Sets in Prikry and Magidor Generic Extensions

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Abstract

We generalize the result of Gitik-Kanovei-Koepke [?] from Prikry forcing over κ to Magidor forcing and characterize all intermediate extensions of Magidor generic extensions. We also investigate how the cofinality of κ is effected when adding a set from a Prikry or Magidor extension.

Introduction

Menachem Magidor introduced "Magidor forcing" in his paper *Changing the cofinality of cardinals* [?]. This forcing was designated to change the cofinality of a measurable cardinal to a regular cardinal larger than ω. Formerly, the main method to change cofinality of measurables was using Prikry forcing, which injects an ω-sequence to that measurable [?].
The process of determining a generic set in both forcings, describes a formation of a cofinal sequence in a target measurable. Partial information about the final sequence yields intermediate extensions. Naturally, the question which arises:

Are these all possible intermediate extensions?

It is well known that if $\mathbb{P}$ is a forcing notion and G is $\mathbb{P}$-generic, then any intermediate ZFC model $V \subseteq N \subseteq V[G]$ is of the form $N = V[X]$ where $X \in V[G]$ is a generic set for some forcing in V. Therefore, the question can be reduced to

Is there $C' \subseteq C_G$ such that $V[X] = V[C']$?

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Where C_G is a Magidor sequence corresponding to the generic set G. As proved in 2010 by Gitik-Kanovei-Koepke [**?**], if the forcing subjected is Prikry forcing the answer to this question is positive. In some sense, Magidor forcing is a generalization of Prikry forcing, one may conjecture that it is possible to generalize the theorem. Asserting the conjecture is the main result of this paper.

Theorem 3.3 *Let $\vec{U}$ be a coherent sequence in V, $\langle\kappa_1,...\kappa_n\rangle$ be a sequence such that $o^{\vec{U}}(\kappa_i) < \min(\nu \mid 0 < o^{\vec{U}}(\nu))$, let G be $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$-generic*[1] *and let $A \in V[G]$ be a set of ordinals. Then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$.*

One of the main methods used in the proof was the construction of a forcing $\mathbb{M}_I[\vec{U}] \in V$, which is a projection of Magidor forcing $\mathbb{M}[\vec{U}]$. This forcing is a Magidor type forcing which uses only measures from $\vec{U}$ with index $i \in I$. Moreovere, $\mathbb{M}_I[\vec{U}]$ adds a prescribed subsequence $C_I := (C_G) \restriction I$ as a generic object, where $I \subseteq \lambda_0$ is a set of indexes in $\lambda_0 = \text{otp}(C_G)$. Hence, we may examine the intermediate extensions $V \subseteq V[C_I] \subseteq V[C_G]$ as an iteration of two forcing, which resemble $\mathbb{M}[\vec{U}]$ and behave well.

An important consequence of this theorem is the classification of all complete subforcings of $\mathbb{M}[\vec{U}]$, this will be discussed in chapter 5.

By Theorem 3.3, if $A \in V[G] \setminus V$ then $V[A] \models \kappa$ *is singular*. When we don't assume that the measures involved are normal, the situation is more complex, chapter 6 is devoted for this investigation. The main theorem of this chapter is

Theorem 6.7 Let $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega}\rangle$ consists of P-point ultrafilters over κ. Then for every new set of ordinals A in $V^{P(\mathbb{U})}$, κ has cofinality ω in $V[A]$[2].

In chapter 7 we give an example for a set A such that κ stays regular in $V[A]$ (even measurable).

[1] $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ is Magidor forcing with the coherent sequence $\vec{U}$ above a condition which has $\langle\kappa_1,...,\kappa_n\rangle$ as it's ordinal sequence

[2] $P(\mathbb{U})$ is the Prikry tree forcing, a detailed definition can be found in chapter 6

Notations

- V denotes the ground model.
- For any set A, $V[A]$ denote the minimal model of ZFC containing V and $\{A\}$
- $\prod_{j=1}^{n} A_j$ increasing sequences $\langle a_1, ..., a_n\rangle$ where $a_i \in A_i$
- $\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{n} A_{i,j}$ left-lexicographically increasing sequences (which is denoted by $\leq_{LEX}$)
- $[\kappa]^{\alpha}$ increasing sequences of length α
- $[\kappa]^{<\omega} = \bigcup\limits_{n<\omega} [\kappa]^n$
- ${}^{\alpha}[\kappa]$ not necessarily increasing sequences, i.e functions with domain α and range κ
- ${}^{\omega>}[\kappa] = \bigcup\limits_{n<\omega} {}^{n}[\kappa]$
- $\langle\alpha, \beta\rangle$ an ordered pair of ordinals. (α, β) the interval between α and β.
- $\vec{\alpha} = \langle\alpha_1, ..., \alpha_n\rangle$, $|\vec{\alpha}| = n$, $\vec{\alpha} \setminus \langle\alpha_i\rangle = \langle\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n\rangle$
- For every $\alpha < \beta$, The Cantor normal form (abbreviated C.N.F) equation is $\alpha + \omega^{\nu_1} + ... + \omega^{\nu_m} = \beta$, $\nu_1 \geq ... \geq \nu_m$ are unique. If $\alpha = 0$ this is the C.N.F of β, otherwise, this is the C.N.F difference of α, β.
- $o(\alpha) = \gamma$ where $\alpha = \omega^{\gamma_1} + ... + \omega^{\gamma_n} + \omega^{\gamma}$ (C.N.F).
- $\mathrm{Lim}(A) = \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$
- $\mathrm{Succ}(A) = \{\alpha \in A \mid \sup(A \cap \alpha) < \alpha\}$
- $\biguplus\limits_{i\in I} A_i$ is the union of $\{A_i \mid i \in I\}$ with the requirement that A_i's are pairwise disjoint.
- If $f : A \to B$ is a function then for every $A' \subseteq A$, $B' \subseteq B$

$$f''A' = \{f(x) \mid x \in A'\} \;,\; f^{-1}{}''B' = \{x \in A \mid f(x) \in B'\}$$

- Let $B \subseteq \langle\alpha_\xi \mid \xi < \delta\rangle = A$ be sequences of ordinals,

$$\mathrm{Index}(B, A) = \{\xi < \delta \mid \exists b \in B \;\alpha_\xi = b\}$$

- Let $\mathbb{P}$ be a forcing notion, σ a formula in the forcing language and $p \in \mathbb{P}$. If $\underset{\sim}{A}$ is a $\mathbb{P}$-name, then

$$p \,||\, \underset{\sim}{A} \text{ means "there is } a \in V \text{ such that } p \Vdash \check{a} = \underset{\sim}{A}\text{"}$$

- Let $p, q \in \mathbb{P}$ then "p, q are compatible in $\mathbb{P}$" if there exists $r \in \mathbb{P}$ such that $p, q \leq_{\mathbb{P}} r$. Otherwise, if they are incompatible denote it by $p \perp q$.
- In any forcing notion, $p \leq q$ means "q extends p".
- The notion of complete subforcing, complete embedding and projection is used as defined in [?]

1 Magidor forcing

Definition 1.1 *A coherent sequence is a sequence* $\vec{U} = \langle U(\alpha,\beta) \mid \beta < o^{\vec{U}}(\alpha)\ ,\alpha \leq \kappa\rangle$ *such that:*

1. $U(\alpha,\beta)$ *is a normal ultrafilter over* α.
2. *Let* $j : V \to Ult(U(\alpha,\beta),V)$ *be the corresponding elementary embedding , then* $j(\vec{U}) \restriction \alpha = \vec{U} \restriction \langle \alpha,\beta\rangle$.

Where

$$\vec{U} \restriction \alpha = \langle U(\gamma,\delta) \mid \delta < o^{\vec{U}}(\gamma)\ ,\gamma \leq \alpha\rangle$$
$$\vec{U} \restriction \langle\alpha,\beta\rangle = \langle U(\gamma,\delta) \mid (\delta < o^{\vec{U}}(\gamma),\ \gamma < \alpha) \vee (\delta < \beta,\ \gamma = \alpha)\rangle$$

■

Fix $\vec{U}$, a coherent sequence of ultrafilters with maximal element κ. We shall assume that $o^{\vec{U}}(\kappa) < \min(\nu \mid o^{\vec{U}}(\nu) > 0) := \delta_0$. Let $\alpha \leq \kappa$ with $o^{\vec{U}}(\alpha) > 0$, define

$$\bigcap U(\alpha,i) = \bigcap_{i<o^{\vec{U}}(\alpha)} U(\alpha,i)$$

We will follow the description of Magidor forcing as presented in [**?**].

Definition 1.2 $\mathbb{M}[\vec{U}]$ *consist of elements* p *of the form* $p = \langle t_1,...,t_n,\langle\kappa,B\rangle\rangle$. *For every* $1 \leq i \leq n$, t_i *is either an ordinal* κ_i *if* $o^{\vec{U}}(\kappa_i) = 0$ *or a pair* $\langle\kappa_i,B_i\rangle$ *if* $o^{\vec{U}}(\kappa_i) > 0$.

1. $B \in \bigcap_{\xi<o^{\vec{U}}(\kappa)} U(\kappa,\xi)$, $\quad \min(B) > \kappa_n$
2. *for every* $1 \leq i \leq n$
   (a) $\langle\kappa_1,...,\kappa_n\rangle \in [\kappa]^{<\omega}$
   (b) $B_i \in \bigcap_{\xi<o^{\vec{U}}(\kappa_i)} U(\kappa_i,\xi)$
   (c) $\min(B_i) > \kappa_{i-1}$ $\quad$ *(*$i > 1$*)*

$\blacksquare$

We shall adopt the following notations:

- $t_0 = 0, t_{n+1} = \langle \kappa, B \rangle$
- $o^{\vec{U}}(t_i) = o^{\vec{U}}(\kappa(t_i))$
- $o^{\vec{U}}(t_i) > 0$ then $t_i = \langle \kappa_i, B_i \rangle = \langle \kappa(t_i), B(t_i) \rangle$
- $o^{\vec{U}}(t_i) = 0$ then $t_i = \kappa_i = \kappa(t_i)$
- $\kappa(p) = \{\kappa(t_1), ..., \kappa(t_n)\}$
- $B(p) = \biguplus_{i=1}^{n+1} B(t_i)$

The ordinals κ_i are designated to form the eventual Magidor sequence and candidates for the sequence's missing elements in the interval $(\kappa(t_{i-1}), \kappa(t_i))$ (where $t_0 = 0,\ \kappa(t_{n+1}) = \kappa$) are provided by the sets $B(t_i)$.

Definition 1.3 *For $p = \langle t_1, t_2, ..., t_n, \langle \kappa, B \rangle \rangle, q = \langle s_1, ..., s_m, \langle \kappa, C \rangle \rangle \in \mathbb{M}[\vec{U}]$, define $p \leq q$ (q extends p) iff:*

1. *$n \leq m$*
2. *$B \supseteq C$*
3. *$\exists 1 \leq i_1 < ... < i_n \leq m$ such that for every $1 \leq j \leq m$:*
   (a) *If $\exists 1 \leq r \leq n$ such that $i_r = j$ then $\kappa(t_r) = \kappa(s_{i_r})$ and $C(s_{i_r}) \subseteq B(t_r)$*
   (b) *Otherwise $\exists\ 1 \leq r \leq n+1$ such that $i_{r-1} < j < i_r$ then*
      i. *$\kappa(s_j) \in B(t_r)$*
      ii. *$o^{\vec{U}}(s_j) < o^{\vec{U}}(t_r)$*
      iii. *$B(s_j) \subseteq B(t_r) \cap \kappa(s_j)$*

We also use p directly extends q, $p \leq^* q$ if:

1. $p \leq q$

2. $n = m$

$\blacksquare$

Remarks:

1. Let $p = \langle t_1, ..., t_n, \langle \kappa, B\rangle\rangle$. Assume we would like to add an element s_j to p between t_{r-1} and t_r. It is possible only if $o^{\vec{U}}(t_r) > 0$. Moreover, let $\xi = o^{\vec{U}}(s_j)$, then

$$s_j \in \{\alpha \in B(t_r) \mid o^{\vec{U}}(\alpha) = \xi\}$$

If $s_j = \kappa(s_j)$ (i.e. $\xi = 0$), then any s_j satisfying this requirement can be added. If $s_j = \langle \kappa(s_j), B(s_j)\rangle$ (i.e. $\xi > 0$), Then according to definition 1.3 (3.b.iii) s_j can be added iff

$$B(t_r) \cap \kappa(s_j) \in \bigcap_{\xi' < \xi} U(\kappa(s_j), \xi')$$

2. If $p = \langle t_1, ..., t_n, \langle \kappa, B\rangle\rangle \in \mathbb{M}[\vec{U}]$. Fix some $1 \le j \le n$ with $o^{\vec{U}}(t_j) > 0$. Then t_j yields a Magidor forcing in the interval $(\kappa(t_{j-1}), \kappa(t_j))$ with the coherent sequence $\vec{U} \restriction \kappa(t_j)$. t_j acts autonomously in the sense that the sequence produced by it is independent of how the sequence develops in other parts. This observation becomes handy when manipulating p, since we can make local changes at t_j with no impact on the t_i's.

Let $Y = \{\alpha \le \kappa \mid o^{\vec{U}}(\alpha) < \delta_0\}$. From Coherency of $\vec{U}$ it follows that $Y \in \bigcap U(\kappa, i)$. For every $\beta \in Y$ with $o^{\vec{U}}(\beta) > 0$ and $i < \delta_0$ define

$$Y(i) = \{\alpha < \kappa \mid o^{\vec{U}}(\alpha) = i\} \text{ and } Y[\beta] = \biguplus_{i < o^{\vec{U}}(\beta)} Y(i)$$

It follows that for every $\beta \in Y$ and $i < o^{\vec{U}}(\beta)$, $Y(i) \cap \beta \in U(\beta, i)$. To see this take $\beta \le \kappa$ in Y and $j_{\beta i} : V \to Ult(U(\beta, i), V)$.

$$Y(i) \cap \beta \in U(\beta, i) \;\Leftrightarrow \beta \in j_{\beta i}(Y(i) \cap \beta)$$

By coherency, $o^{j_{\beta i}(\vec{U})}(\beta) = i$ and therefore

$$\beta \in j_{\beta i}(Y(i) \cap \beta) = \{\alpha < j_{\beta i}(\beta) \mid o^{j_i(\vec{U})}(\alpha) = j_{\beta i}(i) = i\}.$$

Consequently, $Y[\beta] \cap \beta \in \bigcap_{i<o^{\vec{U}}(\beta)} U(\beta,i)$.
For $B \in \bigcap_{i<o^{\vec{U}}(\beta)} U(\beta,i)$ define recursively, $B^{(0)} = B$

$$B^{(n+1)} = \{\alpha \in B^{(n)} \mid (o^{\vec{U}}(\alpha) = 0) \vee (B^{(n)} \cap \alpha \in \cap U(\alpha,i))\}$$

Let $B^{\star} = \bigcap_{n<\omega} B^{(n)}$ it follows by induction that for all $n < \omega$

$$\mathrm{B}^{(n)} \in \bigcap_{i<o^{\vec{U}}(\beta)} U(\beta,i)$$

By β-completeness $B^{\star} \in \bigcap_{i<o^{\vec{U}}(\beta)} U(\beta,i)$. $B^{\star}$ has the feature that

$$\forall\alpha \in B^{\star}\ \alpha \cap B^{\star} \in \bigcap_{i<o^{\vec{U}}(\alpha)} U(\alpha,i)$$

The previous paragraph indicates that by restricting to a dense subset of $\mathbb{M}[\vec{U}]$ we can assume that given $p = \langle t_1, t_2, ..., t_n, \langle\kappa, B\rangle\rangle \in \mathbb{M}[\vec{U}]$, every choice of ordinal in $B(t_r)$ automatically satisfies the requirement that we discussed in remark (2). Formally, we work above $\langle\langle\rangle, \langle\kappa, Y\rangle\rangle$ and we directly-extend any $p = \langle t_1, t_2, ..., t_n, \langle\kappa, B\rangle\rangle$ as follows:
For every $1 \leq r \leq n+1$ and $i < o^{\vec{U}}(t_r)$ define

$$B(t_r, i) := Y(i) \cap B(t_r)^{\star} \in U(\kappa(t_r), i)$$

It follows that

$$B^{\star}(t_r) := \biguplus_{i<o^{\vec{U}}(t_r)} B(t_r,i) \in \bigcap_{i<o^{\vec{U}}(t_r)} U(\kappa(t_r),i).$$

Shrink $B(t_r)$ to $B^{\star}(t_r)$ to obtain

$$p \leq^{*} p^{*} = \langle t'_1, ..., t'_n, \langle\kappa, B^{\star}\rangle\rangle$$

$$t'_r = \begin{cases} t_r & o^{\vec{U}}(t_r) = 0 \\ \langle\kappa(t_r), B^{\star}(t_r)\rangle & \mathit{otherwise} \end{cases}$$

This dense subset also simplifies $\leq$ to

$$p \leq q \text{ iff } \kappa(p) \subseteq \kappa(q)\ ,\ B(p) \subseteq B(q)$$

When applying the revised approach regarding the large sets, it is apparent that $B(t_r, i)$ provide candidates, precisely, for the i-limit indexes in the final sequence C_G (defined in p.10) i.e. of indexes γ such that $o(\gamma) = i$ (for the definition of $o(\gamma)$ see Notations). This is stated formally in proposition 1.5.
Recall that:

- $\mathbb{M}[\vec{U}]$ satisfies $\kappa^+ - c.c.$
- Let $p = \langle t_1, ..., t_n, \langle \kappa, B \rangle \rangle \in \mathbb{M}[\vec{U}]$ and denote $\nu = \kappa(t_j)$ where j is the minimal such that $o^{\vec{U}}(t_j) > 0$. Then above p there is $\nu -_{\leq^*}$closure.
- $\mathbb{M}[\vec{U}]$ satisfies the Prikry condition.

Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic, define

$$C_G = \bigcup \{\kappa(p) \mid p \in G\}$$

We will abuse notation by considering C_G as a the canonical enumeration of the set C_G. C_G is closed and unbounded in κ. Therefore, The order type of C_G determines the cofinality of κ in $V[G]$. The next propositions can be found in [**?**].

Proposition 1.4 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic. Then G can be reconstructed from C_G as follows*

$$G = \{p \in \mathbb{M}[\vec{U}] \mid (\kappa(p) \subseteq C_G) \wedge (C_G \setminus \kappa(p) \subseteq B(p))\}$$

Therefore $V[G] = V[C_G]$.

■

Proposition 1.5 *Let G be $\mathbb{M}[\vec{U}]$-generic and C_G the corresponding Magidor sequence. Let $\langle t_1, ..., t_n, \langle \kappa, B \rangle \rangle \in G$, then*

$$\mathrm{otp}((\kappa(t_i),\kappa(t_{i+1}))\cap C_G)=\omega^{o^{\vec{U}}(\kappa(t_{i+1}))}$$

Thus if $\kappa(t_{i+1})=C_G(\gamma)$ *then* $o(\gamma)=o^{\vec{U}}(t_{i+1})$.

■

Corollary 1.6 $cf^{V[G]}(\kappa)=cf(o^{\vec{U}}(\kappa))$

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Let $p=\langle t_1,...,t_n,\langle\kappa,B\rangle\rangle\in G$. By proposition 1.5 , for each $i\leq n$ one can determine the position of $\kappa(t_i)$ in C_G. Namely, $C_G(\gamma)=\kappa(t_i)$ where

$$\gamma=\sum_{j\leq i}\omega^{o^{\vec{U}}(t_j)}=:\gamma(t_i,p)\in\omega^{o^{\vec{U}}(\kappa)}\quad(*)$$

Addition and power are of ordinals. The equetion (*) induces a C.N.F equation

$$\gamma=\textstyle\sum_{r=1}^{m}\omega^{o^{\vec{U}}(t_{jr})}\quad\text{(C.N.F)}$$

This indicates the close connection between Cantor normal form of the index γ in $\mathrm{otp}(C_G)$ and the important elements $t_{j_1},...,t_{j_m}$ to determine that $\gamma(t_i,p)=\gamma$. Now let $q=\langle s_1,...,s_m,\langle\kappa,B'\rangle\rangle$ be another condition, by definition 1.3 (3.b.ii), if s_j is an element of q which was added to p in the interval $(\kappa(t_r),\kappa(t_{r+1}))$ then $o^{\vec{U}}(s_j)<o^{\vec{U}}(t_{r+1})$. Consequently

$$p\leq q\Rightarrow\gamma(t_r,p)=\gamma(s_{i_r},q)$$

2 Combinatorial properties

The combinatorial nature of $\mathbb{M}[\vec{U}]$ is most clearly depicted through the language of step-extensions as presented below.
To perform a one step extension of $p = \langle t_1, t_2, ..., t_n, \langle \kappa, B \rangle \rangle$

1. choose $1 \leq r \leq n+1$ with $0 < o^{\vec{U}}(t_r)$
2. choose $i < o^{\vec{U}}(t_r)$
3. choose an ordinal $\alpha \in B(t_r, i)$
4. shrink the $B(t_s, j)$'s to $C(t_s, j) \in U(t_s, j)$ for every $1 \leq s \leq n+1$ and $C(t_s) = \biguplus_{j < o^{\vec{U}}(t_i)} C_s(j)$
5. For $j < o^{\vec{U}}(\alpha)$ pick $C(\alpha, j) \in U(\alpha, j)$, $C(\alpha, j) \subseteq B(t_r, j) \cap \alpha$ to obtain $C(\alpha) = \biguplus_{j < o^{\vec{U}}(\alpha)} C(\alpha, j)$
6. cut $C(t_r)$ above α

Extend p to

$$p^\frown \langle \alpha, (C(t_s))_{s=1}^{n+1}, C(\alpha) \rangle = \langle t'_1, ..., t'_{i-1}, \langle \alpha, C(\alpha) \rangle, t'_i, ..., t'_n, \langle \kappa, C(t_{n+1}) \rangle \rangle$$

$$t'_r = \begin{cases} t_r & o^{\vec{U}}(t_r) = 0 \\ \langle \kappa(t_r), C(t_r) \rangle & o.w. \end{cases}$$

It is clear that every extension of p with only one ordinal added is a one step extension. Next we introduce some notations which will describe a general step extension. The idea is simply to classify extensions according to the order of the measures the new elements of the sequence are chosen from.

Definition 2.1 *Let* $p = \langle t_1, t_2, ..., t_n, \underbrace{\langle \kappa, B \rangle}_{t_{n+1}} \rangle \in \mathbb{M}[\vec{U}]$

1. *For* $1 \leq i \leq n+1$ *define the tree* $T_i(p) = {}^{\omega>}[O^{\vec{U}}(t_i)]$, *with the ordering* $\langle x_1, ..., x_m \rangle \preceq \langle x'_1, ..., x'_{m'} \rangle$ *iff* $\exists 1 \leq i_1 < ... < i_m \leq m'$ *such that for every* $1 \leq j \leq m'$*:*

(a) if $\exists 1 \leq r \leq m$ *such that* $i_r = j$ *then* $x_r = x'_j$

(b) otherwise $\exists\ 1 \leq r \leq n+1$ *such that if* $i_{r-1} < j < i_r$ *then* $x'_j < x_r$

We think of x_r*'s as placeholders of ordinals from* $B(t_i, x_r)$*. With this in mind, the ordering is induced by definition 1.3 (3).*

2. $T(p) = \prod_{i=1}^{n+1} T_i(p)$ *with* $\preceq$ *as the product order.*

3. *Let* $X_i \in T_i(p)$ $\ 1 \leq i \leq n+1$, $|X_i| = l_i$, $X = \langle X_1, ..., X_{n+1}\rangle \in T(p)$.

4. *Let*

$$\vec{\alpha}_i = \langle \alpha_1, ..., \alpha_{l_i}\rangle \in \prod_{j=1}^{l_i} B(t_i, X_i(j)) =: B(p, X_i)$$

X_i *is called an extension-type below* t_i *and* $\langle \alpha_1, ..., \alpha_{l_i}\rangle$ *is of type* X_i.

5. *Let*

$$\vec{\alpha} = \langle \vec{\alpha_1}, ..., \vec{\alpha_{n+1}}\rangle \in \prod_{i=1}^{n+1}\prod_{j=1}^{l_i} B(t_i, X_i(j)) =: B(p, X)$$

X *is called an extension-type of* p *and* $\vec{\alpha}$ *is of type* X.

■

Notice that by our assumption $|T(p)| < \min(\nu | 0 < o^{\vec{U}}(\nu)) = \delta_0$. We also use:

- $|X_i| = l_i$
- $l_x = \max(i \mid X_i \neq \emptyset)$
- $x_{i,j} = X_i(j)\ \alpha_{i,j} = \vec{\alpha}_i(j)$
- $x_{i,l_i+1} = o^{\vec{U}}(t_i)$ and $\alpha_{i,n+1} = \kappa(t_i)$
- $x_{mc} = x_{l_X, l_{l_X}}$ (i.e. the last element of X)
- $o^{\vec{U}}(\vec{\alpha}) = \langle o^{\vec{U}}(\alpha_{i,j}) \mid x_{i,j} \in X\rangle$ is the type of $\vec{\alpha}$.

A general extension of p of type X would be of the form:

$$p^\frown\langle \vec{\alpha}, (C(x_{i,j}))_{x_{i,j}\in X}, (C(t_r))_{r=1}^{n+1}\rangle = p^\frown\langle \vec{\alpha}, (C(x_{i,j}))_{\substack{i\leq n+1\\ j\leq l_i+1}}\rangle$$

where

$$p^\frown\langle\vec{\alpha},(C(x_{i,j}))_{\substack{i\leq n+1\\ j\leq l_i+1}}\rangle=\langle\vec{s_1},t'_1,...,\vec{s_n},t'_n,\vec{s_{n+1}},\langle\kappa,C\rangle\rangle$$

1. $\vec{\alpha}\in B(p,X)$ (X is uniquely determined by $\vec{\alpha}$).
2. $t'_s=\begin{cases} t_s & o^{\vec{U}}(t_s)=0\\ \langle\kappa(t_s),C(t_s)\rangle & o.w.\end{cases}$
   For some pre-chosen sets $C(t_s)\in\bigcap\limits_{\xi<o^{\vec{U}}(t_s)}U(\kappa(t_s),\xi)$, $C(t_s)\subseteq B(t_s)$.
3. $\vec{s}_i(j)=\begin{cases} \alpha_{i,j} & x_{i,j}=0\\ \langle\alpha_{i,j},C(x_{i,j})\rangle & o.w.\end{cases}$
   For some pre-chosen sets $C(x_{i,j})\in\bigcap\limits_{\xi<x_{i,j}}U(\alpha_{i,j},\xi)$, $C(x_{i,j})\subseteq B(t_i)\cap\alpha_{i,j}$.
4. $C\in\bigcap\limits_{\xi<o^{\vec{U}}(\kappa)}U(\kappa,\xi)$ and $\min(C)>\max(\vec{s}_{n+1})$

Keeping in mind the development succeeding definition 1.3,

$$p^\frown\langle\vec{\alpha},(C(x_{i,j}))_{\substack{i\leq n+1\\ j\leq l_i+1}}\rangle\in\mathbb{M}[\vec{U}]$$

holds due to the α's being meticulously handpicked. We will more frequently use $p^\frown\langle\vec{\alpha}\rangle$ with the same definition as above except we do not shrink any sets and simply take $\alpha_{i,j}\cap B(t_i)=C(x_{i,j})$. Define

$$p^\frown X=\{p^\frown\langle\vec{\alpha}\rangle\mid\vec{\alpha}\in B(p,X)\}$$

The $p^\frown X$'s induces a partition of $\mathbb{M}[\vec{U}]$ above p as stated in the next proposition which is well known and follows directly from definition 1.3.

Proposition 2.2 *Let $p\in\mathbb{M}[\vec{U}]$ be any condition and $p\leq q\in\mathbb{M}[\vec{U}]$. Then there exists a unique $\vec{\alpha}\in B(p,X)$ such that $p^\frown\langle\vec{\alpha}\rangle\leq^* q$.*

■

Example:

Let

$$p = \langle \underbrace{\langle \kappa(t_1), B(t_1) \rangle}_{t_1}, \underbrace{\kappa(t_2)}_{t_2}, \underbrace{\langle \kappa(t_3), B(t_3) \rangle}_{t_3}, \underbrace{\langle \kappa(t_4), B(t_4) \rangle}_{t_4}, \underbrace{\langle \kappa, B \rangle}_{t_5} \rangle$$

$$o^{\vec{U}}(t_1) = 1,\ o^{\vec{U}}(t_2) = 0,\ o^{\vec{U}}(t_3) = 2,\ o^{\vec{U}}(t_4) = 1,\ o^{\vec{U}}(\kappa) = 3$$

Let

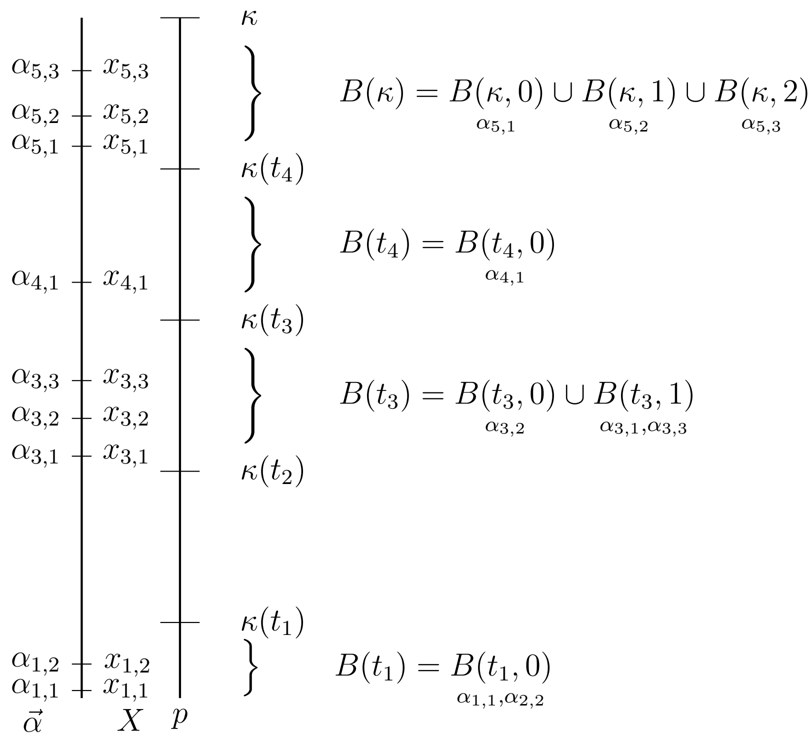
$$q = p^\frown \langle \underbrace{\langle \alpha_{1,1}, \alpha_{1,2} \rangle}_{\vec{\alpha_1}}, \underbrace{\langle \rangle}_{\vec{\alpha_2}}, \underbrace{\langle \alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3} \rangle}_{\vec{\alpha_3}}, \underbrace{\langle \alpha_{4,1} \rangle}_{\vec{\alpha_4}}, \underbrace{\langle \alpha_{5,1}, \alpha_{5,2}, \alpha_{5,3} \rangle}_{\vec{\alpha_5}} \rangle$$

$$o^{\vec{U}}(\alpha_{i,j}) = \begin{cases} 0 & \langle i, j \rangle = \langle 1,1 \rangle, \langle 1,2 \rangle, \\ & \quad \langle 3,2 \rangle, \langle 4,1 \rangle, \langle 5,1 \rangle \\ 1 & \langle i, j \rangle = \langle 3,1 \rangle, \langle 3,3 \rangle, \\ & \quad \langle 5,2 \rangle \\ 2 & \langle i, j \rangle = \langle 5,3 \rangle \end{cases}$$

Then the extention-type of q is

$$X = \langle \underbrace{\langle 0, 0 \rangle}_{X_1}, \underbrace{\langle \rangle}_{X_2}, \underbrace{\langle 1, 0, 1 \rangle}_{X_3}, \underbrace{\langle 0 \rangle}_{X_4}, \underbrace{\langle 0, 1, 2 \rangle}_{X_5} \rangle$$

This can be illustrated as following:



As presented in proposition 2.2, a choice from the set $p^\frown X$ is essentially a choice from some $\prod_{i=1}^{n} A_i$, $A_i \in U_i$ and $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_n$ are measurable cardinals with normal measures $U_1, ..., U_n$, Namely, $\prod_{i=1}^{n} A_i = B(p, X)$. We will need some properties of those sets.

Lemma 2.3 *Let $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_n$ be any collection of measurable cardinals with normal measures $U_1, ..., U_n$ respectively. Assume $F : \prod_{i=1}^{n} A_i \longrightarrow \nu$ where $\nu < \kappa_1$ and $A_i \in U_i$. Then there exists $H_i \subseteq A_i$ $H_i \in U_i$ such that $\prod_{i=1}^{n} H_i$ is homogeneous for F.*

Proof: By induction on n, the case $n = 1$ is known. Assume that the lemma holds for $n - 1$, and fix $\vec{\eta} = \langle \eta_1, ..., \eta_{n-1} \rangle \in \prod_{i=1}^{n-1} A_i$. Define

$$F_{\vec{\eta}} : A_n \setminus (\eta_{n-1} + 1) \longrightarrow \nu$$
$$F_{\vec{\eta}}(\xi) = F(\eta_1, ..., \eta_{n-1}, \xi)$$

By the case n=1 there exists a homogeneous $A_n \supseteq H(\vec{\eta}) \in U_n$ with color $C(\vec{\eta}) < \nu$. Define

$$\Delta_{\vec{\eta} \in \prod_{i=1}^{n-1} A_i} H(\vec{\eta}) =: H_n$$

By the induction hypotheses, $C : \prod_{i=1}^{n-1} A_i \to \nu$ has a homogeneous set of the form $\prod_{i=1}^{n-1} H_i$ where $A_i \supseteq H_i \in U_i$. To see that $\prod_{i=1}^{n} H_i$ is homogeneous for F,
let $\vec{\eta'} = \langle \eta'_1, ..., \eta'_n \rangle, \vec{\eta} = \langle \eta_1, ..., \eta_n \rangle \in \prod_{i=1}^{n} H_i$. We have

$$F(\vec{\eta}) = F_{\vec{\eta} \setminus \langle \eta_n \rangle}(\eta_n) \underset{\eta_n \in H(\vec{\eta} \setminus \langle \eta_n \rangle)}{\underset{\uparrow}{=}} F'(\vec{\eta} \setminus \langle \eta_n \rangle) \underset{\vec{\eta} \setminus \langle \eta_n \rangle, \vec{\eta'} \setminus \langle \eta'_n \rangle \in \prod_{i=1}^{n-1} H_i}{\underset{\uparrow}{=}}$$

$$= F'(\vec{\eta'} \setminus \langle \eta'_n \rangle) = ... = F(\vec{\eta'}).$$

$\blacksquare$

Lemma 2.4 *Let $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_n$ be a non descending finite sequence of measurable cardinals with normal measures $U_1, ..., U_n$ respectively. Assume $F : \prod_{i=1}^{n} A_i \longrightarrow B$ where B is any set, and $A_i \in U_i$. Then there exists $H_i \subseteq A_i$ $H_i \in U_i$ and set of important coordinates $I \subseteq \{1, ..., n\}$ such that $F \restriction \prod_{i=1}^{n} H_i$ is well defined modulo the equivalence relation:*

$$\langle \alpha_1, ..., \alpha_n \rangle \sim \langle \alpha'_1, ..., \alpha'_n \rangle \quad \textit{iff} \ \forall i \in I \ \alpha_i = \alpha'_i$$

and the induced function, $\bar{F}$, is injective.

