Waves in Nonlinear Lattices: Ultrashort Optical Pulses and Bose-Einstein Condensates

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The nonlinear Schrödinger equation $i\partial_z A(z, x, t) + \nabla^2_{x,t} A + [1 + m(\kappa x)]|A|^2 A = 0$ models the propagation of ultrashort laser pulses in a planar waveguide for which the Kerr nonlinearity varies along the transverse coordinate x, and also the evolution of 2D Bose-Einstein condensates in which the scattering length varies in one dimension. Stability of bound states depends on the value of $\kappa =$ beamwidth/lattice period. Wide ($\kappa \gg 1$) and $\kappa = O(1)$ bound states centered at a maximum of $m(x)$ are unstable, as they violate the slope condition. Bound states centered at a minimum of $m(x)$ violate the spectral condition, resulting in a drift instability. Thus, a nonlinear lattice can only stabilize narrow bound states centered at a maximum of $m(x)$. Even in that case, the stability region is so small that these bound states are "mathematically stable" but "physically unstable."

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The nonlinear Schrödinger equation (NLS) is central to the understanding of many classical wave and quantum phenomena. For example, it governs the propagation of paraxial laser beams in a homogeneous medium with a Kerr nonlinearity. Recent advances in fabrication technology make it possible to manufacture transparent media with rapidly varying, high-contrast refractive properties [1]. Until recently, most research focused on the variations of the linear index of refraction. However, these fabrication techniques also induce variations in the nonlinear refractive index. Moreover, these nonlinear variations can be significantly larger than those of the linear refractive index [2]. Therefore, in this study, we focus on the effect of a nonlinear lattice on the propagation. This study is also a first stage towards a unified theory for the combined effects of linear and nonlinear lattices.

The NLS also governs the temporal evolution of Bose-Einstein condensates (BEC) where it is usually referred to as the Gross-Pitaevskii equation. In this context it is possible to induce a spatially varying scattering length $g(x) = -[1 + m(\kappa x)]$ via the Feshbach resonance. Although this technique is mainly used to change the scattering length in time, spatial variations of the scattering length can be induced by varying the local external magnetic field near the resonant value [3–5]. Spatial variations of the scattering length can also be optically induced, as was predicted theoretically [6] and verified experimentally [7]. These techniques have the potential to enable new technologies in information storage as well as means to explore fundamental physical phenomena such as cavity QED and quantum information science [8]. Note that in the context of BEC, the nonlinear lattice is not necessarily accompanied by a linear lattice.

In contrast to the large body of research on linear lattices (see, e.g., [9,10] and references therein), much less research has been devoted to the effect of nonlinear lattices; see [11] and references therein. In [11], we used a combination of rigorous analysis, asymptotic analysis, and numerical simulations to study the structure and dynamic stability of bound states of the *one*-dimensional NLS with a transverse periodic *nonlinear* lattice. In this Letter, we extend and apply the methods of [11] to study the structure and dynamic stability of nonlinear bound states in a two-dimensional, anisotropic setting. It is well known that in the absence of a nonlinear lattice, all bound states of the 2D cubic NLS are unstable, and when perturbed they either become singular (collapse, blowup) or diffract to zero [12]. Therefore, an important open question, both theoretically and for various applications, is whether a nonlinear lattice can stabilize the bound states.

In the course of studying this question, we make the following four general observations, which are also relevant to other physical setups, in particular, to the effect of linear lattices: (1) the bound state structure and (in)stability properties strongly depend on whether it is wider, of the same order, or narrower than the lattice period. Specifically, the same lattice may stabilize beams of a certain width while destabilizing beams of a different width. (2) A bound state of the NLS is dynamically stable if and only if it satisfies (a) a *slope (Vakhitov-Kolokolov) condition* on the power (in nonlinear optics) or particle number (in BEC) versus frequency curve, and (b) a *spectral condition*

