SIAM J. APPL. MATH.
Vol. 61, No. 5, pp. 1680–1705

SELF-FOCUSING IN THE DAMPED NONLINEAR SCHRÖDINGER EQUATION*

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Abstract. We analyze the effect of damping (absorption) on critical self-focusing. We identify a threshold value δ_{th} for the damping parameter δ such that when $\delta > \delta_{th}$ damping arrests blowup. When $\delta < \delta_{th}$, the solution blows up at the same asymptotic rate as the undamped nonlinear Schrödinger equation.

Key words. damping, absorption, singularity, collapse, blowup, modulation theory, critical power

AMS subject classifications. 78A60, 35Q55

PII. S0036139999362609

1. Introduction.

The critical nonlinear Schrödinger equation (NLS)

$$i\psi_t + \Delta\psi + |\psi|^2\psi = 0, \quad \psi(0, x, y) = \psi_0(x, y) \tag{1}$$

is the model equation for the propagation of an intense laser beam through a medium with Kerr nonlinearity. In this model $\psi(t, x, y)$ is electric field amplitude, t is distance in the direction of propagation, x and y are the transverse spatial coordinates, and $\Delta = \partial_{xx} + \partial_{yy}$ is the two-dimensional Laplacian. It is well known that if the initial beam power $||\psi_0||_2^2$ is above a threshold value N_c, solutions of (1) can self-focus and become singular in a finite time. Since physical quantities do not become infinite, this implies that the validity of (1) breaks down near the singularity and that additional physical mechanisms, which are initially small, become important there and prevent the singularity formation.

In this study we analyze the effect of small damping on NLS self-focusing and singularity formation. In physical self-focusing, an electromagnetic wave is absorbed by the medium through which it propagates, an effect which is neglected in (1) which models propagation under "ideal transparency." When damping (absorption) is included, the model equation becomes

$$i\psi_t + \Delta\psi + |\psi|^2\psi + i\delta\psi = 0\ , \quad \psi(0, x, y) = \psi_0(x, y)\ . \tag{2}$$

In the nonlinear optics context, the nondimensional expression for δ is (see Appendix A)

$$\delta = r_0^2 k_0^2 \frac{\mathrm{Im}(n_0^2)}{\mathrm{Re}(n_0^2)} = L_{DF} k_0 \frac{\mathrm{Im}(n_0^2)}{\mathrm{Re}(n_0^2)}\ ,$$

where r_0 is the transverse width of the input beam, k_0 is the (real part of the) wavenumber, $L_{DF} := r_0^2 k_0$ is the diffraction length, and n_0 is the linear index of

*Received by the editors October 25, 1999; accepted for publication (in revised form) October 24, 2000; published electronically March 7, 2001. This research was supported by grant 97-00127 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.
http://www.siam.org/journals/siap/61-5/36260.html

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refraction of the media. By definition, transparency means that damping is small. For example, for water in the visible regime [17],

$$\frac{\mathrm{Im}(n_0^2)}{\mathrm{Re}(n_0^2)} \sim 10^{-7} \ .$$

Since physical damping is always positive, $\delta > 0$. Equation (2) with δ positive also arises in the study of the collapse of Langmuir waves with collisional damping [16]. Our analysis, however, applies also to negative values of δ, which is of interest in the context of the complex Ginzburg–Landau equation where δ plays the role of the "instability parameter" [10].

A unique feature of singularity formation in the critical NLS, (1), is that it is extremely sensitive to the addition of small perturbations to the equation [12, 13]. In particular, it has been shown that various small perturbations (e.g., defocusing quintic nonlinearity, nonparaxiality) arrest self-focusing *regardless of how small they initially are*. In fact, these perturbations remain small compared with the Laplacian and the focusing nonlinearity terms even at the time of arrest, when focusing reaches its peak [7, 12, 26]. Therefore, the question arises as to whether small damping can also arrest blowup, and whether it always does so.

At present, there is no definite answer to this question based on rigorous analysis. However, previous numerical and asymptotical studies [2, 3, 16, 31, 32] suggest that the answer depends on the magnitude of the damping parameter δ in the following way: For a given initial condition that leads to blowup in the undamped NLS, there is a threshold value δ_{th}, which depends on the initial condition, such that collapse is arrested when $\delta > \delta_{th}$ and a singularity forms when $\delta < \delta_{th}$.

Our results in this paper agree with this description. In addition, using rigorous, asymptotic and numerical analysis,

1. we prove a necessary condition for blowup in the damped critical NLS, both for functions in $\Re^2$ and for functions in a bounded domain $\Omega \in \Re^2$ (section 2).
2. We derive a reduced system of two ordinary-differential equations that describe self-focusing in the presence of small damping (35).
3. We identify the two nondimensional parameters which govern damped self-focusing (section 3.4).
4. We provide analytical and asymptotical estimates for δ_{th} (sections 2.1 and 3.4).
5. We show that when the solution of the damped critical NLS becomes singular in finite time, its blowup rate is the same as that of the undamped NLS (section 3.5).
6. We show that the radially-symmetric profile ψ_R remains an attractor in damped self-focusing under perturbations which break up the radial symmetry of the initial condition. Nonisotropy of the initial condition, however, reduces δ_{th} (section 4).
7. We show that δ_{th} is small in critical NLS, contrary to the case of the supercritical NLS (sections 2.1 and 6).
8. We identify the existence of a critical exponent for the effect of nonlinear damping: When the exponent of nonlinear damping is cubic or higher, damping always arrests blowup. However, at lower exponents of damping, the picture is qualitatively similar to the case of linear damping, i.e., there exists a threshold value δ_{th} such that when $\delta > \delta_{th}$, there is no blowup, and when $\delta < \delta_{th}$, the solution blows up at the same rate as the undamped NLS (section 5).

9. We extend the numerical method of *dynamical rescaling* to the damped NLS (section 7).
10. We prove that the optimal constant in the Gagliardo–Nirenberg inequality in critical dimension on a bounded domain is equal to that in free-space (Appendix B).
11. We prove that the lower bound for critical power for singularity formation in the undamped NLS on a bounded domain is equal to that in free-space (Appendix B).

2. Rigorous analysis. We begin with a few notations. The L^p norm of a function $f(x,y)$ in $\Re^2$ is

$$||f||_p := \left(\int |f|^p \, dxdy \right)^{1/p} .$$

In the nonlinear optics context, $||f||_2^2$ is called the *power* of f. The H^1 norm of f is defined as

$$||f||_{H^1} := \sqrt{||f||_2^2 + ||\nabla f||_2^2} .$$

The natural space for analyzing existence and blowup of NLS solutions is H^1. From the local existence theory for solutions of the Cauchy problem (2) it follows that if $\psi_0 \in H^1$, the solution exists in H^1 for $t \in [0, \ \ T_{loc}]$, where T_{loc} is a function of $||\psi_0||_{H^1}$ [15, 18]. Therefore, when $||\psi||_{H^1}$ can be (formally) bounded, the solution exists for all t. In addition, let T^δ be defined such that the maximal interval of existence of the solution in H^1 is $[0, T^\delta)$. Then, either $T^\delta = \infty$ or

$$\lim_{t \to T^\delta} ||\psi||_{H^1} = \infty .$$

In the latter case, the solution is said to blowup at time T^δ.

2.1. Conditions for blowup. Let us define

$$u(t,x,y) := \exp(\delta t) \psi(t,x,y) .$$

Then, the equation for u is

$$iu_t + \Delta u + \exp(-2\delta t)|u|^2 u = 0 , \quad u(0,x,y) = \psi_0(x,y) . \tag{3}$$

Multiplying (3) by u^* (the complex conjugate of u) and subtracting the conjugate equation gives

$$\exp(\delta t) ||\psi||_2 = ||u||_2 \equiv ||\psi_0||_2 . \tag{4}$$

Since, in addition, $\exp(\delta t)||\nabla \psi||_2 = ||\nabla u||_2$, ψ and u blow up at the same finite time T^δ, if at all. In that case

$$\lim_{t \to T^\delta} ||\nabla u||_2 = \lim_{t \to T^\delta} ||\nabla \psi||_2 = \infty .$$

Multiplying (3) by u_t^*, adding the conjugate equation and integrating by parts gives

$$-(||\nabla u||_2^2)_t + \frac{1}{2} e^{-2\delta z} (||u||_4^4)_t = 0$$

or

$$H(t) := ||\nabla u||_2^2 - \frac{1}{2}\exp(-2\delta t)||u||_4^4 - \delta\int_0^t \exp(-2\delta\tau)||u(\tau,\cdot)||_4^4\,d\tau \equiv H(0)\ .$$

Therefore,

$$||\nabla u||_2^2 = H(0) + \frac{1}{2}\exp(-2\delta t)||u||_4^4 + \delta\int_0^t \exp(-2\delta\tau)||u(\tau,\cdot)||_4^4\,d\tau\ . \tag{5}$$

