Critical power for self-focusing in bulk media and in hollow waveguides

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We determine the threshold power for self-focusing collapse both in a bulk medium and in a hollow-core waveguide for various spatial profiles. We find that the threshold power for collapse in the waveguide is always equal to the lower-bound prediction for a bulk medium.

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The nonlinear optical process of self-focusing sets an upper limit on the amount of laser power that can be propagated through a medium with an intensity-dependent refractive index (i.e., $n = n_0 + n_2 I$, where n_0 is the linear refractive index, n_2 is the nonlinear refractive index, and I is the intensity). For powers above this threshold the beam can undergo collapse, with the peak intensity becoming sufficiently high that damage to the material can occur. Although self-focusing was one of the first phenomena studied in nonlinear optics, it continues to be of importance in recent studies, for example, of filamentation in air[1] and of parametric generation[2] and pulse compression[3] with hollow-core waveguides.

Many studies have investigated the power for catastrophic self-focusing or self-trapping[4] in bulk media, and numerous expressions and values have been derived by use of various arguments or through numerical calculations. Generally, the analytic form of the expression has been shown to be of the form[5]

$$P_{\mathrm{cr}} = \alpha(\lambda^2/4\pi n_0 n_2)\,, \tag{1}$$

where λ is the wavelength in free space and α is a constant that is independent of the material parameters. For example, Boyd[6] gives the value of $\alpha = (1.22\pi^2)/8 \approx 1.8362$. Within a different context, Weinstein[7] investigated the conditions of catastrophic collapse of the nonlinear Schrödinger equation (NLSE) and calculated analytically the conditions under which singularity formation occurs.

More recently, the self-focusing process in hollow-core waveguides filled with a gas was interpreted as the transfer of energy from the fundamental mode to the next-higher-order mode.[8] It was estimated from this analysis that the critical power for self-focusing in the waveguide could be as much as five times higher than that of free space.

In this Letter we show that the lower bound for the critical power for either bulk media or a hollow waveguide is the same and is given by the power of the Townes soliton in a bulk medium. We perform numerical simulations for various input profiles in a bulk medium, including Gaussian and super-Gaussian. In all these cases we find that the critical power for self-focusing in bulk media is no more than 10% higher than the power of the Townes soliton. Our simulations for various input profiles in a hollow waveguide show that the critical power in a hollow waveguide is equal to that of the Townes soliton. We also point out that the distinction between critical powers for self-trapping and for catastrophic self-focusing is outdated, as there is only one threshold power, which is that for catastrophic self-focusing.

For a scalar monochromatic field $E(r, z, t) = A(r, z)\exp(ik_0 z - \omega_0 t)$ the propagation of a laser beam in a Kerr medium under the assumptions of cylindrical symmetry and the slowly varying envelope approximation is governed by the following paraxial equation for the amplitude:

$$2ik_0 A_z + \Delta_\perp A + 4\epsilon_0 c k_0{}^2 n_2 |A|^2 A = 0,$$
$$A(r,0) = A_0(r)\,, \tag{2}$$

where $k_0 = \omega_0 n_0/c$ is the wave-vector amplitude and $\Delta_\perp = \partial_{rr} + (1/r)\partial_r$. The input power of the beam is given by

$$P = 4\pi\epsilon_0 n_0 c \int |A_0|^2 r \mathrm{d}r\,. \tag{3}$$

We change to the nondimensional variables

$$\tilde{r} = r/r_0\,, \qquad \tilde{z} = z/2L_{\mathrm{df}}\,, \qquad \psi = 2k_0 r_0 \sqrt{c\epsilon_0 n_2}\, A\,,$$

where r_0 is the initial beam width and $L_{\mathrm{df}} = k_0 r_0{}^2$ is the diffraction length. When we drop the tildes, the equation for the nondimensional amplitude ψ is the NLSE:

$$i\psi_z + \Delta_\perp \psi + |\psi|^2 \psi = 0\,, \qquad \psi(r,0) = \psi_0(r)\,. \tag{4}$$