Proof: By induction on n, if $n = 1$ then it is immediate since for any $f : A \to B$ such that $A \in U$ where U is a normal measure on a measurable cardinal κ, B is any set, then there exists $A \supseteq A' \in U$ for which $F \restriction A'$ is either constant or injective. Assume that the lemma holds for $n - 1$, $n > 1$ and let $F : \prod_{i=1}^{n} A_i \longrightarrow B$ be a function satisfying the conditions of the lemma. Define for every $x_1 \in A_1$, $F_{x_1} : \prod_{i=2}^{n} A_i \setminus (x_1 + 1) \longrightarrow B$

$$F_{x_1}(x_2, ..., x_n) = F(x_1, x_2, ..., x_n)$$

By the induction hypothesis , for every $x_1 \in A_1$ there are $A_i \supseteq A_i(x_1) \in U_i$ and set of important coordinates $I(x_1) \subseteq \{2,...,n\}$. The function
$I : A_1 \to P(\{2,...,n\})$ is constant on $A_1' \in U_1$ with value I'. For every $i = 2,...,n$ define $A_i' = \underset{x_1 \in A_1}{\Delta} A_i(x_1)$. So far, $\prod_{i=1}^{n} A_i'$ has the property:

(1) for any $\langle x_1, x_2, ..., x_n\rangle, \langle x_1, x_2', ..., x_n'\rangle \in \prod_{i=1}^{n} A_i'$ (same first coordinate)

$$F(x_1, x_2, ..., x_n) = F(x_1, x_2', ..., x_n') \text{ iff } \forall i \in I' \; x_i = x_i'$$

In particular, $\bar{F}$ is a well defined function modulo $I' \cup \{1\}$. Next we determine if 1 is important. For every $\langle \alpha, \alpha' \rangle \in A_1' \times A_1'$, define $t_{\langle \alpha,\alpha' \rangle} : \prod_{i=2}^{n} A_i' \setminus (\alpha' + 1) \to 2$

$$t_{\langle \alpha,\alpha' \rangle}(x_2, ..., x_n) = 1 \Leftrightarrow F(\alpha, x_2, ..., x_n) = F(\alpha', x_2, ..., x_n)$$

By lemma 2.3, for $i = 2, ..., n$ there are $A_i' \supseteq A_i(\alpha, \alpha') \in U_i$ such that $\prod_{i=2}^{n} A_i(\alpha, \alpha')$ is homogeneous for $t_{\langle \alpha,\alpha' \rangle}$ with color $C(\alpha, \alpha')$. Taking the diagonal intersection over $A_1' \times A_1'$ of the sets $A_i(\alpha, \alpha')$ at each coordinate $i = 2, ..., n$, we obtain $H_i \in U_i$ such that $\prod_{i=2}^{n} H_i$ is homogeneous for every $t_{\langle \alpha,\alpha' \rangle}$. Finally, the function $C : A_1' \times A_1' \to 2$ yield a homogeneous $A_1' \supseteq H_1 \in U_1$ with color C'.

case 1: $C' = 1$. Let us show that the important coordinates are I'. If $\langle x_1, ..., x_n\rangle, \langle x_1', ..., x_n'\rangle \in \prod_{i=1}^{n} H_i$ then $F(x_1, x_2', ..., x_n') = F(x_1', x_2', ..., x_n')$

$$F(x_1,...,x_n) = F(x_1',...,x_n') \Leftrightarrow F(x_1,x_2,...,x_n) = F(x_1,x_2',...,x_n') \Leftrightarrow \forall i \in I' \; x_i = x_i'$$

case 2: $C' = 0$. We then have a second property:

(2) For every $x_1, x_1' \in H_1$ and $\langle x_2, ..., x_n\rangle \in \prod_{i=2}^{n} H_i$

$$x_1 = x_1' \text{ iff } F(x_1, x_2..., x_n) = F(x_1', x_2, ..., x_n)$$

We would like to claim that in this case the important coordinates are $I = I' \cup \{1\}$ but the H_i's defined, may not be the sets we seek for, since there can still be an counter example for $\bar{F}$ not being injective i.e.

$$\langle x_1, ..., x_n\rangle \neq \langle x_1', ..., x_n'\rangle \text{ mod-}I \text{ such that } F(x_1, ..., x_n) = F(x_1', ..., x_n')$$

Let us prove that if we eliminate all counter examples from H_i's , we are left with a large set. Take Any counter example and set

$$\{x_1, ..., x_n\} \cup \{x'_1, ..., x'_n\} = \{y_1, ..., y_k\} \text{ (increasing enumeration)}$$

To reconstruct $\{x_1, ..., x_n\}, \{x'_1, ..., x'_n\}$ from $\{y_1, ..., y_k\}$ is suffices to know for example how $\{x_1, ..., x_n\}$ are arranged between $\{x'_1, ..., x'_n\}$. There are finitely many ways [3] for Such an arrangement. Therefore, if we succeed with eliminating examples of a fixed arrangement, then by completeness of the measures we will be able to eliminate all counter example.
Fix such an arrangement, the increasing sequence $\langle y_1, ..., y_k\rangle$ is in the product of some k large sets $\prod_{i=1}^{k} H_{n_i}$. We have to be careful since the sequence of measurables induced by $n_1, ..., n_k$ is not necessarily non descending. To fix this we can cut the sets H_i such that in the sequence $\langle \kappa_i \mid i = 1, ..., n\rangle$, wherever $\kappa_i < \kappa_{i+1}$ then $\min(H_{i+1}) > \kappa_i = \sup(H_i)$. Therefore, assume that $\langle \kappa_{n_i} \mid i = 1, ..., k\rangle$ is non descending. Define $G : \prod_{i=1}^{k} H_{n_i} \to 2$

$$G(y_1, ..., y_k) = 1 \Leftrightarrow F(x_1, ..., x_n) = F(x'_1, ..., x'_n)$$

By lemma 2.3 there must be $U_i \ni H'_i \subseteq H_i$ homogeneous for G with value D. If $D = 0$ we have eliminated from H_i's all counter examples of that fixed ordering. Assume $D = 1$, then every $y_1, ..., y_k$ yield a counter example $\langle x_1, ..., x_n\rangle, \langle x'_1, ..., x'_n\rangle$ (different modulo I). $x_1 = x'_1$ is impossible by property (1). If $x_1 < x'_1$, Fix $x < w < y_2 < ... < y_n$, where $x, w \in H'_1$ and $y_i \in H'_{n_i}$ $i = 2, ..., k$. Then $G(x, y_2, ..., y_k) = G(w, y_2, ..., y_k) = 1$ and

$$F(x, x_2, ..., x_n) = F(x'_1, x'_2, ..., x'_n) = F(w, x_2, ..., x_n)$$

contradiction to (2). $x_1 < x'_1$ is symmetric.

■

[3]In general, the number of possibilities to arrange two counter examples into one increasing sequence depends on I. Nevertheless, there is an upper bound: Think of x_i's as balls we would like to divide into $n+1$ cells. The cells are represented by the intervals $(x'_{i-1}, x'_i]$ plus the cell for elements above x'_n. There are $\binom{2n}{n}$ such divisions. For any such division, we decide either the cell is $(x'_{i-1}, x'_i]$ or (x'_{i-1}, x'_i). Hence, there are at most $\binom{2n}{n} \cdot 2^n$ such arrangements.

3 The main result up to κ

As stated in corollary 1.6, Magidor forcing adds a closed unbounded sequence of length $\omega^{o^{\vec{U}}(\kappa)}$ to κ. It is possible to obtain a family of forcings that adds a sequence of any limit length to some measurable cardinal, using a variation of Magidor forcing as we defined it[4]. Namely, let $\vec{U}$ be a coherent sequence and $\lambda_0 < \min(\nu \mid o^{\vec{U}}(\nu) > 0)$ a limit ordinal

$$\text{(not necessarily C.N.F) } \lambda_0 = \omega^{\gamma_1} + ... + \omega^{\gamma_n} \quad ,\gamma_n > 0$$

Let $\langle\kappa_1, ...\kappa_n\rangle$ be an increasing sequence such that $o^{\vec{U}}(\kappa_i) = \gamma_i$. Define the forcing $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ as follows:
The root condition will be

$$0_{\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]} = \langle\langle\kappa_1, B_1\rangle, ..., \langle\kappa_n, B_n\rangle\rangle$$

where $B_1, ..., B_n$ are as in the discussion following definition 1.3. The conditions of this forcing are any finite sequence that extends $0_{\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]}$ in the sense of definition 1.3. Since each $\langle\kappa_i, B_i\rangle$ acts autonomously, this forcing is essentially the same as $\mathbb{M}[\vec{U}]$. In fact, $\mathbb{M}[\vec{U}]$ is just $\mathbb{M}_{\langle\kappa\rangle}[\vec{U}]$. The notation we used for $\mathbb{M}[\vec{U}]$ can be extended to $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ since the conditions are also of the form $\langle t_1, ..., t_r, \langle\kappa, B\rangle\rangle$. Let

$$\langle\langle\nu_1, C_1\rangle, ..., \langle\nu_m, C_m\rangle\rangle \in \mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$$

then $\mathbb{M}_{\langle\nu_1,...,\nu_m\rangle}[\vec{U}]$ is an open subset of $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ (i.e. $\leq$-upwards closed). Moreover, if $G \subseteq \mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ is any generic set with $\langle\langle\nu_1, C_1\rangle, ..., \langle\nu_m, C_m\rangle\rangle \in G$ then

$$(G)_{\langle\nu_1,...,\nu_m\rangle} = G \cap \mathbb{M}_{\langle\nu_1,...,\nu_m\rangle}[\vec{U}] = \{p \in G \mid p \geq \langle\langle\nu_1, C_1\rangle, ..., \langle\nu_m, C_m\rangle\rangle\}$$

is generic for $\mathbb{M}_{\langle\nu_1,...,\nu_m\rangle}[\vec{U}]$. $(G)_{\vec{\nu}}$ is essentially the same generic as G since it yield the same Magidor sequence, in particular $V[(G)_{\vec{\nu}}] = V[G]$.

From now on the set B in $\langle t_1, ..., t_r, \langle\kappa, B\rangle\rangle$ will be suppressed and replaced by $t_{r+1} = \langle\kappa, B\rangle$ where $\kappa_n = \kappa$. An alternative way to describe $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ is through the following product

$$\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}] \simeq \mathbb{M}[\vec{U}]_{\langle\kappa_1\rangle} \times (\mathbb{M}[\vec{U}]_{\langle\kappa_2\rangle})_{>\kappa_1} \times ... \times (\mathbb{M}[\vec{U}]_{\langle\kappa_n\rangle})_{>\kappa_{n-1}}$$
$$(\mathbb{M}_{\langle\nu_1,...,\nu_m\rangle}[\vec{U}])_{>\alpha} = \{\langle t_1, ..., t_{r+1}\rangle \in \mathbb{M}_{\langle\nu_1,...,\nu_m\rangle}[\vec{U}] \mid \kappa(t_1) > \alpha\}$$

[4] Magidor's original formulation of $\mathbb{M}[\vec{U}]$ in [?] gives such a family

This isomorphism is induced by the embeddings

$$i_r : ((\mathbb{M}[\vec{U}]_{\langle\kappa_r\rangle})_{>\kappa_{r-1}} \to \mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}] \quad , r = 1, ..., n$$
$$i_r(\langle s_1, ..., s_{k+1}\rangle) = \Big\langle {\scriptstyle \langle\kappa_1,B_1\rangle,...,\langle\kappa_{r-1},B_{r-1}\rangle,s_1,...,s_k,\underbrace{\langle\kappa_r,B(s_{k+1})\rangle}_{s_{k+1}},...,\langle\kappa_n,B_n\rangle} \Big\rangle$$

From this embeddings, it is clear that the generic sequence produced by $(\mathbb{M}[\vec{U}]_{\langle\kappa_r\rangle})_{>\kappa_{r-1}}$ is just $C_G \cap (\kappa_{r-1}, \kappa_r)$.

The formula to compute coordinates holds in this context:
Let $p = \langle t_1, ..., t_m, t_{m+1}\rangle \in \mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$. For each $1 \leq i \leq m$, the coordinate of $\kappa(t_i)$ in any Magidor sequence extending p is $C_G(\gamma) = \kappa(t_i)$, where

$$\gamma = \sum_{j\leq i} \omega^{o^{\vec{U}}(t_j)} =: \gamma(t_i, p) < \lambda_0$$

Lemma 3.1 *Let G be generic for $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ and the sequence derived*

$$C_G = \bigcup \{\{\kappa(t_1), ..., \kappa(t_l)\} \mid \langle t_1, ..., t_l, t_{l+1}\rangle \in G\}$$

1. $\mathrm{otp}(C_G) = \lambda_0$
2. *If $\kappa_i < C_G(\gamma) < \kappa_{i+1}$ where γ is limit, then there exists $\vec{\nu} = \langle \nu_1, ..., \nu_m\rangle$ such that $(G)_{\vec{\nu}^\frown\langle\kappa_{i+1},...,\kappa_n\rangle}$ is generic for $\mathbb{M}_{\vec{\nu}^\frown\langle\kappa_{i+1},...,\kappa_n\rangle}[\vec{U}]$, $C_G = C_{(G)_{\vec{\nu}^\frown\langle\kappa_{i+1},...,\kappa_n\rangle}}$ and the sequences obtained by the split*

$$\mathbb{M}_{\vec{\nu}}[\vec{U}] \times (\mathbb{M}_{\langle\kappa_{i+1},...,\kappa_n\rangle}[\vec{U}])_{>\nu_m} \simeq \mathbb{M}_{\vec{\nu}^\frown\langle\kappa_{i+1},...,\kappa_n\rangle}[\vec{U}]$$

   are $C_G \cap C_G(\gamma), C_G \setminus C_G(\gamma)$. More accurately, if

$$\gamma = \underbrace{\omega^{\gamma_1} + ... + \omega^{\gamma_i}}_{\xi} + \omega^{\gamma'_{i+1}} + ... + \omega^{\gamma'_m} \quad \textit{(C.N.F)}$$

   then

$$\vec{\nu} = \langle \nu_1, ..., \nu_m\rangle = \langle \kappa_1, ..., \kappa_i, C_G(\xi + \omega^{\gamma'_{i+1}}), ..., C_G(\gamma)\rangle$$

Proof: For (1), the same reasoning as in lemmas 1.5-1.6 should work. For (2), notice that by proposition 1.4, $0_{\mathbb{M}_{\vec{\nu}^\frown\langle\kappa_{i+1},...,\kappa_n\rangle}} \in G$. Thus $(G)_{\vec{\nu}^\frown\langle\kappa_{i+1},...,\kappa_n\rangle}$ is generic for $\mathbb{M}_{\vec{\nu}^\frown\langle\kappa_{i+1},...,\kappa_n\rangle}[\vec{U}]$. The embeddings

$$i_1 : \mathbb{M}_{\langle \nu_1,...,\nu_m\rangle}[\vec{U}] \to \mathbb{M}_{\vec{\nu}^\frown\langle \kappa_{i+1},...,\kappa_n\rangle}[\vec{U}]$$
$$i_1(\langle t_1,...,t_{r+1}\rangle) = \langle t_1,...,t_{r+1},\langle \kappa_{i+1},B_{i+1}\rangle,...,\langle \kappa_n,B_n\rangle\rangle$$

and

$$i_2 : (\mathbb{M}_{\langle \kappa_{i+1},...,\kappa_n\rangle}[\vec{U}])_{>\nu_m} \to \mathbb{M}_{\vec{\nu}^\frown\langle \kappa_{i+1},...,\kappa_n\rangle}[\vec{U}]$$
$$i_2(\langle s_1,...,s_{k+1}\rangle) = \langle \langle \kappa_1,B_1\rangle,...,\langle \kappa_i,B_i\rangle, s_1,...,s_{k+1}\rangle$$

induces the isomorphism of $\mathbb{M}_{\vec{\nu}^\frown\langle \kappa_{i+1},...,\kappa_n\rangle}[\vec{U}]$ with the product. Therefore, $i_1^{-1}(G)$, $i_2^{-1}(G)$ are generic for $\mathbb{M}_{\langle \nu_1,...,\nu_m\rangle}[\vec{U}]$, $(\mathbb{M}_{\langle \kappa_{i+1},...,\kappa_n\rangle}[\vec{U}])_{>\nu_m}$ respectively. By the definition of i_1, i_2 this generics obviously yield the sequences $C_G \cap C_G(\gamma)$ and $C_G \setminus C_G(\gamma)$.

∎

In general we will identify G with $(G)_{\vec{\nu}}$ when using lemma 3.1.

