SOME APPLICATIONS OF SUPERCOMPACT EXTENDER BASED FORCINGS TO HOD.

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ABSTRACT. Supercompact extender based forcings are used to construct models with HOD cardinal structure different from those of V. In particular, a model with all regular uncountable cardinals measurable in HOD is constructed.

1. INTRODUCTION

In [2] the following result was proved:

Theorem. *Suppose $\kappa < \lambda$ are cardinals such that $\mathrm{cf}(\kappa) = \omega$, λ is inaccessible, and κ is a limit of λ-supercompact cardinals. Then there is a forcing poset Q that adds no bounded subsets of κ, and if G is Q-generic then:*
- $\lambda = (\kappa^+)^{V[G]}$.
- *Every cardinal $\geq \lambda$ is preserved in $V[G]$.*
- *For every $x \subseteq \kappa$ with $x \in V[G]$, $(\kappa^+)^{HOD_{\{x\}}} < \lambda$.*

The supercompact extender based Prikry forcing, developed by the second author in [7], is applied to reduce largely the initial assumptions of this theorem and to give a simpler proof. Namely, we show the following:

Theorem 1. *Suppose κ is a $<\lambda$-supercompact cardinal*[1]*, and λ is an inaccessible cardinal above κ. Then there is a forcing poset Q that adds no bounded subsets of κ, and if G is Q-generic then:*
- $\lambda = (\kappa^+)^{V[G]}$.
- *Every cardinal $\geq \lambda$ is preserved in $V[G]$.*
- *For every $x \subseteq \kappa$ with $x \in V[G]$, $(\kappa^+)^{HOD_{\{x\}}} < \lambda$.*

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[1] A cardinal κ is said to be $<\lambda$-supercompact if there is an elementary embedding $j : V \to M$ such that M is transitive, $\mathrm{crit}\, j = \kappa$, $j(\kappa) \geq \lambda$, and $M \supseteq {}^{<\lambda}M$.

- $\mathrm{cf}^{HOD_{\{x\}}}\,\kappa = \omega$

Actually, assuming the measurability (or supercompactness) of λ in V, we obtain that $(\kappa^+)^{V[G]}$ is measurable (or supercompact) in $\mathrm{HOD}_{\{x\}}$.

In [1], a model with the property $(\alpha^+)^{\mathrm{HOD}} < \alpha^+$, for every infinite cardinal α was constructed. We extend this result, using the extender based Magidor forcing of the second author [5], and show the following:

Theorem 2. *Assume there is a Mitchell increasing sequence of extenders $\langle E_\xi \mid \xi < \lambda\rangle$ such that λ is measurable, and for each $\xi < \lambda$, $\mathrm{crit}(j_\xi) = \kappa$, $M_\xi \supseteq {}^{<\lambda}M_\xi$, and $M_\xi \supseteq V_{\lambda+2}$, where $j_\xi : V \to \mathrm{Ult}(V, E_\xi) \simeq M_\xi$ is the natural embedding. Then there is a model of ZFC where all regular uncountable cardinals are measurable in HOD.*

This may be of some interest due to the following result of H. Woodin [8]:

Theorem (The HOD dichotomy theorem)**.** Suppose δ *is an extendible cardinal. Then exactly one of the following holds:*

(1) For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $\gamma^+ = (\gamma^+)^{HOD}$
(2) Every regular cardinal greater than δ is measurable in HOD.

However, we do not have even inaccessibles in the model of theorem 2. It is possible to modify the construction in order to have measurable cardinals (and bit more) in the model. We do not know how to get supercompacts and it is very unlikely the method used will allow model with supercompacts.

The structure of this work is as follows. In section 2 we give definitions and claims about HOD and homogeneous forcing notions which are well know. In section 3 we prove theorem 1. In section 4 we prove theorem 2.

We assume knowledge of large cardinals and forcing. In particular this work depends on the supercompact extender based Prikry-Magidor-Radin forcing.

2. HOD things

Definition 2.1. Let M be a class. The class OD_M contains the sets definable using ordinals and sets from M, i.e., $A \in \mathrm{OD}_M$ iff there is a formula $\varphi(x, x_1, \dots, x_k, y_1, \dots, y_m)$, ordinals $\beta, \alpha_1, \dots, \alpha_k \in \mathrm{On}$, and sets $a_1, \dots, a_m \in M$, such that $A = \{a \in V_\beta \mid V_\beta \vDash \varphi(a, \alpha_1, \dots, \alpha_k, a_1, \dots, a_m)\}$.

The class HOD_M contains sets which are hereditarily in OD_M, i.e., $A \in \mathrm{HOD}_M$ iff $\mathrm{tc}(\{A\}) \subseteq \mathrm{HOD}_M$.

We write OD and HOD for $\mathrm{OD}_\emptyset$ and $\mathrm{HOD}_\emptyset$, respectively.

Note, if $A \in \mathrm{OD}$ is a set of ordinals then $A \in \mathrm{HOD}$.

We will work in HOD of generic extensions, hence the relation between $V[G]$ and $\mathrm{HOD}^{V[G]}$, where $V[G]$ is a generic extension, will be our main machinery.

Our main tool will be forcing notions which are homogeneous in some sense. A forcing notion P is said to be cone homogeneous if for each pair of conditions $p_0, p_1 \in P$ there is a pair of conditions $p_0^*, p_1^* \in P$ such that $p_0^* \leq p_0$, $p_1^* \leq p_1$, and $P/p_0^* \simeq P/p_1^*$.

A forcing notion P is said to be weakly homogeneous if for each pair of conditions $p_0, p_1 \in P$ there is an automorphism $\pi : P \to P$ so that $\pi(p_0)$ and p_1 are compatible. It is evident a weakly homogeneous forcing notion is cone homogeneous.

An automorphism $\pi : P \to P$ induces an automorphism on P-terms by setting recursively $\pi(\langle \dot{\tau}, p\rangle) = \langle \pi(\dot{\tau}), \pi(p)\rangle$.