We now briefly review the rigorous theory on blowup (singularity formation) in the NLSE. For more details, see Ref. 9. The NLSE has a waveguide solution of the form

$$\psi = \exp(iz)R(r)\,, \tag{5}$$

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where $R(r)$, the Townes soliton, is the positive, monotonically decreasing solution of

$$\Delta_\perp R - R + R^3 = 0\,, \qquad R'(0) = 0\,, \qquad R(\infty) = 0\,.$$

A plot of the Townes profile is shown in Fig. 1. Weinstein[7] proved that solutions of NLSE (4) do not blow up if their initial power ($\int |\psi_0|^2 r\mathrm{d}r$) is below the critical power N_{cr}, which is given by

$$N_{\mathrm{cr}} = \int |R|^2 r\mathrm{d}r \approx 1.86225\,.$$

Weinstein's result, which is independent of the initial width and convergence angle of the beam, provides a lower bound for the critical power for blowup. This result is sharp in the case of the Townsian initial conditions $\psi_0 = (1+\epsilon)R(r)$; i.e., there is no blowup for $\epsilon \le 0$ and there is blowup for $\epsilon > 0$. Thus the lower bound for the critical power, in physical units, is given by

$$P_{\mathrm{cr}}{}^{\mathrm{lb}} = N_{\mathrm{cr}} \frac{\lambda^2}{4\pi n_0 n_2}\,. \tag{6}$$

One can calculate a rigorous upper bound for the critical power from the analytic condition that solutions of NLSE (4) blow up if their initial Hamiltonian is negative:

$$H_0 = \int |\nabla\psi_0|^2 r\mathrm{d}r - 1/2 \int |\psi_0|^4 r\mathrm{d}r < 0\,.$$

Therefore, for a given initial profile $\psi_0 = cf(r)$, there is blowup if $c^2 > 2\int |\nabla f|^2 r\mathrm{d}r / \int |f|^4 r\mathrm{d}r$ or

$$\int |\psi_0|^2 r\mathrm{d}r > G[f]\,,$$

where

$$G[f] = \frac{2\int |f|^2 r\mathrm{d}r \int |\nabla f|^2 r\mathrm{d}r}{\int |f|^4 r\mathrm{d}r}\,.$$

Thus the upper bound for the critical power, in physical units, is given by

$$P_{\mathrm{cr}}{}^{\mathrm{ub}} = G[f] \frac{\lambda^2}{4\pi n_0 n_2}\,. \tag{7}$$

Upper bound (7) implies that blowup will occur for any initial profile $f(r)$ with sufficiently high power. We recall that $\min_{f(r)} G[f] = N_{\mathrm{cr}}$ is attained when $f = R(r)$, whereas for other profiles $G[f]$ is higher.[7] For example, for a Gaussian profile $G[\exp(-r^2/2)] = 2$.

The theoretical lower and upper bounds for the critical power can be written in terms of α as

$$N_{\mathrm{cr}} = \alpha^{\mathrm{lb}} \le \alpha \le \alpha^{\mathrm{ub}} = G[f]\,. \tag{8}$$

Inasmuch as, at present, there is no known analytic technique with which to calculate the critical power for these profiles, in Table 1 we give numerical values of α as defined in Eq. (1) for various profiles in a bulk medium as determined from numerically integrating the NLSE. Although collapse is defined as the point at which the beam intensity becomes infinite in a finite distance, in the nonlinear optics context a more realistic definition is the point at which the beam power exceeds the material breakdown threshold. In our simulations we took initially collimated beams and defined collapse when the beam intensity reached 10^4 times the input peak intensity. As can be seen, all the values for the critical power are greater but are within ~10% of the value predicted by the Townes soliton. The increase in threshold power for non-Townesian initial profiles is due to the initial stage of self-focusing, during which the beam profile changes to a modulated Townes soliton. During this reorganization stage, some of the energy is shed away from the inner part of the beam. Therefore the greater the overlap of the initial profile with the Townes profile, the smaller the increase in threshold power is. For example, because a Gaussian profile is relatively similar to a Townesian profile (see Fig. 1), the threshold power for a Gaussian beam is less than 2% higher than that of a Townes profile.

The NLSE in the unbounded domain $0 \le r < \infty$ has no self-trapping solutions, except for waveguide solution (5), which is unstable. Other solutions of the NLSE either blow up in a finite distance or go through a single focusing–defocusing cycle.

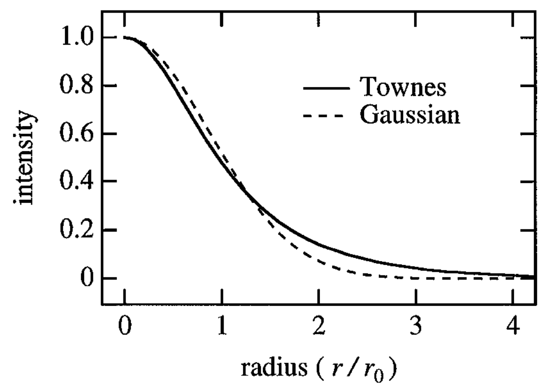

Fig. 1. Townes profile $R(r)$. Also shown is a Gaussian with the same power and on-axis intensity.