Notice that, the information used in order to compute $\gamma(t_i,p)$ is just $o^{\vec{U}}(t_i)$. Let X be an extension type of p, then X provides this information, therefore, one can compute the coordinates of any extension $\vec{\alpha}$ of type X. In particular, for any $\alpha_{i.r}$ substituting $x_{i,r} \in X$ the coordinate of $\alpha_{i,r}$ is

$$\gamma = \gamma(t_{i-1},p) + \omega^{x_{i,1}} + ... + \omega^{x_{i,r}} =: \gamma(x_{i,r}, p^\frown X)$$

In this situation we say that X *unveils the γ-th coordinate*. If $x_{i,r} = x_{mc}$, we say that X *unveils γ as maximal coordinate.*

Proposition 3.2 *Let $p = \langle t_1,...,t_n,t_{n+1}\rangle \in \mathbb{M}_{\langle \kappa_1,...\kappa_n\rangle}[\vec{U}]$ and γ such that for some $0 \le i \le n$, $\gamma(t_i,p) < \gamma < \gamma(t_{i+1},p)$. Then there exists an extension-type X unveiling γ as maximal coordinate. Moreover, if*

$$\gamma(t_i,p) + \sum_{j\le m} \omega^{\gamma_j} = \gamma \text{ (C.N.F)}$$

then the extension type is $X = \langle X_i\rangle$ where $X_i = \langle \gamma_1,...,\gamma_m\rangle$.

∎

Example: Assume $\lambda_0 = \omega_1 + \omega^2 \cdot 2 + \omega$, let $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 = \kappa$ be such that $o^{\vec{U}}(\kappa_1) = \omega_1$, $o^{\vec{U}}(\kappa_2) = o^{\vec{U}}(\kappa_3) = 2$ and $o^{\vec{U}}(\kappa) = 1$. Let

$$p = \Big\langle \underbrace{\langle \nu_1, B(\nu_1)\rangle}_{t_1}, \underbrace{\nu_2}_{t_2}, \underbrace{\langle \kappa_1, B(k_1)\rangle}_{t_3}, \underbrace{\langle \nu_4, B(\nu_3)\rangle}_{t_4}, \underbrace{\langle \kappa_2, B(\kappa_2)\rangle}_{t_5}, \underbrace{\langle \kappa_3, B(\kappa_3)\rangle}_{t_6}, \underbrace{\langle \kappa, B\rangle}_{t_7} \Big\rangle$$
$$o^{\vec{U}}(t_1) = \omega,\ o^{\vec{U}}(t_2) = 0,\ o^{\vec{U}}(t_4) = 1$$

Let G be any generic with $p \in G$. Calculating $\gamma(t_i, p)$ for $i = 1, ..., 7$ we get

1. $\gamma(t_1, p) = \omega^{o^{\vec{U}}(t_1)} = \omega^\omega \ \Rightarrow \ \ C_G(\omega^\omega) = \nu_1$
2. $\gamma(t_2, p) = \omega^\omega + \omega^{o^{\vec{U}}(t_2)} = \omega^\omega + 1 \ \Rightarrow \ \ C_G(\omega^\omega + 1) = \nu_2$
3. $\gamma(t_3, p) = \omega^\omega + 1 + \omega^{\omega_1} = \omega^{\omega_1} = \omega_1$
4. $\gamma(t_4, p) = \omega_1 + \omega \ \Rightarrow \ \ C_G(\omega_1 + \omega) = \nu_3$
5. $\gamma(t_5, p) = \omega_1 + \omega + \omega^2 = \omega_1 + \omega^2$

To demonstrate proposirion 3.2 let $\gamma = \omega^\omega + \omega^5 \cdot 3 + 5$ therefore

$$\gamma(t_2, p) = \omega^\omega + 1 < \gamma < \omega_1 = \gamma(t_3, p)$$
$$(\omega^\omega + 1) + \omega^5 \cdot 3 + 5 = \gamma$$

The extension-type unveiling γ as maximal coordinate is then

$$X = \langle \langle\rangle, \langle\rangle, X_3 \rangle \ X_3 = \langle 5, 5, 5, 0, 0, 0, 0, 0\rangle$$

i.e. every extension $\vec{\alpha} = \langle \alpha_{3,1}, ...\alpha_{3,8}\rangle \in B(p, X)$ will satisfy that

$$\gamma(\alpha_{mc}, p^\frown\vec{\alpha}) = \gamma(\alpha_{3,8}, p^\frown\alpha) = \gamma(x_{3,8}, p^\frown X) = \gamma$$

This concludes the example. Let us state the main theorem of this paper.

Theorem 3.3 *Let $\vec{U}$ be a coherent sequence in V, $\langle \kappa_1, ...\kappa_n\rangle$ be a sequence such that $o^{\vec{U}}(\kappa_i) < \min(\nu \mid 0 < o^{\vec{U}}(\nu)) =: \delta_0$, let G be $\mathbb{M}_{\langle \kappa_1, ...\kappa_n\rangle}[\vec{U}]$-generic and let $A \in V[G]$ be a set of ordinals. Then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$.*

We will prove Theorem 3.3 by induction on $\mathrm{otp}(C_G)$. For $\mathrm{otp}(C_G) = \omega$ it is just the Prikry forcing which is know by [**?**]. Let $\mathrm{otp}(C_G) = \lambda_0$ be a limit ordinal,

$$\lambda_0 = \omega^{\gamma_n} + ... + \omega^{\gamma_1} \quad \text{(C.N.F)}$$

If $\sup(A) < \kappa$, then by lemma 5.3 in [**?**], $A \in V[C \cap \sup(A)]$. By lemma 3.1, $V[C \cap \sup(A)]$ is a generic extension of some $\mathbb{M}_{\langle \nu_1,...,\nu_m \rangle}[\vec{U}]$ with order type smaller the λ_0, thus by induction we are done. In fact, if there exists $\alpha < \kappa$ such that $A \in V[C \cap \alpha]$ then the induction hypothesis works. Let us assume that $A \notin V[C \cap \alpha]$ whenever $\alpha < \kappa$, this kind of set will be called *recent set*. Since $\kappa_1, ..., \kappa_n$ will be fixed through the rest of this chapter we shall abuse notation and denote $\mathbb{M}_{\langle \kappa_1,...\kappa_n \rangle}[\vec{U}] = \mathbb{M}[\vec{U}]$. First let us show that for A with small enough cardinality the theorem holds regardless of the induction.

Lemma 3.4 *Let $\underset{\sim}{x}$ be a $\mathbb{M}[\vec{U}]$-name and $p \in \mathbb{M}[\vec{U}]$ such that $p \Vdash \underset{\sim}{x}$ is an ordinal. Then there exists $p \leq^* p^* \in \mathbb{M}[\vec{U}]$ and an extension-type $X \in T(p)$ such that*

$$(*) \qquad\qquad \forall p^{*\frown}\langle \vec{\alpha} \rangle \in p^{*\frown}X \quad p^{*\frown}\langle \vec{\alpha} \rangle || \underset{\sim}{x}$$

Proof: Let $p = \langle t_1, ..., t_n, t_{n+1} \rangle \in \mathbb{M}[\vec{U}]$.

<u>Claim:</u> *There exists $p \leq^* p'$ such that for some extension type X*

$$\forall \vec{\alpha} \in B(p', X)\ \exists C(x_{i,j}) \text{ s.t. } p'^\frown\langle \vec{\alpha}, (C(x_{i,j}))_{i,j} \rangle \ || \ \underset{\sim}{x}$$

Proof of Claim: Define sets $B_X(t_i, j)$, for any fixed $X \in T(p)$ as follows: Recall the notation $l_X,\ x_{mc}$ and let $\vec{\alpha} \in B(p, X \setminus \langle x_{mc} \rangle)$. Define

$$B^{(0)}_X(\vec{\alpha}) = \{\theta \in B(t_{l_X}, x_{mc}) \mid \exists (C(x_{i,j}))_{i,j} \ \ p^\frown\langle \vec{\alpha}, \theta, (C(x_{i,j}))_{i,j} \rangle || \underset{\sim}{x}\}$$

and $B^{(1)}_X(\vec{\alpha}) = B(t_{l_X}, x_{mc}) \setminus B^{(0)}_X(\vec{\alpha})$. One and only one of $B^{(0)}_X(\vec{\alpha})$, $B^{(1)}_X(\vec{\alpha})$ is in $U(\kappa(t_{l_X}), x_{mc})$. Set $B_X(\vec{\alpha})$ and $F_X(\vec{\alpha}) \in \{0, 1\}$:

$$B_X(\vec{\alpha}) = B^{(F_X(\vec{\alpha}))}_X(\vec{\alpha}) \in U(\kappa(t_{l_X}), x_{mc})$$

Define

$$B'_X(t_{l_X}, x_{mc}) = \underset{\vec{\alpha}\in B(p,X\setminus\langle x_{mc}\rangle)}{\Delta} B_X(\vec{\alpha})$$

Consider the function $F : B(p, X \setminus \langle x_{mc}\rangle) \to \{0,1\}$. Applying lemma 2.3 to F, we get a homogeneous $\prod_{x_{i,j}\in X\setminus\langle x_{mc}\rangle} B'_X(t_i, x_{i,j})$ where

$$B'_X(t_i, x_{ij}) \subseteq B(t_i, x_{ij}),\ B'_X(t_i, x_{ij}) \in U(t_i, x_{i,j}),\ x_{ij} \in X \setminus \langle x_{mc}\rangle$$

For $\xi \notin X_i$, Set

$$B'_X(t_i, \xi) = B(t_i, \xi)$$

Since $|T(p)| < \kappa(t_1)$, for each $1 \le i \le n+1$ and $\xi < o^{\vec{U}}(t_i)$

$$B'(t_i, \xi) := \bigcap_{X\in T(p)} B'_X(t_i, \xi) \in U(\kappa(t_i), \xi)$$

Finally, let $p' = \langle t'_1, ..., t'_n, t'_{n+1}\rangle$ where

$$t'_i = \begin{cases} t_i & o^{\vec{U}}(t_i) = 0 \\ \langle \kappa(t_i), B'(t_i)\rangle & otherwise \end{cases}$$

It follows that $p \le^* p' \in \mathbb{M}[\vec{U}]$.
Let H be $\mathbb{M}[\vec{U}]$-generic, $p' \in H$. By the assumption on p, there exists $\delta < \kappa$ such that $V[H] \models (\underset{\sim}{x})_H = \delta$. Hence , there is $p' \le q \in M[\vec{U}]$ such that $q \Vdash \underset{\sim}{x} = \check{\delta}$. By proposition 2.2 there is a unique $p'^\frown\langle\vec{\alpha}, \theta\rangle \in p'^\frown X$ for some extension type X, such that $p'^\frown\langle\vec{\alpha},\theta\rangle \le^* q$. X, p' are as wanted:
By the definition of p' it follows that $\vec{\alpha} \in B(p', X \setminus \langle x_{mc}\rangle)$ and $\theta \in B_X(\vec{\alpha})$. Since $q \Vdash \underset{\sim}{x} = \check{\delta}$, we have that $F_X(\vec{\alpha}) = 0$. Fix $\langle\vec{\alpha'}, \theta'\rangle$ of type X. $\vec{\alpha'}$ and $\vec{\alpha}$ belong to the same homogeneous set, thus $F(\vec{\alpha'}) = F(\vec{\alpha}) = 0$ and

$$\theta' \in B^{(0)}_X(\vec{\alpha'}) \Rightarrow \exists (C(x_{i,j}))_{i,j}\ s.t.\ p'^\frown\langle\vec{\alpha'}, \theta', (C(x_{i,j}))_{i,j}\rangle || \underset{\sim}{x}$$

$\blacksquare_{of\ \ claim}$

For every $\vec{\alpha} \in B(p', X)$, fix some $(C_{i,j}(\vec{\alpha}))_{\substack{i\le n+1\\ j\le l_i+1}}$ such that

$$p'^\frown\langle\vec{\alpha},(C_{i,j}(\vec{\alpha}))_{\substack{i\le n+1\\ j\le l_i+1}}\rangle|| \underset{\sim}{x}$$

It suffices to show that we can find $p'\le^* p^*$ such that for every $\vec{\alpha}\in B(p^*,X)$

$$B(t_i^*)\cap(\alpha_s,\alpha_{i,j})\subseteq C_{i,j}(\vec{\alpha})\ ,\quad 1\le i\le n+1,\ 1\le j\le l_i+1$$

Where α_s is the predecessor of $\alpha_{i,j}$ in $\vec{\alpha}$. In order to do that, define $p'\le^* p_{i,j}$ $i\le n+1,\ j\le l_i+1$ then $p^*\ge^* p_{i,j}$ will be as wanted. Define $p_{i,j}$ as follows:
Fix $\vec{\beta}\in B(p',\langle x_{1,1},...,x_{i,j}\rangle)$, by lemma 2.3, the function

$$C_{i,j}(\vec{\beta},*):B(p',X\setminus\langle x_{1,1},...,x_{i,j}\rangle)\to P(\beta_{i,j})$$

has homogeneous sets $B^*(\vec{\beta},x_{r,s})\subseteq B(p',x_{r,s})$ for $x_{r,s}\in X\setminus\langle x_{1,1},...,x_{i,j}\rangle$. Denote the constant value by $C^*_{i,j}(\vec{\beta})$. Define

$$B^*(t_r,x_{r,s})=\underset{\vec{\beta}\in B(p',\langle x_{1,1},...,x_{i,j}\rangle)}{\Delta}B^*(\vec{\beta},x_{r,s}),\ \ x_{r,s}\in X\setminus\langle x_{1,1},...,x_{i,j}\rangle$$

Next, fix $\alpha\in B(t'_i,x_{i,j})$ and let

$$C^*_{i,j}(\alpha)=\underset{\vec{\alpha'}\in B(p',\langle x_{1,1},...,x_{i,j-1}\rangle)}{\Delta}C^*_{i,j}(\vec{\alpha'},\alpha)$$

Thus $C^*_{i,j}(\alpha)\subseteq\alpha$. Moreover, $\kappa(t_i)$ is in particular an ineffable cardinal and therefore there are $B^*(t_i,x_{i,j})\subseteq B(t'_i,x_{i,j})$ and $C^*_{i,j}$ such that

$$\forall\alpha\in B^*(t_i,x_{i,j})\quad C^*_{i,j}\cap\alpha=C^*_{i,j}(\alpha)$$

By coherency, $C^*_{i,j}\in\bigcap U(t_i,\xi)$. Finally, define $p_{i,j}=\langle t_1^{(i,j)},...,t_n^{(i,j)},t_{n+1}^{(i,j)}\rangle$

$$B(t_i^{(i,j)})=B^*(t_i)\cap(\bigcap_j C^*_{i,j})\qquad 1\le i\le n+1$$

To see that p^* is as wanted, let $\vec{\alpha}\in B(p^*,X)$ and fix any i,j. Then $\vec{\alpha}\in B(p_{i,j},X)$ and $\alpha_{i,j}\in B^*(t_i,x_{i,j})$. Thus

$$B(t_i^*)\cap(\alpha_s,\alpha_{i,j})\subseteq C^*_{i,j}\cap\alpha_{i,j}\setminus\alpha_s=C^*_{i,j}(\alpha_{i,j})\setminus\alpha_s\subseteq C^*_{i,j}(\alpha_{1,1},...,\alpha_{i,j})=C_{i,j}(\vec{\alpha})$$

■

Lemma 3.5 *Let G be $\mathbb{M}[\vec{U}]$-generic and $A \in V[G]$ be any set of ordinals, such that $|A| < \delta_0$. Then there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

proof: Let $A = \langle a_\xi \mid \xi < \delta\rangle \in V[G]$, where $\delta < \min(\nu \mid 0 < o^{\vec{U}}(\nu))$ and $\underset{\sim}{A} = \langle \underset{\sim}{a_\xi} \mid \xi < \delta\rangle$ be a name in G for $\langle a_\xi \mid \xi < \delta\rangle$. Let $q \in G$ such that $q \Vdash \underset{\sim}{A} \subseteq Ord$. We proceed by a density argument, fix $q \leq p \in \mathbb{M}[\vec{U}]$. By lemma 3.5 , for each $\xi < \delta$ there exists $X(\xi)$ and $p \leq^* p^*_\xi$ satisfying $(*)$. By $\delta^+ - _{\leq^*}$closure above p we have $p^* \in \mathbb{M}[\vec{U}]$ such that $\forall \xi < \delta$ $p^*_\xi \leq p^*$. For each ξ, define $F_\xi : B(p^*, X(\xi)) \longrightarrow \kappa$

$$F_\xi(\vec{\alpha}) = \gamma \text{ for the unique } \gamma \text{ such that } p^{*\frown}\langle\vec{\alpha}\rangle \Vdash \underset{\sim}{a_\xi} = \check{\gamma}.$$

Using lemma 2.4, we obtain for every $\xi < \delta$ a set of important coordinates

$$I_\xi \subseteq \{\langle i, j\rangle \mid 1 \leq i \leq n+1 \ , \ 1 \leq j \leq l_i\}$$

Example: Assume $o^{\vec{U}}(k) = 3$,$C_G = \langle C_G(\alpha) \mid \alpha < \omega^3\rangle$.

$$a_0 = C_G(80), a_1 = C_G(\omega+2) + C_G(3), a_2 = C_G(\omega^2 \cdot 2 + \omega + 1)$$

and

$$p = \langle \nu_0, \langle \nu_\omega, B(\nu_\omega, 0)\rangle, \langle \kappa, \underbrace{B(\kappa,0) \cup B(\kappa,1) \cup B(\kappa,2)}_{B(\kappa)}\rangle\rangle$$

We use as index the coordinate in the final sequence to improve clarity. To determine a_0, unveil the first 80 elements of the Magidor sequence i.e. any element of the form

$$p_0 = \langle \nu_0, \nu_1, ..., \nu_{80}, \langle \nu_\omega, B(\nu_\omega, 0) \setminus \nu_{80}+1\rangle, \langle \kappa, B(\kappa)\rangle\rangle$$

will decide the value of a_0. Thus the extension type X(0) is

$$X(0) = \langle\langle \underbrace{0, ..., 0}_{80 \ times}\rangle, \langle\rangle\rangle$$