Note ground model terms are fixed by automorphisms, i.e., $\pi(\check{x}) = \check{x}$, in particular for each ordinal α, $\pi(\check{\alpha}) = \check{\alpha}$.

An essential fact about a cone homogeneous forcing notion P is that for each formula φ, either $\Vdash_P \varphi(\alpha_1, \ldots, \alpha_l)$ or $\Vdash_P \neg\varphi(\alpha_1, \ldots, \alpha_l)$. If in addition the forcing P is ordinal definable then we get $\mathrm{HOD}^{V[G]} \subseteq V$, where G is P-generic.

In [3] it was shown that an arbitrary iteration of weakly (cone) homogeneous forcing notions is weakly (cone) homogeneous under the very mild assumption that the iterand is fixed by automorphisms. For the sake of completeness we show here a special case of this theorem, which is enough for our purpose.

Theorem 2.2 (Special case of Dobrinen-Friedman [3]). *Assume $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$ is a backward Easton iteration such that for each $\beta < \kappa$, $\Vdash_{P_\beta}$ "$\dot{Q}_\beta$ is cone homogeneous" and for each $p_0, p_1 \in P_\beta$ and automorphism $\pi : P_\beta/p_0 \to P_\beta/p_1$, we have $\Vdash_{P_\beta/p_0}$ "$\pi^{-1}(\dot{Q}_\beta) = \dot{Q}_\beta$". Then P_κ is cone homogeneous.*

Proof. Fix two conditions $p_0, p_1 \in P_\kappa$. We will construct two conditions $p_0^* \leq p_0$ and $p_1^* \leq p_1$ such that $P_\kappa/p_0^* \simeq P_\kappa/p_1^*$, by which we will be done. The construction is done by induction on $\alpha \leq \kappa$ as follows.

Assume $\alpha = \beta + 1$, $p_0^* \restriction \beta$, $p_1^* \restriction \beta$, and $\pi_\beta : P_\beta/p_0^* \restriction \beta \simeq P_\beta/p_1^* \restriction \beta$ were constructed. We know $\Vdash_{P_\beta/p_0^* \restriction \beta}$ "$\dot{Q}_\beta = \pi_\beta^{-1}(\dot{Q}_\beta)$ is cone homogeneous". Let $\rho_\beta : \dot{Q}_\beta \to \dot{Q}_\beta$ be a function for which $\dot{\tau}[G] = \rho_\beta(\dot{\tau})[\pi''_\beta G]$ holds, whenever $G \subseteq P_\beta$ is generic and $\dot{\tau}[G] \in \dot{Q}[G]$. If both $p_0(\beta)$ and $p_1(\beta)$ are the maximal element of $\dot{Q}_\beta$ then let $p_0^*(\beta)$ and $p_1^*(\beta)$ be the maximal element of $\dot{Q}_\beta$ and let $\sigma_\beta = \mathrm{id}$ be the trivial automorphism of $\dot{Q}_\beta$. If either $p_0(\beta)$ or $p_1(\beta)$ is not the maximal element of $\dot{Q}_\beta$ then use the

the cone homogeneity of $\dot{Q}_\beta$ to find P_β-names $p_0^*(\beta)$, $p_1^*(\beta)$, and $\dot{\sigma}_\beta$, such that $p_0^*\restriction\beta \Vdash_{P_\beta}$ "$p_0^*(\beta) \leq p_0(\beta)$", $p_1^*\restriction\beta \Vdash_{P_\beta}$ "$p_1^*(\beta) \leq p_1(\beta)$", and $\dot{\sigma}_\beta : \dot{Q}_\beta/p_0^*(\beta) \simeq \dot{Q}_\beta/\rho_\beta^{-1}(p_1^*(\beta))$ is an automorphism. Whatever way $\dot{\sigma}_\beta$ was constructed define the automorphism $\pi_{\beta+1}$ by letting $\pi_{\beta+1}(s) = \langle \pi_\beta(s\restriction\beta), \rho_\beta(\dot{\sigma}_\beta(s(\beta)))\rangle$, for each $s \leq p_0^*\restriction\beta+1$.

Assume α is limit and for each $\beta < \alpha$ we have $p_0^*\restriction\beta \leq p_0\restriction\beta$, $p_1^*\restriction\beta \leq p_1\restriction\beta$, and $\pi_\beta : P_\beta/p_0^*\restriction\beta \simeq P_\beta/p_1^*\restriction\beta$ is an automorphism such that $\pi_\beta\restriction P_{\beta'} = \pi_{\beta'}$, whenever $\beta' \leq \beta$. For each $s \leq p_0^*\restriction\alpha$ let $\pi_\alpha(s) \in P_\alpha$ be the condition defined by setting for each $\beta < \alpha$, $\pi_\alpha(s)(\beta) = \pi_{\beta+1}(s\restriction\beta+1)(\beta)$. □

The following claim is practically the successor case of the previous one. It is useful when we will have automorphism of forcing notions which are not necessarily cone homogeneous.

Claim 2.3. *Assume P_0 and P_1 are forcing notions with $\pi_0 : P_0 \to P_1$ being an isomorphism. Let $\dot{Q}_0$ be a P_0-name of a cone homogeneous forcing notion such that $\Vdash_{P_0}$ "$\dot{Q}_0 = \dot{Q}_1$", where $\dot{Q}_1 = \pi_0(\dot{Q}_0)$.*

Then for each pair $1 * \dot{q}_0 \in P_0 * \dot{Q}_0$ and $1 * \dot{q}_1 \in P_1 * \dot{Q}_1$ there are stronger conditions $1 * \dot{q}_0^* \leq 1 * \dot{q}_0$ and $1 * \dot{q}_1^* \leq 1 * \dot{q}_1$ such that $P_0 * \dot{Q}_0/1 * \dot{q}_0^* \simeq P_1 * \dot{Q}_1/1 * \dot{q}_1^*$.