Table 1. Comparison of Threshold Parameter α with Theoretical Lower Bound $N_{\mathrm{cr}} \approx 1.86225$ and Upper Bound $G[f]$ [Eq. (8)] for Various Initial Profiles $CA_0(r/r_0)$[a]

	α		
Initial Profile $A_0(r/r_0)$	Lower Bound	Numerical Value	Upper Bound
Bulk medium			
$CR(r/r_0)$	N_{cr}	N_{cr}	N_{cr}
$C\exp[-(r/r_0)^2/2]$	N_{cr}	1.8962	2
$C\exp[-(r/r_0)^4/2]$	N_{cr}	2.0267	$2\sqrt{2}$
Hollow waveguide			
$CJ_0(2.405r/r_0)$	N_{cr}	N_{cr}	2.76
$C\cos(\pi r/2r_0)$	N_{cr}	N_{cr}	2.99
$C(1 - r^2/r_0{}^2)$	N_{cr}	N_{cr}	3.33
$C\sin(\pi r/r_0)$	N_{cr}	N_{cr}	6.58

[a] C is a constant that provides a suitable normalization for the power in each profile.

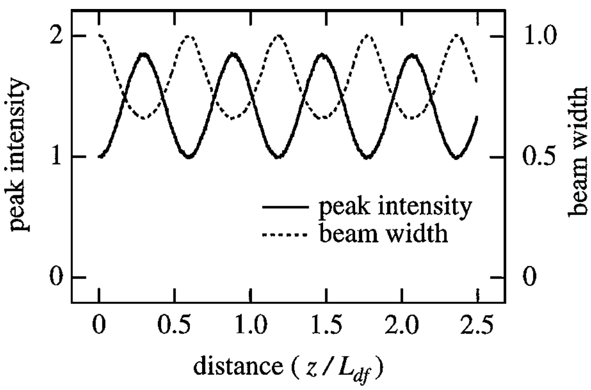

Fig. 2. Plots of the peak intensity and the beam width inside the hollow waveguide as a function of propagation distance for $P/P_{cr}^{lb} = 0.5$.

Therefore, although self-trapping of cw optical beams has been observed,[10] it cannot be explained by use of the NLSE model alone. However, various additional physical mechanisms, such as saturation[11] and nonparaxiality,[12] that are neglected in the NLSE model have the same generic effect, which leads to oscillations of the beam amplitude with propagation,[9,13] which can be interpreted as self-trapping. In all these cases, however, the critical power for self-trapping is the same as that for catastrophic self-focusing.

For the case of a hollow waveguide, the equation that governs propagation of the field is still Eq. (2) in the bounded domain $0 \le r \le r_0$, and we assume, as in previous studies,[8] that $A(r \ge r_0, z) = 0$, where now r_0 is taken to be the radius of the waveguide. The former assumption is valid as long as r_0 is much greater than the wavelength.[8,14] Because the critical power for NLSE (4) in a bounded domain is the same as in unbounded domains,[15] the theoretical lower bound for the critical power for blowup is still given by Eq. (6). As shown in Table 1, from our numerical calculations using the criteria discussed above we find that the parameter α for the critical power for catastrophic collapse for various initial profiles is equal to the Townesian value. The reason for this behavior is that, unlike in bulk media, the walls prevent the shedding of energy and keep the energy localized in the transverse domain. For powers below this value, the peak intensity and the width of the beam oscillate periodically as energy is exchanged between the fundamental mode and the higher-order modes.[8] This behavior, which is unlike that in a bulk medium, is a result of the confinement provided by the waveguide. For powers significantly below the critical power for catastrophic collapse such that nonlinear phase modulation effects can be neglected, the period of oscillation can be estimated from the difference $\Delta\beta = \beta_1 - \beta_2$ in the propagation constants β_1 and β_2 for the fundamental and the next-higher-order modes, respectively,[14] and is found to be approximately equal to $0.5L_{df}$.[8] As the power approaches P_{cr}^{lb} (see Fig. 2), the period becomes longer as a result of the cross-phase modulation of the field in the second-order mode by the field in the fundamental mode.

In conclusion, we have established the threshold condition for catastrophic collapse owing to self-focusing in a bulk medium and in a hollow-core waveguide for various initial beam profiles. We find in all cases studied that in bulk media the critical power for self-focusing is within 10% of the analytic prediction based on the Townes profile and that for a hollow waveguide the critical power is exactly equal to that of Townes profile. In contrast, the upper bound value, which is based on the condition of a negative Hamiltonian, can lead to highly inaccurate predictions. This study should resolve various long-standing discrepancies in this problem and should be relevant for nonlinear optical applications that utilize hollow-core waveguides.

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