The important coordinates to decide the value of a_0 is only the 80th coordinate and it is easily seen to be one to one modulo the irrelevant coordinates. For a_1 the form is

$$p_1 = \langle \nu_0, \nu_1, \nu_2, \nu_3, \langle \nu_\omega, B(\nu_\omega, 0) \setminus \nu_3 + 1 \rangle, \nu_{\omega+1}, \nu_{\omega+2}, \langle \kappa, B(\kappa) \setminus (\nu_{\omega+2} + 1) \rangle \rangle$$

The extension type is

$$X(1) = \langle \langle 0, 0, 0 \rangle, \langle 0, 0 \rangle \rangle$$

The important coordinates are the 3rd and the 5th. For a_2 we have

$$p_2 = \langle \nu_0, \langle \nu_\omega, B(\nu_\omega, 0) \rangle, \langle \nu_{\omega^2}, B(\nu_{\omega^2}) \rangle, \langle \nu_{\omega^2 \cdot 2}, B(\nu_{\omega^2 \cdot 2}) \rangle, \langle \nu_{\omega^2 \cdot 2+\omega}, B(\nu_{\omega^2 \cdot 2+\omega}) \rangle, \langle \kappa, B(\kappa) \setminus \nu_{\omega^2 \cdot 2+\omega} \rangle \rangle$$

$$X(2) = \langle \langle \rangle, \langle 2, 2, 1 \rangle \rangle$$

Back to the proof, since p was generic, there is $\langle t_1, ..., t_n, t_{n+1} \rangle = p^\star \in G$ with such functions. Find $D_\xi \subseteq C_G$ such that

$$D_\xi \in B(p^\star, X_\xi)$$

D_ξ exists by proposition 1.4 and $p^\star \in G$. Since $V[G] \models (\underset{\sim}{a_\xi})_G = a_\xi$ we have

$$p^{\star\frown} \langle D_\xi \rangle \Vdash \underset{\sim}{a_\xi} = \check{a}_\xi \Rightarrow F_\xi(D_\xi) = a_\xi$$

Set $C_\xi = D_\xi \restriction I_\xi$ and $C' = \bigcup\limits_{\xi < \delta} C_\xi$. Let us show that $V[\langle a_\xi | \xi < \delta \rangle] = V[C']$:

In $V[C']$, fix some enumeration of C'. The sequence $\langle C_\xi \mid \xi < \delta \rangle$ can be extracted from C' using the sequence $\langle \text{Index}(C_\xi, C') \mid \xi < \delta \rangle \in V$ ($\text{Index}(C_\xi, C') \subseteq \text{otp}(C_G)$). For every $\xi < \delta$ find

$$D'_\xi \in B(p^\star, X_\xi) \text{ such that } D'_\xi \restriction I_\xi = C_\xi$$

Such D'_ξ exists as D_ξ witnesses (the sequence $\langle D_\xi \mid \xi < \delta \rangle$ may not be in $V[C']$). Since $D'_\xi \sim_{I_\xi} D_\xi$ one sees that

$$F_\xi(D'_\xi) = F_\xi(D_\xi) = a_\xi$$

hence $\langle a_\xi \mid \xi < \delta\rangle = \langle F_\xi(D'_\xi) \mid \xi < \delta\rangle \in V[C']$.
In the other direction, Given $\langle a_\xi \mid \xi < \delta\rangle$, $\forall \xi < \delta$ pick $D'_\xi \in F_\xi^{-1}(a_\xi)$ ($F_\xi^{-1}(a_\xi) \neq \emptyset$ follows from the fact that $D_\xi \in dom(F_\xi)$ and $F_\xi(D_\xi) = a_\xi$). Since F_ξ is 1-1 modulo I_ξ and $F_\xi(D_\xi) = F_\xi(D'_\xi)$ we have

$$D_\xi \sim_{I_\xi} D'_\xi \text{ and } C_\xi = D_\xi \restriction I_\xi = D'_\xi \restriction I_\xi$$

Hence

$$\langle C_\xi \mid \xi < \delta\rangle = \langle D'_\xi \restriction I_\xi \mid \xi < \delta\rangle \in V[\langle a_\xi \mid \xi < \delta\rangle] \text{ and } C' \in V[\langle a_\xi \mid \xi < \delta\rangle].$$

■

We shall proceed by induction on sup(A) for a recent set A. As we have seen in the discussion following Theorem 3.3, if $A \subseteq \kappa$ is recent then sup$(A) = \kappa$. For such A, the next lemma gives a sufficient conditions.

Lemma 3.6 *Let $A \in V[G]$, $\sup(A) = \kappa$. Assume that $\exists C^* \subseteq C_G$ such that*

1. *$C^* \in V[A]$ and $\forall \alpha < \kappa\ A \cap \alpha \in V[C^*]$*
2. *$cf^{V[A]}(\kappa) < \delta_0$*

Then $\exists C' \subseteq C_G$ such that $V[A] = V[C']$.

Proof: Let $cf^{V[A]}(\kappa) = \eta$ and $\langle \gamma_\xi \mid \xi < \eta\rangle \in V[A]$ be a cofinal sequence in κ. Work in $V[A]$, pick an enumerations of $P(\gamma_\xi) = \langle X_{\xi,i} \mid i < 2^{\gamma_\xi}\rangle \in V[C^*]$. Since $A \cap \gamma_\xi \in V[C^*]$, there exists $i_\xi < 2^{\gamma_\xi}$ such that $A \cap \gamma_\xi = X_{\xi,i_\xi}$. The sequences

$$C^*, \quad \langle i_\xi \mid \xi < \eta\rangle, \quad \langle \gamma_\xi \mid \xi < \eta\rangle$$

can be coded in $V[A]$ to a sequence $\langle x_\alpha \mid \alpha < \eta\rangle$. By lemma 3.5, $\exists C' \subseteq C_G$ such that $V[\langle x_\alpha \mid \alpha < \eta\rangle] = V[C']$. To see that $V[A] = V[\langle x_\alpha \mid \alpha < \delta\rangle]$: $V[A] \supseteq V[\langle x_\alpha \mid \alpha < \eta\rangle]$ is trivial and $A = \bigcup_{\xi<\eta} X_{\xi,i_\xi} \in V[\langle x_\alpha \mid \alpha < \eta\rangle]$.

■

We have two sorts of A:

1. $\exists\alpha^* < \kappa$ such that $\forall\beta < \kappa \quad A \cap \beta \in V[A \cap \alpha^*]$ and we say that $A \cap \alpha$ stabilizes. An example of such A can be found in Prikry forcing where A is simply the Prikry sequence ($\alpha^* = 0$).

2. For all $\alpha < \kappa$ there exists $\beta < \kappa$ such that $V[A\cap\alpha] \subsetneq V[A\cap\beta]$ as example we can take Magidor forcing with $o^{\vec{U}}(\kappa) = 2$ and A can be the Magidor sequence $A = \langle \kappa_\alpha \mid \alpha < \omega^2 \rangle$.

We shall first deal with A's such that $A \cap \alpha$ does not stabilize.

Lemma 3.7 *Assume that $A \cap \alpha$ does not stabilize, then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$.*

Proof: Work in $V[A]$, define the sequence $\langle \alpha_\xi \mid \xi < \theta \rangle$:

$$\alpha_0 = \min(\alpha \mid V[A \cap \alpha] \supsetneq V)$$

Assume that $\langle \alpha_\xi \mid \xi < \lambda \rangle$ has been defined and for every ξ, $\alpha_\xi < \kappa$. If $\lambda = \xi + 1$ then set

$$\alpha_\lambda = \min(\alpha \mid V[A \cap \alpha] \supsetneq V[A \cap \alpha_\xi])$$

If the sequence $\alpha_\lambda = \kappa$, then α_λ satisfies that

$$\forall\alpha < \kappa \quad A \cap \alpha \in V[A \cap \alpha_{\lambda^*}]$$

Thus $A \cap \alpha$ stabilizes which by our assumption is a contradiction.
If λ is limit, define

$$\alpha_\lambda = \sup(\alpha_\xi \mid \xi < \lambda)$$

if $\alpha_\lambda = \kappa$ define $\theta = \lambda$ and stop. The sequence $\langle \alpha_\xi \mid \xi < \theta \rangle \in V[A]$ is a continues, increasing unbounded sequence in κ. Therefore, $cf^{V[A]}(\kappa) = cf(\theta)$. We shell first show that $\theta < \delta_0$. Work in $V[G]$, for every $\xi < \theta$ pick $C_\xi \subseteq C_G$ such that $V[A \cap \alpha_\xi] = V[C_\xi]$. This is a 1-1 function from θ to $P(C_G)$. The cardinal δ_0 is still a strong limit cardinal (since there are no new bounded subsets below this cardinal and it is measurable in V). Moreover, $\lambda_0 := \text{otp}(C_G) < \delta_0$, thus

$$\theta \leq |P(C_G)| = |P(\lambda_0)| < \delta_0$$

The only thing left to prove, is that we can find C^* as in Lemma 3.6. Work in $V[A]$, for every $\xi<\theta$, $C_\xi\in V[A]$ (The sequence $\langle C_\xi \mid \xi<\theta\rangle$ may not be in $V[A]$). C_ξ witnesses that

$$\exists d_\xi\subseteq\kappa\ (|d_\alpha|<2^{\lambda_0} \text{ and } V[A\cap\alpha]=V[d_\alpha])$$

So $d=\bigcup\{d_{\alpha_\xi}|\xi<\theta\}\in V[A]$ and $|d|\leq 2^{\lambda_0}$. Finally, by lemma 3.5, there exists $C^*\subseteq C_G$ such that $V[C^*]=V[d]\subseteq V[A]$ and for all $\alpha<\kappa$ $A\cap\alpha\in V[C^*]$. By Lemma 3.6, the theorem holds.

■

For the rest of this chapter we can assume that the sequence $A\cap\alpha$ stabilizes on α^*. Let C^* be such that $V[A\cap\alpha^*]=V[C^*]$ and $\kappa^*=\sup(C^*)$ is limit in C_G. Notice that, $\kappa^*<\kappa$, this follows from the fact that $A\cap\alpha^*\in V[C_G\cap\alpha^*]$. Our final goal is to argue that if A is very new then κ changes cofinality in $V[A]$. To do this, consider the initial segment $C_G\cap\kappa^*$ and assume that $\kappa_{j-1}\leq\kappa^*<\kappa_j$. By lemma 3.1 we can split $\mathbb{M}[\vec{U}]$

$$\mathbb{M}_{\langle\nu_1,...,\nu_i,\kappa^*\rangle}[\vec{U}]\times(\mathbb{M}_{\langle\kappa_j,...,\kappa\rangle}[\vec{U}])_{>\kappa^*}$$
$$\mathbb{M}_{\leq\kappa^*}=\mathbb{M}_{\langle\nu_1,...,\nu_i,\kappa^*\rangle}[\vec{U}]\ ,\ \mathbb{M}_{>\kappa^*}[\vec{U}]=(\mathbb{M}_{\langle\kappa_j,...,\kappa\rangle}[\vec{U}])_{>\kappa^*}$$

such that C_G is generic for $\mathbb{M}_{\leq\kappa^*}[\vec{U}]\times\mathbb{M}_{>\kappa^*}[\vec{U}]$ and $C_G\cap\kappa^*$ is generic for $\mathbb{M}_{\leq\kappa^*}[\vec{U}]$. As we will see in the next chapter, there is a natural projection of $\mathbb{M}_{\leq\kappa^*}[\vec{U}]$ onto some forcing $\mathbb{P}$ such that $V[C^*]=V[G^*]$ for some generic G^* of $\mathbb{P}$. Recall that if $\pi:\mathbb{M}_{\leq\kappa^*}[\vec{U}]\rightarrow\mathbb{P}$ is the projection, then

$$\mathbb{M}_{\leq\kappa^*}[\vec{U}]/G^*=\pi^{-1}(G^*)$$

In $V[G^*]$ define $\mathbb{Q}=\mathbb{M}_{\leq\kappa^*}[\vec{U}]/C^*\subseteq\mathbb{M}_{\leq\kappa^*}[\vec{U}]$. It is well known that $C_G\cap\kappa^*$ is generic for $\mathbb{Q}$ above $V[C^*]$ and obviously $V[C^*][C_G\cap\kappa^*]=V[C_G\cap\kappa^*]$. The reader can refer to chapter 4 to see a formal development of $\mathbb{Q}$, though in this chapter we will only use the existence of such a forcing and the fact that the projection depends only on the part below κ^*, therefore $\mathbb{Q}$ is of small cardinality. The forcing $\mathbb{M}_{>\kappa^*}[\vec{U}]$ has all good properties of $\mathbb{M}[\vec{U}]$ (and more) since in $V[C^*]$ all measurables in $\vec{U}$ above κ^* are unaffected by the existence of C^*. In conclusion, we have managed to find a forcing $\mathbb{Q}\times\mathbb{M}_{>\kappa^*}[\vec{U}]\in V[C^*]$ such that $V[G]$ is one of it's generic extensions and $\forall\alpha<\kappa$ $A\cap\alpha\in V[C^*]$.

Work in $V[C^*]$, let $\underset{\sim}{A}$ be a name for A in $\mathbb{Q}\times\mathbb{M}_{>\kappa^*}[\vec{U}]\in V[C^*]$. By our assumption on C^*, we can find $\langle q,p\rangle\in G$ such that $\langle q,p\rangle\Vdash\forall\alpha<\kappa$ $\underset{\sim}{A}\cap\alpha$ is old (where old means in

$V[C^*]$). Formally, the next argument is a density argument above $\langle q,p\rangle$. Nevertheless, in order to simplify notation, assume that $\langle q,p\rangle = 0_{Q\times\mathbb{M}[\vec{U}]_{>\kappa^*}}$. Lemmas 3.8-3.9 prove that a certain property holds densely often in $\mathbb{M}[\vec{U}]_{>\kappa^*}$. In order to Make these lemmas more clear, we will work with an ongoing parallel example.
Example: Let $\lambda_0 = \mathrm{otp}(C_G) = \omega^2$,

$$A = \{C_G(2n) \mid n \leq \omega\} \cup \{C_G(\omega\cdot n) + C_G(n) \mid 0 < n < \omega\}$$

Therefore

$$C^* = \{C_G(2n) \mid n < \omega\},\ \kappa^* = C_G(\omega)$$

The forcing $\mathbb{Q}$ can be thought of as adding the missing coordinates to $C_G \restriction \omega$ i.e. the odd coordinates. Let

$$p = \langle \underbrace{\langle \nu_{\omega\cdot 2}, B_{\omega\cdot 2}\rangle}_{t_1}, \underbrace{\nu_{\omega\cdot 2+1}}_{t_2}, \underbrace{\langle \kappa, B(\kappa)\rangle}_{t_3}\rangle \in \mathbb{M}[\vec{U}]_{>\kappa^*}$$

Lemma 3.8 *For every $p \in \mathbb{M}[\vec{U}]_{>\kappa^*}$ there exists $p \leq^* p^*$ such that for every extension X of p^* and $q \in \mathbb{Q}$: (Recall that $\vec{\alpha} = \langle \alpha_{11}, ..., \alpha_{mc}\rangle$)*

$$(\exists p^{*\frown}\vec{\alpha} \in p^{*\frown}X\ \exists p^{**} \geq^* p^{*\frown}\vec{\alpha}\ s.t.\langle q, p^{**}\rangle || \underset{\sim}{A} \cap \alpha_{mc}) \Rightarrow$$

$$(*) \qquad (\forall p^{*\frown}\vec{\alpha} \in p^{*\frown}X\ \langle q, p^{*\frown}\vec{\alpha}\rangle || \underset{\sim}{A} \cap \alpha_{mc} =: a(q,\vec{\alpha}))\ \textit{(a propery of } q, X)$$

Example: Let

$$q = \langle \nu_1, \nu_3, \langle \kappa^*, B(\kappa^*)\rangle\rangle\ ,\ X = \langle \underbrace{\langle 0,0\rangle}_{X_1}, \underbrace{\langle\ \rangle}_{X_2}, \underbrace{\langle 1,0\rangle}_{X_3}\rangle\text{-extension of } p$$

Let

$$\vec{\alpha} = \langle \langle \alpha_{\omega+1}, \alpha_{\omega+2}\rangle, \langle\ \rangle, \langle \alpha_{\omega\cdot 3}, \alpha_{\omega\cdot 3+1}\rangle\rangle \in B(p,X)$$

If H is any generic with $\langle q, p^\frown\langle\vec{\alpha}\rangle\rangle \in H$ then all the elements in q and $p^\frown\langle\vec{\alpha}\rangle$ have there coordinates in C_H as specified above, thus

$$(\underset{\sim}{A})_H \cap \alpha_{mc} = (\underset{\sim}{A})_H \cap \alpha_{\omega\cdot 3+1} =$$
$$= \{C_H(2n) \mid n \leq \omega\} \cup \{C_H(\omega \cdot n) + C_H(n) \mid 0 < n < \omega\} \cap C_H(\omega \cdot 3 + 1)$$

If $\alpha_{\omega\cdot 3} + \nu_3 \geq \alpha_{\omega\cdot 3+1}$ then

$$a(q, \vec{\alpha}) = (\underset{\sim}{A})_H \cap \alpha_{mc} = C_H \restriction_{even} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega\cdot 2} + C_H(2)\}$$

If $\alpha_{\omega\cdot 3} + \nu_3 < \alpha_{\omega\cdot 3+1}$ then

$$a(q, \vec{\alpha}) = (\underset{\sim}{A})_H \cap \alpha_{mc} = C_H \restriction_{even} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega\cdot 2} + C_H(2), \alpha_{\omega\cdot 3} + \nu_3\}$$

Anyway, we have that $a(q, \vec{\alpha}) \in V[C^*]$ and therefore $\langle q, p^\frown\vec{\alpha}\rangle || \underset{\sim}{A} \cap \alpha_{mc}$ for every extension $\vec{\alpha}$ of type X. Namely, q, X satisfy (*).