Global existence theory for the NLS is based on the Gagliardo–Nirenberg inequality

$$||u||_4^4 \le C_{1,2}||\nabla u||_2^2||u||_2^2\ . \tag{6}$$

The optimal constant $C_{1,2}$ in (6) is equal to [36]

$$C_{1,2} = \frac{2}{N_c}\ ,$$

where N_c is the critical power for singularity formation in the undamped NLS, whose value is given by

$$N_c = ||R||_2^2 \cong 11.69\ ,$$

$R(r)$, the Townes soliton, is the positive radially-symmetric solution of

$$\Delta R - R + R^3 = 0\ ,\quad R'(0) = 0\ ,\quad R(\infty) = 0\ , \tag{7}$$

and

$$r = \sqrt{x^2+y^2}\ .$$

Combining (4), (5), and (6), we have

$$\begin{aligned} &||\nabla u(t,\cdot)||_2^2 \\ &\le H(0) + \frac{||\psi_0||_2^2}{N_c}\left[\exp(-2\delta t)||\nabla u(t,\cdot)||_2^2 + 2\delta\int_0^t \exp(-2\delta\tau)||\nabla u(\tau,\cdot)||_2^2\,d\tau\right]\ . \end{aligned} \tag{8}$$

As we have already mentioned, the local existence theory for the NLS implies that when $||u||_{H^1}$ can be formally bounded, the solution exists globally. Since $||u||_2 \equiv ||\psi_0||_2$, from inequality (8) we can recover the well-known result that

$$||\psi_0||_2^2 \ge N_c \tag{9}$$

is a necessary condition for singularity formation in the undamped NLS (1), since otherwise $||\nabla u||_2$ remains bounded. We now use (8) to extend this result to the damped NLS.

LEMMA 2.1 (necessary condition for blowup). *If the solution u of* (3) *(hence the solution ψ of* (2)*) blows up at a finite time T^δ, then*

$$||\psi_0||_2^2 \ge \exp\left(2\delta T^\delta\right) N_c\ . \tag{10}$$

Proof. We prove Lemma 2.1 by showing that the negation of condition (10) leads to a contradiction. Let us denote $G(t) := ||\nabla u(t,\cdot)||_2^2$. Therefore,

$$\lim_{t \to T^\delta} G(t) = \infty \ . \tag{11}$$

When $\delta \le 0$, from (8) and the negation of condition (10), we have that for all $0 \le t < T^\delta$,

$$G(t) \le H(0) + \nu G(t) \ , \quad \nu := \frac{||\psi_0||_2^2}{N_c} \exp\left(-2\delta T^\delta\right) < 1 \ ,$$

which is in contradiction with (11).

When $\delta > 0$, let us first show that (11) implies that there exists a monotonically increasing sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \to T^\delta$ and

$$\int_0^{t_n} \exp(-2\delta\tau) G(\tau)\, d\tau = o(G(t_n)) \qquad \text{as} \quad n \to \infty \ . \tag{12}$$

To see why (12) holds, let us define $F(t) := \int_0^t G(\tau)\, d\tau$. When $\delta > 0$,

$$\int_0^t \exp(-2\delta\tau) G(\tau)\, d\tau \le F(t) \ , \quad 0 \le t < T^\delta \ .$$

If $\lim_{t\to T^\delta} F(t) < \infty$, then (11) implies that (12) holds for any sequence $t_n \to T^\delta$. Otherwise, $\lim_{t \to T^\delta} F(t) = \infty$, and therefore,

$$\lim_{t\to T^\delta} \log F = \infty \implies \limsup_{t\to T^\delta}\ (\log F)' = \infty \implies \limsup_{t \to T^\delta} G/F = \infty \implies$$

there exists a sequence $t_n \to T^\delta$ such that $F(t_n) = o(G(t_n))$.

From (8) and (12), it follows that

$$G(t_n) \le H(0) + G(t_n)[\nu + o(1)] \qquad \text{as} \ \ n \to \infty,$$

where $\nu < 1$, which is in contradiction with (11). $\square$

The necessary condition for blowup (10) is a generalization of (9). This result is intuitive, since near T^δ (3) approaches the undamped NLS

$$i\psi_t + \Delta\psi + \kappa|\psi|^2\psi = 0 \ , \tag{13}$$

with $\kappa = \exp(-2\delta T^\delta)$, and the critical power for blowup of solutions of (13) is equal to N_c/κ. Alternatively, in light of (4), Lemma 2.1 implies that the power at the time of blowup has to be above critical

$$||\psi(T^\delta,\cdot)||_2^2 \ge N_c \ .$$

An immediate consequence of Lemma 2.1 is the following.

COROLLARY 2.2. *If* $||\psi_0||_2^2 < N_c$ *and* $\delta \ge 0$, *solutions of* (2) *and* (3) *exist for all* $0 \le t < \infty$.

Lemma 2.1 is not optimal, in the sense that it involves the (unknown) blowup time T^δ which varies with δ. In order for it to involve only the blowup time of the undamped NLS $T^0 := T^{\delta=0}$, we use the following conjecture.

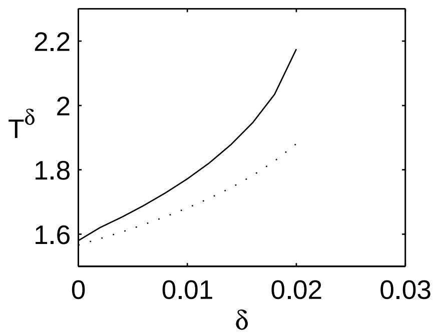

FIG. 1. *Blowup time T^δ is monotonically increasing in δ for $0 \le \delta < \delta_{th}$. When $\delta > \delta_{th}$, there is no blowup. Here, the initial condition is $\psi_0 = \sqrt{1.1}R(r)$ and $\delta_{th} \approx 0.02$. Continuous and dotted lines are simulation results of the damped NLS* (2) *and of the reduced system* (35)*, respectively.*

Conjecture 1. *For a given initial condition ψ_0, $T^\delta = T^\delta(\psi_0)$ is monotonically increasing in δ.*

Conjecture 1 is intuitive from physical considerations, since damping delays self-focusing (as evident, for example, in Figure 1). However, at present there is no rigorous proof of this result. In fact, even a simpler result, that $T^0(\kappa)$ in (13) is monotonically decreasing in κ, has not been proved.

The following Lemma follows directly from Lemma 2.1 and Conjecture 1.

LEMMA 2.3 (necessary condition for blowup). *Assuming that Conjecture* 1 *holds, if the solution u of* (3) *(hence the solution ψ of* (2)*) blows up at a finite time, then*

$$||\psi_0||_2^2 \ge \exp\left(2\delta T^0\right) N_c \ . \tag{14}$$

Lemmas 2.1 and 2.3 provide *lower bounds* for the critical input power for blowup, as a function of δ. In Figure 2 we compare these theoretical lower bounds with the actual critical power for various initial profiles, which we calculate numerically. As expected, condition (14) is not sharp, and for non-Townesian initial conditions (10) is also not sharp.[1] However, condition (10) does seem to be sharp for Townesian initial conditions $cR(r)$ in the limit as $\delta \to 0$.

The results of Lemmas 2.1 and 2.3 can also be interpreted as a condition on δ for global existence for a given initial condition ψ_0. For example, (14) implies that when

$$\delta > \delta_{ub} := \frac{1}{2T^0(\psi_0)} \log \frac{||\psi_0||_2^2}{N_c} \ , \tag{15}$$

damping arrests self-focusing and the solution exists globally.

The previous results can be summarized in the following theorem.

THEOREM 2.4 (threshold damping δ_{th}). *Assume that Conjecture* 1 *holds. Then, for a given initial condition ψ_0 there is a threshold value $\delta_{th} = \delta_{th}(\psi_0)$ such that when $\delta < \delta_{th}$, the solution of* (2) *blows up at a finite time, and when $\delta > \delta_{th}$, the solution of* (2) *exists globally. In addition, $\delta_{th} \le \delta_{ub}$.*

[1]The theoretical lower bound N_c for the critical power in the undamped NLS, (9), is sharp for Townesian initial conditions $cR(r)$ [36]. However, for all non-Townesian initial conditions, the critical power is strictly higher than N_c [28, 29], although typically not by much [8].

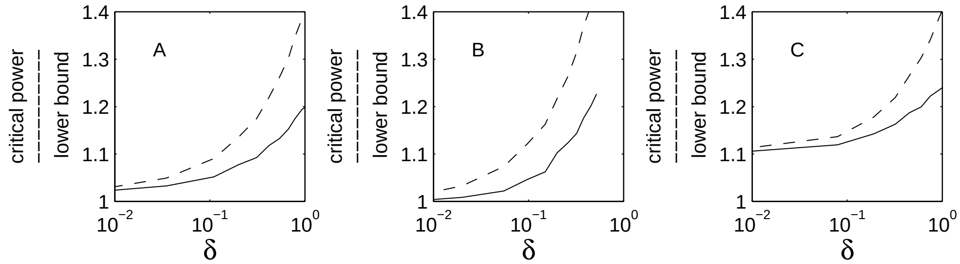

FIG. 2. *Ratio of calculated critical power in the damped NLS* (2) *over theoretical lower bound estimates as a function of δ: $N_c \exp(2\delta T^\delta)$ (solid line) and $N_c \exp(2\delta T^0)$ (dashed line). Initial conditions are* (A) ce^{-r^2}, (B) $cR(r)$, *and* (C) ce^{-r^4}.

