Extender based forcings, fresh sets and Aronszajn trees

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Abstract

Extender based forcings are studied with respect of adding branches to Aronszajn trees. We construct a model with no Aronszajn tree over $\aleph_{\omega+2}$ from the optimal assumptions. This answers a question of Friedman and Halilović [1].

The reader interested only in Friedman and Halilović question may skip the first section and go directly to the second.

1 No branches to κ^+- Aronszajn trees.

We deal here with Extender Based Prikry forcing, Long and short extenders Prikry forcing. Let us refer to [2] for definitions.

Theorem 1.1 *Extender based Prikry forcing over κ cannot add a cofinal branch to a κ^+-Aronszajn tree.*

Proof. Let $\langle T, \leq_T \rangle$ be a κ^+-Aronszajn tree. Denote by $\mathcal{P}$ the extender based Prikry forcing over κ. Suppose that $\mathcal{P}$ adds a cofinal branch through T. Let $\underset{\sim}{b}$ be a name of such branch and $0_{\mathcal{P}} \Vdash \underset{\sim}{b}$ is a κ^+-branch through T.

Let $p, q \in \mathcal{P}$ and $n < \omega$. We say that $q$is an n-extension of p iff $q \geq p$ and q is obtained from p by taking n-element sequence $\langle \eta_1, ..., \eta_n \rangle$ from the first n-levels of the tree of sets of measures one over the maximal coordinate of p, adding it to p and projecting to all permitted coordinates of p. Denote such q by $p^\frown \langle \eta_1, ..., \eta_n \rangle$.

For each $\alpha < \kappa^+$ and $p \in \mathcal{P}$ there are $n < \omega$ and $p_\alpha \geq^* p$ such that any n-extension of p_α decides $\underset{\sim}{b}(\alpha)$, as was shown in [5]. Note that the branch $\underset{\sim}{b} \restriction \alpha + 1$ is decided as well, since $T \in V$ and so the value at the level α determines uniquely the branch to it below. Denote by $n(p, \alpha)$ the least such n.

Lemma 1.2 *For each $p \in \mathcal{P}$ there are $p^* \geq^* p$ and $n^* < \omega$ such that for every $q \geq^* p^*$ and for every large enough $\alpha < \kappa^+$ we have $n(q, \alpha) = n^*$.*

Proof. Suppose otherwise. Define by induction a $\geq^*$-increasing sequence $\langle p_k \mid k < \omega \rangle$ of direct extensions of p and an increasing sequence $\langle \alpha_k \mid k < \omega \rangle$ of ordinals below κ^+ such that $n(p_k, \alpha_k) < n(p_{k+1}, \alpha_{k+1})$, for every $k < \omega$.

Find $p_\omega \in \mathcal{P}$ which is $\leq^*$-stronger than every p_k. Extend it to a condition q that decides $\underset{\sim}{b}(\alpha_\omega)$, where $\alpha_\omega = \bigcup_{k<\omega} \alpha_k$. Let m be the length of the normal sequence of q. Denote the decided value by t. Pick k with $n_k > m$. Then there are two n_k-extensions of q q_1, q_2 which decide $\underset{\sim}{b}(\alpha_{n_k})$ differently, say t_1 and t_2. But this is impossible since t must be above both t_1, t_2.

$\square$

Suppose for simplicity that p^* is the empty condition and $n^* = 1$.

We define by induction a sequence of conditions $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$ such that

1. p_α decides $\underset{\sim}{b}(\alpha)$ to be some $t_\alpha \in T$,
2. p_α,$p_{\alpha'}$ are compatible, for every α, α' in some $S \subseteq \kappa^+$ of cardinality κ^+.

Then $\{t \in T \mid \exists \alpha \in S, t \leq_T t_\alpha\}$ is a cofinal branch in T, since $\alpha < \alpha', \alpha, \alpha' \in S$ implies by (2) that $t_\alpha <_T t_{\alpha'}$. This is a contradiction because T is a κ^+-Aronszajn tree.

Let q_0 be a direct extension of $0_{\mathcal{P}}$ such that every 1-extension of it decides $\underset{\sim}{b}(0)$. Set p_0 to be a 1-extension of q_0 by some ν_0 from the measure one set of its maximal coordinate. Consider a condition p_0' which is obtained from p_0 by removing the projection $(\nu_0)^0$ of ν_0 to the normal measure of the extender, by creating a new maximal coordinate and moving to it the tree of p_0 putting the first level to be κ.

Let q_1 be a direct extension of p_0' such that every 1-extension of it decides $\underset{\sim}{b}(1)$. Set p_1 to be a 1-extension of q_1 by some ν_1 from the measure one set of its maximal coordinate.

Continue further by induction. Suppose that $\langle p_\beta \mid \beta < \alpha \rangle$ is defined. Define p_α.

If α is a successor ordinal then proceed as above. Suppose that α is a limit ordinal. If $\mathrm{cof}(\alpha) < \kappa$, then pick a cofinal in α sequence $\langle \alpha_i \mid i < \mathrm{cof}(\alpha) \rangle$ and combine conditions $\langle p'_{\alpha_i} \mid i < \mathrm{cof}(\alpha) \rangle$ into one condition q'_α. Then pick a direct extension q_α of q'_α such that every 1-extension of it decides $\underset{\sim}{b}(\alpha)$. Set p_α to be a 1-extension of q_α by some ν_α from the measure one set of its maximal coordinate.

Suppose that $\mathrm{cof}(\alpha) = \kappa$. Pick a cofinal in α sequence $\langle \alpha_i \mid i < \kappa \rangle$. Combine conditions $\langle p'_{\alpha_i} \mid i < \kappa \rangle$ into a single condition q'_α as follows. For each $\eta \in \mathrm{supp}(p'_{\alpha_i})$ put a barrier i, add a new maximal coordinate, move trees of p'_{α_i} to it, take the diagonal intersection of them leaving the first level to be κ. Now proceed as before and define q_α,p'_α, ν_α and p_α.

This completes the construction.

There are a stationary $S \subseteq \kappa^+$ and $\nu^* < \kappa$ such that for every $\alpha \in S$ we have $(\nu_\alpha)^0 = \nu^*$.
$\square$

J. Hamkins defined in [7] the following two useful notions:

Definition 1.3 (Hamkins) Let $V \subseteq V_1$. δ-approximation property holds between V and V_1 iff for every set A of ordinals in V_1, if $A \cap a \in V$ for all $a \in V$ with $V \models |a| < \delta$, then $A \in V$.

Definition 1.4 (Hamkins) Let $V \subseteq V_1$, $A \subseteq \lambda, A \in V_1$. A is called fresh iff for each $\alpha < \lambda$, $A \cap \alpha \in V$.

Theorem 1.5 *Let $Cohen(\omega)$ be the Cohen real forcing, $\langle \mathcal{P}, \leq, \leq^* \rangle$ be long or short extenders Prikry or extender based Prikry forcing over κ over $V^{Cohen(\omega)}$. Then in $V^{Cohen(\omega) * \mathcal{R}}$ there is no new fresh subsets of ordinals of cofinality bigger than κ.*

Proof. We refer to [3], sections 1,2 for definitions and basic properties of Long (and short) extenders forcing , a more detailed account may be found in [6]. Let us give here only a brief description. Conditions in this forcings are of the form
$p = \langle p_n \mid n < \omega \rangle$ such that there is $\ell(p) < \omega$ with p_n being a Cohen condition in $Cohen(\lambda, \kappa_n) = \{f \mid |f| \leq \kappa, f : \lambda \to \kappa_n\}$, for each $n < \ell(p)$. If $n \geq \ell(p)$, then $p_n = \langle a_n, f_n, A_n \rangle$, where a_n is a partial order preserving function from λ to κ_n (or just a subset of λ in Long extenders forcing) of cardinality $< \kappa_n$, A_n is a set of measure one for the measure of E_n which corresponds to the maximal element of $\operatorname{ran}(a_n)$ and f_n is in $Cohen(\lambda, \kappa_n)$.

Let $\mathcal{P}$ be the long extenders Prikry forcing over $\kappa = \bigcup_{n<\omega} \kappa_n$. Other two forcing notions are treated similar. Assume for simplicity that $\lambda = \kappa^+$. Let r be a Cohen real and $\underset{\sim}{X}$ is a $\mathcal{P}$-name of a subset of κ^+ with every initial segment in V. We will show that then $\underset{\sim}{X}$ is in V as well.

Let us work in $V[r]$. Consider the following $\mathcal{P}$-name:

$$\underset{\sim}{Y} := \{\langle p, \check{\alpha} \rangle \mid p \Vdash \check{\alpha} \in \underset{\sim}{X}\}.$$

Then for every G generic for the forcing $\langle \mathcal{P}, \leq^* \rangle$ and every $a \subseteq \kappa^+, a \in V, |a| < \kappa_0$ (one may take $|a| < \aleph_2$ instead) we have $a \cap \underset{\sim}{Y}_G \in V$. This holds, since for every $q \in \mathcal{P}$ there is $p \geq^* q$

such that for all $\alpha \in a$, $p || \alpha \in \underset{\sim}{X}$. Hence there is such $p \in G$. Then

$$a \cap \underset{\sim}{Y}_G = \{\alpha \in a \mid p \Vdash \alpha \in \underset{\sim}{X}\}.$$

Now the forcing $\langle \mathcal{P}, \leq^* \rangle$ is κ_0-closed, hence by Hamkins, the κ_0-approximation property holds between V and $V[r, G]$. In particular, $\underset{\sim}{Y}_G \in V$.