Proof. Note there is a function ρ taking P_0-names to P_1-names such that $\dot{q}_0[G_0] = \rho(\dot{q}_0)[G_1]$, where $G_0 \subseteq P_0$ is generic and $G_1 = \pi_0'' G_0$.

Set $\dot{q}_1' = \rho^{-1}(\dot{q}_1)$. By the cone homogeneity of $\dot{Q}_0$ in V^{P_0} there are stronger conditions $\dot{q}_0^* \leq \dot{q}_0$ and $\dot{q}_1'^* \leq \dot{q}_1'$, for which there is (a name of) an automorphism $\pi_1 : \dot{Q}_0/\dot{q}_0^* \to \dot{Q}_0/\dot{q}_1'^*$. Set $\dot{q}_1^* = \rho(\dot{q})_1'^*$. Since for generics G_0, G_1 as above we have $\dot{Q}_0/\dot{q}_1'^*[G_0] = \dot{Q}_1/\dot{q}_1^*[G_1]$ we get $\pi(p * \dot{q}) = \pi_0(p) * (\rho \circ \pi_1(\dot{q}))$ is the required automorphism. □

While the forcing notions we will use are cone homogeneous we will deliberately break some of their homogeneity. The relation between $\mathrm{HOD}^{V[G]}$ and V will be as follows.

Claim 2.4. *Assume P is an ordinal definable cone homogeneous forcing notion. Let $\pi : P \to P$ be a projection. Assume for each condition $p \in P$ and ordinals $\alpha_1, \ldots, \alpha_l \in On$, if $p \Vdash_P \varphi(\alpha_1, \ldots, \alpha_l)$ then $\pi(p) \Vdash_P \varphi(\alpha_1, \ldots, \alpha_l)$. Then $HOD^{V[G]} \subseteq V[\pi'' G]$.*

Proof. Assume $\Vdash_P$ "$\dot{A} \subseteq \mathrm{On}$ and $\dot{A} \in \mathrm{HOD}$". Let $G \subseteq P$ be generic. Then in $V[G]$ there are ordinals $\alpha_1, \ldots, \alpha_l, \beta$ such that for each $\alpha \in \mathrm{On}$,

$$\alpha \in \dot{A}[G] \iff V_\beta \vDash \varphi(\alpha, \alpha_1, \ldots, \alpha_l).$$

Let $X_0^\alpha \cup X_1^\alpha \subseteq P$ be a maximal antichain such that for each $p \in X_0^\alpha$,

$$p \Vdash V_\beta \models \neg\varphi(\alpha, \alpha_1, \dots, \alpha_l),$$

and for each $p \in X_1^\alpha$,

$$p \Vdash V_\beta \models \varphi(\alpha, \alpha_1, \dots, \alpha_l).$$

Let $\dot{A}'$ be a $\pi''P$-name defined by setting for each $p \in X_0^\alpha \cup X_1^\alpha$.

$$\pi(p) \Vdash_{\pi''P} \text{“}\alpha \in \dot{A}'\text{”} \iff p \Vdash_P \text{“}\alpha \in \dot{A}\text{”}.$$

Since $\pi''(X_0 \cup X_1)$ is predense in $\pi''P$ we get $\dot{A}'[\pi''G] = \dot{A}[G]$, by which we are done. $\square$

Let $C(\tau, \mu)$ be the Cohen forcing for adding μ subsets to τ, i.e., $C(\tau, \mu) = \{f : a \to 2 \mid a \subseteq \mu,\ |a| < \tau\}$. The following is well known.

Claim 2.5. $\mathrm{C}(\tau, \mu)$ *is cone homogeneous.*

Proof. Assume $f, g \in \mathbb{C}(\tau, \mu)$ are conditions. Choose stronger conditions, $f^* \leq f$ and $g^* \leq g$, such that $\operatorname{dom} f^* = \operatorname{dom} g^* = \operatorname{dom} f \cup \operatorname{dom} g$. Define $\pi : \mathbb{C}(\tau, \mu)/f^* \to \mathbb{C}(\tau, \mu)/g^*$ by setting $\pi(f') = g^* \cup (f' \setminus f^*)$ for each $f' \leq f^*$. It is obvious π is an automorphism. $\square$

The following is immediate from the previous claim and theorem 2.2.

Claim 2.6. *The Easton product of Cohen forcing notions is cone homogeneous.*

3. The cofinality ω case

Let us switch to the cone-homogeneity of the Extender Based Prikry forcing ([4]). Let E be an extender as in [7] or [5]. Let $\mathbb{P}_E$ be the extender based Prikry forcing derived from E. We show $\mathbb{P}_E$ is cone homogeneous.

Claim 3.1. *For a pair of conditions $p_0, p_1 \in \mathbb{P}_E$ there are direct extensions $p_0^* \leq^* p_0$ and $p_1^* \leq^* p_1$ such that $\mathbb{P}_E/p_0^* \simeq \mathbb{P}_E/p_1^*$.*

Proof. Set $d = \operatorname{dom} f^{p_0} \cup \operatorname{dom} f^{p_1}$. Set $f_0^* = f^{p_0} \cup \{\langle \alpha, \langle\rangle \rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_0}\}$ and $f_1^* = f^{p_1} \cup \{\langle \alpha, \langle\rangle \rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_1}\}$. Choose a set $A \subseteq \pi^{-1}_{d, \operatorname{dom} f^{p_0}}(A^{p_0}) \cap \pi^{-1}_{d, \operatorname{dom} f^{p_1}}(A^{p_1})$ so that both $p_0^* = \langle f_0^*, A\rangle$ and $p_1^* = \langle f_1^*, A\rangle$ are conditions. Define $\pi : \mathbb{P}_E/p_0^* \to \mathbb{P}_E/p_1^*$ by setting for each $p \leq p_0^*$, $\pi(p) = \langle f^{p_1^*}_{\langle \nu_0, \dots, \nu_{n-1}\rangle} \cup (f^p \restriction (\operatorname{dom} f^p \setminus d)), A^p\rangle$, where $\langle \nu_0, \dots, \nu_{n-1}\rangle \in {}^{<\omega}A^p$ and $p \leq^* p^*_{0\langle \nu_0, \dots, \nu_{n-1}\rangle}$. It is evident π is an automorphism of $\mathbb{P}_E$. $\square$

For a generic filter $G \subseteq \mathbb{P}_E$ define the function f_G by setting $f_G(\alpha) = \bigcup\{f^p(\alpha) \mid p \in G, \alpha \in \operatorname{dom} f^p\}$.