[13,14]. In particular, it is possible for a bound state to be unstable even though it satisfies the slope condition, a point which has often been overlooked in the physics literature. Violation of the spectral condition implies that the bound state exhibits a drift instability, rather than the blowup instability, associated with a failure to satisfy the slope condition. (3) The size of the stability region, i.e., the region in function space where the initial condition leads to a stable solution, depends on the *magnitude* of the slope of the power (particle number) versus frequency curve. (4) A bound state can be "mathematically stable" (i.e., satisfy the two conditions for stability) but "physically unstable" (i.e., cannot be realized experimentally), due to a small stability region. Therefore, for "physical stability" one should also consider the *magnitude* of the slope of the power (particle number) versus frequency curve, and not only its sign.

Consider the propagation of ultrashort pulses in a planar waveguide for which the linear refractive index $\bar{n}_0$ is constant but the nonlinear Kerr coefficient n_2 is periodically modulated in the transverse spatial coordinate x, i.e., $n = \bar{n}_0 + n_2(x)|A|^2$, where $n_2(x) = \bar{n}_2[1 + m(\frac{x}{x_{\rm ms}})]$, $\bar{n}_2 = \langle n_2(x)\rangle$ is the arithmetic average of $n_2(x)$ over one lattice period, and $m(x/x_{\rm ms})$ is a mean-zero periodic function with period $x_{\rm ms}$. If we rescale x by the input beamwidth $x_{\rm beam}$, the retarded time t by the pulse duration, and z by twice the diffraction length, then the pulse propagation in the anomalous dispersion regime is governed by the dimensionless NLS [11]

$$i\partial_z A(z, x, t) + \nabla^2_{x,t} A + [1 + m(\kappa x)]|A|^2 A = 0, \quad (1)$$

where $\nabla^2_{x,t} = \partial^2_x + \partial^2_t$. Equation (1) with t replaced by y describes the dynamics of laser beams in a 2D medium in which the nonlinear refractive index is modulated in only one dimension.

The parameter $\kappa = x_{\rm beam}/x_{\rm ms}$ is the ratio of the incident beamwidth to the lattice period. Therefore, the case $\kappa \gg 1$ ($\kappa \ll 1$) corresponds to beams which are wide (narrow) compared with the lattice period. Note that the lattice is *anisotropic*, as it varies in x but not on t. In BEC, Eq. (1) models the temporal (z = time) evolution of 2D condensates in which the scattering length varies along one spatial coordinate x, but is independent of the second spatial coordinate t.

Bound states of Eq. (1) are of the form $A(z, x, t) = e^{i\nu z} R^{(\kappa)}_\nu(x, t)$, where $R^{(\kappa)}_\nu$ is the solution of

$$\nabla^2_{x,t} R^{(\kappa)}_\nu + [1 + m(\kappa x)][R^{(\kappa)}_\nu]^3 - \nu R^{(\kappa)}_\nu = 0, \qquad \lim_{r\to\infty} R^{(\kappa)}_\nu = 0, \quad (2)$$

and $r = \sqrt{x^2 + t^2}$. Since the spatial coordinate x was normalized by the beamwidth $x_{\rm beam}$, then $\nu = \mathcal{O}(1)$, see also Remark 2 in [11]. The necessary and sufficient conditions for the (orbital) stability of the $R^{(\kappa)}_\nu$ were found in [13]. In the case of a *positive* bound state these conditions are [11] (i) the *slope condition* $\partial_\nu \int [R^{(\kappa)}_\nu]^2 dxdt > 0$, (ii) the *spectral condition* that the linearized operator $L^{(\kappa)}_{+,\nu} \equiv -\nabla^2_{x,t} + \nu - 3[1 + m(\kappa x)][R^{(\kappa)}_\nu]^2$ has exactly one negative eigenvalue. In a homogeneous Kerr medium ($m \equiv 0$), Eq. (2) reduces to $\nabla^2_{x,t} R_\nu + R^3_\nu - \nu R_\nu = 0$. By the dilation scaling, $R_\nu(r) = \sqrt{\nu} R(\sqrt{\nu} r)$, where R satisfies $\nabla^2_{x,t} R + R^3 - R = 0$. The ground state solution of this equation is known as the *Townes profile* [15] and its power is equal to the critical power for self-focusing, i.e., $\mathcal{P}_{\rm cr} = \mathcal{P}[R] = \int |R|^2 dxdt$ [15].