We apply this observation to conditions of $\mathcal{P}$ with different lengthes of trunks. Let us start with $0_{\mathcal{P}}$. Pick $p_0 \geq^* 0_{\mathcal{P}}$ and $Y_0 \in V$ such that

$$p_0 \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \underset{\sim}{Y} = \check{Y}_0.$$

Now let us construct by induction on $\nu \in Lev_0(p_0)$ a sequence $\langle p_0(\nu) \mid \nu \in Lev_0(p_0) \rangle$ of extensions of p_0 and a sequence $\langle Y_0(\nu) \mid \nu \in Lev_0(p_0) \rangle$ of subsets of κ^+ such that

1. $Y_0(\nu) \in V$,
2. $p_0(\nu) \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \underset{\sim}{Y} = \check{Y}_0(\nu)$,
3. the sequence $\langle p_0(\nu) \setminus \nu \mid \nu \in Lev_0(p_0) \rangle$ is $\leq^*$-increasing, where $p_0(\nu) \setminus \nu \geq^* p_0$ is the condition obtained from $p_0(\nu)$ by removing ν and its projections to $\mathrm{supp}(p_0)$ but leaving all the rest.

Let p_1 be a $\leq^*$-upper bound of $\langle p_0(\nu) \setminus \nu \mid \nu \in Lev_0(p_0) \rangle$.

Continue similar (starting with p_1) to the second level (dealing with pairs) and define p_2. Proceed further and define p_3, p_4, etc. Let p_ω be a $\leq^*$-upper bound of $\langle p_n \mid n < \omega \rangle$.

Claim 1 Suppose that $q \geq p_\omega$ and for some $\alpha < \kappa^+$,

$$q \Vdash \alpha \in \underset{\sim}{X} \quad (\text{or } q \Vdash \alpha \notin \underset{\sim}{X}).$$

Then

$$p_\omega{}^\frown \langle \nu_1, ..., \nu_n \rangle \Vdash \alpha \in \underset{\sim}{X} \quad (\text{or } q \Vdash \alpha \notin \underset{\sim}{X}),$$

where $\langle \nu_1, ..., \nu_n \rangle$ are such that

$$p_\omega \leq p_\omega{}^\frown \langle \nu_1, ..., \nu_n \rangle \leq^* q$$

and the measure one sets of p_ω are intersected with projections of the measure one sets of q.

Proof. Assume that $q \Vdash \alpha \in \underset{\sim}{X}$.
Suppose for simplicity that $n = 1$. We have

$$p_\omega \leq p_\omega{}^\frown\nu \leq^* q.$$

Note that for any $s \geq^* p_\omega{}^\frown\nu$, if

$$s||\alpha \in \underset{\sim}{X},$$

then

$$s \Vdash \alpha \in \underset{\sim}{X}.$$

Suppose otherwise, i.e.

$$s \Vdash \alpha \notin \underset{\sim}{X}.$$

Recall that $p_\omega{}^\frown\nu \geq^* p_0(\nu)$ and

$$p_0(\nu) \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \underset{\sim}{Y} = \check{Y}_0(\nu).$$

We have $q \Vdash \alpha \in \underset{\sim}{X}$. Then $\langle q, \check{\alpha} \rangle \in \underset{\sim}{Y}$. But $q \geq^* p_\omega{}^\frown\nu \geq^* p_0(\nu)$, hence

$$p_0(\nu) \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \alpha \in \underset{\sim}{Y}.$$

Let now G be a generic set for $\langle \mathcal{P}, \leq^* \rangle$ with $s \in G$. Then in $V[r, G]$ we have $\alpha \in Y$. This means that there is $t \in G$ with $\langle t, \check{\alpha} \rangle \in \underset{\sim}{Y}$. So, by the definition of $\underset{\sim}{Y}$,

$$t \Vdash \alpha \in \underset{\sim}{X}.$$

But $s, t \in G$ so they are compatible also as conditions of $\langle \mathcal{P}, \leq \rangle$, which is impossible, since they force a contradictory information about α.

Assume now that

$$p_\omega{}^\frown\nu \nVdash \alpha \in \underset{\sim}{X}.$$

There is $t \geq p_\omega{}^\frown\nu$ such that

$$t \Vdash \alpha \notin \underset{\sim}{X}.$$

Find $\eta_1, ..., \eta_k$ such that

$$t \geq^* p_\omega{}^\frown\nu^\frown\langle \eta_1, ..., \eta_k \rangle \geq p_\omega{}^\frown\nu.$$

The argument above shows that any $\leq^*$-extension of $p_\omega{}^\frown\nu^\frown\langle \eta_1, ..., \eta_k \rangle$ forces "$\alpha \notin \underset{\sim}{X}$". But take $q_1 \geq q$ be so that $q_1 \geq^* p_\omega{}^\frown\nu^\frown\langle \eta_1, ..., \eta_k \rangle$. Note $\eta_1, ..., \eta_k$ have pre-images in measure one sets of q so such q_1 exists. We have

$$q_1 \Vdash \alpha \in \underset{\sim}{X},$$

as $q_1 \geq q$. Contradiction.
$\square$ of the claim.

Now let G be a generic for $\langle \mathcal{P}, \leq \rangle$ with $p_\omega \in G$.
Clearly for every $\alpha < \kappa^+$ there is $q \in G, q \geq p_\omega$ which decides "$\alpha \in \underset{\sim}{X}$". Then, by the claim, an extension of a form $p_\omega{}^\frown\langle \eta_1, ..., \eta_n\rangle$ of p_ω which is $\leq^*$-weaker than q already decides the statement and does it the same way. We have then $p_\omega{}^\frown\langle \eta_1, ..., \eta_n\rangle \in G$ and the coordinates of measure one sets of $p_\omega{}^\frown\langle \eta_1, ..., \eta_n\rangle$ are the same (above the trunk) as those of p_ω.
The forcing $\langle \mathcal{P}, \leq \rangle$ preserves κ^+. Then for κ^+-many α's we will have q's in G which decide "$\alpha \in \underset{\sim}{X}$" and have trunks of the same length. Then there will be a single sequence $\langle \eta_1, ..., \eta_n\rangle$ with $p_\omega{}^\frown\langle \eta_1, ..., \eta_n\rangle$ making the same decisions.

Consider now the subforcing

$$\mathcal{P}/p_\omega := \{s \in \mathcal{P} \mid s \geq p_\omega, \forall n \geq \ell(p_\omega), a_n(s) = a_n(p_\omega)\}.$$

Then $\mathcal{P}/p_\omega$ is just equivalent to the usual tree Prikry forcing. We can deal with coordinates of sets of measure one of p_ω and ignore the rest.

The argument of the previous paragraph implies that only $G \restriction \mathcal{P}/p_\omega$ is needed in order to decide $\underset{\sim}{X}$ completely.

It is easy to finish now. Recall that every initial segment of X is in V. Let $H \subseteq \mathcal{P}/p_\omega$ be generic. Find $n < \omega$ such that for κ^+-many β's there is a condition in H with a trunk of the length n which decides $\underset{\sim}{X} \restriction \beta$. Let $\langle \eta_1, ..., \eta_n\rangle$ be this trunk (it should be the same due to compatibility of members of H). Then

$$X = \bigcup\{Z \subseteq \kappa^+ \mid \exists \beta < \kappa^+ \exists q \geq^* p_\omega{}^\frown\langle \eta_1, ..., \eta_n\rangle \quad q \Vdash \underset{\sim}{X} \cap \beta = Z\},$$

since any two conditions with the same trunk are compatible in $\mathcal{P}/p_\omega$. The right side of the equality is obviously in V. Hence $X \in V$.
$\square$

Remark 1.6 1. Note that the forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$ over V adds fresh subsets to κ^+. Thus let $G \subseteq \mathcal{P}$ be a generic subset. Consider a set

$$A := \{\alpha < \kappa^+ \mid \exists p \in G, \ell(p) > 0, p = \langle p_n \mid n < \omega\rangle, p_0(\alpha) = 0\}.$$

Then for each $\beta < \kappa^+$ the set $Acap\beta$ is in V, since a single one extension decides it completely.

