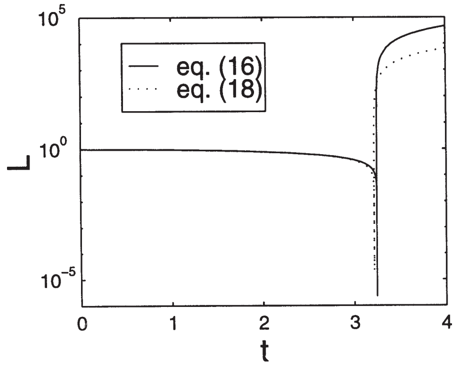

Fig. 1. Evolution of radial beam width L according to the reduced system (16) and to its leading order approximation (18).

The required values of β_0 and $L(t)$ were recovered using the asymptotic relations (9) and

$$L(t) \sim \frac{R(0)}{|\psi(t,0,0)|} .$$

Both relations are only $\mathrm{O}(\beta)$ accurate. However, as can be seen from (26), the results of the reduced system have an exponentially large sensitivity to errors in recovering the initial values β_0 and L_0. Due to that sensitivity, one can expect difficulties in matching the results obtained from solving both equations. Therefore, as evident below, for a general initial condition the numerical solutions of the two equations agree qualitatively, while for Townes-based initial conditions the two numerical solutions match also quantitatively.

We begin by verifying numerically that (18) is indeed valid as an approximation to (16). In Fig. 1 we present the results obtained by integrating both equations with the initial conditions

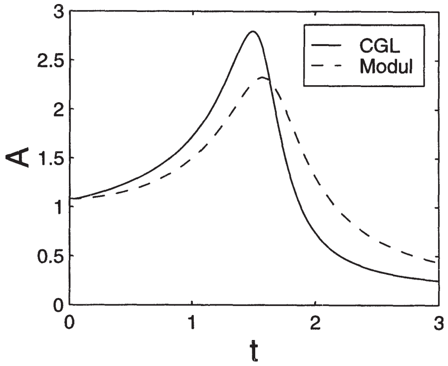

Fig. 2. Evolution of the on-axis amplitude $A = 1/L$ for the initial condition (27) with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.01$ and $\delta = 0.08$.

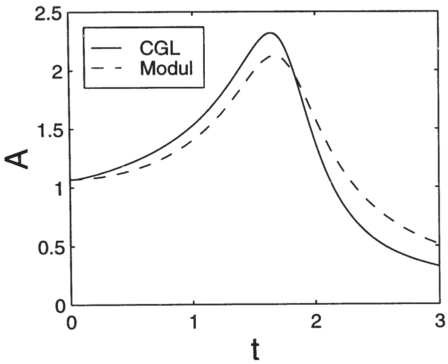

Fig. 3. Evolution of the on-axis amplitude for the initial condition (27) with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.01$ and $\delta = 0.07$.

$$\beta(0) = 0.1\,, \quad L(0) = 1\,, \quad L_t(0) = 0\,,$$

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.0001\,.$$

It is clear that (18) is a good approximation to (16) and that both solutions have one focusing event followed by complete defocusing.

In Figs. 2 and 3 we present results obtained with a Townes-based initial data with power slightly above critical,

$$\psi(t = 0, r) = (1 + \delta) R(r). \tag{27}$$

Here $R(r)$ is the Townes soliton, i.e. the ground-state solution of (4). In Fig. 2, $\delta = 0.08$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.01$ while in Fig. 3, $\delta = 0.07$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.01$. In Fig. 4 we added focusing to the initial condition by adding a quadratic phase factor