Let us define the Easton products we are going to work with. Let $A \subseteq \mathrm{On}$ be a set of ordinals. Let $\mathbb{C}_{\chi,A}$ be the Easton product of the Cohen forcing notions yielding, in the generic extension, for each $\xi < \sup A$,

$$2^{\chi^{+\xi+1}} = \begin{cases} \chi^{+\xi+3} & \xi \in A, \\ \chi^{+\xi+2} & \xi \notin A. \end{cases}$$

When forcing with $\mathbb{C}_{\chi,A}$ we will choose χ to be large enough so as not to interfere with our intended usage. Due to the (cone) homogeneity of $\mathbb{P}_E$, the seuqences forced by $\mathbb{P}_E$ are not in $\mathrm{HOD}^{V^{\mathbb{P}_E}}$. We would like to break the homogeneity of $\mathbb{P}_E$ so as to have the Prikry sequence enter $\mathrm{HOD}^{V^{\mathbb{P}_E}}$. We will achieve this by coding the Prikry sequence into the power set function. We will want the Cohen forcing used to be stabilized by reasonable automorphisms of $\mathbb{P}_E$. Thus define the projection $s : \mathbb{P}_E \to \mathbb{P}_E$ by setting $s(p) = \langle f^p \restriction \{\kappa\}, A^p \restriction \{\kappa\} \rangle$, where $A^p \restriction \{\kappa\} = \{\nu \restriction \{\kappa\} \mid \nu \in A^p\}$. Note $s''\mathbb{P}_E$ is the usual Prikry forcing based on $E(\kappa)$. Moreover if $G \subseteq \mathbb{P}_E$ is generic then $s''G$ is $s''\mathbb{P}_E$-generic.

Claim 3.2. *Let $\mathbb{P} = \mathbb{P}_E * \dot{\mathbb{C}}_{\chi,\dot{f}_G(\kappa)}$. Assume $\langle p_0, \dot{q}_0\rangle, \langle p_1, \dot{q}_1\rangle \in \mathbb{P}$ are conditions such that $s(p_0)$ and $s(p_1)$ are compatible. Then there are stronger conditions, $\langle p_0^*, \dot{q}_0^*\rangle \leq \langle p_0, \dot{q}_0\rangle$ and $\langle p_1^*, \dot{q}_1^*\rangle \leq \langle p_1, \dot{q}_1\rangle$, such that $\mathbb{P}/\langle p_0^*, \dot{q}_0^*\rangle \simeq \mathbb{P}/\langle p_1^*, \dot{q}_1^*\rangle$.*

Proof. Since $s(p_0)$ and $s(p_1)$ are compatible, we can choose conditions $p_0' \leq p_0$ and $p_1' \leq p_1$ such that $f^{p_0'} \restriction \{\kappa\} = f^{p_1'} \restriction \{\kappa\}$. By claim 3.1 there are direct extensions $p_0^* \leq^* p_0'$ and $p_1^* \leq^* p_1'$ such that $\pi_0 : \mathbb{P}_E/p_0^* \simeq \mathbb{P}_E/p_1^*$ is an automorphism. Since $\mathbb{C}_{\chi,f_G(\kappa)} = \pi(\mathbb{C}_{\chi,f_G(\kappa)})$, where $G \subseteq \mathbb{P}_E$ is generic, we are done by claim 2.3. ☐

The following is immediate from the previous claim.

Corollary 3.3. *Assume $\alpha, \alpha_1, \ldots, \alpha_n \in On$ and $\langle p, q\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \ldots, \alpha_n)$. Then $\langle s(p), 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \ldots, \alpha_n)$.*

Proof. In order to show $\langle s(p), 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \ldots, \alpha_n)$ we will show a dense subset of conditions below $\langle s(p), 1\rangle$ forces $\varphi(\alpha, \alpha_1, \ldots, \alpha_n)$. Let $\langle p_0, \dot{q}_0\rangle \leq \langle s(p), 1\rangle$ be an arbitrary condition. By claim 3.2 there is $\langle p_0', \dot{q}_0'\rangle \leq \langle p_0, \dot{q}_0\rangle$ and $\langle p_1', \dot{q}_1'\rangle \leq \langle p, \dot{q}\rangle$ such that $\mathbb{P}/p_0' * \dot{q}_0' \simeq \mathbb{P}/p_1' * \dot{q}_1'$. Thus $\langle p_0', \dot{q}_0'\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \ldots, \alpha_n)$. ☐

The previous corollary together with claim 2.4 yields the following.

Corollary 3.4. *Assume $G * H$ is $\mathbb{P}$-generic. Then $\mathrm{cf}^{V[G*H]}\kappa = \omega$ and $f_G(\kappa) \in HOD^{V[G*H]} \subseteq V[s''G]$.*

We will get a special case of theorem 1 by invoking the last corollary in a model of the form $L[A]$.

Corollary 3.5. *Assume $V = L[A]$, where $A \subseteq On$ is a set of ordinals, and E is an extender witnessing κ is a $<\lambda$-supercompact cardinal. There is a forcing notion R preserving the extender E such that in $V[I][G*H]$, where $I * G * H$ is $R * \mathbb{P}$-generic, $\kappa^+ = \lambda$, $\mathrm{cf}\,\kappa = \omega$, and $HOD^{V[I][G][H]} = V[I][s''G]$.*

Proof. We will begin by defining the forcing notion R so that for an R-generic filter I we will have $\mathrm{HOD}^{V[I]} = V[I]$.