2. By [4], the Prikry forcing does not add new fresh subsets to κ^+ (or to ordinals of cofinality $\geq \kappa^+$.

Theorem 1.7 *Let Q be a forcing of cardinality $\leq \kappa$ and $\mathcal{P}$ be a forcing in V^Q that preserves κ^+ and does not add new fresh (relatively to V) subsets to κ^+. Suppose $\langle T, \leq_T \rangle$ is a κ^+-Aronszajn tree in V^Q. Then $\mathcal{P}$ does not add κ^+-branches to T.*

Proof. Suppose otherwise. Assume without loss of generality that $Lev_\alpha(T) = [\kappa\cdot\alpha, \kappa\cdot\alpha+\kappa)$. Let $\underset{\sim}{b}$ be a $Q * \mathcal{P}$ name such that

$$\langle 0_Q, 0_{\mathcal{P}} \rangle \Vdash \underset{\sim}{b} \text{ is a } \kappa^+ - \text{ branch through } \underset{\sim}{T}.$$

Let $G * H$ be a generic subset of $Q * \mathcal{P}$. There are $s \in G$,$\nu < \kappa$ and $S \subseteq \kappa^+$ stationary such that for every $\alpha \in S$ there is $p \in H$ with

$$\langle s, p \rangle \Vdash \kappa \cdot \alpha + \nu \in \underset{\sim}{b}.$$

Then for every $\alpha < \beta, \alpha, \beta \in S$ we have

$$s \Vdash_Q \kappa \cdot \alpha + \nu \leq_T \kappa \cdot \beta + \nu,$$

since otherwise there will be some $s' \geq s$ which forces "$\kappa \cdot \alpha + \nu, \kappa \cdot \beta + \nu$ are incompatible in T". Pick $p \in H$ such that

$$\langle s, \underset{\sim}{p} \rangle \Vdash \kappa \cdot \alpha + \nu \in \underset{\sim}{b}$$

and

$$\langle s, \underset{\sim}{p} \rangle \Vdash \kappa \cdot \beta + \nu \in \underset{\sim}{b}.$$

But then $\langle s', \underset{\sim}{p} \rangle \geq \langle s, \underset{\sim}{p} \rangle \Vdash \kappa \cdot \alpha + \nu \in \underset{\sim}{b}, \kappa \cdot \beta + \nu \in \underset{\sim}{b}$ and this is impossible.

Consider now the set T_* which consists of all ordinals $\kappa\cdot\alpha+\nu$ such that $\langle s, 0_P \rangle \nVdash \kappa\cdot\alpha+\nu \notin \underset{\sim}{b}$. Define an order $\leq_*$ on T_* (in V) as follows:

$$\kappa \cdot \alpha + \nu \leq_* \kappa \cdot \beta + \nu$$

iff

$$s \Vdash_Q \kappa \cdot \alpha + \nu \leq_T \kappa \cdot \beta + \nu.$$

The tree T_* need not be a κ^+-Aronszajn tree since its levels may have cardinality κ^+, but still its hight is κ^+ and it has no κ^+-branches. Thus for a given $\delta < \kappa^+$ pick $\beta < \kappa^+$ such that $otp(\beta \cap S) \geq \delta$ and for some $p \in H$,

$$\langle s, p \rangle \Vdash \kappa \cdot \beta + \nu \in \underset{\sim}{b}.$$

Then $\kappa \cdot \beta + \nu \in T_*$ and for each $\alpha \in S \cap \beta$ we have

$$s \Vdash_Q \kappa \cdot \alpha + \nu \leq_T \kappa \cdot \beta + \nu,$$

as was observed above. Hence the level of $\kappa \cdot \beta + \nu$ in T_* is at least δ.
T_* cannot have κ^+-branches, since any such branch will generate a κ^+-branch in T.

Now, a branch b translates easily into κ^+-branch c of T_*. Just set

$$c = \{\xi \in T_* \mid \exists \beta \in S, \kappa \cdot \beta + \nu \in b, \xi \leq_* \kappa \cdot \beta + \nu\}.$$

Then c will be a new fresh subset of κ^+. Contradiction.

$\square$

Corollary 1.8 *Let $Cohen(\omega)$ be the Cohen real forcing, $\langle \mathcal{P}, \leq, \leq^* \rangle$ be long or short extenders Prikry or extender based Prikry forcing over κ over $V^{Cohen(\omega)}$. Then $\langle \mathcal{P}, \leq, \leq^* \rangle$ does not add κ^+-branches to κ^+-Aronszajn trees in $V^{Cohen(\omega)}$.*

Let us show directly that Extender based forcings for a singular κ do not add κ^+-branches to κ^+-Aronszajn trees.

Theorem 1.9 *Extender based forcings for a singular κ do not add κ^+-branches to κ^+-Aronszajn trees.*

Proof. Let $\langle \mathcal{P}, \leq, \leq^* \rangle$ be such a forcing, $\langle T, \leq_T \rangle$ a κ^+-Aronszajn tree and $\underset{\sim}{b}$ a name of a κ^+-branch. The idea will be to find ω-levels $\langle \alpha_n \mid n < \omega \rangle$ and for every $n < \omega$ some δ_n points over the level α_n which are potential elements of $\underset{\sim}{b}$ with $\{\delta_n \mid n < \omega\}$ cofinal in κ. Then $|\prod_{n<\omega} \delta_n| = \kappa^+$ which allows to argue that the level $\cup_{n<\omega}\alpha_n$ has κ^+ many points.

Recall that each $p \in \mathcal{P}$ is of the form $\langle p_n \mid n < \omega \rangle$ and there is $\ell(p) < \omega$ such that p_n's with $n < \ell(p)$ are Cohen conditions which are κ^+-closed. For every $n \geq \ell(p)$, p_n is of the form $\langle a_n, A_n, f_n \rangle$ with A_n's being sets of measure one for the measure of the extender corresponding to $\max(\mathrm{ran}(a_n))$.
Denote by $A_{\leq m}(p)$ the product of the first m measure one sets of p. Given $m < \omega$, by *an m-extension of p* we mean an extension of p obtained by choosing some $\langle \eta_1, ..., \eta_m \rangle \in A_{\leq m}(p)$ and extending p by adding it. Denote such extension by $p^\frown\langle \eta_1, ..., \eta_m \rangle$.

Use Lemma 1.2 and fix $n^* < \omega$,p^* such that for every $q \geq^* p^*$ and $\alpha < \kappa^+$ there is $q_\alpha \geq^* q$ with any n^*-extension of it deciding $\underset{\sim}{b} \restriction \alpha$.

For $k, 1 \leq k < \omega$, let $r \geq^{*k} q$ means that $r \geq^* q$ and for every level $i \leq k$ the conditions r, q have the same maximal coordinate at level i.

Now by the standard argument for each q, $k, 1 \leq k < \omega$, and $\alpha < \kappa^+$ there are $n < \omega$ and $q_{\alpha k} \geq^{*k} q$ such that any n-extension of $q_{\alpha k}$ decides $\underset{\sim}{b} \restriction \alpha$. Denote by $n(q,k,\alpha)$ the least such n.

Lemma 1.10 *$n^* = n(q,k,\alpha)$, for any $k, 1 \leq k < \omega$.*

Proof. Clearly $n^* \leq n(q,k,\alpha)$. Let us show the equality. Run the standard argument trying to decide $\underset{\sim}{b} \restriction \alpha$ starting with q for each of its k-extensions, i.e. for any choice of $\langle \eta_1, ..., \eta_k \rangle$ in sets of measure one of the first k-levels of q. Then shrink this first k sets in order to have the same conclusion (n-extension decides or not $\underset{\sim}{b} \restriction \alpha$). Finally we will have $r \geq^{*k} q$ and $n < \omega$ such that

1. every n-extension of r decides $\underset{\sim}{b} \restriction \alpha$.
2. for every $\langle \eta_1, ..., \eta_k \rangle, \langle \eta'_1, ..., \eta'_k \rangle$ two sequences from the first k sets of measure one of r and any $m < \omega$, if there is an m-extension of some $t \geq^* r^\frown \langle \eta_1, ..., \eta_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$ then any m-extension of $r^\frown \langle \eta'_1, ..., \eta'_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$.

Suppose for a moment that $n > n^*$. Pick some $s \geq^* r$ such that every n^*-extension of s decides $\underset{\sim}{b} \restriction \alpha$. Clearly, s need not be $*k$-extension, i.e. maximal coordinates of levels $\leq k$ may increase. Let $\langle \rho_1, ..., \rho_k \rangle$ be a sequence from the first k sets of measure one of s. Denote its projection to the first k maximal coordinates of r by $\langle \eta_1, ..., \eta_k \rangle$. Then $s^\frown \langle \rho_1, ..., \rho_k \rangle \geq^* r^\frown \langle \eta_1, ...\eta_k \rangle$ and $n^* - k$-extension of $s^\frown \langle \rho_1, ..., \rho_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$. So already $n^* - k$-extension of $r^\frown \langle \eta_1, ..., \eta_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$ and then the same holds if we replace $\langle \eta_1, ..., \eta_k \rangle$ by any other sequence from the first k-sets of measures one of r. But this means that any n^*-extension of r already decides $\underset{\sim}{b} \restriction \alpha$.
$\square$ of the lemma.

Suppose for simplicity that p^* is just $0_{\mathcal{P}}$.

Fix a cofinal in κ sequence of regular cardinals $\langle \delta_n \mid n < \omega \rangle$ such that for some increasing sequence $\langle k_n \mid n < \omega \rangle$ of numbers above n^* we have $2^{\kappa_{k_n}} < \delta < \kappa_{k_n+1}$.
Let $p \in \mathcal{P}$, $\delta = \delta_m$, for some $m < \omega$ and $k = k_m$.
Define by induction a continuous $\in$-chain of elementary submodels $\langle M_i \mid i \leq \delta \rangle$ and a sequence of conditions $\langle q_i \mid i \leq \delta \rangle$ so that

1. $p \in M_0$,
2. $\tilde{q}_i \geq^{*k} p$, where $\tilde{q}_i$ is obtained from q_i by intersecting its first n^* sets of measure one with those of p.