We now study the profiles and dynamic stability of $R^{(\kappa)}_\nu$ in the three regimes: $\kappa \gg 1$, $\kappa = \mathcal{O}(1)$, and $\kappa \ll 1$.

Wide bound states.—For $\kappa \gg 1$, Eq. (2) can be solved with a multiple scales expansion, which exploits the scale separation between the slow $\mathcal{O}(x)$ beamwidth and the fast $\mathcal{O}(X = \kappa x)$ variations of the lattice. Since $m(X) = m(X + 1)$, we can expand $m(X) = \sum_{n\neq 0} m_n e^{i2\pi nX}$. Following [11], it can be shown that the bound state is given by

$$R^{(\kappa)}_\nu(x, t) = R_\nu(r) + \frac{\nu}{\kappa^2}\Bigg[\underbrace{3\bigg(\int_0^1 [\partial^{-1} m]^2 dX\bigg) L_+^{-1}[R^5(r)]}_{\text{isotropic in } (x,t)} - \underbrace{[\partial^{-2} m(X)]R^3(r)}_{\text{anisotropic}}\Bigg] + \mathcal{O}(\kappa^{-4}), \quad (3)$$

where $\partial^{-k} m(X) = \sum_{n\neq 0} (i2\pi n)^{-k} m_n e^{i2\pi nX}$, $L_{+,\nu} = -\nabla^2_{x,t} + \nu - 3R^2_\nu$, and $L_+ = L_{+,\nu=1}$. The multiple scales solution (3) shows that to leading order, a wide bound state experiences only a homogeneous Kerr effect with effective Kerr nonlinearity of $\bar{n}_2 = \langle n_2(x)\rangle$. Moreover, the deviation of wide bound states from the rescaled Townes profile R_ν is small even if the lattice modulations themselves are of $\mathcal{O}(1)$. This is indeed confirmed in our simulations [see Fig. 1(a)] where $R^{(\kappa=2)}_\nu$ is nearly indistinguishable from the homogeneous medium bound state R_ν.

By Eq. (3), the power of $R^{(\kappa)}_\nu$ for $\kappa \gg 1$ is given by (see [11])

$$\mathcal{P}(R^{(\kappa)}_\nu) = \mathcal{P}_{\rm cr} - \frac{\nu}{\kappa^2} C_{\rm wide} + \mathcal{O}(\kappa^{-4}), \quad (4)$$

where $C_{\rm wide} = 2(\int_0^1 [\partial^{-1} m]^2 dX) \iint R^6(r) dxdt > 0$ is a lattice-dependent positive constant. Hence, our multiple scales analysis shows that a *periodic nonlinear lattice always reduces the power of wide bound states* below $\mathcal{P}_{\rm cr}$.

In the absence of a nonlinear lattice, the Townes profile is unstable, since $\partial_\nu \mathcal{P}(R_\nu) = 0$. This violation of the slope condition results in a blowup or total-diffraction instability. In the presence of a lattice, it follows from Eq. (4) that $\partial_\nu \mathcal{P}(R^{(\kappa)}_\nu) < 0$, i.e., a nonlinear lattice further destabilizes wide bound states. To see this instability, in Fig. 1(b) we solve Eq. (1) with the initial conditions $A_0 = (1 \pm 0.01)R^{(\kappa)}_\nu$. As in the homogeneous medium case, the instability is manifested either by blowup (singularity formation) or by diffractive spreading to zero.