Define by induction the forcing notions $\langle R_n \mid n \leq \omega\rangle$ and sets $\langle A_n \mid n < \omega\rangle$, as follows. Set $R_0 = 1$ and $A_0 = A$. For each $n < \omega$ define R_{n+1} as follows. In $V[G_n]$, where $G_n \subseteq R_n$ is generic over V, let $\mathbb{C}_n$ be the forcing notion $\mathbb{C}_{\chi_n, A_n}$. Let A_{n+1} be $\mathbb{C}_n$-generic over $V[G_n]$, i.e., A_{n+1} is a code for A_n. Set $R_{n+1} = R_n * \dot{\mathbb{C}}_n$, where $\dot{\mathbb{C}}_n$ is an R_n-name for $\mathbb{C}_n$. Let R be the inverse limit of $\langle R_n \mid n < \omega\rangle$. Let $I \subseteq R$ be generic.

Invoking *corollary* 3.4 inside $V[I]$ and calculating $\mathrm{HOD}^{V[I][G][H]}$ we get $f_G(\kappa) \in \mathrm{HOD}^{V[I][G][H]} \subseteq V[I][s''G]$. For each $n < \omega$, $A_n \in \mathrm{HOD}^{V[I][G][H]}$, thus $\mathrm{HOD}^{V[I][G][H]} \supseteq L[A][I][s''G] = V[I][s''G]$. □

Hence we get:

Corollary 3.6. *Assume λ is measurable and κ is $<\lambda$-supercompact. Then there is a generic extension in which $\mathrm{cf}^{HOD}\kappa = \omega$, and κ^+ (of the generic extension) is HOD-measurable.*

In order to analyze $\mathrm{HOD}_{\{a\}}$, where $a \subseteq \kappa$, let us derive another line of corollaries stemming from claim 3.2. The problem we face when dealing with $\mathrm{HOD}_{\{a\}}$ is an automorphism π of $\mathbb{P}$ might move $\dot{a}$, the name of a. Thus we will need to fine tune the projection s.

First we recall the notion of good pair from [5]. We say the pair $\langle N, f\rangle$ is a good pair if $N \prec H_\chi$ is a κ-internally approachable elementary substructure and there is a sequence $\langle\langle N_\xi, f_\xi\rangle \mid \xi < \kappa\rangle$ such that $\langle N_\xi \mid \xi < \kappa\rangle$ witnesses the κ-internal approachablity of N, $f = \bigcup\{f_\xi \mid \xi < \kappa\}$, $\langle f_\xi \mid \xi < \kappa\rangle$ is a $\leq^*$-decreasing continuous sequence in $\mathbb{P}^*_f$, and for each $\xi < \kappa$, $f_\xi \in \bigcap\{D \in N_\xi \mid D \text{ is a dense open subset of } \mathbb{P}^*_f\}$, $f_\xi \subseteq N_{\xi+1}$, and $f_\xi \in N_{\xi+1}$.

Define the projection $s_N : \mathbb{P}_E \to \mathbb{P}_E$ by setting for each $p \in \mathbb{P}_E$, $s_N(p) = \langle f^p \restriction N, A^p \restriction N\rangle$.

Corollary 3.7. *Assume $N \prec H_\chi$ is an elementary substructure such that p^* is an $\langle N, \mathbb{P}_E\rangle$-generic condition and $\langle N, f^{p^*}\rangle$ is a good pair. Let $\dot{a} \in N$ be a $\mathbb{P}_E$-name such that $\Vdash_{\mathbb{P}_E}$ "$\dot{a} \subseteq \kappa$". If $\alpha, \alpha_1, \dots, \alpha_n \in On$, $p \leq p^*$, and $\langle p, \dot{q}\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$, then $\langle s_N(p), 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$.*

Proof. In order to show $\langle s_N(p), 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$ we will show a dense subset of conditions below $\langle s_N(p), 1\rangle$ forces $\varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$.

Let $\langle p_0, \dot{q}_0\rangle \leq \langle s_N(p), 1\rangle$ be arbitrary condition. We can choose $p_1 \leq p$ such that $s_N(p_0) = s_N(p_1)$. By claim 3.1 there is $p^*_0 \leq^* p_0$ and $p^*_1 \leq^* p_1$ such that $\mathbb{P}_E/p^*_0 \simeq \mathbb{P}_E/p^*_1$.

Recall that if $r \leq p^*$, $\alpha < \kappa$, and $r \Vdash_{\mathbb{P}_E}$ "$\alpha \in \dot{a}$", then $p^*_{\langle \nu_0,\dots,\nu_{l-1}\rangle} \Vdash$ "$\alpha \in \dot{a}$", where $\langle \nu_0, \dots, \nu_{l-1}\rangle \in {}^{<\omega}A^{p^*}$ is such that $r \leq^* p^*_{\langle \nu_0,\dots,\nu_{l-1}\rangle}$. Thus for each $\langle \nu_0, \dots, \nu_{l-1}\rangle \in A^{p^*_0} = A^{p^*_1}$, $\alpha < \kappa$, and $r \in \mathbb{P}_E/p^*_0$,

$$
\begin{aligned}
& r \leq^* p^*_{0\langle \nu_0,\dots,\nu_{l-1}\rangle} \text{ and } r \Vdash_{\mathbb{P}_E} \text{“}\alpha \in \dot{a}\text{”} \iff \\
& p_{\langle \nu_0,\dots,\nu_{l-1}\rangle \restriction \operatorname{dom} f^p} \Vdash_{\mathbb{P}_E} \text{“}\alpha \in \dot{a}\text{”} \iff \\
& \pi(r) \leq^* p^*_{1\langle \nu_0,\dots,\nu_{l-1}\rangle} \text{ and } \pi(r) \Vdash_{\mathbb{P}_E} \text{“}\alpha \in \pi(\dot{a})\text{”}.
\end{aligned}
$$