This means that the first n^*-sets of measure one of q_i may be different (larger) than those of p.

3. If $i \leq j$, then $\tilde{q}_j \geq^{*k} q_i$, where $\tilde{q}_j$ is obtained from q_j by intersecting its first k sets of measure one with those of q_i,

4. $|M_i| < \delta$, for every $i < \delta$,

5. ${}^{\kappa_k}M_i \subseteq M_i$, for $i = 0$ or i a successor ordinal,

6. each n^*-extension of q_i decides $\underset{\sim}{b} \restriction \mu_i$, where $\mu_i = \sup(M_i \cap \kappa^+)$.

7. (maximality condition)
   If for some n^*-sequence $\langle \eta_1, ..., \eta_{n^*} \rangle$ we have

   (a) $q_i{}^\frown\langle \eta_1, ..., \eta_{n^*} \rangle \in \mathcal{P}$,

   (b) some $t \geq^{*k} q_i{}^\frown\langle \eta_1, ..., \eta_{n^*} \rangle$ decides $\underset{\sim}{b} \restriction \mu_i$,

   then $\langle \eta_1, ..., \eta_{n^*} \rangle \in A_{\leq n^*}(q_i)$ and already $q_i{}^\frown\langle \eta_1, ..., \eta_{n^*} \rangle$ decides $\underset{\sim}{b} \restriction \mu_i$

8. $q_i \in M_{i+1}$ and it is the least possible in some fixed in advance well order.

Note that the sequence $\langle q_i \mid i \leq \delta \rangle$ remains un-effected once we replace in p its first n^* sets of measure one but keep all the rest unchanged.

Denote the final M_δ by M, μ_δ by μ and q_δ by q. Note that $\delta > 2^{\kappa_k}$ so sets of measures one of the first k-levels of δ many of q_i's are the same. Assume then that for all i's they are the same. Just shrink to i's with a constant value otherwise. Assume that the same sets stand in q as well.
Then each n^*-extension of q decides $\underset{\sim}{b} \restriction \mu$. Denote by $X = \{t_\xi \mid \xi < \kappa_{n^*}\}$ the set of all such decisions. Note that $|X| \leq \kappa_{n^*} < \delta$.
The set of decisions of n^*-extensions of q_i will be $X_i = \{t_\xi \restriction \mu_i \mid \xi < \kappa_{n^*}\}$.
Note that X and each particular t_ξ need not be in M, but X_i's and the initial segments of t_ξ are in M.
Let us fix $i^* < \delta$ such that all the branches in X already split before the level μ_{i^*}. It is possible since $\delta > \kappa_n$.
Suppose that
(*) there is $i, i^* < i < \delta$ such that for every $\xi \in \kappa^+ \cap M$ and $r \geq^{*k} q_i, r \in M$ all of which n^*-direct extensions decide $\underset{\sim}{b} \restriction \xi$ we have the following:

for all $\langle \eta_1, ..., \eta_{n^*} \rangle$ an n^*-sequence from the first n^* measure one sets of r,

$$r^\frown \langle \eta_1, ..., \eta_{n^*} \rangle \Vdash \underset{\sim}{b} \restriction \xi = s,$$

and s is an initial segment of $t_\zeta \in X$, where the n^*-extension of q_{i^*}

$$q_{i^*}{}^\frown \langle \eta_1, ..., \eta_{n^*} \rangle \Vdash \underset{\sim}{b} \restriction \mu_{i^*} = t_\zeta \restriction \mu_{i^*}.$$

Let us choose (in M) for each $\xi \in \kappa^+$ one $r_\xi \geq^{*k} q_i$ all of which n^*-direct extensions decide $\underset{\sim}{b} \restriction \xi$.

Then there are $\zeta < \kappa_{n^*}$, an unbounded $Z \subseteq \kappa^+, Z \in M$ and $\langle \eta_1^*, ..., \eta_{n^*}^* \rangle$ an n^*-sequence in M such that for every $\xi \in Z \cap M$ we have $r_\xi \leq r_\xi{}^\frown \langle \eta_1^*, ..., \eta_{n^*}^* \rangle$ forces "$\underset{\sim}{b} \restriction \mu_{i^*} = t_\zeta \restriction \mu_{i^*}$". Set

$$e := \{s \mid \exists \xi \in Z, r_\xi{}^\frown \langle \eta_1^*, ..., \eta_{n^*}^* \rangle \Vdash s = \underset{\sim}{b} \restriction \xi\}.$$

Then e will be a κ^+-branch in T. Which is impossible. Hence (*) is falls.
So, the following holds:
(**) for every $i, i^* < i < \delta$ there will be $\xi \in \kappa^+ \cap M$ and $r \geq^{*k} q_i, r \in M$ all of which n^*-direct extensions decide $\underset{\sim}{b} \restriction \xi$, such that

> for some $\langle \eta_1, ..., \eta_{n^*} \rangle$ an n^*-sequence from the first n^* measure one sets of r(or equivalently from q_i or from p), let
>
> $$r^\frown \langle \eta_1, ..., \eta_{n^*} \rangle \Vdash \underset{\sim}{b} \restriction \xi = s_{\langle \eta_1, ..., \eta_{n^*} \rangle}.$$
>
> Then $s_{\langle \eta_1, ..., \eta_{n^*} \rangle}$ is not an initial segment of $t_\zeta \in X$, where $q_{i^*}{}^\frown \langle \eta_1, ..., \eta_{n^*} \rangle$ forces "$\underset{\sim}{b} \restriction \mu_{i^*} = t_\zeta \restriction \mu_{i^*}$". By the choice of i^*, hence $s_{\langle \eta_1, ..., \eta_{n^*} \rangle}$ is not an initial segment of any other member of X as well.

Define a partition $F : A_{\leq n^*}(q) \to 2$ as follows.
Set $F(\vec{\eta}) = 0$ iff there is $i(\vec{\eta}), i^* < i(\vec{\eta}) < \delta$, such for every $i, i(\vec{\eta}) \leq i < \delta$, there are $\xi \in \kappa^+ \cap M, \xi > \mu_i$ and $r \geq^{*k} q_i, r \in M$ all of which n^*-direct extensions decide $\underset{\sim}{b} \restriction \xi$, $\vec{\eta} \in A_{\leq n^*}(r)$, such that

> if $r^\frown \vec{\eta} \Vdash \underset{\sim}{b} \restriction \xi = s_{\vec{\eta}}$, then $s_{\vec{\eta}}$ is not an initial segment of $t_\zeta \in X$, where $q_{i^*}{}^\frown \vec{\eta}$ forces "$\underset{\sim}{b} \restriction \mu_{i^*} = t_\zeta \restriction \mu_{i^*}$".

Set $X_0(p, \delta) = \{\vec{\eta} \in A_{\leq n^*}(p) \mid F(\vec{\eta}) = 0\}, X_1(p, \delta) = \{\vec{\eta} \in A_{\leq n^*}(p) \mid F(\vec{\eta}) = 1\}$. Once δ is fixed let us omit it.

Lemma 1.11 *$X_0(p)$ is of measure one (relatively to the first n^* measures of p).*

Proof. Otherwise $X_1(p)$ is of measure one. By the definition of F we have the following:

for every $\vec{\eta} \in X_1(p)$, for every $j, i^* < j < \delta$, there is $i(j, \vec{\eta}), j \leq i(j, \vec{\eta}) < \delta$, for every $\xi \in \kappa^+ \cap M, \xi > \mu_{i(j,\vec{\eta})}$ and $r \geq^{*k} q_{i(j,\vec{\eta})}, r \in M$ all of which n^*-direct extensions decide $\underset{\sim}{b} \restriction \xi$

> if $\vec{\eta} \in A_{\leq n^*}(r)$ and $r^\frown\vec{\eta} \Vdash \underset{\sim}{b} \restriction \xi = s_{\vec{\eta}}$, then $s_{\vec{\eta}}$ is an initial segment of $t_\zeta \in X$, where $q_{i^*}{}^\frown\vec{\eta}$ forces "$\underset{\sim}{b} \restriction \mu_{i^*} = t_\zeta \restriction \mu_{i^*}$".

Let $\langle \vec{\eta}_\tau \mid \tau < \kappa_{n^*} \rangle$ be an enumeration of $X_1(p)$. Define by induction an increasing continuous sequence $\langle j_\tau \mid \tau < \kappa_{n^*} \rangle$:
$j_0 = i^* + 1, j_1 = i(j_0, \vec{\eta}_0), ..., j_{\tau+1} = i(j_\tau, \vec{\eta}_\tau), ...$ Let $i^{**} = \bigcup_{\tau < \kappa_{n^*}} j_\tau$. Then for every $\vec{\eta} \in X_1(p)$, for every $\xi \in \kappa^+ \cap M$ and every $r \geq^{*k} q_{i^{**}}, r \in M$ all of which n^*-direct extensions decide $\underset{\sim}{b} \restriction \xi$

> if $\vec{\eta} \in A_{\leq n^*}(r)$ and $r^\frown\vec{\eta} \Vdash \underset{\sim}{b} \restriction \xi = s_{\vec{\eta}}$, then $s_{\vec{\eta}}$ is an initial segment of $t_\zeta \in X$, where $q_{i^*}{}^\frown\vec{\eta}$ forces "$\underset{\sim}{b} \restriction \mu_{i^*} = t_\zeta \restriction \mu_{i^*}$".