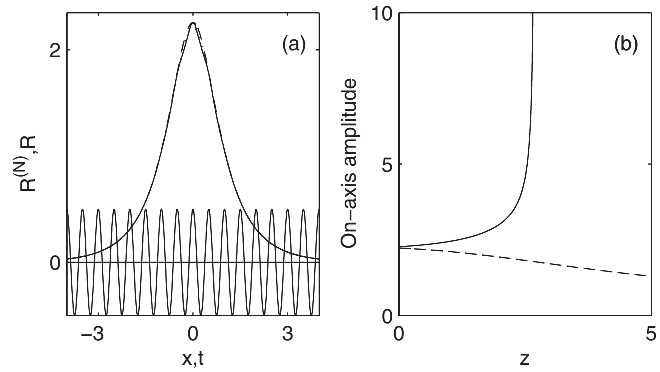

FIG. 1. (a) Cross sections $R_\nu^{(\kappa)}(x, t = 0)$ (solid line), $R_\nu^{(\kappa)}(x = 0, t)$ (dashed line) of the solution of Eq. (2) with $\kappa = 2$, $\nu = 1$, and the lattice $m(\kappa x) = 0.5\cos(2\pi\kappa x)$ (thin solid line). Cross sections are nearly indistinguishable from $R_\nu(r)$ (dotted line). (b) Amplitude of the solutions of Eq. (1) with the initial condition $A_0 = (1 + \delta)R_\nu^{(\kappa=2)}$ for $\delta = 0.01$ (solid line) and $\delta = -0.01$ (dashed line).

$\kappa = \mathcal{O}(1)$ *bound states.*—When the beamwidth is comparable to the lattice period, the bound states $R_\nu^{(\kappa)}$ cannot be approximated analytically but can be evaluated numerically [11,16]. Our numerical simulations with various lattices ranging from the smooth lattice $m = \alpha\cos(2\pi\kappa x)$ to a step-function lattice show that when $\kappa = \mathcal{O}(1)$, the sign of $\partial_\nu \mathcal{P}(R_\nu^{(\kappa)})$ is negative (positive) when the beam is centered at a lattice maximum (minimum), see Fig. 2(a). Hence, $\mathcal{O}(1)$ bound states centered at a local maximum of the lattice are unstable, as they do not satisfy the slope condition. Indeed, our simulations show that $\mathcal{O}(1)$ beams centered at a local maximum exhibit blowup/diffraction instability, similar to Fig. 1(b) (data not shown). On the other hand, while $\mathcal{O}(1)$ bound states centered at a lattice minimum satisfy the *slope condition*, they are also unstable

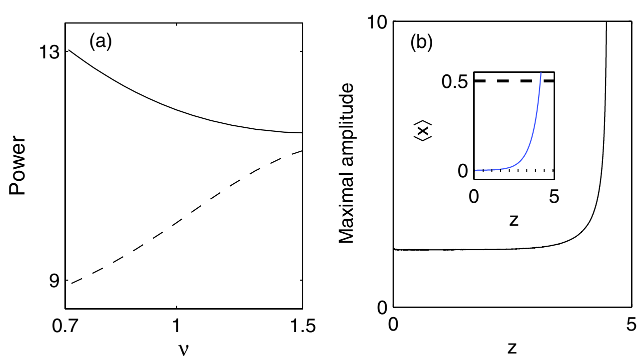

FIG. 2 (color online). (a) Power of the solutions of Eq. (2) centered at a local maximum (dashed line) or minimum (solid line) of the lattice $m = 0.5\cos(2\pi\kappa x)$ with $\kappa = 1$. (b) Same as Fig. 1(b) with the initial condition $A_0 = [1 + 0.05\text{rand}(x, t)]R_\nu^{(\kappa)}$ centered at a lattice minimum for a (mean-zero) random noise. Inset shows that the spatial center of mass of the beam (solid line) moves from a lattice minimum (dotted line) toward a lattice maximum (dashed line).