Proof of 3.8: Let $p = \langle t_1, ..., t_n, t_{n+1}\rangle$. For every

$$X = \langle X_1, ..., X_{n+1}\rangle\text{- extension of } p \quad , q \in \mathbb{Q} \quad , \vec{\alpha} \in B(p, X \setminus \langle x_{mc}\rangle)$$

Recall that $l_X = \min(i \mid X_i \neq \emptyset)$ and define $B^X_{(0)}(q, \vec{\alpha})$ to be the set

$$\{\theta \in B(t_{l_X}, x_{mc}) \mid \exists a \exists (C(x_{i,j}))_{x_{i,j}} \quad \langle q, p^\frown\langle\vec{\alpha}, \theta, C(x_{i,j})\rangle \Vdash \underset{\sim}{A} \cap \theta = a\}$$

Also let $B^X_{(1)}(q, \vec{\alpha}) = B(t_{l_X}, x_{mc}) \setminus B^X_{(0)}(q, \vec{\alpha})$. One and only one of $B^X_{(1)}(q, \vec{\alpha}), B^X_{(0)}(q, \vec{\alpha})$ is in $U(t_{l_X}, x_{mc})$. Define $B^X(q, \vec{\alpha})$ and $F^X_q(\vec{\alpha}) \in \{0, 1\}$ such that

$$B^X(q, \vec{\alpha}) = B^X_{(F^X_q(\vec{\alpha}))}(q, \vec{\alpha}) \in U(t_{l_X}, x_{mc})$$

Since $|\mathbb{Q}| \leq 2^{\kappa^*} < \kappa(t_{l_X})$ we have $B^X(\vec{\alpha}) = \bigcap\limits_q B^X(q, \vec{\alpha}) \in U(t_{l_X}, x_{mc})$. Define

$$B^X(t_{l_X}, x_{mc}) = \underset{\vec{\alpha}}{\Delta} B^X(\vec{\alpha}) \in U(t_{l_X}, x_{mc})$$

Use lemma 2.3 to find $B^X(t_i, x_{i,j}) \subseteq B(t_i, x_{i,j})$, $B^X(t_i, x_{i,j}) \in U(t_i, x_{i,j})$ homogeneous for every F^X_q. As before, if $\lambda \notin X_i$ set $B^X(t_i, \lambda) = B(t_i, \lambda)$. Let

$$p^* = p^\frown\langle(B^*(t_i))^{n+1}_{i=1}\rangle, \; B^*(t_i, \lambda) = \bigcap\limits_X B^X(t_i, \lambda)$$

So far what we have managed to do is the following: Assuming they exist, let $q, \vec{\alpha}, (C(x_{i,j}))_{i,j}, a$ be such that $\langle q, p^{*\frown}\langle \vec{\alpha}, (C(x_{i,j}))_{i,j}\rangle\rangle \Vdash \underset{\sim}{A} \cap \alpha_{mc} = a$. Since $\alpha_{mc} \in B^X(q, \vec{\alpha} \setminus \langle \alpha_{mc}\rangle)$ we most have that $F_q^X(\vec{\alpha} \setminus \langle \alpha_{mc}\rangle) = 0$. Let $\vec{\alpha}'$ be another extension of type X, then $\vec{\alpha}' \setminus \langle \alpha'_{mc}\rangle$ and $\vec{\alpha} \setminus \langle \alpha_{mc}\rangle$ belong to the same homogeneous set, thus

$$F_q^X(\vec{\alpha}' \setminus \langle \alpha'_{mc}\rangle) = F_q^X(\vec{\alpha} \setminus \langle \alpha_{mc}\rangle) = 0$$

By the definition of $F_q^X(\vec{\alpha}' \setminus \langle \alpha'_{mc}\rangle)$ it follows that $\alpha'_{mc} \in B^X_{(0)}(q, \vec{\alpha}' \setminus \langle \alpha'_{mc}\rangle)$ as wanted. For every $\vec{\alpha} \in B(p', X)$ and $q \in \mathbb{Q}$ fix some $(C_{i,j}(q, \vec{\alpha}))_{\substack{i \leq n+1 \\ j \leq l_i+1}}$ such that

$$, p^{*\frown}\langle \vec{\alpha}, (C_{i,j}(q, \vec{\alpha}))_{\substack{i \leq n+1 \\ j \leq l_i+1}}\rangle\rangle || \underset{\sim}{A} \cap \alpha_{mc}$$

Prove that we can extend p^* to p^{**} such that for all $1 \leq i \leq n+1$, $1 \leq j \leq l_i + 1$ and $\vec{\alpha} \in B(p^*, X)$,

$$B(t_i^{**}) \cap (\alpha_s, \alpha_{i,j}) \subseteq C_{i,j}(\vec{\alpha})$$

Where α_s is the predecessor of $\alpha_{i,j}$ in $\vec{\alpha}$. In order to do that, fix $i,\ j$ and stabilize $C_{i,j}(\vec{\alpha})$ as follows:
Fix $\vec{\beta} \in B(p^*, \langle x_{1,1}, ..., x_{i,j}\rangle)$ By lemma 2.3 , the function

$$C_{i,j}(q, \vec{\beta}, *) : B(p^*, X \setminus \langle x_{1,1}, ..., x_{i,j}\rangle) \to P(\beta_{i,j})$$

has homogeneous sets $B'(\vec{\beta}, x_{r,s}, q) \subseteq B(t_r^*, x_{r,s})$ for $x_{r,s} \in X \setminus \langle x_{1,1}, ..., x_{i,j}\rangle$. Denote the constant value by $C^*_{i,j}(q, \vec{\beta})$. Define

$$B'(t_r^*, x_{r,s}) = \underset{\substack{\vec{\beta} \in B(p^*, \langle x_{1,1}, ..., x_{i,j}\rangle) \\ q \in \mathbb{Q}}}{\Delta} B'(\vec{\beta}, x_{r,s}, q), \quad x_{r,s} \in X \setminus \langle x_{1,1}, ..., x_{i,j}\rangle$$

Next, fix $\alpha \in B(t_i^*, x_{i,j})$ and let

$$C^*_{i,j}(\alpha) = \underset{\substack{\vec{\alpha'} \in B(p^*, \langle x_{1,1}, ..., x_{i,j-1}\rangle) \\ q \in \mathbb{Q}}}{\Delta} C^*_{i,j}(q, \vec{\alpha'}, \alpha)$$

Thus $C^*_{i,j}(\alpha) \subseteq \alpha$. $\kappa(t_i)$ is ineffable thus, there is $B'(t_i^*, x_{i,j}) \subseteq B(t_i^*, x_{i,j})$ and $C^*_{i,j}$ such that for every $\alpha \in B'(t_i^*, x_{i,j})$, $C^*_{i,j} \cap \alpha = C^*_{i,j}(\alpha)$. By coherency, $C^*_{i,j} \in \bigcap U(t_i, \xi)$. Finally, define $p^{**} = \langle t_1^{**}, ..., t_n^{**}, t_{n+1}^{**}\rangle$

$$B(t_i^{**}) = B'(t_i^*) \cap (\bigcap_j C_{i,j}^*) \qquad 1 \leq i \leq n+1$$

To see that p^{**} is as wanted, let $\vec{\alpha} \in B(p^{**}, X)$ and fix any i, j. Then $\vec{\alpha} \in B(p^{**}, X)$ and $\alpha_{i,j} \in B(t_i^{**}, x_{i,j})$ thus for any $i.j$

$$B(t_i^{**}) \cap (\alpha_s, \alpha_{i,j}) \subseteq C_{i,j}^* \cap \alpha_{i,j} \setminus \alpha_s = C_{i,j}^*(\alpha_{i,j}) \setminus \alpha_s \subseteq C_{i,j}^*(\alpha_{1,1}, ..., \alpha_{i,j}) = C_{i,j}(\alpha)$$

■

Lemma 3.9 *Let p^* be as in lemma 3.8 There exist $p^* \leq p^{**}$ such that for every extension X of p^{**} and $q \in \mathbb{Q}$ that satisfies (*) there exists sets $A(q, \vec{\alpha}) \subseteq \kappa$ $\vec{\alpha} \in B(p^{**}, X \setminus \langle x_{mc} \rangle)$ such that for all $\alpha \in B(p^{**}, x_{mc})$*

$$A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha)$$

Example: Recall that we have obtained the sets

$$a(q, \vec{\alpha}) = C_H \restriction_{even} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega \cdot 2} + C_H(2)\} \cup b(q, \vec{\alpha})$$

$$b(q, \vec{\alpha}) = \begin{cases} \emptyset & \alpha_{\omega \cdot 3} + \nu_3 \geq \alpha_{mc} \\ \{\alpha_{\omega \cdot 3} + \nu_3\} & \alpha_{\omega \cdot 3} + \nu_3 < \alpha_{mc} \end{cases}$$

The element α_{mc} is chosen from the set $B(t_3, x_{mc}) = B(t_3, 0)$, by shrinking this set, we can directly extend p to p^* such that for every $\vec{\alpha} \in B(p^*, X)$, $\alpha_{\omega \cdot 3} + \nu_3 < \alpha_{mc}$. Therefore,

$$A(q, \vec{\alpha}) = C_H \restriction_{even} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega \cdot 2} + C_H(2), \alpha_{\omega \cdot 3} + \nu_3\}$$

Proof of 3.9: Fix q, X satisfying (*) and $\vec{\alpha} \in B(p^*, X \setminus \langle x_{mc} \rangle)$, since $\kappa(t_i)$ is ineffable we can shrink the set $B(t_{l_X}^*, x_{mc})$ to $B'(q, \vec{\alpha})$ to find sets $A(q) \subseteq t_i$ such that

$$\forall \alpha \in B'(q, \vec{\alpha}) \;\; A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha)$$

define $B_q(t_i^*, x_{mc}) = \underset{\vec{\alpha} \in B(p^*, X \setminus \langle x_{mc} \rangle)}{\Delta} B^{**}(q, \vec{\alpha})$ intersect over all X, q and defines p^{**} as before.

■

Thus there exists $p_* \in G_{>\kappa^*}$ with the properties described in Lemma's 3.8-3.9. Next we would like to claim that for some sufficiently large family of $q \in \mathbb{Q}$ and extension-type X we have q, X satisfy (*).

Lemma 3.10 *Let $p_* \in G_{>\kappa^*}$ be as above and let X be any extension-type of p_*. Then there exists a maximal antichain $Z_X \subseteq \mathbb{Q}$ and extension-types $X \preceq X_q$ for $q \in Z_X$, unveiling the same maximal coordinate as X such that for every $q \in Z_X$, q, X_q satisfy (*).*

Example: For our X, the correct anti chain Z_X is : For any possible ν_1, ν_3 choose a condition $\langle \nu_1, \nu_3, \langle \kappa^*, B^* \rangle \rangle \in \mathbb{Q}$. This set definitely form a maximal anti chain, and by the same method of the previous examples taking $X_q = X$ works. In general, if the maximal coordinate of X is some $\omega \cdot (2n+1)$, Z_X will be the anti chain consisting of representative conditions for the $2n+1$ first coordinates.

Proof: The existence of Z_X will follow from Zorn's Lemma and the method proving existence of X_q for some q. Fix any $\vec{\alpha} \in B(p_*, X)$, there exists a generic $H \subseteq \mathbb{Q} \times \mathbb{M}_{>\kappa^*}[\vec{U}]$ with $\langle 1_{\mathbb{Q}}, p_*^\frown \vec{\alpha} \rangle \in H = H_{\leq \kappa^*} \times H_{>\kappa^*}$. Consider the decomposition of $\mathbb{M}[\vec{U}]_{>\kappa^*}$ above $p_*^\frown \vec{\alpha}$ induced by α_{mc} and let $p_*^\frown \vec{\alpha} = \langle p_1, p_2 \rangle$, i.e. $\langle p_1, p_2 \rangle \in (\mathbb{M}[\vec{U}]_{>\kappa^*})_{\leq \alpha_{mc}} \times (\mathbb{M}[\vec{U}]_{>\kappa^*})_{>\alpha_{mc}}$. H stays generic for the forcing $\mathbb{Q} \times (\mathbb{M}[\vec{U}]_{>\kappa^*})_{\leq \alpha_{mc}} \times (\mathbb{M}[\vec{U}]_{>\kappa^*})_{>\alpha_{mc}}$. Define $H_1 = H_{\leq \kappa^*} \times (H_{>\kappa^*})_{\leq \alpha_{mc}}$ and $H_2 = H_{>\alpha_{mc}}$. Then $(\underset{\sim}{A})_{H_1} \in V[H_1]$ is a name of A in the forcing $\mathbb{M}[\vec{U}]_{>\alpha_{mc}}$. Above p_2 we have sufficient closure to determine $(\underset{\sim}{A})_{H_1} \cap \alpha_{mc}$

$$\exists p_2^* \geq^* p_2 \ s.t. \ p_2^* \Vdash_{\mathbb{M}[\vec{U}]_{>\alpha_{mc}}} (\underset{\sim}{A})_{H_1} \cap \alpha_{mc} = a$$

for some $a \in V[C^*]$. Hence there exists $\langle 1_{\mathbb{Q}_{\leq \kappa^*}}, p_1 \rangle \leq \langle q, p_1^* \rangle$ such that

$$\langle q, p_1^* \rangle \Vdash_{\mathbb{Q} \times \mathbb{M}_{\leq \alpha_{mc}}[\vec{U}]} \overset{\vee}{p_2^{**}} \Vdash_{\mathbb{M}[\vec{U}]_{>\alpha_{mc}}} \underset{\sim}{A} \cap \alpha_{mc} = a$$

It is clear that $\langle q, p_1^*, p_2^* \rangle ||_{\mathbb{Q} \times \mathbb{M}_{>\kappa^*}[\vec{U}]} \underset{\sim}{A} \cap \alpha_{mc}$. Finally, X_q is simply the extension type of p_1^*. Since $p_1^* \in \mathbb{M}_{\leq \alpha_{mc}}[\vec{U}]$, X_q unveils the same maximal coordinate as X. By lemma 3.8, X_q, q satisfies $(*)$.

■

Lemma 3.11 *κ changes cofinality in $V[A]$.*

Proof: Let $p_* = \langle t_1^*, ..., t_n^*, t_{n+1}^* \rangle \in G_{>\kappa^*}$ be as before, $\lambda_0 = \text{otp}(C_G)$ and $\langle C_G(\xi) \mid \xi < \lambda_0 \rangle$ be the Magidor sequence corresponding to G. Work in V[A], define a sequence $\langle \nu_i \mid \gamma(t_n^*, p_*) \leq i < \lambda_0 \rangle \subset \kappa$:

$$\nu_{\gamma(t_n^*,p_*)} = C_G(\gamma(t_n^*, p_*)) + 1 = \kappa(t_n^*) + 1$$

Assume that $\langle \nu_{\xi'} \mid \xi' < \xi < \lambda_0 \rangle$ is defined such that it is increasing and $\nu_{\xi'} < \kappa$. If ξ is limit define

$$\nu_\xi = \sup(\nu_{\xi'}) + 1.$$

If $\sup(\nu_{\xi'}) = \kappa$ we are done, since κ changes cofinality to $cf(\xi) < \lambda_0$ (which is actually a contradiction for regular λ_0). Therefore, $\nu_\xi < \kappa$. If $\xi = \xi' + 1$, by proposirion 3.2, there exist an extension type X_ξ of p_* unveiling ξ as maximal coordinate. By lemma 3.10 we can find Z_ξ and $X_\xi \preceq X_q$ unveiling ξ as maximal coordinate such that q, X_q satisfies (*). By lemma 3.9 there exists

$$A(q, \vec{\alpha})\text{'s for } q \in Z_\xi \quad \vec{\alpha} \in B(p^*, X_q \setminus \langle x_{mc} \rangle).$$

Since $A \notin V[C^*]$, $A \neq A(q, \vec{\alpha})$. Thus define $\eta(q, \vec{\alpha}) = \min(A(q, \vec{\alpha}) \Delta A) + 1$

$$\beta_\xi = \sup(\eta(q, \vec{\alpha}) \mid \vec{\alpha} \in [\nu_{\xi'}]^{<\omega} \cap B(p^*, X_q \setminus \langle x_{mc} \rangle),\ q \in Z_\xi)$$

It follows that $\beta_\xi \leq \kappa$. Assume $\beta_\xi = \kappa$, then κ changes cofinality but it might be to some other cardinal larger than δ_0, this is not enough (actually, by Theorem 3.3 this can not happen). Continue toward a contradiction, fix an unbounded and increasing sequence $\langle \eta(q_i, \vec{\alpha_i}) \mid i < \theta < \kappa \rangle$. Notice that since $\eta(q_i, \vec{\alpha_i}) < \eta(q_{i+1}, \vec{\alpha_{i+1}})$ it must be that $A(q_i, \vec{\alpha_i}) \neq A(q_{i+1}, \vec{\alpha_{i+1}})$ and

$$A(q_i, \vec{\alpha_i}) \cap \eta(q_i, \vec{\alpha_i}) = A \cap \eta(q_i, \vec{\alpha_i}) = A(q_{i+1}, \vec{\alpha_{i+1}}) \cap \eta(q_i, \vec{\alpha_i})$$

Define $\eta_i = \min(A(q_i, \vec{\alpha_i}) \Delta A(q_{i+1}, \vec{\alpha_{i+1}})) \geq \eta(q_i, \vec{\alpha_i})$. It follows that $\langle \eta_i \mid i < \theta \rangle$ is a short cofinal sequence in κ. This definition is independent of A an only involve $\langle \langle q_i, \vec{\alpha_i} \rangle \mid i < \theta < \kappa \rangle$, which can be coded as a bounded sequence of κ. By the induction hypothesis there is $C'' \subseteq C$, bounded in κ such that $V[C''] = V[\langle \langle q_i, \vec{\alpha_i} \rangle \mid i < \theta < \kappa \rangle]$. Define $C' = C^* \cup C''$, the model $V[C']$ should keep κ measurable but also has the sequence $\langle \eta_i \mid i < \theta \rangle$, contradiction.
Therefore, $\beta_\xi < \kappa$, set $\nu_\xi = \beta_\xi + 1$. This concludes the construction of the sequence ν_ξ . To see that it is indeed unbounded in κ, let us show that $C_G(\xi) < \nu_\xi$: We have $C_G(\gamma(t_n^*, p_*)) < \nu_{\gamma(t_n^*,p_*)}$ Assume that $C_G(i) < \nu_i,\ \gamma(t_n^*, p_*) \leq i < \xi)$. If ξ is limit then by closureness of the Magidor sequence

$$C_G(\xi) = \sup(C_G(i) \mid i < \xi) \leq \sup(\nu_i \mid \gamma(t_n^*, p_*) \leq i < \xi) < \nu_\xi$$

If $\xi = \xi' + 1$ is successor, let $\{q_\xi\} = Z_\xi \cap G_{\leq \kappa^*}$

$$p_\xi = p_*^\frown \langle C_G(i_1), ..., C_G(i_n), C_G(\xi)\rangle \in p_*^\frown X_\xi \cap G_{>\kappa^*}$$

By induction $C_G(i_r) < \nu_{\xi'}$, therefore, $\eta(q_\xi, \langle C_G(i_1), ..., C_G(i_n)\rangle) < \nu_\xi$. Finally, $\langle q_\xi, p_\xi\rangle \in G$, $\langle q_\xi, p_\xi\rangle \Vdash \underset{\sim}{A} \cap C_G(\xi) = A(q_\xi, \langle C_G(i_1), ..., C_G(i_n)\rangle) \cap C_G(\xi)$, thus

$$A \cap C_G(\xi) = A(q_\xi, \langle C_G(i_1), ..., C_G(i_n)\rangle) \cap C_G(\xi) \; C_G(\xi) \leq \eta(q_\xi, \langle C_G(i_1), ..., C_G(i_n)\rangle) < \nu_\xi.$$

■

4 The main result above κ

In order to push the induction to sets above κ we will need a projection of $\mathbb{M}[\vec{U}]$ onto some forcing that adds a subsequence of C_G. The majority of this chapter is the definition of this projection and some of it's properties. The induction argument will continue at lemma 4.13.