Now the contradiction follows similar to (*) above.
Let us choose (in M using elementarity) for each $\xi \in \kappa^+$ one $r_\xi \geq^{*k} q_{i^{**}}$ all of which n^*-direct extensions decide $\underset{\sim}{b} \restriction \xi$.

Then there are $\zeta < \kappa_{n^*}$, an unbounded $Z \subseteq \kappa^+, Z \in M$ and $\langle \eta_1^*, ..., \eta_{n^*}^* \rangle$ an n^*-sequence in $X_1(p)$ such that for every $\xi \in Z \cap M$ we have $\langle \eta_1^*, ..., \eta_{n^*}^* \rangle \in A_{\leq n^*}(r_\xi)$ and $r_\xi \leq r_\xi{}^\frown\langle \eta_1^*, ..., \eta_{n^*}^* \rangle$ forces "$\underset{\sim}{b} \restriction \mu_{i^*} = t_\zeta \restriction \mu_{i^*}$".
Set

$$e := \{s \mid \exists \xi \in Z, r_\xi{}^\frown\langle \eta_1^*, ..., \eta_{n^*}^* \rangle \Vdash s = \underset{\sim}{b} \restriction \xi\}.$$

Then e will be a κ^+-branch in T. Which is impossible.
□

Set $X_0(p) = \bigcap_{m<\omega} X_0(p, \delta_m)$.

Let us define $r \succeq^{n^*k} s$ iff $\tilde{r} \geq^{*k} s$, where $\tilde{r}$ is the condition obtained from r be intersecting its first n^* sets of measure one with those of s.

Lemma 1.12 *For every $p \in \mathcal{P}$ there is $p^* \geq^{*k} p$ such that for every $s \succeq^{n^*k} p^*$ there is $r \succeq^{n^*k} s$ with $X_0(r) = X_0(p^*)$.*

Proof. Suppose otherwise. Then

$$\exists p \forall p^* \geq^{*k} p \exists s(p^*) \succeq^{n^*k} p^* \forall r \succeq^{n^*k} s(p^*), X_0(r) \neq X_0(p^*).$$

Use completeness of $\langle \mathcal{P}, \succeq^{n^*k} \rangle$ to construct an $\succeq^{n^*k}$-increasing sequence $\langle p_\alpha \mid \alpha < (2^{\kappa_{n^*}})^+\rangle$ such that $p_{\alpha+1} = s(\tilde{p}_\alpha)$, where $\tilde{p}_\alpha$ is obtained from p_α by intersecting its first n^* sets of measure one with those of p. In particular, $\tilde{p}_\alpha \geq^{*k} p$ and so $s(\tilde{p}_\alpha)$ is defined. The sequence of q_i's which corresponds to p_α is effected by replacing p_α with $\tilde{p}_\alpha$. Hence $X_0(p_\alpha) = X_0(\tilde{p}_\alpha)$. Then there will be $\alpha < \beta < (2^{\kappa_{n^*}})^+$ with $X_0(p_{\alpha+1}) = X_0(p_{\beta+1})$. But this is impossible since $p_{\beta+1} \succeq^{n^*k} p_{\alpha+2} = s(\tilde{p}_{\alpha+1})$.
$\square$

Replace now the original p by p^*. Still denote it further by p. For each $\vec{\eta} \in A_{\leq n^*}(p) \cap X_0(p^*)$ and $i \geq i(\vec{\eta})$ pick $\xi(i, \vec{\eta})$ and $r(i, \vec{\eta})$ as in the definition of $F(\vec{\eta})$.

Build by induction sequences $\langle i_j \mid j < \delta\rangle$,$\langle \xi_j \mid j < \delta\rangle$,$\langle r_j(\vec{\eta}) \mid j < \delta\rangle$ such that

1. $r_j(\vec{\eta}) \geq^{*k} q_{i_j}$,
2. all n^*-direct extensions of $r_j(\vec{\eta})$ decide $\underset{\sim}{b} \restriction \xi_j$,
3. $i_j, \xi_j, r_j(\vec{\eta})$ satisfy the definition of $F(\vec{\eta})$,
4. $\mu_{i_j} > \xi_j$.

Finally we find a sequence $\langle p_j(\vec{\eta}) \mid j < \delta\rangle$, such that

1. $p_j(\vec{\eta}) \geq^{*k} r_j(\vec{\eta})$,
2. $X_0(p_j(\vec{\eta})) = X_0(p^*)$,
3. $\vec{\eta} \in A_{\leq n^*}(p_j(\vec{\eta}))$,
4. all of n^*-direct extensions of $p_j(\vec{\eta})$ decide $\underset{\sim}{b} \restriction \mu_\delta$.

It is possible by Lemma 1.12 and for the item 3 use the maximality condition (7) of the definition of M_i's and q_i's.

Now, using the above, we can find an increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ of levels of T and $\langle p_u^{\vec{\eta}} \mid u \in \prod_{n<k} \kappa_n, k < \omega, \vec{\eta} \in A_{\leq n^*}(p) \cap X_0(p^*)\rangle$ such that

1. if u is an initial segment of v, then $p_u^{\vec{\eta}} \leq^* p_u^{\vec{\eta}}$, for every $\vec{\eta} \in A_{\leq n^*}(p) \cap X_0(p^*)$,
2. all n^*-direct extensions of $p_u^{\vec{\eta}}$ decide $\underset{\sim}{b} \restriction \alpha_{|u|}$,
3. if u, v are incompatible then $p_u^{\vec{\eta}} {}^\frown \vec{\eta}$ and $p_v^{\vec{\eta}} {}^\frown \vec{\eta}$ provide incompatible decisions of $\underset{\sim}{b} \restriction \alpha_{|u|}, \underset{\sim}{b} \restriction \alpha_{|v|}$.

Fix some $\vec{\eta} \in A_{\leq n^*}(p) \cap X_0(p^*)$. Let $f \in \prod_{n<\omega} \kappa_n$. Let p_f be a $\leq^*$-upper bound of $\langle p^{\vec{\eta}}_{f\restriction n} {}^\frown \vec{\eta} \mid n < \omega\rangle$. Then p_f will decide $\underset{\sim}{b} \restriction \alpha_\omega + 1$, where $\alpha_\omega := \cup_{n<\omega}\alpha_n$. Different f's will give different decisions. But the number of f's is κ^+. So the level α_ω of T will have cardinality κ^+, which is impossible. Contradiction.
□

2 A model with no Aronszajn tree over κ^{++}, for singular κ.

Our aim in this section will be to prove the following theorem:

Theorem 2.1 *Assume GCH. Let $\kappa = \cup_{n<\omega}\kappa_n$ with $o(\kappa_n) = \kappa_n^{+n+2}$ and $\lambda > \kappa$ is a weakly compact. Then there is a generic extension $V^{\mathcal{P}}$ such that*

1. *GCH holds below κ,*
2. *κ^+ is preserved,*
3. *$\lambda = \kappa^{++}$,*
4. *$2^\kappa = \lambda$,*
5. *there is no κ^{++}-Aronszajn trees.*

Proof.

Suppose $\kappa = \cup_{n<\omega}\kappa_n$, $\langle \kappa_n \mid n < \omega\rangle$ increasing, $\lambda > \kappa$ and each of κ_n's carries an extender E_n (like in the long or short extender based forcings).
We refer to [3], sections 1,2 for definitions and basic properties of Short (and long) extenders forcing , a more detailed account may be found in [6]. Let us give here only a brief description. Conditions in this forcings are of the form
$p = \langle p_n \mid n < \omega\rangle$ such that there is $\ell(p) < \omega$ with p_n being a Cohen condition in $Cohen(\lambda, \kappa_n) = \{f \mid |f| \leq \kappa, f : \lambda \to \kappa_n\}$, for each $n < \ell(p)$. If $n \geq \ell(p)$, then $p_n = \langle a_n, f_n, A_n\rangle$, where a_n is a partial order preserving function from λ to κ_n (or just a subset of λ in Long extenders forcing) of cardinality $< \kappa_n$, A_n is a set of measure one for the measure of E_n which corresponds to the maximal element of ran(a_n) and f_n is in $Cohen(\lambda, \kappa_n)$.

We would like to make a small change here and allow Cohen parts of a condition to be names which depend on Prikry sequences added before. Namely, if $\alpha \in \mathrm{dom}(a_n)$, for each

$n > n_0$, then Cohen functions on $\lambda \setminus \alpha + 1$ may depend on the Prikry sequence of α. Still we require that $\langle dom(f_n) \mid n < \omega \rangle \in V$ as well as $\langle a_n \mid n \geq \ell(p) \rangle$. Also for each $\alpha < \lambda$, $\langle f_n(\alpha) \mid n < \omega \rangle$ should be in V, where α is a common point of domains of functions f_n (it is possible to present this forcing in a way that there is no such α's at all).