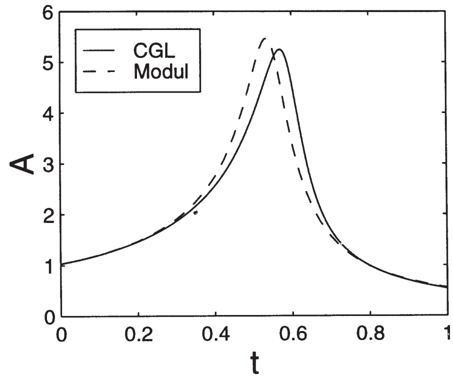

Fig. 4. Evolution of the on-axis amplitude for the focused initial condition (28) with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.008$, $\delta = 0.03$ and $F = 0.7$.

$$\psi(t = 0, r) = (1 + \delta) R(r) \exp\left(-\mathrm{i}\frac{r^2}{4F}\right). \tag{28}$$

Here, $F = 0.7$, $\delta = 0.03$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.008$. Each figure displays two graphs: The amplitude of the solution of CGL

$$A_{[\text{Eq. (1)}]}(t) = \frac{|\psi(t, 0, 0)|}{R(0)}\,,$$

and that of the solution of the corresponding reduced system

$$A_{[\text{Eq. (16)}]}(t) = \frac{1}{L(t)}\,.$$

As predicted by the analysis, in all three figures there is a single focusing–defocusing event and there is a reasonable quantitative agreement between the two solutions. The maximal focusing (AF) is larger in Fig. 2 than in Fig. 3, since the initial power is larger there, in agreement with (26). In Fig. 4, the maximal focusing is even larger than in Figs. 2 and 3 although the initial power is smaller there, due to the addition of the initial focusing factor (which corresponds to the A_0' term in (25)). Finally, we note that in Fig. 1, where the initial power is smaller than in Figs. 2–4 (in this case $\delta \sim \beta M/2N_c \sim 0.015$), the maximal focusing is much higher, due to the exponential dependence of AF on ϵ (relation (26)).

It is quite surprising that in Fig. 4 the two solutions, after separating during the focusing stage, seem to merge again at later times. This phenomena was observed in numerous simulations that we carried out for various values of $\epsilon_1, \epsilon_2, \epsilon_3, \delta$ and F. We have no explanation for this behavior.

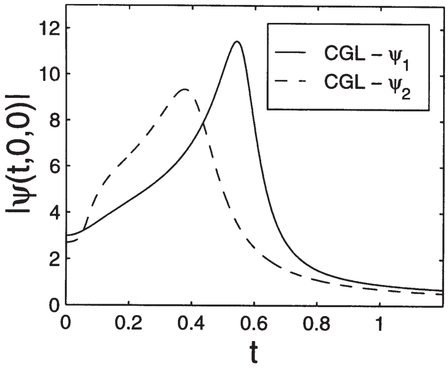

Fig. 5. Evolution of $|\psi(t,0,0)|$ for Gaussian and super-Gaussian initial conditions (29) with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.01$.

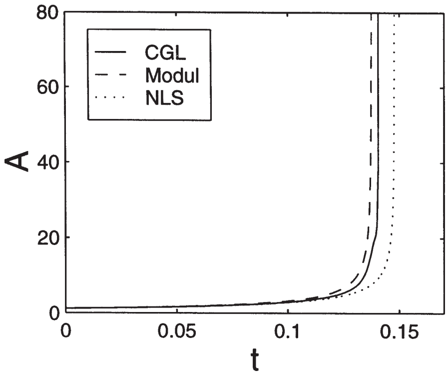

Fig. 6. Blowup time for CGL at the non-physical domain ($\epsilon_1 = \epsilon_2 = \epsilon_3 = -0.01$) is earlier than for NLS. Here, $\delta = 0.2$ and $F = 0.2$.

We also verified that a similar focusing–defocusing behavior is observed for other types of initial conditions. For example, in Fig. 5 we present numerical results obtained with Gaussian and super-Gaussian initial data

$$\psi_1(t=0,r) = 3\exp(-r^2),$$
$$\psi_2(t=0,r) = 2.7\exp(-r^4). \tag{29}$$

As in the previous examples, there is a single focusing–defocusing event.