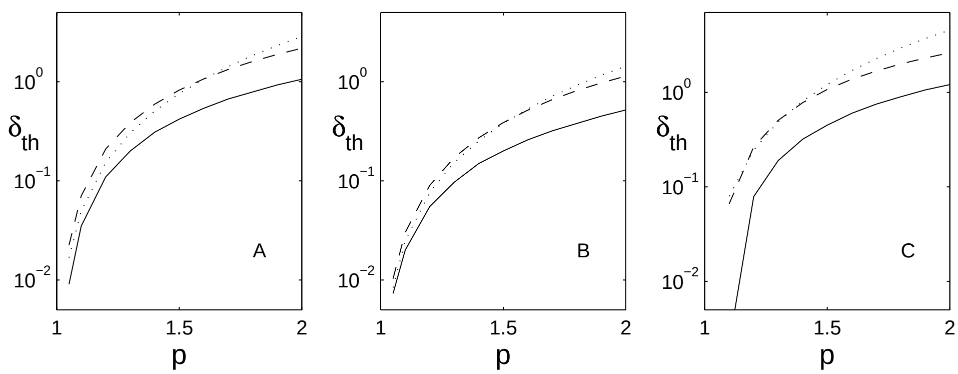

FIG. 3. *Simulation values of threshold damping δ_{th} in the damped NLS* ((2), *solid line*) *as a function of initial power $p = ||\psi_0||_2^2/N_c$. Also plotted are the theoretical upper bound δ_{ub}* ((15), *dashed line*) *and the asymptotic estimate* ((44), *dots*). *Initial conditions are* (A) ce^{-r^2}, (B) $cR(r)$, *and* (C) ce^{-r^4}.

Proof. From Conjecture 1 it follows that if the solution blows up when $\delta = \delta_0$, then it blows up for all $0 \le \delta \le \delta_0$. In addition, from Lemma 2.3 it follows that there is no blowup for $\delta > \delta_{ub}$. Therefore,

$$\delta_{th} = \sup\left\{\delta \mid u^\delta \text{ blows up}\right\} \le \delta_{ub} . \qquad \square$$

In Figure 3 we present the results of Figure 2 in terms of the threshold damping δ_{th}. In agreement with our earlier discussion, the upper bound δ_{ub} for δ_{th} is not sharp, except for the case of Townesian initial conditions $\psi_0 = (1+\epsilon)R(r)$ in the limit as $\epsilon \to 0+$.

Remark. Since

$$\exp(2\delta_{th} T_0) \le \exp(2\delta_{ub} T_0) = \frac{||\psi_0||_2^2}{N_c} ,$$

we see that δ_{th} is typically small,[2] as indeed can be seen in Figure 3.

[2]This property is unique to the damped *critical* NLS (see section 6).

2.1.1. Maximal time of blowup. In the undamped NLS (1), for given initial conditions $\psi_0^\epsilon(x,y) = (1+\epsilon)\phi_0(x,y)$ such that blowup occurs for $\epsilon > 0$ and no blowup occurs for $\epsilon < 0$, we can define the corresponding time of blowup by $T^0(\epsilon; \phi_0(x,y))$. Then, depending on $\phi_0(x,y)$, $\lim_{\epsilon\to 0+} T^0(\epsilon; \phi_0(x,y))$ can be either finite or infinite. From a continuity argument, it is clear that these two possibilities correspond to whether the NLS solution with $\epsilon = 0$ become singular in finite time or exists globally, respectively. Indeed, we recall that if ψ is a solution of (1), and if L depends linearly on t,

$$L(t) = 1 + \frac{t}{F} \ , \quad F \text{ constant} \ ,$$

then $\tilde{\psi}$, defined by the lens (pseudoconformal) transformation[3] [33]

$$\tilde{\psi}(t,x,y) = \frac{1}{L(t)}\psi(\tau,\xi,\eta)\exp\left(i\frac{L_t}{L}\frac{r^2}{4}\right) \ ,$$

where

$$\xi = \frac{x}{L} \ , \quad \eta = \frac{y}{L} \ , \quad \tau = \int_0^t \frac{1}{L^2(t')}\,dt' \ ,$$

is an exact solution of (1) with the initial condition

$$\tilde{\psi}_0(x,y) = \psi_0(x,y)\exp\left(i\frac{r^2}{4F}\right) \ .$$

In addition, t and τ are related by

$$\frac{1}{t} + \frac{1}{F} = \frac{1}{\tau} \ .$$

Therefore,

$$\frac{1}{\lim_{\epsilon\to 0+} T^0(\epsilon; \phi_0(x,y))} + \frac{1}{F} = \frac{1}{\lim_{\epsilon\to 0+} T^0(\epsilon; \phi_0(x,y)\exp(ir^2/4F))} \ ,$$

implying that with proper choice of F, $\lim_{\epsilon\to 0+} T^0(\epsilon; \phi_0(x,y)\exp(ir^2/4F))$ can be always made either finite or infinite.

We note that numerical simulations suggest the following.

Conjecture 2. *When $\phi_0(x,y)$ is real, then* $\lim_{\epsilon\to 0+} T^0(\epsilon;\phi_0) = \infty$.

We are not aware of a rigorous[4] proof for this conjecture, except for the special case $\phi_0 = R(r)$ (see Figure 4), where the solution exists globally for $\epsilon \le 0$ but becomes singular for $\epsilon > 0$ [36].

In contrast, in the case of the undamped NLS, the following corollary shows that T^δ always has a finite limit as $\delta \nearrow \delta_{th}$ (see Figure 1) and provides an upper bound for it.

Corollary 2.5. *Assuming that Conjecture* 1 *holds, then*

$$\sup_{0\le\delta<\delta_{th}} T^\delta = \lim_{\delta\to\delta_{th}-} T^\delta \le \frac{1}{2\delta_{th}}\log\frac{||\psi_0||_2^2}{N_c} < \infty \ .$$

[3]For additional information on the role of the lens transformation in NLS theory, see, e.g., sections 2.2 and 2.3 in [12].

[4]An asymptotical "proof" is given is section 3.1.1.

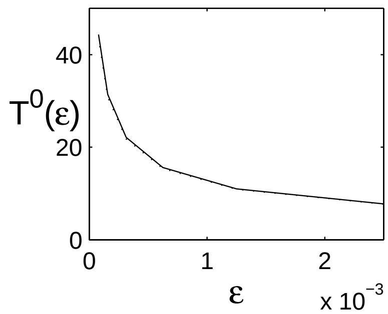

FIG. 4. *Time of blowup in the undamped NLS ((1), solid line) for the initial conditions $\psi_0 = (1+\epsilon)R(r)$ is well approximated by the adiabatic law prediction ((34) with $\beta \approx (2\epsilon+\epsilon^2)N_c/M$, dots).*

Proof. From Lemma 2.1,

$$T^\delta < \frac{1}{2\delta}\log\frac{||\psi_0||_2^2}{N_c} \qquad \text{for} \quad \delta < \delta_{th} \ ,$$

and by Conjecture 1, T^δ is monotonically increasing in δ. ∎

2.2. A variance identity. Let us define the variance of u as

$$V(t) = ||ru||_2^2 \ .$$

Then, differentiating V twice with respect to t, using (3) and integrating by parts gives

$$V_{tt} = 8\left[||\nabla u||_2^2 - \frac{1}{2}e^{-2\delta t}||u||_4^4\right] \ .$$

Therefore, the variance identity for the damped NLS can be written as

$$V_{tt} = 8\left[H(0) + \delta\int_0^t e^{-2\delta\tau}||u(\tau)||_4^4\,d\tau\right] \ ,$$

or

$$V_{ttt} = 8\delta e^{-2\delta t}||u(t)||_4^4 \ .$$

From these identities we see that, unlike the undamped critical case and the damped supercritical case (section 6), the condition $H(0) < 0$ does not guarantee blowup when $\delta > 0$. In fact, at present there is no rigorous proof that solutions of (2) can indeed become singular in finite time.

The following conclusions can be made from the variance identity:

1. When $\delta < 0$, the condition $H(0) < 0$ is sufficient for blowup.
2. $V^\delta(t) > V^{\delta=0}(t) > V^{-\delta}(t)$ for all $\delta > 0$ and $t > 0$.