Denote such forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$. Define the following refinement $\leq^{*k}$ of $\leq$, for $k < \omega$: set $p \leq^{*k} q$ iff $p \leq q$ and $\mathrm{dom}(a_i(p)) = \mathrm{dom}(a_i(q))$, for every $i, \ell(p) \leq i \leq k$.

The next lemma is similar to the standard Prikry condition lemma for this types of forcings:

Lemma 2.2 *Let $p \in \mathcal{P}, k < \omega$ and σ is a statement of the forcing language. Then there is $q \geq^{*k} p$ which decides σ.*

Suppose now $\nu < \lambda$ is an ordinal of cofinality $\geq \kappa^{++}$. Let $\mathcal{P} \restriction \nu$ be the natural restriction of $\mathcal{P}$ to ν. Note that we need first to restrict ourself to a dense subset of $\mathcal{P}$ which consists of conditions p such that $a_n(p) \cap \nu$ has a maximal element which is also a maximal in the extender (E_n) order, and then restrict everything to ν.

Let $\langle \underset{\sim}{T}, \leq_{\underset{\sim}{T}} \rangle$ be a $\mathcal{P} \restriction \nu$-name of a $(\kappa^{++})^{V^{\mathcal{P} \restriction \nu}} = \nu$-Aronszajn tree. Suppose that the rest of the forcing, i.e. $\mathcal{P}/\mathcal{P} \restriction \nu$, adds a κ^{++}-branch. Denote it $\mathcal{P}$-name by $\underset{\sim}{b}$. Assume without loss of generality that the α-th level of T is just $[\kappa^+ \cdot \alpha, \kappa^+ \cdot \alpha + \kappa^+)$.

Lemma 2.3 *Let $\alpha < \nu, p \in \mathcal{P}$. Then there are $q_\alpha \geq^* p$, $n_\alpha < \omega$ and such that every n_α-extension $q_\alpha{}^\frown\langle \eta_1, ..., \eta_{n_\alpha} \rangle$ of q_α forces "$\underset{\sim}{b} \restriction \alpha = \underset{\sim}{t}_{\alpha, \langle \eta_1, \ldots, \eta_{n_\alpha} \rangle}$," for some $\mathcal{P} \restriction \nu$-name $\underset{\sim}{t}_{\alpha, \langle \eta_1, \ldots, \eta_{n_\alpha} \rangle}$.*

Similar the following slight strengthening of the previous lemma holds:

Lemma 2.4 *Let $\alpha < \nu, p \in \mathcal{P}$, $k < \omega$. Then there are $q_{\alpha k} \geq^{*k} p$, $n_{\alpha k} < \omega$ and such that every $n_{\alpha k}$-extension $q_{\alpha k}{}^\frown\langle \eta_1, ..., \eta_{n_\alpha} \rangle$ of $q_{\alpha k}$ forces "$\underset{\sim}{b} \restriction \alpha = \underset{\sim}{t}_{\alpha k, \langle \eta_1, \ldots, \eta_{n_\alpha} \rangle}$," for some $\mathcal{P} \restriction \nu$-name $\underset{\sim}{t}_{\alpha k, \langle \eta_1, \ldots, \eta_{n_\alpha} \rangle}$.*

Let us denote the least n_α for which there is q_α as in 2.3 by $n(p, \alpha)$ and the least $n_{\alpha k}$ for which there is $q_{\alpha k}$ as in 2.4 by $n(p, \alpha, k)$.

Lemma 2.5 *For each $p \in \mathcal{P}$ there are $p^* \geq^* p$ and $n^* < \omega$ such that for every $q \geq^* p^*$ and for every large enough $\alpha < \kappa^{++}$ we have $n(q, \alpha) = n^*$.*

Proof. Suppose otherwise. Define by induction a $\geq^*$-increasing sequence $\langle p_k \mid k < \omega \rangle$ of direct extensions of p and an increasing sequence $\langle \alpha_k \mid k < \omega \rangle$ of ordinals below κ^{++} such that $n(p_k, \alpha_k) < n(p_{k+1}, \alpha_{k+1})$, for every $k < \omega$.

Find $p_\omega \in \mathcal{P}$ which is $\leq^*$-stronger than every p_k. Extend it to a condition q that decides $\underset{\sim}{b} \restriction \alpha_\omega$, where $\alpha_\omega = \bigcup_{k<\omega} \alpha_k$. Let $m = \ell(q)$. Denote the decided value by $\underset{\sim}{t}$. Pick k with $n(p_k, \alpha_k) > m$. Then this contradicts the definition of $n(p_k, \alpha_k)$ being the least possible.
$\square$

Suppose that $p \in \mathcal{P}$ and p^*, n^* are given by the lemma.

Lemma 2.6 *$n^* = n(p^*, k, \alpha)$, for any $k < \omega$.*

Proof. Clearly $n^* \leq n(q, k, \alpha)$. Let us show the equality. Run the standard argument trying to decide $\underset{\sim}{b} \restriction \alpha$ starting with p^* for each of its k-extensions, i.e. for any choice of $\langle \eta_1, ..., \eta_k \rangle$ in sets of measure one of the first k-levels of q. Then shrink this first k sets in order to have the same conclusion (n-extension decides or not $\underset{\sim}{b} \restriction \alpha$). Finally we will have $r \geq^{*k} p^*$ and $n < \omega$ such that

1. every n-extension of r decides $\underset{\sim}{b} \restriction \alpha$,
2. for every $\langle \eta_1, ..., \eta_k \rangle, \langle \eta'_1, ..., \eta'_k \rangle$ two sequences from the first k sets of measure one of r and any $m < \omega$, if there is an m-extension of some $t \geq^* r^\frown \langle \eta_1, ..., \eta_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$ then any m-extension of $r^\frown \langle \eta'_1, ..., \eta'_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$.

Suppose for a moment that $n > n^*$. Pick some $s \geq^* r$ such that every n^*-extension of s decides $\underset{\sim}{b} \restriction \alpha$. Clearly, s need not be $*k$-extension, i.e. maximal coordinates of levels $\leq k$ may increase. Let $\langle \rho_1, ..., \rho_k \rangle$ be a sequence from the first k sets of measure one of s. Denote its projection to the first k maximal coordinates of r by $\langle \eta_1, ..., \eta_k \rangle$. Then $s^\frown \langle \rho_1, ..., \rho_k \rangle \geq^* r^\frown \langle \eta_1, ...\eta_k \rangle$ and $n^* - k$-extension of $s^\frown \langle \rho_1, ..., \rho_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$. So already $n^* - k$-extension of $r^\frown \langle \eta_1, ..., \eta_k \rangle$ decides $\underset{\sim}{b} \restriction \alpha$ and then the same holds if we replace $\langle \eta_1, ..., \eta_k \rangle$ by any other sequence from the first k-sets of measures one of r. But this means that any n^*-extension of r already decides $\underset{\sim}{b} \restriction \alpha$.
$\square$ of the lemma.

Suppose for simplicity that p^* is just $0_{\mathcal{P}}$.

Lemma 2.7 *Let $p \in \mathcal{P}$ and let δ be a regular cardinal above $2^{\kappa_{n^*}}$ and $2^{\kappa_k} < \delta < \kappa_{k+1}$ for some $k < \omega$. Then there are $\alpha < \kappa^+$, $q \geq^{*k} p \restriction \nu$ and a sequence of $(*, k)$-direct extensions $\langle p_\xi \mid \xi < \delta \rangle$ of p with the same sequence of sets of measures one up to the level k such that*

1. *$p_\xi \restriction \nu = q$,*
2. *every n^*-extension of p_ξ decides $\underset{\sim}{b} \restriction \alpha + 1$,*

3. *if $\xi \neq \xi'$, $p_\xi{}^\frown\langle \eta_1, ..., \eta_{n^*}\rangle$ an n^*-extension of p_ξ and $p_{\xi'}{}^\frown\langle \eta'_1, ..., \eta'_{n^*}\rangle$ an n^*-extension of $p_{\xi'}$ then q forces (in $\mathcal{P} \restriction \nu$) that decisions made by $p_\xi{}^\frown\langle \eta_1, ..., \eta_{n^*}\rangle$ and $p_{\xi'}{}^\frown\langle \eta'_1, ..., \eta'_{n^*}\rangle$ are different (incompatible).*

Proof. Suppose otherwise. Let p and δ be a counterexample. Assume δ is a regular cardinal above κ_{n^*} and $2^{\kappa_k} < \delta < \kappa_{k+1}$ for some $k < \omega$.
Define by induction a continuous $\in$-chain of elementary submodels $\langle M_i \mid i \leq \delta\rangle$ and a sequence of direct extensions $\langle q_i \mid i \leq \delta\rangle$ of p so that

1. $q_i \geq^{*k} p$,
2. if $i \leq j$, then $\tilde{q}_j \geq^{*k} q_i$, where $\tilde{q}_j$ is obtained from q_j by intersecting its first k sets of measure one with those of q_i,
3. $|M_i| < \delta$, for every $i < \delta$,
4. ${}^{\kappa_k}M_i \subseteq M_i$, for $i = 0$ or i a successor ordinal,
5. each n^*-extension of q_i decides $\underset{\sim}{b} \restriction \mu_i + 1$, i.e. for some $\mathcal{P} \restriction \nu$-name $\underset{\sim}{t}$ of $\underset{\sim}{T} \restriction \mu_{i+1} + 1$ and $\gamma < \kappa^+$ we have
$$q_i \Vdash \underset{\sim}{b} \restriction \mu_i + 1 = \underset{\sim}{t} \text{ and } \underset{\sim}{b}(\mu_i) = \kappa^+ \cdot \mu_i + \gamma,$$
where $\mu_i = \sup(M_i \cap \kappa^+)$.
6. $q_i \in M_{i+1}$.