Thus $p^*_0 \Vdash$ "$\dot{a} = \pi^{-1}(\dot{a})$". Use claim 3.2 to find stronger conditions $\langle p'_0, \dot{q}'_0\rangle \leq \langle p^*_0, \dot{q}_0\rangle$ and $\langle p'_1, \dot{q}'_1\rangle \leq \langle p^*_0, \dot{q}\rangle$ such that $\tilde{\pi} : \mathbb{P}/p'_0 * \dot{q}'_0 \simeq \mathbb{P}/p'_1 * \dot{q}'_1$ is an automorphism. Since $\langle p'_1, \dot{q}'_1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \dots, \alpha_n, \dot{a})$ we get $\langle p'_0, \dot{q}'_0\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \dots, \alpha_n, \pi^{-1}(\dot{a}))$. We are done since $p'_0 \Vdash$ "$\dot{a} = \pi^{-1}(\dot{a})$". □

Corollary 3.8. *Assume $G * H$ is $\mathbb{P}$-generic, $a \in V[G * H]$, and $a \subseteq \kappa$. Then $\operatorname{cf}^{V[G*H]} \kappa = \omega$ and $f_G(\kappa) \in HOD^{V[G*H]}_{\{a\}} \subseteq V[s''_X G]$ for a set $X \subseteq \operatorname{dom} E$ such that $|X| < \lambda$.*

We will get theorem 2 by beginning with a model where HOD $\supseteq V_{\lambda+2}$. For this let us define the following coding. Let $\mathfrak{A} = \langle A_\alpha \mid \alpha < \lambda^{+3}\rangle$ be an enumeration of all subsets of λ^{++}. Let $\mathbb{C}_{\chi,\mathfrak{A}}$ be the Easton product of the Cohen forcing notions yielding, in the generic extension, for each $\alpha < \lambda^{+3}$ and $\xi < \lambda^{++}$,

$$
2^{\chi^{+\lambda^{++}\cdot\alpha+\xi+1}} = \begin{cases} \chi^{\lambda^{++}\cdot\alpha+\xi+3} & \xi \in A_\alpha, \\ \chi^{\lambda^{++}\cdot\alpha+\xi+2} & \xi \notin A_\alpha. \end{cases}
$$

Corollary 3.9. *Let E is an extender witnessing κ is a $<\lambda$-supercompact cardinal. In $V[I][G * H]$, where $I * G * H$ is $\mathrm{C}_{\chi,\mathfrak{A}} * \mathbb{P}$-generic, $\kappa^+ = \lambda$, and for each set $a \subseteq \kappa$, $\operatorname{cf}^{HOD^{V[I][G*H]}_{\{a\}}} \kappa = \omega$ and λ is $HOD^{V[I][G][H]}_{\{a\}}$-measurable.*

Proof. Let $U \in V$ be a measure on λ. Then $U \in V_{\lambda+2}$, hence $U \in \mathrm{HOD}^{V[I]}$, where I is $\mathrm{C}_{\chi,\mathfrak{A}}$-generic.

Working in $V[I]$ let $G * H$ be $\mathbb{P}$-generic. By corollary 3.8 there is $X \subseteq \operatorname{dom} E$ such that $|X| < \lambda$, $X \in V[I]$, and $f_G(\kappa) \in \mathrm{HOD}^{V[I][G*H]}_{\{a\}} \subseteq V[I][s''_X G]$. The filter $s''_X G$ is $s''_X \mathbb{P}_E$-generic. Since $|X| < \lambda$ we have $|s''_X \mathbb{P}_E| < \lambda$, hence any V-measure over λ trivially lifts to a $V[s''_X G]$-measure over λ. In particular U lifts to $\bar{\bar{U}}$, which is definable by $\bar{\bar{U}} = \{B \in V[I][s''_X G] \cap \mathcal{P}(\lambda) \mid \exists A \in U\, B \supseteq A\}$. Since $U \in \mathrm{HOD}^{V[I][G*H]}_{\{a\}}$ we can define in $\mathrm{HOD}^{V[I][G*H]}_{\{a\}}$, $\bar{U} = \{B \in \mathrm{HOD}^{V[I][G*H]}_{\{a\}} \cap \mathcal{P}(\lambda) \mid \exists A \in U\, B \supseteq A\}$. Since $\mathrm{HOD}^{V[I][G*H]}_{\{a\}} \subseteq V[I][s''_X G]$ we necessarily have $\bar{U} \subseteq \bar{\bar{U}}$. Thus $\bar{U}$ is a measure on λ in $\mathrm{HOD}^{V[I][G*H]}_{\{a\}}$. □

4. The global result

In this section we prove theorem 2. Thus throughout this section assume $\vec{E} = \langle E_\xi \mid \xi < \lambda \rangle$ is a Mitchell increasing sequence of extenders such that λ is measurable, and for each $\xi < \lambda$, $\operatorname{crit}(j_\xi) = \kappa$, $M_\xi \supseteq {}^{<\lambda} M_\xi$, and $M_\xi \supseteq V_{\lambda+2}$, where $j_\xi : V \to \operatorname{Ult}(V, E_\xi) \simeq M_\xi$ is the natural embedding. (We demand $M_\xi \supseteq V_{\lambda+2}$ since we want λ to be measurable in all ultrapowers, not only in V).