2.3. Damped NLS in $H_0^1(\Omega)$. Let us consider the damped NLS on a domain $\Omega \in \Re^2$ which is smooth and simply-connected:

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^2\psi + i\delta\psi = 0, & t \geq 0\ , \quad (x,y) \in \Omega\ , \\ \psi(t,x,y) = 0, & t \geq 0\ , \quad (x,y) \in \partial\Omega\ , \\ \psi(0,x,y) = \psi_0(x,y), & (x,y) \in \Omega\ . \end{cases} \tag{16}$$

We would like to know whether the results of section 2.1 remain valid for solutions of (16) which are in $H_0^1(\Omega)$. In order to answer this, we note that we can extend the results of section 2.1 to (16) provided that the following three conditions hold.

Condition I. A priori bounds of $||\psi||_{H_0^1(\Omega)}$ imply global existence.

Condition II. The Gagliardo–Nirenberg inequality, which now reads

$$||u||^4_{L^4(\Omega)} \le C_{1,2}(\Omega)||u||^2_{L^2(\Omega)}||\nabla u||^2_{L^2(\Omega)} \ . \tag{17}$$

Note that, in general, the optimal constant in Sobolev inequalities depends on the domain. However, in Appendix B we prove that $C_{1,2}$ is independent of the domain Ω, i.e.,

$$C_{1,2}(\Omega) = 2/||R||^2_{L^2(\Re^2)} \ .$$

Condition III. Vanishing of boundary terms in integration by parts.

Condition I follows from the global existence proof of [35]. Conditions II and III clearly hold for a bounded smooth domain Ω. Therefore, we have the following.

PROPOSITION 2.6. *All results obtained in section* 2.1 *for* (2) *remain true for* (16)*, with the obvious changes that all norms are over* Ω *instead of over* $\Re^2$*.*

For example, combining Corollary 2.2 with Proposition 2.6 gives the following.

COROLLARY 2.7. *If* $||\psi_0||^2_{L^2(\Omega)} < N_c$ *and* $\delta \ge 0$*, the solution of* (16) *exists for all* $0 \le t < \infty$*.*

The result of Corollary 2.7 is stated (without proof) in [30]. Tsutsumi proved that there exists a constant $M_\Omega > 0$ such that solutions of (16) on a bounded domain Ω exist globally when $||\psi_0||_{H_0^1(\Omega)} \le M_\Omega$ [35]. Therefore, Corollary 2.7 provides a lower bound for the optimal constant M_Ω: $M_\Omega \ge N_c^{1/2}$.

Note, in particular, that Corollary 2.7 (and Appendix B) implies that *the lower bound for the critical power in the undamped NLS on a bounded domain is the same as in free-space*:

$$N_c(\Omega) = N_c(\Re^2) \ .$$

In fact, numerical simulations suggest that on bounded domains the lower bound N_c is generically sharp [8]. For more on self-focusing in bounded domains, see [11].

3. Asymptotic analysis. The (rigorous) analysis in section 2 on the effect of damping on critical self-focusing leaves many questions open. For example,

- what is the dynamics of damped self-focusing?
- What are the nondimensional parameters which govern damped self-focusing?
- What is the effect of the initial focusing angle?
- When blowup occurs, what is its asymptotic rate?

In order to address these questions, we turn to asymptotic analysis of (2) using *modulation theory*, which is a systematic perturbation method for analyzing the effect of small perturbations on self-focusing in the critical NLS.

3.1. Modulation theory: Review. In this section we provide a short review of modulation theory. For more details, see [12, 13]. Modulation theory is based on the observations that near the singularity,

1. the self-focusing part of the solution, ψ_s, is of the form

$$\psi_s(t, x, y) \sim \psi_R(t, r) \ ,$$

where

$$(18)\ \psi_R = \frac{1}{L(t)} R(\rho)\exp(iS)\ , \quad \rho = \frac{r}{L}\ , \quad S = \tau(t) + \frac{L_t}{L}\frac{r^2}{4}\ , \quad \frac{\partial \tau}{\partial t} = \frac{1}{L^2}\ ,$$

and $R(r)$ is the Townes soliton (7).

2. The key parameter of the problem,

$$\beta(t) := -L^3 L_{tt} \tag{19}$$

is small.

Averaging over the transverse (x, y) coordinates leads to the following result.

PROPOSITION 3.1 (modulation theory). *If*

$$\psi \sim \psi_R\ , \quad |\beta(t)| \ll 1\ ,$$

and the perturbation is small, i.e.,

$$|\epsilon F| \ll |\Delta\psi| \quad \text{and} \quad |\epsilon F| \ll |\psi|^3\ ,$$

then self-focusing in the perturbed NLS

$$i\psi_t + \Delta\psi + |\psi|^2\psi + \epsilon F(\psi, \psi_t, \nabla\psi, \ldots) = 0\ , \quad |\epsilon| \ll 1\ , \tag{20}$$

is given to leading order by the reduced system

$$\beta_t(t) + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_t - \frac{2\epsilon}{M} f_2\ , \quad L_{tt}(t) = -\frac{\beta}{L^3}\ . \tag{21}$$

The auxiliary functions f_1 and f_2 are given by

$$f_1(t) = 2L(t) Re\left[\int F(\psi_R)\exp(-iS)[R(\rho) + \rho R'(\rho)]\, dxdy\right]\ , \tag{22}$$

$$f_2(t) = Im\left[\int \psi_R^* F(\psi_R)\, dxdy\right]\ , \tag{23}$$

$\nu(\beta)$ is defined as

$$\nu(\beta) \sim \begin{cases} \frac{4\pi A_R^2}{M}\exp\left(-\pi/\beta^{1/2}\right), & \beta > 0, \\ 0, & \beta \le 0, \end{cases}$$

and

$$M = \frac{1}{4}||rR||_2^2 \cong 3.46\ , \quad A_R = \lim_{r\to\infty} R(r) r^{1/2}\exp(r) \cong 3.52\ .$$

From (18) we can see that the modulation variable $L(t)$ is proportional to the transverse width of the self-focusing part of the solution, as well as to $1/||\psi||_{H^1}$ and to $1/|\psi(t, 0, 0)|$. Therefore, blowup corresponds to $L \searrow 0$ and complete defocusing to $L \nearrow \infty$.

The power of ψ_s can be expanded asymptotically as [12]

$$||\psi_s||_2^2 \sim N_c + \beta M - \frac{\epsilon}{2} f_1\ .$$

Therefore, when $f_1 \equiv 0$ (as in the case of (2)), β is proportional to the excess power above critical of ψ_s :

$$\beta \sim \frac{||\psi_s||_2^2 - N_c}{M}\ . \tag{24}$$

3.1.1. Adiabatic law of blowup: Review. The search for the rate of blowup of the undamped NLS (1) has a long history and was believed to have ended with the derivation of the *loglog law* [14, 19, 25]

$$L(t) \sim \left(\frac{2\pi(T^0 - t)}{\log\log(T^0 - t)^{-1}} \right)^{\frac{1}{2}} \qquad \text{as} \quad t \to T^0. \tag{25}$$

The loglog law can be derived from the reduced system (21) with $\epsilon = 0$:

$$\beta_t(t) = -\frac{\nu(\beta)}{L^2} \ , \tag{26}$$

$$L_{tt}(t) = -\frac{\beta}{L^3} \ . \tag{27}$$

To do that, (26) is decoupled from (27) by rewriting it in terms of τ

$$\beta_\tau = -\nu(\beta) \ . \tag{28}$$

The leading order solution of (28) is

$$\beta \sim \frac{\pi^2}{\log^2 \tau} \ . \tag{29}$$

Expressing relation (29) in terms of L and t leads to (25).

After the derivation of the loglog law, it turned out that it does not become valid even after L becomes as small as 10^{-90}. In [6, 12] it was shown that the reason for this "failure" of the loglog law is that even at these huge focusing levels the leading order approximation (29) for (28) is not valid. However, one can solve the reduced equations (26), (27) in the domain of physical interest with a multiple-scales approach. To do that, we note that when β is small, changes in β (26) are slow compared with changes in L (27). Therefore, to leading order one can neglect nonadiabatic effects and solve the system

$$L_{tt} = -\beta L^{-3} \ , \quad \beta \equiv \beta_0 \ , \quad \beta_0 := \beta(0) \ . \tag{30}$$

Integrating (30) twice leads to Fibich's adiabatic law [6, 12]

$$L(t) \sim \sqrt{2\sqrt{\beta}\,(T^0 - t) + C(0)\,(T^0 - t)^2} \ , \tag{31}$$

where

$$C(t) = L_t^2 - \frac{\beta}{L^2} \ .$$

Numerical simulations show that the adiabatic law (31) becomes valid almost from the onset of self-focusing [6, 12]. Nonadiabatic effects can be added to this leading order approximation by adding the slow variation in time of β and C.

As focusing progresses, (31) reduces to Malkin's adiabatic law [26],

$$L(t) \sim \sqrt{2\sqrt{\beta}(T^0 - t)} \ , \tag{32}$$

which becomes valid after some focusing has occurred. In the far-far asymptotic limit Malkin's law reduces to the loglog law. Thus, the three laws are consistent with each other, but their domain of validity differ considerably.