Denote the final M_δ by M, μ_δ by μ and q_δ by q. Note that $\delta > 2^{\kappa_k}$ so sets of measures one of the first k-levels of δ many of q_i's are the same. Assume then that for all i's they are the same in order to insure $q \geq^{*k} q_i$ for every i. Then each n^*-extension of q decides $\underset{\sim}{b} \restriction \mu$. Denote by $X = \{t_\xi \mid \xi < \kappa_{n^*}\}$ the set of all such decisions. Note that $|X| \leq \kappa_{n^*} < \delta$. Shrink the sets of measure one of the maximal coordinates of each $q_i, i < \delta$ for every level $\leq n^*$ to the projection of the corresponding set of measure one of q. Denote still the resulting conditions by q_i. We have M_{i+1} is closed under κ_{n^*}-sequences, so such new q_i will be in M_{i+1}. Then the set of decisions of n^*-extensions of q_i will be $X_i = \{t_\xi \restriction \mu_i \mid \xi < \kappa_{n^*}\}$.
We have that X and each particular t_ξ need not be in M, but X_i's and the initial segments of t_ξ are in M.
Note that for each $i \leq \delta$ any n^*-extension $q_i{}^\frown\langle \eta_1, ..., \eta_{n^*}\rangle$ of q_i gives the values $\langle \underset{\sim}{b}(\mu_j) \mid j < i\rangle$,

by the item (5) above. Denote the sequence of the values by $\langle \gamma_{j\langle \eta_1,...,\eta_{n^*}\rangle} \mid j < i\rangle$. Then $j' < j$ implies

$$q_i \restriction \nu \Vdash \gamma_{j'\langle \eta_1,...,\eta_{n^*}\rangle} \leq_{\underset{\sim}{T}} \gamma_{j\langle \eta_1,...,\eta_{n^*}\rangle}.$$

Recall that $\delta > 2^{\kappa_{n^*}}$. Hence there will be $i^* < \delta$ such that all the branches in X already split before the level μ_{i^*}. Actually for every $j \leq \delta$ of cofinality $\geq \kappa_{n^*}$ the same is true, i.e. all the branches of X_j split already at some $j^* < j$.

Assume that we are at a stage $i+1$ of the construction. So $q_i \in M_{i+1}$ and we have q_{i+1} and the list $X_{i+1} = \{\underset{\sim}{t}_{i+1\xi} \mid \xi < \kappa_{n^*}\}$.

Suppose for a moment that the following holds:

(*) there are $i, i^* \leq i < \delta$ and $r \in \mathcal{P} \restriction \nu, r \geq q_i \restriction \nu, r \in M_{i+1}$ such that for all $\xi \in \nu \cap M_{i+1}$, for all $\tilde{r}$ if $\tilde{r} \geq q_i$,$\tilde{r} \geq r$ and $\tilde{r}$ decides $\underset{\sim}{b} \restriction \xi$ then $\tilde{r} \Vdash \underset{\sim}{b} \restriction \xi$ is an initial segment of one of $\underset{\sim}{t}_{i+1\zeta}$, for some $\zeta < \kappa_{n^*}$.

By extending r if necessary we may assume that $\ell(r) \geq n^*$.

Let us choose (in M) for each $\xi \in \kappa^+$ one r_ξ that witnesses (*) for ξ.

Then there are an unbounded $Z \subseteq \kappa^+, Z \in M$ and $\langle \eta_1^*, ..., \eta_{n^*}^*\rangle$ an n^*-sequence in M such that for every $\xi \in Z \cap M$ the sequence $\langle \eta_1^*, ..., \eta_{n^*}^*\rangle$ is the projection to the first n^*-maximal coordinates of q_i, i.e. $q_i{}^\frown\langle \eta_1^*, ..., \eta_{n^*}^*\rangle \leq r_\xi$. Then $r_\xi \Vdash \underset{\sim}{b} \restriction \xi$ is an initial segment of $\underset{\sim}{t}_{i+1\zeta}$, where ζ corresponds to the μ_{i+1}-branch determined by $q_i{}^\frown\langle \eta_1^*, ..., \eta_{n^*}^*\rangle$ (remember that we are above i^* so it does not split further in q_{i+2}, q_{i+3}, etc.).

Set

$$e := \{\underset{\sim}{s} \mid \underset{\sim}{s} \text{is a } \mathcal{P} \restriction \nu \text{ name and } \exists \xi \in Z, r_\xi \Vdash \underset{\sim}{s} = \underset{\sim}{b} \restriction \xi\}.$$

Then e will be forced by r to be a κ^{++}-branch in $\underset{\sim}{T}$. Which is impossible. Hence (*) is falls. So,

(**) for every $i, i^* \leq i < \delta$ and $r \in \mathcal{P} \restriction \nu, r \geq q_i \restriction \nu, r \in M_{i+1}$ there will be $\xi \in \nu \cap M_{i+1}$ and $\tilde{r}$ such that $\tilde{r} \geq q_i$,$\tilde{r} \geq r$, $\tilde{r}$ decides $\underset{\sim}{b} \restriction \xi$ and $\tilde{r} \Vdash \underset{\sim}{b} \restriction \xi$ is not an initial segment of none of $\underset{\sim}{t}_{i+1\zeta}$, for $\zeta < \kappa_{n^*}$.

Start with $i = i^*$.

Our next tusk will be to extend $q_i \restriction \nu$ by finding a $(*k)$-direct extension $q_i^* \in \mathcal{P} \restriction \nu$ such that

(**i) for some $n_i, n^* \leq n_i < \omega$ for every n_i-extension $q_i^*{}^\frown\langle \eta_1, ..., \eta_{n_i}\rangle$ of q_i^* there is some $r_{i\langle \eta_1,...,\eta_{n_i}\rangle} \in \mathcal{P} \restriction \lambda \setminus \nu$ such that $q_i^*{}^\frown\langle \eta_1, ..., \eta_{n_i}\rangle^\frown r_{i\langle \eta_1,...,\eta_{n_i}\rangle} \in \mathcal{P}_\nu$, $r_{i\langle \eta_1,...,\eta_{n_i}\rangle}$ decides $\underset{\sim}{b} \restriction \xi$ and $q_i^*{}^\frown\langle \eta_1, ..., \eta_{n_i}\rangle^\frown r_{i\langle \eta_1,...,\eta_{n_i}\rangle} \Vdash \underset{\sim}{b} \restriction \xi$ is not an initial segment of none of $\underset{\sim}{t}_{i+1\zeta}$, for $\zeta < \kappa_{n^*}$.

Let us combine such $r_{i\langle \eta_1,...,\eta_{n_i}\rangle}$'s into a $\mathcal{P} \restriction \nu$ name $\underset{\sim}{r_i}$.

Note that $q_i^{*\frown}\langle\eta_1,...,\eta_{n_i}\rangle^\frown r_{i\langle\eta_1,...,\eta_{n_i}\rangle} \Vdash \underset{\sim}{b} \restriction \xi$ is an end extension of $\underset{\sim}{t}_{i\zeta}$ which corresponds to $\langle\eta_1,...,\eta_{n^*}\rangle$, since the last condition extends $q_i{}^\frown\langle\eta_1,...,\eta_{n^*}\rangle$.

We run a standard Prikry type argument running over all the possibilities over the levels, shrinking each of them in order to have same decisions.

It is easy to combine q_i^* and the part of q_i above ν into one condition. Just increase maximal coordinates of the part of q_i above ν and shrink the resulting measure one sets. Denote such combination by q_i^{**}. Also this definition can be carried out inside M_{i+1}.
Note that q_i^{**} need not be compatible with q_{i+1}. So we just replace q_i by q_i^{**} and define new q_j's, $(i < j \leq \delta)$ and new i^*. Denote them the same.
At the next stage go to the next i above the previous one and $\geq$ the new i^* and define its q_i^{**} and so on. At limit stages there may be not enough completeness in order to intersect sets of measure one at certain levels. In this case we put this sets to be as large as possible (i. e. over the level n just κ_n) and then continue the process.
Denote the final q_δ^{**} by q.