Let $\mathbb{P}_{\vec{E}}$ be the supercompact extender based Radin forcing using $\vec{E}$. (see [6, 5]). Let us deal with the homogeneity of the Extender Based Radin forcing

Lemma 4.1. *For a pair of conditions $p_0, p_1 \in \mathbb{P}^*_{\vec{E}}$ there are direct extensions $p^*_0 \leq^* p_0$ and $p^*_1 \leq^* p_1$ such that $\mathbb{P}_{\vec{E}}/p^*_0 \simeq \mathbb{P}_{\vec{E}}/p^*_1$.*

Proof. Set $d = \operatorname{dom} f^{p_0} \cup \operatorname{dom} f^{p_1}$. Set $f^*_0 = f^{p_0} \cup \{\langle \alpha, \langle\rangle \rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_0}\}$ and $f^*_1 = f^{p_1} \cup \{\langle \alpha, \langle\rangle \rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_1}\}$. Choose a set $A \subseteq \bigcup_{\xi < \mathrm{o}(\vec{E})} \pi^{-1}_{\xi, d, \operatorname{dom} f^{p_0}}(A^{p_0}) \cap \bigcup_{\xi < \mathrm{o}(\vec{E})} \pi^{-1}_{\xi, d, \operatorname{dom} f^{p_1}}(A^{p_1})$ so that both $p^*_0 = \langle f^*_0, A \rangle$ and $p^*_1 = \langle f^*_1, A \rangle$ are conditions. Define $\pi : \mathbb{P}_{\vec{E}}/p^*_0 \to \mathbb{P}_{\vec{E}}/p^*_1$ by setting $\pi(p^0)$ for each $p^0 \leq p^*_0$ as follows. Let $\langle \nu_0, \dots, \nu_{n-1} \rangle \in {}^{<\omega} A^{p^*_0}$ such that $p^0 \leq^* p^*_{0\langle \nu_0, \dots, \nu_{n-1} \rangle}$. Let $p^0 = p^0_0 \frown \cdots p^0_n$ and $p^*_{1\langle \nu_0, \dots, \nu_{n-1} \rangle} = p^{1*}_0 \frown \cdots \frown p^{1*}_n$. Let $\pi(p^0) = p^1_0 \frown \cdots \frown p^1_n$, where $p^1_i = \langle f^{p^{1*}_i} \cup f^{p^0_i} \restriction (\operatorname{dom} f^{p^0_i} \setminus \operatorname{dom} f^{p^{1*}_i}), A^{p^0_i} \rangle$. It is evident π is an automorphism. □

Recall that for a condition $p = p_0 \frown \cdots \frown p_n$ we have $\mathbb{P}_{\vec{E}}/p \simeq \mathbb{P}_{\vec{e}_0}/p_0 \cdots \frown \mathbb{P}_{\vec{e}_n}/p_n$, where $p_i \in \mathbb{P}^*_{\vec{e}_i}$ and $\vec{e}_n = \vec{E}$. Thus the following is an immediate corollary of the above lemma by recursion.

Corollary 4.2. *Assume $p^0, p^1 \in \mathbb{P}_{\vec{E}}$ are conditions such that $p^0, p^1 \in \prod_{0 \leq i \leq n} P^*_{\vec{e}_i}$. Then there are direct extensions $p^{0*} \leq^* p^0$ and $p^{1*} \leq^* p^1$ such that $\mathbb{P}_{\vec{E}}/p^{0*} \simeq \mathbb{P}_{\vec{E}}/p^{1*}$.*

For a condition $p \in \mathbb{P}^*_{\vec{E}}$ define its projection $s(p)$ to the normal measure by setting $s(p) = \langle f^p \restriction \{\kappa\}, A^p \restriction \{\kappa\} \rangle$. Define by recursion the projection of arbitrary condition $p = p_0 {}^\frown \cdots {}^\frown p_n \in \mathbb{P}_{\vec{E}}$ by setting $s(p) = s(p_0 {}^\frown \cdots {}^\frown p_{n-1}) {}^\frown s(p_n)$. It is obvious $s''\mathbb{P}_{\vec{E}}$ is the Radin forcing using the measures $\langle E_\xi(\kappa) \mid \xi < \mathrm{o}(\vec{E}) \rangle$. Moreover, if G is $\mathbb{P}_{\vec{E}}$-generic then $s''G$ is $s''\mathbb{P}_{\vec{E}}$-generic.

Let G be $\mathbb{P}_{\vec{E}}$-generic. Work in $V[G]$. Let $\langle \kappa_\alpha \mid \alpha < \kappa \rangle$ be the increasing enumeration of $f_G(\kappa)$. Define the sequence $\langle \mu_\alpha, U_\alpha \mid \alpha < \kappa \rangle$ by setting for each $\alpha < \kappa$,

$$\mu_\alpha = \begin{cases} \kappa_\alpha^+ & \alpha \text{ is limit,} \\ \kappa_\alpha & \alpha \text{ is successor.} \end{cases}$$

Note: If α is limit, then $\mu_\alpha = \kappa_\alpha^+$ is V-measurable since it is a reflection of λ being measurable in one of the V-ultrapowers. On the other hand, if α is successor then $\mu_\alpha = \kappa_\alpha$ is V-measurable since E_0 concentrates on measurables. Thus for each $\alpha < \kappa$ we can choose $U_\alpha \in V$ which is a V-measure over μ_α. Define the backward Easton iteration $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa \rangle$ by setting for each $\alpha < \kappa$, $\dot{Q}_\alpha = \mathrm{Col}(\mu_\alpha, {<}\kappa_{\alpha+1})$. By theorem 2.2 the iteration P_κ is cone homogeneous. Let $H \subseteq P_\kappa$ be generic.

Working in $V[G * H]$ we want to pull into the HOD of a generic extension the V-measures U_α's. Define the backward Easton iteration $\langle R_\alpha, \dot{S}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa \rangle$ by setting for each $\beta < \kappa$, $\dot{S}_\beta = \mathrm{C}_{\chi_\beta, \mathfrak{A}_\beta}$, where, $\mathfrak{A}_\beta = \{A \in V \mid A \subseteq (\mu_\beta^{++})_V\}$ and $\sup_{\gamma < \beta} \chi_\gamma < \chi_\beta < \kappa$. By theorem 2.2 R_κ is cone homogeneous.