The adiabatic law (31) can be used to estimate T^0 from the initial conditions [6, 12]:

$$T^0 \sim \frac{L_0^2}{\beta_0^{1/2} - L_0 L_t(0)} \ . \tag{33}$$

For example, in the case of an initially-collimated beam (ψ_0 real), $L_t(0) = 0$ and (33) becomes

$$T^0 \sim \frac{L_0^2}{\sqrt{\beta_0}} \ . \tag{34}$$

Therefore,

$$\lim_{\beta_0 \to 0+} T^0 = \infty \ ,$$

which, in light of (24), can be viewed as an asymptotical "proof" of Conjecture 2.

3.2. Derivation of reduced equations for damped critical self-focusing. If we apply Proposition 3.1 to (2), then $F = i\psi$,

$$f_1 \equiv 0, \quad \text{and} \quad f_2 = \mathrm{Im} \int \psi_R^* (i\psi_R) = N_c \ ,$$

and to leading order, self-focusing in the damped critical NLS (2) is given by

$$\beta_t(t) = -\frac{\nu(\beta)}{L^2} - \frac{2N_c\delta}{M} \ , \quad L_{tt}(t) = -\frac{\beta(t)}{L^3} \ . \tag{35}$$

The first equation in (35) can be interpreted as follows: Power losses of the collapsing part of the beam ψ_s (left-hand side), are due to nonadiabatic radiation to the noncollapsing part of the beam and damping effects (first and second terms on the right-hand side). Self-focusing dynamics in (35) depends on whether nonadiabatic radiation is dominant over damping or vice versa. As we shall see, when damping dominates over nonadiabatic radiation, to leading order self-focusing dynamics is given by (36). In this case, either damping dominates over the focusing nonlinearity and self-focusing is arrested, or self-focusing dominates over damping and the solution blows up (section 3.4). In the later case, damping effects become negligible compared with nonadiabatic radiation near the singularity and self-focusing is described there by the system (26)–(27) (section 3.5). This is, of course, also the case when nonadiabatic radiation dominates over damping from the onset of self-focusing. Note, however, that when self-focusing starts in the nonadiabatic regime where β is not small, it is possible for the relative sizes of damping and nonadiabatic radiation to change more than once (Figure 8B).

3.3. "Proof" of Conjecture 1. When nonadiabatic power radiation is small compared with power loss due to damping, (35) can be approximated with

$$L_{tt}(t) = -\frac{\beta(t)}{L^3} \ , \qquad \beta = \beta_0 - \frac{2N_c\delta}{M} t \ . \tag{36}$$

We now prove the following result, which is the asymptotic analogue of Conjecture 1.

LEMMA 3.2. *Let $L(t;\delta)$ be the solution of* (36)*, and let $T(\delta)$ be the time when L vanishes, i.e., $L(T(\delta);\delta) = 0$. Then, for given initial conditions L_0 and $L_t(0)$, the function $T(\delta)$ is monotonically increasing in δ.*

Proof. We first note that β is monotonically decreasing in δ. Therefore, for $t > 0$ and sufficiently small, L_{tt}, and hence L_t and L, are monotonically increasing in δ. Therefore, $L_{tt} = -\beta/L^3$ remains monotonically increasing in δ, and hence also L_t and L, for as long as the solution exists. ∎

3.4. Threshold damping for blowup. At the time of blowup T^δ, the excess power above critical of ψ_s should be positive. Therefore, in light of (24), (36), a necessary condition for blowup is

$$\beta_0 \geq \frac{2N_c\delta}{M}T^\delta \ . \tag{37}$$

This necessary condition for blowup is the asymptotic analogue of the analytic condition (10). To see that, we note that from (37) we have that $\delta T^\delta = O(\beta_0) \ll 1$. Therefore, we can expand the exponential in (10) and use (24) to see that the two conditions agrees with $O(\beta)$ accuracy (which is the order of accuracy of modulation theory).

From Conjecture 1 we have that $T^\delta > T^0$. Therefore, the necessary condition for blowup (37) can be rewritten as

$$\beta_0 \geq \frac{2N_c\delta}{M}T^0 \ , \tag{38}$$

which is the asymptotic analogue of condition (15).

In order to find out the dynamics of damped critical self-focusing, we change to the rescaled variables

$$\tilde{L} = \frac{L}{L_0} \ , \qquad \tilde{t} = \frac{t}{T_{SF}} \ ,$$

where $L_0 = L(0)$ is the initial transverse width of the solution and $T_{SF} = L_0^2/\sqrt{\beta_0}$ is the adiabatic law estimate for the time of blowup of an initially-collimated beam undergoing self-focusing (see (34)). Under this transformation, the system (36) becomes

$$-\tilde{L}^3\tilde{L}_{\tilde{t}\tilde{t}} = 1 - \theta\tilde{t} \ , \qquad \tilde{L}(0) = 1 \ , \qquad \tilde{L}_{\tilde{t}}(0) = v_0 \ , \tag{39}$$

where

$$\theta := \frac{2N_c}{M}\frac{\delta L_0^2}{\beta^{3/2}(0)} \ , \qquad v_0 := \frac{L_0 L_t(0)}{\beta_0^{1/2}} \ . \tag{40}$$

We thus see that the dynamics of the damped NLS is determined by the two nondimensional parameters θ and v_0. These two parameters depend on the initial condition ψ through L_0, β_0, and $L_t(0)$ which correspond to the width, power, and divergence angle of the input beam, respectively.

The parameter θ can be written as

$$\theta = \frac{T_{SF}}{T_{damp}} \ , \qquad T_{damp} := \frac{\beta_0 M}{2N_c\delta} \ .$$

Therefore, θ is equal to the ratio of the blowup time T_{SF} due to nonlinear self-focusing (in the absence of damping and initial focusing), to the characteristic time for damping to reduce the power below critical (37). From this interpretation, it is clear that when θ is small, nonlinearity dominates over damping and blowup occurs, while when θ

is large, damping dominates and the power goes below critical before catastrophic self-focusing can occur.

The second nondimensional parameter can be written as

$$v_0 = \frac{T^{SF}}{T_{LF}} , \qquad T_{LF} := \frac{L(0)}{L_t(0)} ,$$

i.e., the ratio of the characteristic time for nonlinear self-focusing with no initial focusing T_{SF}, to the characteristic time for initial focusing to focus the solution to a point in the absence of diffraction and nonlinearity.

Although (39) cannot be solved explicitly, it is clear that there is a threshold value $\theta_{th} = \theta_{th}(v_0)$ such that

- when $\theta < \theta_{th}$, $\tilde{L}$ vanishes at a finite time (i.e., blowup).
- When $\theta > \theta_{th}$, blowup is arrested and there is a single focusing-defocusing cycle.

These two possibilities for the dynamics of damped self-focusing can be seen in Figure 5. Thus, the nondimensional system (39) together with the interpretation of the two nondimensional parameters θ and v_0 provide a complete qualitative description of damped critical self-focusing. We note that this qualitative picture is consistent with numerical simulations of damped critical self-focusing [2, 16, 31].

The critical damping parameter is given by (40)

$$\delta_{th} \approx \frac{\theta_{th} M}{2N_c} \frac{\beta_0^{3/2}}{L_0^2} . \tag{41}$$

The fact that $\delta_{th} \sim L_0^{-2}$ is evident from a simple rescaling argument applied to (2). In addition, if we denote by p the initial power, normalized by N_c,

$$p := \frac{||\psi_0||_2^2}{N_c} ,$$

then, in light of (24), relation (41) implies that

$$\delta_{th} \sim (p-1)^{3/2} . \tag{42}$$

The validity of relation (42) is supported by numerical simulations (Figure 6). Finally, we note that (41) is consistent with our earlier results that a solution with higher power can blow up in the presence of larger damping and that in critical self-focusing δ_{th} is relatively small.

In Figure 7, which we obtain by solving (39) numerically, we plot θ_{th} as a function of v_0. To a good approximation,

$$\theta_{th} \approx 0.78 - 0.82 v_0 . \tag{43}$$

Substituting (43) in (41) gives

$$\delta_{th} \approx \left(0.78 - 0.82 \frac{L_0 L_t(0)}{\beta_0^{1/2}}\right) \frac{M}{2N_c} \frac{\beta_0^{3/2}}{L_0^2} . \tag{44}$$

We can also adopt a somewhat different approach, based on the condition for global existence (38) and the adiabatic law estimate for the blowup time (33). This yields an estimate for the upper bound δ_{ub}:

$$\delta_{ub} \sim \left(1 - \frac{L_0 L_t(0)}{\beta_0^{1/2}}\right) \frac{M \beta_0^{3/2}}{2N_c L_0^2} .$$

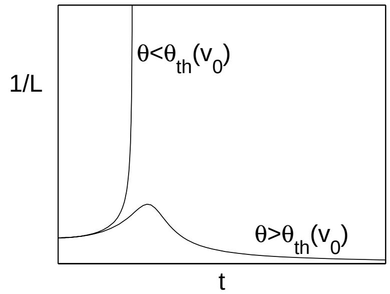

FIG. 5. *Damped self-focusing, according to the reduced system* (39).