since they violate the *spectral condition*. To see that, we recall that in the absence of the nonlinear lattice, $L_{+,\nu}^{(\kappa)}$ reduces to $L_{+,\nu}$ which has one simple negative eigenvalue, a double eigenvalue $\lambda_{0,x} = \lambda_{0,t} = 0$ with the corresponding eigenfunctions $\partial_x R_\nu = \frac{x}{r}R_\nu'$ and $\partial_t R_\nu = \frac{t}{r}R_\nu'$, and a positive continuous spectrum $[\nu, \infty)$ [14]. In the presence of a spatial nonlinear lattice $L_{+,\nu}^{(\kappa)}[\partial_t R_\nu^{(\kappa)}] = 0$, so that the corresponding eigenvalue $\lambda_{0,t}^{(\kappa)}$ remains at 0. However, $\partial_x R_\nu^{(\kappa)}$ is no longer an eigenfunction of $L_{+,\nu}^{(\kappa)}$. Numerical calculations (not shown) of the eigenvalues of the discretized operator $L_{+,\nu}^{(\kappa)}$ show that the corresponding eigenvalue $\lambda_{0,x}^{(\kappa)}$ always becomes positive (negative) for a bound state centered at a lattice maximum (minimum). Therefore, the spectral condition is not satisfied for $\mathcal{O}(1)$ beams centered at a lattice minimum so that these beams are also unstable. In that case, the instability manifests itself as a drift of the beam toward the nearest lattice maximum, a behavior we designated as a *drift instability* [11]. Intuitively, this dynamics can be explained by an Ehrenfest-like argument [11,17] showing that the spatial *center of mass* of the beam, $\langle x\rangle = \iint x|A|^2 dxdt / \iint |A|^2 dxdt$, drifts toward regions with higher index of refraction. Indeed, numerical solutions of Eq. (1) show that a *randomly perturbed* $\mathcal{O}(1)$ bound state centered at a local minimum of the lattice drifts towards the lattice maximum and then blows up [Fig. 2(b)]. A similar drift instability occurs also for wide beams ($\kappa \gg 1$) centered at a local minimum. However, since in this case $\lambda_{0,x}^{(\kappa)}$ is exponentially small [11,18], this drift is negligible compared with the blowup instability that arises from violation of the slope condition.

Narrow bound states.—For $\kappa \ll 1$, the solutions of Eq. (2) are yet different from the solutions in the regimes $\kappa \gg 1$ and $\kappa = \mathcal{O}(1)$ discussed so far. In this case, the solution is affected only by the local variation of the lattice and not by the global periodic structure. Following the perturbation analysis of [11,19], we can derive the following asymptotic solution

$$R_\nu^{(\kappa)}(x, t) = \sqrt{\frac{1}{1 + m(0)}}\left(R_\nu + \kappa^2 \frac{m''(0)}{2[1 + m(0)]} L_{+,\nu}^{-1}[x^2 R_\nu^3]\right) + \mathcal{O}(\kappa^4).$$

Thus, the lattice has an $\mathcal{O}(\kappa^2)$ effect on the bound state profile. However, the lattice effect on the bound state power is only $\mathcal{O}(\kappa^4)$ [11,19], i.e.,

$$\mathcal{P}(R_\nu^{(\kappa)}) = \frac{\mathcal{P}_{\text{cr}}}{1 + m(0)} - \frac{\kappa^4}{\nu^2} C_{\text{narrow}} + \mathcal{O}(\kappa^6), \qquad (5)$$

where $C_{\text{narrow}} = [[m''(0)]^2 G - m^{(4)}(0)[1 + m(0)]] \times \iint x^4 R^4(r) dxdt / (48[1 + m(0)]^3)$ and $G = -12 \times \frac{\iint x^2 R^3 \mathcal{L}_+^{-1}[x^2 R^3]}{\iint x^4 R^4} \cong -1.085$. Equation (4) shows that narrow beams satisfy the slope condition if and only if $C_{\text{narrow}} > 0$.

TABLE I. (In)stability of the bound states of Eq. (1). Source for instability is marked by a $*$ or $\dagger$ for a failure to satisfy the slope condition and the spectral condition, respectively.