Finally let p_i be a $(*k)$-extension of $q^\frown \underset{\sim}{r}_i$ which decides $\underset{\sim}{b} \restriction \mu_\delta$.
The choice of $\underset{\sim}{r}_i$'s provides the desired conclusion.
$\square$

Fix now a cofinal in κ sequence $\langle\delta_n \mid n < \omega\rangle$ of measurable cardinals with $2^{\kappa_n} < \delta_n < \kappa_{n+1}$. Let us use the previous lemma (Lemma 2.7) to find an increasing sequence $\langle\alpha_n \mid n < \omega\rangle$ of levels of $\underset{\sim}{T}$, $\langle Y_n \mid n < \omega\rangle$ and sequences of conditions $\langle q_n \mid n < \omega\rangle$,$\langle p_u \mid u \in \prod_{n<k} Y_n, k < \omega\rangle$ such that

1. $Y_n \subseteq \delta_n$ of cardinality δ_n,
2. $q_n \in \mathcal{P} \restriction \nu$,
3. $q_n \leq^{*n} q_{n+1}$,
4. $p_u \restriction \nu = q_n$, for each u with $|u| = n$,
5. if u is an initial segment of v, then $p_u \leq^* p_v$,
6. all n^*-direct extensions of p_u decide $\underset{\sim}{b} \restriction \alpha_{|u|}$,
7. if u, v are incompatible then any n^*-extensions of p_u and of p_v provide incompatible decisions of $\underset{\sim}{b} \restriction \alpha_{|u|}, \underset{\sim}{b} \restriction \alpha_{|v|}$.

Apply Lemma 2.7 to δ_0. This will produce q_0 and $\langle p_{\langle\xi\rangle} \mid \xi < \delta_0\rangle$. Now let $\xi < \delta_0$. Apply Lemma 2.7 to δ_1 and $p_{\langle\xi\rangle}$. We will obtain $\alpha_{\langle\xi\rangle}$, $q'_{\langle\xi\rangle} \geq^{*1} q_0$ and $\langle p'_{\langle\xi\zeta\rangle} \mid \zeta < \delta_1\rangle$.

Set $\alpha_1 = \max(\alpha_0+1, \cup_{\xi<\delta_0}\alpha_{\langle\xi\rangle})$. We stretch each of $p'_{\langle\xi\zeta\rangle}$ to its $(*1)$-extension $p''_{\langle\xi\zeta\rangle}$ with all n^*-extensions deciding $\underset{\sim}{b} \restriction \alpha_1$. Do this by induction on ξ keeping the parts in $\mathcal{P} \restriction \nu$ increasing, except possibly for the first 2 sets of measures one (we do not have enough completeness in order to intersect this sets). At the final stage let us use the measurability of δ_1. Let U_1 be a normal measure on it. Pick a set $Z_\xi \in U_n$ on which all first 2 sets of measures one are the same. Use this constant value in the final condition and restrict ourself only to conditions with indexes in Z_ξ. Denote the result by $\langle p'''_{\langle\xi\zeta\rangle} \mid \zeta \in Z_\xi\rangle$ and let $q'''_{\langle\xi\rangle}$ be the restriction of this conditions to $\mathcal{P} \restriction \nu$.
We preform the process described above by induction on $\xi < \delta_0$ and require that the sequence $\langle q'''_{\langle\xi\rangle} \mid \xi < \delta_0$ be $(*1)$-increasing, again except possibly for the first set of measures one. Set $Z_1' = \cap_{\xi<\delta_0} Z_\xi$. Then $Z_1 \in U_1$. Combine $\langle q'''_{\langle\xi\rangle} \mid \xi < \delta_0\rangle$ into one condition. We need to stabilize the first set of measure one. Pick a normal ultrafilter U_0 on δ_0. There is $Z_0 \in U_0$ on which we will have this sets the same. The result taking this common set of the measure one on the first level will be our q_1. Shrink to Z_0 and deal further only with $\langle p_{\langle\xi\rangle} \mid \xi \in Z_0\rangle$. Then let $p_{\xi\zeta}$ be the final condition after stabilizing the set of measure over the first level. So we constructed $\langle p_{\langle\xi\zeta\rangle} \mid \xi \in Z_0, \zeta \in Z_1\rangle$.
Continue further in same fashion. At the final stage we will take Y_n to be intersections of ω-many sets Z corresponding to u's of the length n. Such Y_n will be still in U_n and in particular will have cardinality δ_n.

Let $f \in \prod_{n<\omega} Y_n$. Let p_f be a $\leq^*$-upper bound of $\langle p_{f\restriction n} \mid n < \omega\rangle$. Let us pick an n^*-sequence $\langle \eta_1^f, ..., \eta_{n^*}^f\rangle$ in the first n^* sets of measure one of p_f. Then $p_f{}^\frown\langle \eta_1^f, ..., \eta_{n^*}^f\rangle$ will decide $\underset{\sim}{b} \restriction \alpha_\omega + 1$, where $\alpha_\omega := \cup_{n<\omega}\alpha_n$. Different f's will give different decisions. But the number of f's in $V^{\mathcal{P}\restriction\nu}$ is $\nu = (\kappa^{++})^{V^{\mathcal{P}\restriction\nu}}$. So the level α_ω of T will have cardinality κ^{++}, which is impossible since we assumed that T is a κ^{++}-Aronszajn tree in $V^{\mathcal{P}\restriction\nu}$. Contradiction.
$\square$

3 Down to $\aleph_{\omega+2}$.

We will move κ of the previous section to $\aleph_\omega$. The process will be rather standard and so will concentrate only on few new points. In general the idea in this sort of constructions is to use collapses together with main things that are done and not afterwards.
We work here entirely with Short extenders forcing. Long extender forcing does not allow to move down to $\aleph_\omega$.
In our setting there will be two sort of collapses: $Col(\rho_n^{+n+4}, < \kappa_n)$ and $Col(\kappa_n^{+n+8}, < \rho_{n+1})$,

for each $n < \omega$, where ρ_n denotes a generic one element Prikry sequence for the normal measure of the extender E_n. The first collapse will be guided by a function F_n defined on the projection of the maximal coordinate of a level n of a condition to one corresponding to the normal measure of E_n, i.e. κ_n. For each $\nu \in \mathrm{dom}(F_n)$, we require that $F_n(\nu) \in Col(\nu^{+n+4}, < \kappa_n)$.

Let now $\nu < \lambda$ be as in the previous section. The collapses will not involve names and so they will be in $\mathcal{P} \restriction \nu$.

The analog of Lemma 2.7 will deal with δ such that $2^{\kappa_k} = \kappa_k^+ < \delta < \kappa^{+4}$. Note that the collapse over κ_k starts only further up from κ_k^{+8}, which provides enough completeness for running the argument.

Let us give the definition of the forcing used here.

Definition 3.1 $\mathcal{P}$ consists of sequences $p = \langle p_n \mid n < \ell(p)\rangle^\frown \langle p_n \mid \ell(p) \leq n < \omega\rangle$ such that

1. $\ell(p) < \omega$,
2. for every $n < \ell(p), p_n$ is of the form $\langle \rho_n, h_{<n}, h_{>n}, f_n\rangle$ where
   - (a) ρ_n is the one element Prikry sequence for the normal measure of E_n (i.e. an indiscernible for it),
   - (b) $h_{<n} \in Col(\rho_n^{+n+4}, < \kappa_n)$,
   - (c) $h_{>n} \in Col(\kappa_n^{+n+8}, < \rho_{n+1})$, if $n+1 < \ell(p)$ and $h_{>n} \in Col(\kappa_n^{+n+8}, < \kappa_{n+1})$, if $n+1 = \ell(p)$
   - (d) f_n is a partial function of cardinality at most κ from λ to κ_n.
3. For every $n \geq \ell(p)$, p_n is of the form $\langle a_n, A_n, S_n, h_{>n}, f_n\rangle$ where
   - (a) a_n,A_n and f_n are as in Section 2,
   - (b) S_n is a function with domain the projection of A_n to the normal measure of E_n such that for each $\nu \in \mathrm{dom}(S_n)$ we have $S_n(\nu) \in Col(\nu^{+n+4}, < \kappa_n)$,
   - (c) $\min(\mathrm{dom}(S_n)) > \sup(\mathrm{ran}(h_{>n-1}))$,
   - (d) $h_{>n} \in Col(\kappa_n^{+n+8}, < \kappa_{n+1})$.

At the final step of the argument of 2.1, we cannot use measurable δ_n's which differ from κ_n's. The use of κ_n's seems problematic, since the forcing $Col(\kappa_n^{+n+8}, < \kappa_{n+1})$ of cardinality κ_{n+1} is involved and its degree of completeness is only κ_n^{+n+8}.

Let us overcome the difficulty by taking in advance (before the forcing with short extenders) δ_n's to be successor cardinals but carrying precipitous ideals I_n's for which the forcing with positive,i.e. $\mathcal{P}(\delta_n)/I_n$, being δ_n^--strategically closed.
Let us construct such I_n's in advance by collapsing measurables. Namely, we fix, for every $n < \omega$, a measurable cardinal $\delta_{n+1}, \kappa_n < \delta_{n+1} < \kappa_{n+1}$ and a normal ultrafilter W_{n+1} over it. Then force with the full support iteration of $Col(\kappa_n^{+n+5}, < \delta_{n+1})$. The filter W_{n+1} will be as desired in this generic extension, i.e. the forcing with its positive sets will be isomorphic to $Col(\kappa_n^{+n+5}, < i_{W_{n+1}}(\delta_{n+1}))$ which is κ_n^{+n+5}-closed and κ_n^{+n+5} is the immediate predecessor of δ_{n+1} in this generic extension, where $i_{W_{n+1}}$ is the corresponding to W_{n+1} elementary embedding.

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