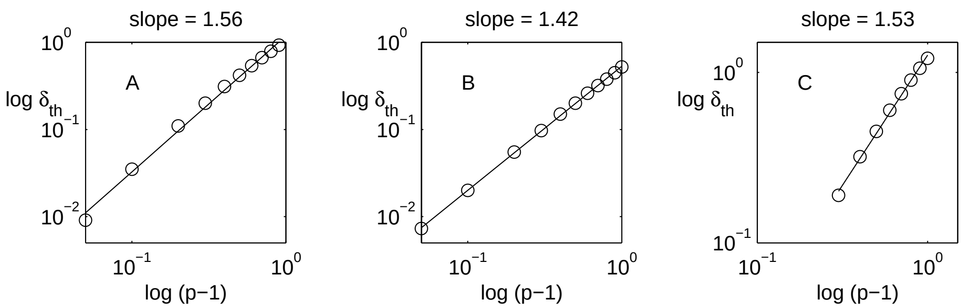

FIG. 6. *Simulation results of* $\log \delta_{th}$ *as a function of* $\log(p-1)$ *(circles). Initial conditions are* (A) ce^{-r^2} *with* $1.05 \le p \le 2$, (B) $cR(r)$ *with* $1.05 \le p \le 2$, *and* (C) ce^{-r^4} *with* $1.3 \le p \le 2$. *Slope of interpolating lines is close to the* $3/2$ *prediction of relation* (42).

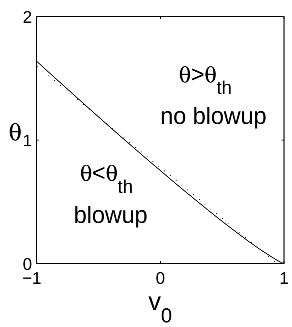

FIG. 7. *Value of* θ_{th} *as a function of* v_0 *((39), solid) and its linear approximation ((43), dots).*

Note the similarity between this upper bound estimate and (44). For example, in the case of an initially-collimated beam this bound is larger than the estimate (44) by roughly 30%. Finally, if we substitute (24) and (33) in (15), we get

$$\delta_{ub} \sim \frac{\beta_0^{1/2}}{2L_0^2}\left(1 - \frac{L_0 L_t(0)}{\beta_0^{1/2}}\right) \log\left(1 + \frac{\beta M}{N_c}\right) ,$$

which, for small β, agrees asymptotically with the previous estimate.

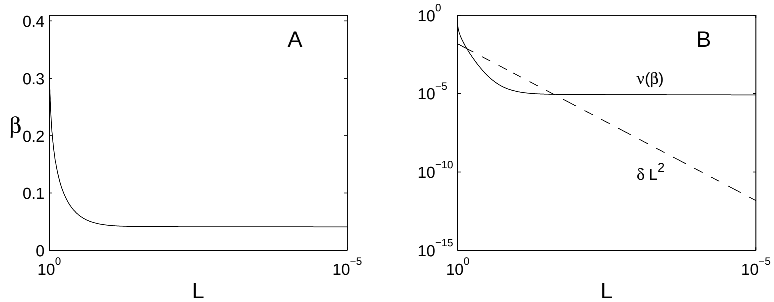

FIG. 8. *Solution of the reduced system* (35) *for damped self-focusing with nonadiabatic effects.* (A) *Changes in β are very small compared with the focusing rate, i.e., damped self-focusing is adiabatic.* (B) *Effect of damping, δL^2, becomes negligible compared with nonadiabatic losses, $\nu(\beta)$, near the blowup point. Here, $\beta_0 = 0.4$, $L_0 = 1$, $L'(0) = 0$, and $\delta = 0.015$.*

3.5. Blowup rate. Near the singularity the damped NLS (2) approaches the undamped NLS (13). Therefore, it may seem reasonable to assume that the blowup rate in (2) is the same as that of the undamped NLS. However, blowup in the critical NLS is characterized by the near-balance between the focusing nonlinearity and diffraction, and is highly sensitive to small perturbations [12]. As a result, it is unclear a priori which of the two small mechanisms, nonadiabatic losses and damping, is dominant near the blowup point. In order to determine this, we note that by following the multiple-scales approach that leads to the adiabatic law (section 3.1.1), the first order correction to β is given by

$$\beta_\tau \sim -\nu(\beta_0) - \frac{2N_c}{M}\delta L^2 \; . \tag{45}$$

Therefore, the effect of damping becomes negligible as $L \to 0$ and damped self-focusing is governed asymptotically by the reduced system (26)–(27) of the undamped NLS. As a result, the rate of blowup is the same as the one of the undamped NLS: It is given by the adiabatic law (31), which, as focusing goes on, reduces to (32) and eventually in the far-far asymptotic limit reduces to the loglog law (25). It should be noted, however, that while damping does not change the blowup rate, it does delay the time of blowup, as evident in Figure 1.

In Figure 8 we present a numerical solution of the reduced nonadiabatic system (35). As can be seen, when β is small, damped self-focusing is essentially adiabatic: While L decreases by five orders of magnitude, β does not even decrease by one order of magnitude. In addition, one can see that the small nonadiabatic term $\nu(\beta)$ remains relatively unchanged, while the damping term δL^2 quickly becomes negligible in comparison, thus providing a support to the validity of the above analysis of the blowup rate of the damped NLS.

In Figure 9 we compare the blowup rate of solutions of the damped NLS (2) with the predictions of the three asymptotic laws. As can be seen, only the adiabatic law (31) provides a reasonable approximation for the blowup rate in the domain of physical interest $1 \le 1/L \le 10^5$.

4. Stability of isotropic self-similar dynamics. A key assumption of modulation theory (which we use in the asymptotic analysis in section 3) is that near the singularity the solution approaches the radially-symmetric attractor ψ_R. While it is clear that ψ remains radially-symmetric when ψ_0 is radially-symmetric, it is not clear

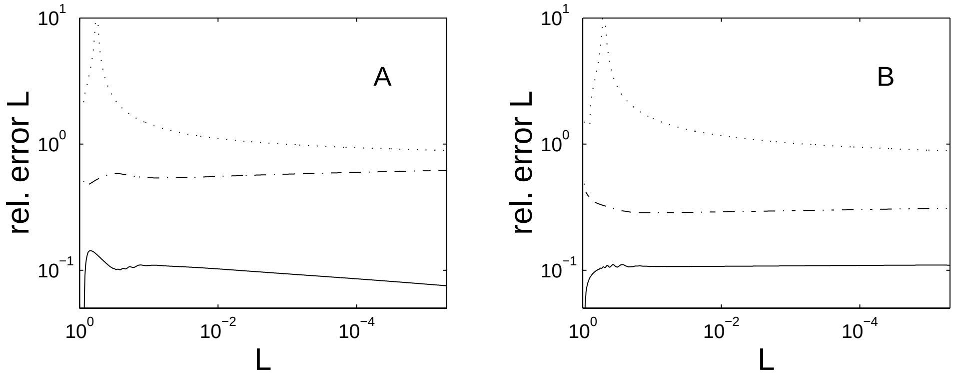

FIG. 9. *Relative error in the predictions of Fibich's adiabatic law ((31), solid line), Malkin's adiabatic law ((32), dash-dot), and the loglog law ((25), dots) for the blowup rate L of the solution of the damped NLS (2). Initial conditions are:* (A) $\psi_0 = \sqrt{1.3}R(r)$ *with* $\delta = 0.06$, (B) $\psi_0 = \sqrt{1.1}R(r)$ *with* $\delta = 0.01$.

a priori that ψ_R remains an attractor under symmetry-breaking perturbations. We note that at present, there is no rigorous theory to support the assumption of the stability of radially-symmetric self-similar dynamics. This assumption is supported, however, in numerical studies of self-focusing in NLS and perturbed NLS [9, 20, 21].

In order to test the effect of nonisotropic initial conditions on damped self-focusing, as well as the validity of the assumption of radial symmetry in the asymptotic analysis[5] of the damped NLS, we integrate the damped NLS (2) with highly nonisotropic initial conditions

$$\psi_0 = c\exp(-(0.4x)^2 - y^2) \ , \tag{46}$$

where

$$c = 2\sqrt{0.52N_c} \approx 1.97 \tag{47}$$

(i.e., $||\psi_0||_2^2 = 1.3N_c$). We monitor the solution nonisotropy with the ratio of the widths of the solution in the x and y directions

$$\frac{L_x}{L_y} = \sqrt{\frac{\int |(\psi|^2)_y|\, dxdy}{\int |(\psi|^2)_x|\, dxdy}} \ .$$

The relative increase in amplitude is monitored with $||\psi||_\infty/||\psi_0||_\infty$.