Let G be generic and C_G the corresponding Magidor sequence. Let $C^* \subseteq C_G$ be a subsequence and $I = \text{Index}(C^*, C_G)$. Then I is a subset of λ_0, hence $I \in V$. Assume that $\kappa^* = \sup(C^*)$ is a limit point in C_G and that C^* is closed i.e. containing all of it's limit points below κ^*. As we will see in the next lemma, one can find a forcing $\mathbb{M}_{\langle \nu_1,...,\nu_m \rangle}[\vec{U}]$ for which G is still generic and will be easier to project.

Proposition 4.1 *Let G be $\mathbb{M}_{\langle \kappa_1,...\kappa_n \rangle}[\vec{U}]$-generic and $C^* \subseteq C_G$ such that C^* is closed and $\kappa^* = \sup(C^*)$ is a limit point of C_G. Then there exists $\langle \nu_1, ..., \nu_m \rangle$ such that G is generic for $\mathbb{M}_{\langle \nu_1,...,\nu_m \rangle}[\vec{U}]$ and for all $1 \leq i \leq m$, $C^* \cap (\nu_{i-1}, \nu_i)$ is either empty or a club in ν_i. (as usual we have the convention $\nu_0 = 0$)*

Example: Assume that $\lambda_0 = \omega_1 + \omega^2 \cdot 2 + \omega$, C^* is

$$C_G \restriction (\omega_1 + 1) \cup \{C_G(\omega_1 + \omega + 2), C_G(\omega_1 + \omega + 3)\} \cup \{C_G(\omega_1 + \alpha) \mid \omega^2 \cdot 2 < \alpha < \lambda_0\}$$

Let $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 = \kappa$ be such that $o^{\vec{U}}(\kappa_1) = \omega_1$, $o^{\vec{U}}(\kappa_2) = o^{\vec{U}}(\kappa_3) = 2$ and $o^{\vec{U}}(\kappa) = 1$. We have

1. $(0, \kappa_1) \cap C^* = C_G \restriction \omega_1$
2. $(\kappa_1, \kappa_2) \cap C^* = \{C_G(\omega_1 + \omega + 2), C_G(\omega_1 + \omega + 3)\}$
3. $(\kappa_2, \kappa_3) \cap C^* = \emptyset$
4. $(\kappa_3, \kappa_4) \cap C^* = \{C_G(\omega_1 + \alpha) \mid \omega^2 \cdot 2 < \alpha < \lambda_0\}$

Then (1),(3),(4) are either empty or a club but (2) isn't. To fix this we shall simply add $\{C_G(\omega_1 + \omega + 2), C_G(\omega_1 + \omega + 3)\}$ to $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4$.

Proof of 4.1: By induction on m, we shall define a sequence

$$\vec{\nu_m} = \langle \nu_{1,m}, ..., \nu_{n_m,m} \rangle$$

such that for every m, G is generic for $\mathbb{M}_{\vec{\nu}_m}[\vec{U}]$. Define $\vec{\nu_0} = \langle\kappa_1, ..., \kappa_n\rangle$. Assume that $\vec{\nu_m}$ is defined with G generic, if for every $1 \leq i \leq n_m + 1$ we have $C^* \cap (\nu_{i-1,m}, \nu_{i,m})$ is either empty or unbounded (and therefore a club), stabilize the sequence at m. Otherwise, let i be maximal such that $C^* \cap (\nu_{i-1,m}, \nu_{i,m})$ is nonempty and bounded. Thus,

$$\nu_{i-1,m} < \sup(C^* \cap (\nu_{i-1,m}, \nu_{i,m})) < \nu_{i,m}$$

Since C^* is closed, $C_G(\gamma) = \sup(C^* \cap (\nu_{i-1,m}, \nu_{i,m})) \in C^*$ for some γ. As in lemma 3.1 we can find

$$\vec{\nu_{m+1}} = \langle\nu_{1,m}, ..., \nu_{i,m}, \xi_1, ..., \xi_k, \nu_{i+1,m}, ..., \nu_{n_m,m}\rangle \subseteq C_G$$

such that $C_G(\gamma) = \xi_k$ is unveiled and the forcing $\mathbb{M}_{\vec{\nu_{m+1}}}[\vec{U}] \subseteq \mathbb{M}_{\vec{\nu_m}}[\vec{U}]$ is a subforcing of $\mathbb{M}_{\vec{\nu_m}}[\vec{U}]$ with G one of it's generic sets. It is important that the maximal ordinal in the sequence $\vec{\nu_{m+1}}$ such that $C^* \cap (\nu_{j-1,m+1}, \nu_{j,m+1})$ is nonempty and bounded is strictly less than $\nu_{i,m}$. Therefore this iteration stabilizes at some $N < \omega$. Consider the forcing $\mathbb{M}_{\vec{\nu}_N}[\vec{U}]$, by the construction of the $\vec{\nu}_r$'s, we necessarily have that for every $1 \leq i \leq n_N + 1$ $C^* \cap (\nu_{i-1,N}, \nu_{i,N})$ is either empty or unbounded (Since $\vec{\nu}_{N+1} = \vec{\nu}_N$).

■

By this proposition, we can assume that $\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}]$ and C^* satisfy the property of 4.1. If one wishes to define a projection of $\mathbb{M}[\vec{U}]$ onto some forcing $\prod_{i=1}^n \mathbb{P}_i$, the decomposition

$$\mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}] = \prod_{i=1}^n (\mathbb{M}_{\kappa_i})_{>\kappa_{i-1}}$$

permits us to derive a projection $\pi : \mathbb{M}_{\langle\kappa_1,...\kappa_n\rangle}[\vec{U}] \to \prod_{i=1}^n \mathbb{P}_i$ through projections

$$\pi_i : (\mathbb{M}_{\kappa_i})_{>\kappa_{i-1}} \to \mathbb{P}_i \quad (1 \leq i \leq n)$$

First, if $C^* \cap (\kappa_{i-1}, \kappa_i)$ is empty, the projection is going to be to the trivial forcing. Otherwise, $C^* \cap (\kappa_{i-1}, \kappa_i)$ is a club. In order to simplify notation, we will assume that $(\mathbb{M}_{\kappa_i})_{>\kappa_{i-1}} = \mathbb{M}[\vec{U}]_{\langle\kappa\rangle} = \mathbb{M}[\vec{U}]$ and $C^* = C^* \cap (\kappa_{i-1}, \kappa_i)$ is a club in κ. It seems natural that the projection will keep only the coordinates in I i.e. let $p = \langle t_1, ..., t_{n+1}\rangle$ then $\pi_I(p) = \langle t'_i \mid \gamma(t_i, p) \in I\rangle^\frown\langle t_{n+1}\rangle$ where

$$t'_i = \begin{cases} \kappa(t_i) & \gamma(t_i, p) \in \text{Succ}(I) \\ t_i & \gamma(t_i, p) \in \text{Lim}(I) \end{cases}$$

Let us define a forcing notion $\mathbb{P}_i = \mathbb{M}_I[\vec{U}]$ (the range of the projection π_I) that will add the subsequence C^*, such that the forcing $\mathbb{M}[\vec{U}]$ (more precisely, a dense subset of $\mathbb{M}[\vec{U}]$) projects onto $\mathbb{M}_I[\vec{U}]$ via the projection π_I as we have just defined.

$\underline{\mathbb{M}_I[\vec{U}]}$

Thinking of C^* as a function with domain I, we would like to have a function similar to $\gamma(t_i, p)$ that tells us which coordinate are we unveiling. Given $p = \langle t_1, ..., t_n, t_{n+1}\rangle$, define recursively $I(t_0, p) = 0$ and

$$I(t_i, p) = \min(i \in I \setminus I(t_{i-1}, p) + 1 \mid o(i) = o^{\vec{U}}(t_i))$$

It is tacitly assumed that $\{i \in I \setminus I(t_{i-1}, p) + 1 \mid o(i) = o^{\vec{U}}(t_i)\} \neq \emptyset$.

Example: Work with Magidor forcing adding a sequence of length ω^2 i.e. $C_G = \{C_G(\alpha) \mid \alpha < \omega^2\}$. Assume $C^* = \{C_G(0)\} \cup \{C_G(\alpha) \mid \omega \leq \alpha < \omega^2\}$. Thus $I = \{0\} \cup (\omega^2 \setminus \omega)$, the ω-th element of C_G is no longer limit in C^*. Let

$$p = \langle \underbrace{\langle \kappa(t_1), B(t_1)\rangle}_{t_1}, \underbrace{\langle \kappa, B(t_2)\rangle}_{t_2}\rangle$$

Where $o^{\vec{U}}(t_1) = 1$. Computing $I(t_1, p)$ we have:

$$I(t_1, p) = \omega = \gamma(t_1, p)$$

Therefore $\pi_I(p) = \langle \kappa(t_1), t_2\rangle$.

Definition 4.2 *The conditions of* $\mathbb{M}_I[\vec{U}]$ *are of the form* $p = \langle t_1, ..., t_{n+1}\rangle$ *such that:*

1. $\kappa(t_1) < ... < \kappa(t_n) < \kappa(t_{n+1}) = \kappa$
2. For $i = 1, ..., n+1$
   (a) $I(t_i, p) \in \mathrm{Succ}(I)$
      i. $t_i = \kappa(t_i)$
      ii. $I(t_{i-1}, p)$ *is the predecessor of* $I(t_i, p)$ *in* I

iii. $I(t_{i-1},p)+\sum_{i=1}^{m}\omega^{\gamma_i}=I(t_i,p)$ *(C.N.F) , then*
$Y(\gamma_1)\times ... \times Y(\gamma_{m-1})\bigcap[(\kappa(t_{i-1}),\kappa(t_i))]^{<\omega}\neq\emptyset$
(Reminder: $Y(\gamma)=\{\alpha<\kappa \mid o^{\vec{U}}(\alpha)=\gamma\}$*)*

(b) $I(t_i,p)\in \mathrm{Lim}(I)$

i. $t_i=\langle\kappa(t_i),B(t_i)\rangle$, $B(t_i)\in\bigcap_{\xi<o^{\vec{U}}(t_i)}U(t_i,\xi)$

ii. $I(t_{i-1},p)+\omega^{o^{\vec{U}}(t_i)}=I(t_i,p)$

iii. $\min(B(t_i))>\kappa(t_{i-1})$

■

Definition 4.3 *Let* $p=\langle t_1,...,t_n,t_{n+1}\rangle, q=\langle s_1,...,s_m,s_{m+1}\rangle\in\mathbb{M}_I[\vec{U}]$*. Define* $\langle t_1,...,t_n,t_{n+1}\rangle\leq_I\langle s_1,...,s_m,s_{m+1}\rangle$ *iff* $\exists 1\leq i_1<...<i_n\leq m<i_{n+1}=m+1$ *such that*

$I(s_j,q)\in\mathrm{Lim}(I)$ then $B(s_j)\subseteq B(t_{k+1})\cap\kappa(s_j)$

1. $\kappa(t_r)=\kappa(s_{i_r})$ *and* $B(s_{i_r})\subseteq B(t_r)$

If $i_k<j<i_{k+1}$

1. $\kappa(s_j)\in B(t_{k+1})$
2. $I(s_j,q)\in\mathrm{Succ}(I)$ *then*

$$[(\kappa(s_{j-1}),\kappa(s_j))]^{<\omega}\cap B(t_{k+1},\gamma_1)\times ...\times B(t_{k+1},\gamma_{k-1})\neq\emptyset$$

where $I(s_{i-1},q)+\sum_{i=1}^{k}\omega^{\gamma_i}=I(s_i,q)$ *(C.N.F)*

3. $I(s_j,q)\in\mathrm{Lim}(I)$ *then* $B(s_j)\subseteq B(t_{k+1})\cap\kappa(s_j)$

■

Definition 4.4 *Let* $p=\langle t_1,...,t_n,t_{n+1}\rangle, q=\langle s_1,...,s_m,s_{m+1}\rangle\in\mathbb{M}_I[\vec{U}]$*, q is a direct extension of p, denoted* $p\leq_I^* q$ *iff*

1. $p\leq_I q$
2. $n=m$

■

Remarks:

1. In definition 4.2 (b.i), although it seems superfluous to take all the measures corresponding to t_i as well as those which do not take an active part in the development of C^*, the necessity is apparent when examining definition 4.3 (2.b)- the γ_i's may not be the measures taking active part in C^*. In lemma 4.8 this condition will be crucial when completing C^* to C_G.

2. As we have seen in earlier chapters, the function $\gamma(t_i, p)$ returns the same value when extending p. $I(t_i, p)$ have the same property, let $p = \langle t_1, ..., t_n, t_{n+1}\rangle$, $q = \langle s_1, ..., s_m, s_{m+1}\rangle \in \mathbb{M}_I[\vec{U}]$, $p \leq_I q$, use 4.2 (2.b.ii) to see that $I(t_r, p) = I(s_{i_r}, q)$.

3. In definition 4.4, since $n = m$ we only have to check (1) of definition 4.3.

4. Let $p = \langle t_1, ..., t_{n+1}\rangle \in \mathbb{M}_I[\vec{U}]$ be any condition. Assume we would like to unveil a new index $j \in I$ between $I(t_i, p)$ and $I(t_{i+1}, p)$. It is possible if for example j is the successor of $I(t_i, p)$ in I:
   Assume $I(t_i, p) + \sum_{l=1}^{m}\omega^{\gamma_l} = j$ (C.N.F), then $\gamma_l < o^{\vec{U}}(t_{i+1})$. Extend p by choosing $\alpha \in B(t_{i+1}, \gamma_m)$ above some sequence

$$\langle\vec{\beta}_1, ..., \vec{\beta}_k\rangle \in B(t_{i+1}, \gamma_1) \times ... \times B(t_{i+1}, \gamma_{m-1})$$
$$I(\alpha, p^\frown\langle\alpha\rangle) = \min(r \in I \setminus I(t_i, p) \mid o(r) = o(j)) = j$$

   Another possible index is any $j \in \mathrm{Lim}(I)$ such that $I(t_i, p) + \omega^{o(j)} = j$. For such j, extend p by picking $\alpha \in B(t_{i+1}, o(j))$ above some sequence $\langle\vec{\beta}_1, ..., \vec{\beta}_k\rangle$, to obtain

$$p \leq_I \langle t_1, ..., t_i, \langle\alpha, \bigcap_{\xi < o(j)} B(t_{i+1}, \xi) \cap \alpha\rangle, \langle\kappa(t_{i+1}), B(t_{i+1}) \setminus (\alpha + 1)\rangle, ..., t_{n+1}\rangle$$

   Checking definition 4.2 we see that in both cases the extension of p is in $\mathbb{M}_I[\vec{U}]$.

The forcing $\mathbb{M}_I[\vec{U}]$ has lots of the properties of $\mathbb{M}[\vec{U}]$, however, they are irrelevant for the proof. Therefore, we will state only few of them.