One final definition is in order before the following claim. If $p \in \mathbb{P}^*_{\vec{E}}$ then set $\kappa(p) = \operatorname{ran} f^p(\kappa)$. If $p = p_0 {}^\frown \cdots {}^\frown p_n \in \mathbb{P}_{\vec{E}}$ then set by recursion $\kappa(p) = \kappa(p_0 {}^\frown \cdots {}^\frown p_{n-1}) {}^\frown \kappa(p_n)$. Note $\kappa(p)$ is the subset of $f_G(\kappa)$ decided by the condition p.

Claim 4.3. *Let $\mathbb{P} = \mathbb{P}_{\vec{E}} * \dot{P}_\kappa * \dot{R}_\kappa$. Assume $\langle p_0, \dot{q}_0, \dot{q}_0 \rangle, \langle p_1, \dot{q}_1, \dot{r}_1 \rangle \in \mathbb{P}$ are conditions such that $s(p_0)$ and $s(p_1)$ are compatible. Then there are stronger conditions, $\langle p_0^*, \dot{q}_0^*, \dot{r}_0^* \rangle \leq \langle p_0, \dot{q}_0, \dot{r}_0 \rangle$ and $\langle p_1^*, \dot{q}_1^*, \dot{q}_1^* \rangle \leq \langle p_1, \dot{q}_1, \dot{r}_1 \rangle$, such that $\mathbb{P}/\langle p_0^*, \dot{q}_0^*, \dot{r}_0^* \rangle \simeq \mathbb{P}/\langle p_1^*, \dot{q}_1^*, \dot{r}_1^* \rangle$.*

Proof. Since $s(p_0)$ and $s(p_1)$ are compatible there are stronger conditions $p_0' \leq p_0$ and $p_1' \leq p_1$ and Mithcell increasing sequences $\{\vec{e}_i \mid i \leq k\}$ such that $p_0', p_1' \in \prod_{i \leq k} \mathbb{P}_{\vec{e}_i}$ and $\kappa(p_0') = \kappa(p_1')$. By the previous corollary there are direct extensions $p_0^* \leq^* p_0'$ and $p_1^* \leq^* p_1'$ such that

$\pi : \mathbb{P}_{\vec{E}}/p_0^* \simeq \mathbb{P}_{\vec{E}}/p_1^*$. Most importantly we have $\pi(\dot{P}_\kappa * \dot{Q}_\kappa) = \dot{P}_\kappa * \dot{Q}_\kappa$ is cone homogeneous. Thus by claim 2.3 we are done. $\square$

Corollary 4.4. *If* $\langle p, \dot{q}, \dot{r}\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \ldots, \alpha_l)$*, then* $\langle s(p), 1, 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \ldots, \alpha_l)$*.*

Proof. We will prove a dense subset of conditions below $\langle s(p), 1, 1\rangle$ force $\varphi(\alpha_0, \ldots, \alpha_l)$. Assume $\langle p^0, \dot{q}^0, \dot{r}^0\rangle \leq \langle s(p), 1, 1\rangle$. Trivially $s(p^0)$ and $s(p)$ are compatible, hence by the previous corollary there are stronger conditions $\langle p^{0*}, \dot{q}^{0*}, \dot{r}^{0*}\rangle \leq \langle p^0, \dot{q}^0, \dot{r}^0\rangle$ and $\langle p^{1*}, \dot{q}^{1*}, \dot{r}^{1*}\rangle \leq \langle p, \dot{q}, \dot{r}\rangle$ such that $\mathbb{P}/\langle p^{0*}, \dot{q}^{0*}, \dot{r}^{0*}\rangle \simeq \mathbb{P}/\langle p^{1*}, \dot{q}^{1*}, \dot{r}^{1*}\rangle$. Necessarily $\langle p^{0*}, \dot{q}^{0*}, \dot{r}^{0*}\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_0, \ldots, \alpha_l)$. $\square$

Letting I be R_κ-generic over $V[G][H]$ we get the following from the previous corollary together with claim 2.4.

Corollary 4.5. $HOD^{V[G][H][I]} \subseteq V[s''G]$*.*

Claim 4.6. *In* $V_\kappa^{V[G][H][I]}$ *all regulars above* κ_0 *are* $HOD^{V_\kappa^{V[G][H][I]}}$*-measurable.*

Proof. Since the regulars in the range $[\kappa_0, \kappa)$ are $\{\mu_\alpha \mid \alpha < \kappa\}$, we will be done by showing for each $\alpha < \kappa$ the V-measure U_α lifts to a $\mathrm{HOD}^{V_\kappa^{V[G][H][I]}}$-measure. In V, μ_α is measurable. The set $s''\mathbb{P}_{\vec{E}}$ is the plain Radin forcing, hence any measure in V over μ_α lifts trivially to a measure on μ_α in $V[s''G]$. In particular the V-measure U_α lifts to the $V[s''G]$ measure $\bar{\bar{U}}_\alpha$, which is definable by $\bar{\bar{U}}_\alpha = \{B \in V[s''G] \mid \exists A \in U_\alpha\ A \subseteq B \subseteq \mu_\alpha\}$.

Since $\mathrm{HOD}^{V_\kappa^{V[G][H][I]}} \supseteq V_{(\mu_\alpha^{++})_V}$ we get $U_\alpha \in \mathrm{HOD}^{V_\kappa^{V[G][H][I]}} \subseteq \mathrm{HOD}^{V[G][H][I]} \subseteq V[s''G]$. Let $\bar{U}_\alpha = \{B \in \mathrm{HOD}^{V_\kappa^{V[G][H][I]}} \mid \exists A \in U_\alpha\ A \subseteq B \subseteq \mu_\alpha\}$. Then $\bar{U}_\alpha \in \mathrm{HOD}^{V_\kappa^{V[G][H][I]}}$ and $\bar{U}_\alpha \subseteq \bar{\bar{U}}_\alpha$. Necessarily $\bar{U}_\alpha$ is a measure on μ_α. $\square$

We get theorem 2 by forcing in $V[G][H][I]$ with $\mathrm{Col}(\omega, {<}\kappa_0)$.

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