	Local maximum	Local minimum
$\kappa \gg 1$	unstable *	unstable *
$\kappa = \mathcal{O}(1)$	unstable *	unstable†
$\kappa \ll 1$	determined by sgn($C_{\rm narrow}$)	unstable†

Since $C_{\rm narrow}$ depends not only on $m''(0)$ but also on $m^{(4)}(0)$ and $m(0)$, by a proper design of a lattice, the sign of the slope can be positive for beams centered at a lattice maximum and at a lattice minimum. This is different from the cases of $\kappa = \mathcal{O}(1)$ where the slope is positive only at lattice minimum, and $\kappa \gg 1$ where the slope is always negative. A perturbation analysis similar to [11] shows that $\lambda_{0,t}^{(\kappa)} = 0$ and $\lambda_{0,x}^{(\kappa)} \cong -\frac{m''(0)}{1+m(0)}\kappa^2$. Therefore, as in the case of $\mathcal{O}(1)$ beams, $\lambda_{0,x}^{(\kappa)}$ becomes positive (negative) for beams centered at a lattice maximum (minimum).

Our stability results are summarized in Table I and show that a nonlinear lattice can only stabilize beams that are (i) narrow, (ii) centered at a local maximum of a lattice that (iii) satisfies $C_{\rm narrow} > 0$. However, the $\mathcal{O}(\kappa^4)$ small positive slope, see Eq. (4), implies that the stabilization by the lattice is weak. To illustrate that, in Fig. 3 we show the amplitude of solutions of Eq. (1) with the initial condition $A_0 = (1+\delta)R_\nu^{(\kappa)}$ centered at a local maximum. Since $C_{\rm narrow} < 0$ for the lattice $m = 0.5\cos(2\pi\kappa x)$, we use the lattice $m = 0.48\cos(2\pi\kappa x) - 0.1\cos(4\pi\kappa x)$ for which $C_{\rm narrow} > 0$. As predicted, the bound states remain stable under the perturbation $\delta = 0.0001$. However, when $\delta = 0.004$, the beam collapses while for $\delta = -0.004$, the beam undergoes total diffraction. This shows that the beam is

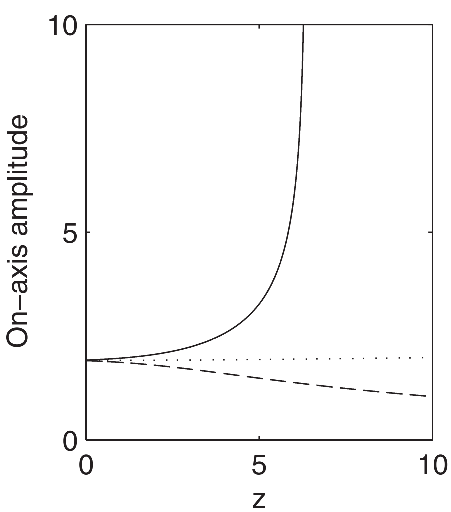

FIG. 3. Amplitude of the solutions of Eq. (1) with the initial condition $A_0 = (1+\delta)R_\nu^{(\kappa=0.2)}$ for $\delta = 0.0001$ (dotted line), $\delta = 0.004$ (solid line), and $\delta = -0.004$ (dashed line).

stable, but that the stability region is smaller than $\pm 0.8\%$ of the beam power. Therefore, narrow bound states that satisfy the three conditions (i), (ii), and (iii) are "mathematically stable", but "physically unstable", since in actual physical setups the profile of the incident beam can be controlled only up to a few percent accuracy.

In conclusion, we have demonstrated that the methods and four general observations of [11] can be easily generalized to an anisotropic two-dimensional nonlinear lattice. Generalizations to an isotropic or a square two-dimensional lattice $[m = m(x, y)]$ or to a three-dimensional setting are also straightforward. Moreover, our methods, results, and observations can be useful to many other problems in nonlinear optics and in BEC, e.g., with a linear lattice [20].

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