In Figure 10A we see that when $\delta = 0.01$ the solution blows up. The initial oscillations in L_x/L_y disappear after focusing by a factor of five (Figure 10B) as the collapsing part of the solution[6] converges to the radially-symmetric profile ψ_R (Figure 11). Singularity formation is arrested, however, when $\delta = 0.015$ (Figure 12A). In this case, the solution also converges to a radially-symmetric profile during the self-focusing stage (Figure 12B). These simulations confirm that the radially-symmetric profile is an attractor during self-focusing.

Simulation results show that $\delta_{th} \approx 0.015$ for the initial conditions (46), (47). For comparison, for isotropic Gaussian initial conditions with the same initial power,

[5]We did not assume radial symmetry in the rigorous analysis in section 2.

[6]The "outer" part of the solution which does not self-focus does remain nonisotropic.

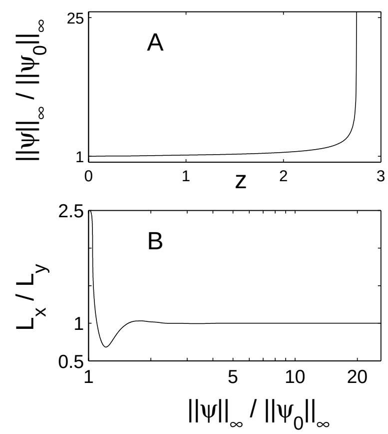

FIG. 10. *Damped self-focusing* (2) *with nonisotropic initial conditions* (46), (47), *and* $\delta = 0.01$.

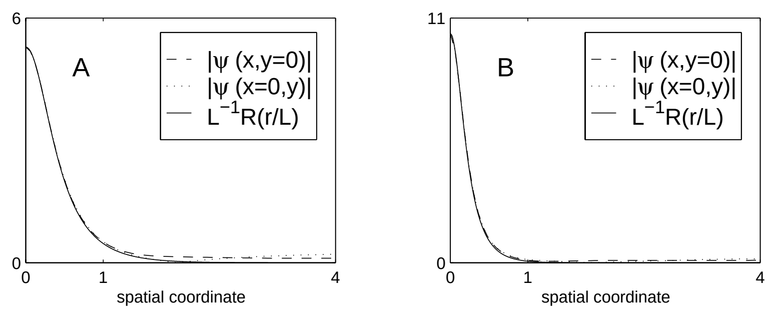

FIG. 11. *Convergence to the radially-symmetric profile* ψ_R *for the solution of Figure* 10. (A) $t = 2.48$, $L = 0.42$, *and* $||\psi||_\infty/||\psi_0||_\infty = 2$, (B) $t = 2.69$, $L = 0.21$, *and* $||\psi||_\infty/||\psi_0||_\infty = 5.2$.

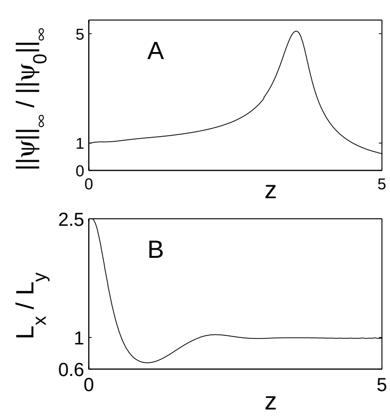

FIG. 12. *Same as in Figure* 10 *with* $\delta = 0.015$.

$\delta_{th} \approx 0.20$ (Figure 3). This drastic change in the value of δ_{th} is due to the increase in the effective critical power for blowup of elliptic beams, compared with radially-symmetric ones. For example, the threshold power for blowup in the undamped NLS (1) for the initial conditions (46) is $||\psi_0||_2^2 \approx 1.20 N_c$ [9]. Thus, the initial power of (46), (47) is only about 8% above the "effective" critical power for (46). Indeed, for isotropic Gaussian initial conditions with $p = (1.3/1.2)N_c$ we calculate numerically that $\delta_{th} \approx 0.025$, which is of the same order as the value of δ_{th} for the initial conditions (46), (47).

5. Nonlinear damping. The damped NLS (2) is a special case of

$$i\psi_t + \Delta\psi + |\psi|^2\psi + i\delta|\psi|^q\psi = 0 \tag{48}$$

with $q = 0$ (linear damping). In the nonlinear optics context, the origin of nonlinear damping is multiphoton absorption, and the damping power q can take on integer values between 2 and 8. For example, in the case of solids the number q corresponds to the number of photons it takes to make a transition from the valence band to the conduction band. Similar behavior can occur with free atoms, in which case q corresponds to the number of photons needed to make a transition from the ground state to some excited state or to the continuum.

Application of modulation theory to (48) with $F = i|\psi|^q\psi$ gives that $f_1 \equiv 0$ and

$$f_2 = \frac{c_q}{L^q} \ , \quad c_q = ||R||_{q+2}^{q+2} > 0 \ .$$

Therefore, self-focusing in (48) is given to leading order by

$$\beta_t = -\frac{\nu(\beta)}{L^2} - \frac{2c_q\delta}{M}\frac{1}{L^q} \ ,$$

or

$$\beta_\tau = -\nu(\beta) - \frac{2c_q\delta}{M}L^{2-q} \ . \tag{49}$$

This equation was already derived in [4].

Equation (49) shows that the effect of nonlinear damping depends on the sign of $(q-2)$:

- When $q < 2$, self-focusing dynamics is qualitatively similar to the case of linear damping:
  - When $\delta > \delta_{th}(\psi_0; q)$, focusing is arrested and there is one focusing-defocusing cycle.
  - When $\delta < \delta_{th}(\psi_0; q)$, damping effects become negligible near the singularity and the blowup rate is the same as that of the undamped NLS.
- When $q = 2$, nonadiabatic effects become negligible compared with damping and to leading order

  $$\beta_\tau \sim -\epsilon \ , \quad \epsilon = \frac{4N_c\delta}{M} \ , \tag{50}$$

  where we have used the relation $c_2 = 2N_c$. Following [10], if we make the change of variables,

  $$A = \frac{1}{L} \ , \quad s = \epsilon^{-2/3}(\beta_0 - \epsilon\tau) \ ,$$

and use the relation $\beta = A_{\tau\tau}/A$, (50) is transformed into Airy's equation

$$A_{ss} = sA \ .$$

The initial condition is given at $s_0 := s(t = 0) = \epsilon^{-2/3}\beta_0 > 0$ and s is monotonically decreasing as t increases. The solution of Airy's equation is a linear combination of the Airy and Bairy functions:

$$A = k_1 \mathrm{Ai}(s) + k_2 \mathrm{Bi}(s),$$

where k_1 and k_2 are constants. We recall that for $s > 0$, as s decreases $\mathrm{Bi}(s)$ decays exponentially while $\mathrm{Ai}(s)$ increases exponentially (e.g., [1]). Therefore, as t increases $A \sim k_1 \mathrm{Ai}(s)$ and focusing is arrested when $\mathrm{Ai}(s)$ attains its global maximum at $s_2 \approx -1.0$. Although $\mathrm{Ai}(s)$ oscillates as $s \to -\infty$, the reduced system is valid only until $s_3 \approx -2.3$ where $\mathrm{Ai}(s)$ vanishes, corresponding to a complete defocusing ($L = \infty$).[7] Therefore, in this case there is a single focusing-defocusing event, *regardless of how small δ is.*

The amount of power loss in a collapsing event can be estimated by[8]

$$\Delta N \sim M \int_{\tau(s_0)}^{\tau(s_3)} \epsilon \, d\tau \sim M(\beta_0 - \epsilon^{2/3} s_3) \ .$$

Therefore, in the limit $\delta \to 0$, the amount of power loss is equal to $M\beta_0$, i.e., the excess power above critical of the solution.

- When $q > 2$, damping effects are even stronger. As a result, damping always arrests blowup *regardless of how small δ is.* When the initial power is slightly above critical, (49) is valid from the onset of self-focusing and self-focusing dynamics consists of a single focusing-defocusing cycle. However, if the initial power is highly above critical, numerical simulations show a pattern of several focusing-defocusing cycles, with abrupt power losses due to dissipation at the times of maximal focusing [23].

6. Supercritical damped NLS. It is interesting to compare the effect of linear damping in the critical NLS with its effect in the D-dimensional supercritical NLS

$$i\psi_t + \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_D^2} \right) \psi + |\psi|^{2\sigma}\psi + i\delta\psi = 0 \ , \quad \sigma D > 2 \ . \tag{51}$$

Tsutsumi proved that the conditions

$$H(0) \le 0$$

and

$$\delta \frac{2\sigma}{\sigma D - 2} V(0) + V'(0) \le 0 \ , \tag{52}$$

where V is the variance, lead to finite-time blowup of the solution of (51) [34]. However, this result does not carry over to the critical case $\sigma D = 2$.

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[7] Strictly speaking, the reduced system is not valid once $L \gg 1$. At that stage, however, the NLS solution simply continues to defocus, as does the solution of the reduced system [10].

[8] Nonadiabatic effects, $\nu(\beta)$, represent power *radiation* and do not contribute to overall power loss.