Finally, we present in Fig. 6 a comparison of the solution of CGL in the non-physical regime $\epsilon < 0$ with the solution of NLS. The parameters used are $\epsilon_1 = \epsilon_2 = \epsilon_3 = -0.01$, and the initial data is (28) with $\delta = 0.2$, $F = 0.2$. As predicted by the analysis, the solution of CGL blows-up in a finite time which is earlier than the blow-up time of the NLS solution with the same initial data. As before, there is good agreement between the solution of CGL and the one of the reduced system.

6. Final remarks

Our analysis of blowup in CGL is based on the assumption that CGL solution is close to the NLS blowup attractor ψ_R. Consequently, we do not claim that there are no other mechanisms of blowup or near-blowup in CGL. This assumption, however, seems reasonable in the context of our analysis of the relation between blowup in CGL and in NLS. The results of our simulations in Section 5 provide a strong support to the validity of this assumption.

We have used the assumption of radial symmetry in the analysis and in the simulations. This assumption is reasonable for initial conditions not exceeding twice the critical power [13]. However, it is well known that NLS solutions with radially symmetric initial conditions whose power is much larger than critical are unstable with respect to symmetry-breaking perturbations, which break the solution into filaments, each of which self-focuses pretty much independently of the others. In that case, our analysis should apply to each filament separately.

We note that a similar analysis can be performed for any CGL equation which is a perturbation of a critical NLS, such as, e.g., the 1D CGL with quintic nonlinearity. Our analysis, however, can not be straightforwardly carried over to the subcritical and the supercritical cases, since the existence of an attractor ψ_R and modulation theory are only valid in the critical case.

A different approach, based on energy estimates, was taken in Ref. [14] where it was shown that the CGL solution exists globally and a bound was derived on the maximal growth of the solution as $t \to \infty$. Our analysis provides a different kind of information: The *dynamics* of the solution (one focusing–defocusing cycle), the *location of the maximal focusing*, a bound on the maximal growth of the solution for $0 \leqslant t < \infty$ and the *seperatix* $\epsilon_2 + 2\epsilon_3$ between global existence and blowup.

In Ref. [15], the 2D CGL on a square cell with Neumann boundary conditions was approximated near the threshold of the modulational instability by a five-dimensional dynamical system. Further analysis showed that this system has a region of chaos and does not preclude the possibility of blowup. Our results (for CGL on an infinite domain) suggest that there is no blowup and we have seen no evidence for chaos.

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Appendix A. Derivation of g_1 and g_2

Below, we outline the derivation of $g_1(t)$ and $g_2(t)$ which are the auxiliary functions corresponding to G defined in (15).

$$g_1(t) = 2L(t)\mathrm{Re}\int [-\mathrm{i}\Delta\psi_R]\,\exp(-\mathrm{i}S)\,[R+\rho\nabla R]$$

$$= 2\,\mathrm{Im}\int [R+\rho\nabla R]\,\exp(-\mathrm{i}S)\,\Delta[R(\rho)\exp(\mathrm{i}S)]$$

$$= 2\int [R+\rho\nabla R]\left[R\frac{L_t}{L} + 2\frac{\nabla R(\rho)}{L}\frac{r}{2}\frac{L_t}{L}\right] r\,\mathrm{d}r$$

$$= 2LL_t\int \nabla R[R+\rho\nabla R]\,\rho\mathrm{d}\rho$$

$$= 2LL_t\left[\int \rho^3[\nabla R(\rho)]^2\rho\,\mathrm{d}\rho - \int [\nabla R(\rho)]^2\rho\,\mathrm{d}\rho\right],$$

$$g_2(t) = \mathrm{Im}\int [-\mathrm{i}\Delta\psi_R]\psi_R^* = \int |\nabla_\perp\psi_R|^2$$

$$= \frac{1}{L^2}\left[\frac{1}{L^2}|\nabla R|^2 + \left(\frac{L_t}{L}\frac{r}{2}\right)^2 R^2\right].$$

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