From the condition (52) it follows that there is blowup if

$$\delta \in 0\left[0, \quad -\frac{V'(0)}{V(0)}\frac{\sigma D-2}{2\sigma}\right] .$$

Since it is reasonable to assume that Conjecture 1 is also true in the supercritical case, one can expect that Theorem 2.4 remains true in the supercritical case, i.e., for a given initial condition ψ_0 which leads to blowup in the undamped supercritical NLS, there exists a threshold value $\delta_{th} = \delta_{th}(\psi_0)$ such that there is finite-time blowup for $\delta \le \delta_{th}$ and global existence for $\delta > \delta_{th}$. However, since blowup in the supercritical NLS is much less sensitive to small perturbations, δ_{th} should be much larger than in the critical case.

The numerical results of Akrivis et al. [2] support this qualitative picture. In this study a Galerkin finite-element method was used to integrate the damped NLS, (2), in two dimensions (critical case) and in three dimensions (supercritical case). The results suggest that in the supercritical damped NLS there is also a critical threshold for δ and that in the two dimensional damped NLS "much smaller values of δ were needed to lead to definite blowup" compared with the three dimensional damped supercritical NLS.

7. Numerical method. We solve (3) in the radially-symmetric case by extending the method of *dynamic rescaling* for the unperturbed NLS [27] to the case of the damped NLS. To do that, we introduce the rescaling transformation for the function u and for the independent variables t and r

$$u(t,r) = \frac{1}{L(t)}V(\tau,\rho) \ , \quad \rho = \frac{r}{L(t)} \ , \quad \tau(t) = \int_0^t \frac{1}{L^2(s)}\,ds \ .$$

The rescaled function $V(\tau,\rho)$ satisfies

$$iV_\tau + \Delta V + e^{-2\delta t}|V|^2 V - ia(\rho V)_\rho = 0 \ ,$$

where

$$a(\tau) = L\frac{dL}{dt} = \frac{1}{L}\frac{dL}{d\tau} \qquad \text{and} \qquad t(\tau) = \int_0^\tau L^2(s)\,ds \ .$$

The choice

$$a(\tau) = -\frac{e^{-2\delta t}}{G(0)}\int_0^\infty |V|^2\,\mathrm{Im}(V\Delta V^*)\,\rho d\rho \ ,$$

where

$$G(\tau) = \int_0^\infty |V_\rho|^2\,\rho d\rho \ ,$$

has the useful property that

$$G_\tau = 2a\Big(G(\tau) - G(0)\Big) \ .$$

Therefore, global smoothness of V is maintained since $G(V) \equiv G_0$ and $a < 0$ when the solution is focusing.

The equation for V is solved on a fixed grid by combining a Crank–Nicholson implicit method for the Laplacian with a predictor-corrector stage for the other terms. For more details, see [5, 24]. As a consistency check, we monitor the conservation of $\int_0^\infty |V|^2 \rho d\rho$ and of G.

We use a bisection approach in order to find δ_{th}. The algorithm determines that blowup occurs when L becomes smaller than 10^{-6} and that blowup is not going to occur if L_t becomes positive after the initial transient.

8. A final remark. The NLS model (1) for beam propagation through a Kerr medium, in which damping is neglected, leads to finite-time singularity. As we have seen, when linear damping is included in the model, it acts as a defocusing mechanism which delays the onset of blowup and may even arrest it. However, linear damping arrests blowup only when it is "sufficiently large" (or, more accurately, sufficiently "nonsmall") and it *does not* prevent the singularity formation when $\delta < \delta_{th}$. This kind of effect on singularity formation distinguishes linear damping from all other defocusing perturbations of NLS analyzed so far using modulation theory [10, 12], which always arrests blowup, regardless of how small they initially are. Therefore, linear damping is not a potential candidate for the regularizing mechanism that would allow extension of solutions on NLS beyond the blowup point. However, the role of "viscosity" can be played by nonlinear damping with power greater or equal to three, since it always arrests blowup, regardless of how small it initially is.

Appendix A. Physical value of $\boldsymbol{\delta}$.

The propagation of laser beams is governed by the vectorial Maxwell equations. In the case of cw (continuous wave) laser beams, Maxwell equations be reduced to the scalar Helmholtz equation [22]

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] E(z,x,y) + k^2 E = 0 \ , \quad k^2 = \frac{\omega_0^2 n^2}{c^2} \ , \tag{53}$$

where ω_0 is frequency, c is speed of light, and n is the index of refraction. The linear Schrödinger equation for the electric field envelope ψ, i.e., (2) without the nonlinear term, is derived from (53) by using the substitution $E = \exp(ik_0 z)\psi(z,x,y)$ with $k_0 = \sqrt{(\mathrm{Re}(k^2)}$, changing to the nondimensional variables

$$\tilde{t} = \frac{z}{2L_{DF}} \ , \quad \tilde{x} = \frac{x}{r_0} \ , \quad \tilde{y} = \frac{y}{r_0}$$

and dropping the nonparaxial term ψ_{zz}. Therefore,

$$\delta = r_0^2 \, \mathrm{Im}\,(k^2) \ .$$

In addition, in transparency, $\mathrm{Re}\,(k_0^2) \sim k_0^2$ and the expression for δ can be rewritten as

$$\delta \sim r_0^2 k_0^2 \frac{\mathrm{Im}(k^2)}{\mathrm{Re}(k^2)} \ .$$

In terms of the absorption coefficient α (e.g., [17])

$$\delta = \alpha L_{DF} \ .$$

For example, for water in the visible spectrum $\alpha < 3 \times 10^{-3}$ cm^{-1} [17]. Therefore, when L_{DF} is on the order of 1 cm, δ is less than one percent.

Appendix B. $C_{1,2}(\Omega) = C_{1,2}(\Re^2)$.

LEMMA B.1. *Let $\Omega \subset \Re^2$ be a smooth simply-connected domain and let $C_{1,2}(\Omega)$ be the optimal constant in the Gagliardo–Nirenberg inequality*

$$||f||^4_{L^4(\Omega)} \le C_{1,2}(\Omega)||f||^2_{L^2(\Omega)}||\nabla f||^2_{L^2(\Omega)} \ , \quad f \in H^1_0(\Omega) \ .$$

Then,

$$C_{1,2}(\Omega) = C_{1,2}(\Re^2) \ .$$

Proof. For any $f \in H^1_0(\Omega)$, we define the corresponding $\tilde f \in H^1(\Re^2)$ by

$$\tilde f = \begin{cases} f, & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

Therefore,

$$\frac{1}{C_{1,2}(\Omega)} = \inf_{f \in H^1_0(\Omega)} \frac{||f||^2_{L^2(\Omega)}||\nabla f||^2_{L^2(\Omega)}}{||f||^4_{L^4(\Omega)}} = \inf_{\tilde f} \frac{||\tilde f||^2_2 ||\nabla \tilde f||^2_2}{||\tilde f||^4_4}$$
$$\ge \inf_{f \in H_1(\Re^2)} \frac{||f||^2_2||\nabla f||^2_2}{||f||^4_4} = \frac{1}{C_{1,2}(\Re^2)} \ .$$

On the other hand, let us define $f_\epsilon \in H^1_0(\Omega)$ by

$$f_\epsilon(r) = \begin{cases} \frac{1}{\epsilon} R\left(\frac{r}{\epsilon}\right), & r \le M/2, \\ g_\epsilon(r), & M/2 \le r \le M, \\ 0, & |x| \ge M, \end{cases}$$

where M is a positive number such that $\{|x| \le M\} \subset \Omega$, R is the Townes soliton (7), and $g_\epsilon(r)$ is a smooth monotonically decreasing function such that $g_\epsilon(M) = 0$ and $g_\epsilon(M/2) = (1/\epsilon)R(M/2\epsilon)$. Since $R(r) \sim c\exp(-r)r^{-1/2}$ for $r \gg 1$, we have that

$$\frac{1}{C_{1,2}(\Omega)} \le \lim_{\epsilon \to 0} \frac{||f_\epsilon||^2_2||\nabla f_\epsilon||^2_2}{||f_\epsilon|^4_4} = \frac{||R||^2_2||\nabla R||^2_2}{||R|^4_4} \ .$$

Therefore, in light of [36]

$$\frac{1}{C_{1,2}(\Re^2)} = \frac{||R||^2_2||\nabla R||^2_2}{||R|^4_4} \ ;$$

the lemma is proved. ∎

Finally, we note that since (as in the free-space case [36])

$$N_c(\Omega) = \frac{2}{C_{1,2}(\Omega)} \ ,$$

we get the following.

COROLLARY B.2. *The lower bound for the critical power in bounded domains is the same as in free-space, i.e.,*

$$N_c(\Omega) = N_c(\Re^2) \ .$$

Acknowledgments. We would like to thank the referees for numerous useful suggestions. We also thank Steve Schochet, Frank Merle, and Alexander Gaeta for useful comments and Boaz Ilan for the (2+1)D simulations in section 4.

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