Lemma 4.5 $\mathbb{M}_I[\vec{U}]$ *satisfy* $\kappa^+ - c.c$

Proof: Let $\{\langle t_{\alpha,1}, ..., t_{\alpha,n_\alpha}\rangle = p_\alpha \mid \alpha < \kappa^+\} \subseteq \mathbb{M}_I[\vec{U}]$. Find $n < \omega$ and $E \subseteq \kappa^+$, $|E| = \kappa^+$ and $\langle\kappa_1, ..., \kappa_n\rangle$ such that $\forall\alpha \in E$,

$$n_\alpha = n \text{ and } \langle \kappa(t_{\alpha,1}), ..., \kappa(t_{\alpha,n_\alpha})\rangle = \langle \kappa_1, ..., \kappa_n\rangle$$

Fix any $\alpha, \beta \in E$. Define $p^* = \langle t_1, ..., t_n, t_{n+1}\rangle$ where

$$B^*(t_i) = B(t_{i,\alpha}) \cap B(t_{i,\beta}) \in \bigcap_{\xi < o^{\vec{U}}(\kappa_i)} U(\kappa_i, \xi)$$

$$t_i = \begin{cases} \langle \kappa_i, B^*(t_i)\rangle & I(t_i, p) \in \mathrm{Lim}(I) \\ \kappa_i & \textit{otherwise} \end{cases}$$

Since $p_\alpha, p_\beta \in \mathbb{M}_I[\vec{U}]$, it is clear that $p^* \in \mathbb{M}_I[\vec{U}]$ and also $p_\alpha, p_\beta \leq_I^* p^*$.

■

Lemma 4.6 *Let $G_I \subseteq \mathbb{M}_I[\vec{U}]$ be generic , define*

$$C_I = \bigcup \{\{\kappa(t_i) | i = 1, ..., n\} \mid \langle t_1, ..., t_n, t_{n+1}\rangle \in G_I\}$$

Then

1. $\mathrm{otp}(C_I) = \mathrm{otp}(I)$ *(thus we may also think of C_I as a function with domain I).*
2. *G_I consist of all conditions $p = \langle t_1, ..., t_n, t_{n+1}\rangle \in \mathbb{M}_I[\vec{U}]$ such that*
   - *(a)* $C_I(I(t_i, p)) = \kappa(t_i)$
   - *(b)* $C_I \cap (\kappa(t_{i-1}), \kappa(t_i)) \subseteq B(t_i) \quad 1 \leq i \leq n+1$
   - *(c)* $\forall i \in \mathrm{Succ}(I) \cap (I(t_r, p), I(t_{r+1}, p))$ *with predecessor $j \in I$ such that* $j + \sum_{l=1}^{k} \omega^{\gamma_l} = i$ *(C.N.F) we have*

     $$[(C_I(j), C_I(i))]^{<\omega} \cap B(t_{r+1}, \gamma_1) \times ... \times B(t_{r+1}, \gamma_{k-1}) \neq \emptyset$$

Proof: For (1) , let us consider the system of ordered sets of ordinals $(\kappa(p), i_{p,q})_{p,q}$ where

$$\kappa(p) = \{\kappa(t_1), ..., \kappa(t_n)\} \text{ for } p = \langle t_1, ..., t_{n+1}\rangle \in G_I$$

$i_{p,q} : \kappa(p) \to \kappa(q)$ are defined for $p = \langle t_1, ..., t_{n+1}\rangle \leq_I \langle s_1, ..., s_{m+1}\rangle = q$ as the inclusion:

$$i_{p,q}(\kappa(t_r)) = \kappa(t_r) = \kappa(s_{i_r}) \text{ } (i_r \text{ are as in the definition of } \leq_I \text{ })$$

Since G_I is a filter, $(\kappa(p), i_{p,q})_{p,q}$ form a directed system with a direct ordered limit $\underrightarrow{\mathrm{Lim}}\, \kappa(p) = \bigcup_{p\in G_I} \kappa(p) = C_I$ and inclusions $i_p : \kappa(p) \to C_I$.
We already defined for $p \leq_I q$, $p, q \in G_I$

$$I(*, p) : \kappa(p) \to I,\ I(*, p) = I(*, q) \circ i_{p,q}$$

Thus $(I(*, p))_{p\in G}$ form a compatible system of functions and by the universal propery of directed limits, we obtain

$$I(*) : C_I \to I,\ I(*) \circ i_p = I(*, p)$$

Let us show that I is an isomorphism of ordered set: Since $I(*, p)$ are injective $I(*)$ is also injective. Assume $\kappa_1 < \kappa_2 \in C_I$, find $p \in G_I$ such that $\kappa_1, \kappa_2 \in \kappa(p)$. Therefore, $I(\kappa_i, p) = I(\kappa_i)$ preserve the order of κ_1, κ_2. Fix $i \in I$, it suffices to show that there exists some condition $p \in G_I$ such that $i \in Im(I(*, p))$. To do this, let us show that the set of all conditions $p \in \mathbb{M}_I[\vec{U}]$ with $i \in Im(I(*, p))$ is a dense subset of $\mathbb{M}_I[\vec{U}]$. Let $p = \langle t_1, ..., t_{n+1}\rangle \in \mathbb{M}_I[\vec{U}]$ be any condition , if $i \in Im(I(*, p))$ then we are done. Otherwise, there exists $0 \leq k \leq n$ such that

$$I(t_k, p) < i < I(t_{k+1}, p)$$

therefore $I(t_{k+1}, p) \in \mathrm{Lim}(I)$. By induction on i, we shall prove that it is possible to extend p to a condition p', such that $i \in Im(I(*, p'))$. If

$$\sum_{l=1}^{k} \omega^{\gamma_l} = i = \min(I) \text{ (C.N.F)}$$

then it must be that $i < I(t_1, p)$. By definition 4.2 (2.b.ii) $I(t_1, p) = \omega^{o^{\vec{U}}(t_1)}$. To extend p just pick any α above some sequence

$$\langle \vec{\beta}_1, ..., \vec{\beta}_k\rangle \in B(t_1, \gamma_1) \times ... \times B(t_1, \gamma_{k-1})$$

and

$$p \leq_I \langle \alpha, \langle \kappa(t_1), B(t_1) \setminus (\alpha+1)\rangle, t_2, ..., t_{n+1}\rangle \in \mathbb{M}_I[\vec{U}]$$

If $i \in \text{Succ}(I)$ with predecessor $j \in I$. By the induction hypothesis, we can assume that for some k, $j = I(t_k, p) \in Im(I(*, p))$. Thus by the remark following definition 4.4 we can extend p by some α such that $i \in Im(I(*, p))$. Finally if $i \in \text{Lim}(I)$, then

$$i = \underbrace{\sum_{i=1}^{m} \omega^{\gamma_i}}_{\alpha} + \omega^{o(i)} \text{ (C.N.F)}$$

Therefore $\forall \beta \in (\alpha, i)$, $\beta + \omega^{o(i)} = i$. Take any $i' \in I \cap (\alpha, i)$. Just as before, it can be assumed that $i' = I(t_k, p)$, thus $I(t_k, p) + \omega^{o(i)} = i$. By the same remark, we can extend p to some $p' \in \mathbb{M}_I[\vec{U}]$ with $j \in Im(I(*, p'))$.

For (2), let $p = \langle t_1, ..., t_{n+1}\rangle \in G_I$. (a) is satisfied by the argument in (1). Fix $\alpha \in C_I \cap (\kappa(t_i), \kappa(t_{i+1}))$, there exists $p \leq_I p' = \langle s_1, ..., s_m\rangle \in G_I$ such that $\alpha \in \kappa(p')$ thus $\alpha \in B(t_{i+1})$ by definition. Moreover, if $I(\alpha, p') \in \text{Succ}(I)$ with predecessor $j \in I$, then by definition 4.2 (2.a.ii), there is s_k such that $j = I(s_k, p')$ and by definition 4.3 (2.b)

$$[(\kappa(s_{k-1}), \kappa(s_k))]^{<\omega} \cap B(t_{i+1}, \gamma_1) \times ... \times B(t_{i+1}, \gamma_{k-1}) \neq \emptyset$$

From (a),

$$\kappa(s_k) = C_I(j) \text{ and } \kappa(s_{k+1}) = C_I(i)$$

In the other direction, if $p = \langle t_1, ..., t_{n+1}\rangle \in \mathbb{M}_I[\vec{U}]$ satisfies (a)-(c). By (a), there exists some $p'' \in G_I$ with $\kappa(p) \subseteq \kappa(p'')$. Set E to be

$$\{\langle w_1, ..., w_{l+1}\rangle \in (\mathbb{M}_I[\vec{U}])_{\geq I p''} \mid \kappa(w_j) \in B(t_i) \cup \{\kappa(t_i)\} \rightarrow \ B(w_j) \subseteq B(t_i)\}$$

E is dense in $\mathbb{M}_I[\vec{U}]$ above p''. Find $p'' \leq_I p' = \langle s_1, ..., s_{m+1}\rangle \in G_I \cap D$. Checking definition 4.3, Let us show that $p \leq_I p'$: For (1), since $\kappa(p) \subseteq \kappa(p')$ there is a natural injection $1 \leq i_1 < ... < i_n \leq m$ which satisfy $\kappa(t_r) = \kappa(s_{i_r})$. Since $p' \in E$, $B(s_{i_r}) \subseteq B(t_r)$. (2a), follows from condition (b), (2b) follows from condition (c). Since $p' \in E$, if $i_r < j < i_{r+1}$ then $\kappa(s_j) \in B(t_{r+1})$, thus, (2c) holds.

■

So given a generic set G_I for $\mathbb{M}_I[\vec{U}]$, we have $V[C_I] = V[G_I]$. Once we will show that π_I is a projection, then for every $G \subseteq \mathbb{M}[\vec{U}]$ generic,

$$\pi_I(G) = \{p \in \mathbb{M}_I[\vec{U}] \mid \exists q \in \pi_I'' G,\ p \leq_I q\}$$

will be generic for $\mathbb{M}_I[\vec{U}]$ and by the definition of π_I on page 45 we have that the corresponding sequence to $\pi_I(G)$ is C^*, as wanted. Let us concentrate on showing π_I is a projection. Let D be the set of all

$$p = \langle t_1, ..., t_n, t_{n+1}\rangle \in \mathbb{M}[\vec{U}]\ ,\ \pi_I(p) = \langle t'_{i_1}, ..., t'_{i_m}, t_{n+1}\rangle$$

such that:

1. $\gamma(t_{i_j}, p) \in \mathrm{Lim}(I) \rightarrow \gamma(t_{i_{j-1}}, p) = \gamma(t_{i_j - 1}, p)$
2. $\gamma(t_{i_j}, p) \in \mathrm{Succ}(I) \rightarrow \gamma(t_{i_j - 1}, p)$ is the predecessor of $\gamma(t_{i_j}, p)$ in I.

Condition (1) is to be compared with definition 4.2 (2.b.ii) and condition (2) with (2.a.ii). The following example justifies the necessity of D.

Example: Assume that

$$\lambda_0 = \omega^2 \text{ and } I = \{2n \mid n \leq \omega\} \cup \{\omega + 2, \omega + 3\} \cup \{\omega \cdot n \mid n < \omega\}$$

let p be the condition

$$\langle \underbrace{\langle \nu_\omega, B_\omega\rangle}_{t_1}, \underbrace{\nu_{\omega+1}}_{t_2}, \underbrace{\langle \nu_{\omega\cdot 2}, B_{\omega\cdot 2}\rangle}_{t_3}, \underbrace{\langle \kappa, B\rangle}_{t_4}\rangle$$

$$\pi_I(p) = \langle \underbrace{\langle \nu_\omega, B_\omega\rangle}_{t_1 \mapsto t'_{i_1}}, \underbrace{\nu_{\omega\cdot 2}}_{t_3 \mapsto t'_{i_2}}, \underbrace{\langle \kappa, B\rangle}_{t_4}\rangle$$

The $\omega+2, \omega+3$-th coordinates cannot be added. On one hand, they should be chosen below $\nu_{\omega\cdot 2}$, on the other hand, there is no large set we can choose them from. The difficulty occurs due to:

$$\omega \cdot 2 \in \mathrm{Succ}(I) \text{ but } \omega + 3 \in I \text{ is the predecessor and } \gamma(t_{i_2) = \omega}$$

Pointing out condition (2). Notice that we can extend p to

$$\langle\langle\nu_\omega, B_\omega\rangle, \nu_{\omega+1}, \nu_{\omega+2}, \nu_{\omega+3}, \langle\nu_{\omega\cdot 2}, B_{\omega\cdot 2}\rangle, \langle\kappa, B\rangle\rangle$$

to avoid this problem.
Next consider

$$I = \{2n \mid n \leq \omega\} \cup \{\omega+2, \omega+3\} \cup \{\omega\cdot n \mid n < \omega,\ n \neq 2\}$$

and let p be the condition

$$\langle\underbrace{\langle\nu_\omega, B_\omega\rangle}_{t_1}, \underbrace{\langle\nu_{\omega\cdot 2}, B_{\omega\cdot 2}\rangle}_{t_2}, \underbrace{\langle\nu_{\omega\cdot 3}, B_{\omega\cdot 3}\rangle}_{t_3}, \underbrace{\langle\kappa, B\rangle}_{t_4}\rangle$$

$$\pi_I(p) = \langle\underbrace{\langle\nu_\omega, B_\omega\rangle}_{t_1\mapsto t'_{i_1}}, \underbrace{\langle\nu_{\omega\cdot 3}, B_{\omega\cdot 3}\rangle}_{t_3\mapsto t'_{i_2}}, \underbrace{\langle\kappa, B\rangle}_{t_4}\rangle$$

Once again the coordinates $\omega+2, \omega+3$ cannot be added since $\min(B_{\omega\cdot 3}) > \nu_{\omega\cdot 2}$. This corresponds to condition (1)

$$\gamma(t_{i_1}, p) = \omega < \omega\cdot 2 = \gamma(t_{i_2-1}, p)$$

As before, we can extend p to avoid this problem.

Proposition 4.7 *D is dense in $\mathbb{M}[\vec{U}]$*

Proof: Fix $p = \langle t_1, ..., t_{n+1}\rangle \in \mathbb{M}[\vec{U}]$, define $\langle p_k \mid k < \omega\rangle$ as follows:
$p_0 = p$. Assume that $p_k = \langle t_1^{(k)}, ..., t_{n_k}^{(k)}, t_{n_k+1}^{(k)}\rangle$ is defined. If $p_k \in D$, define $p_{k+1} = p_k$. Otherwise, there exists a maximal $1 \leq i_j = i_j(k) \leq n'+1$ such that $\gamma(t_{i_j}^{(k)}, p_k) \in I$ which doesn't satisfy $(1) \vee (2)$ of the definition of D.

<u>¬(1)</u>: $\quad \gamma(t_{i_j}^{(k)}, p_k) \in \mathrm{Lim}(I)$ and $\gamma(t_{i_{j-1}}^{(k)}, p_k) < \gamma(t_{i_j-1}^{(k)}, p_k)$

Since $\gamma(t_{i_j}^{(k)}, p_k) \in \mathrm{Lim}(I)$ there exists $\gamma \in I \cap (\gamma(t_{i_j-1}^{(k)}, p_k), \gamma(t_{i_j}^{(k)}, p_k))$. Use proposirion 3.2 to find $p_{k+1} \geq p_k$ with γ added and the only other coordinates added are below γ, thus if $t_{i_j}^{(k)} = t_r^{(k+1)}$ then $\gamma = \gamma(t_{r-1}^{(k+1)}, p_{k+1})$. Thus, every $l \geq r$ satisfies $(1) \vee (2)$. If $p_{k+1} \notin D$ then the problem must accrue below $\gamma(t_{i_j}^{(k)}, p_k)$.

¬(2): $\gamma(t^{(k)}_{i_j},p)\in \mathrm{Succ}(I)$ and $\gamma(t^{(k)}_{i_j-1},p)$ is not the predecessor of $\gamma(t^{(k)}_{i_j},p))$

Let γ be the predecessor in I of $\gamma(t^{(k)}_{i_j},p)$. By proposirion 3.2, there exist $p_{k+1}\geq p_k$ with γ added and the only other coordinates added are below γ. As before, if $t^{(k)}_{i_j}=t^{(k+1)}_r$ then $\gamma=\gamma(t^{(k+1)}_{r-1},p_{k+1})$ and for every $l\geq r$ $\gamma(t^{(k+1)}_l,p_{k+1})$ satisfies $(1)\vee(2)$.

The sequence $\langle p_k \mid k<\omega\rangle$ is defined. It necessarily stabilizes, otherwise then the sequence $\gamma(t^{(k)}_{i_j(k)},p_k)$ form a strictly decreasing infinite sequence of ordinals. Let p_{n^*} be the stabilized condition, it is an extension of p in D.

■

Lemma 4.8 *$\pi_I\restriction D: D\to \mathbb{M}_I[\vec{U}]$ is a projection, i.e:*

1. *π_I is onto.*
2. *$p_1\leq p_2 \Rightarrow \pi_I(p_1)\leq_I \pi_I(p_2)$ (also $\leq^*$ is preserved)*
3. *$\forall p\in \mathbb{M}[\vec{U}]\ \forall q\in \mathbb{M}_I[\vec{U}]\ (\pi_I(p)\leq_I q\to \exists p'\geq p\ \ (q=\pi_I(p'))$*

Proof: Let $p\in D$, such that $\pi_I(p)=\langle t'_{i_1},...,t'_{i_{n'}},t_{n+1}\rangle$

Claim: $\pi_I(p)$ computes I correctly i.e. for every $0\leq j\leq n'$, we have the equality $\gamma(t_{i_j},p)=I(t'_{i,j},\pi_I(p))$.

Proof of claim: By induction on j, for $j=0$, $\gamma(0,p)=0=I(0,\pi_I(p))$. For $j>0$, assume $\gamma(t_{i_{j-1}},p)=I(t'_{i_{j-1}},\pi_I(p))$ and $\gamma(t_{i_j},p)\in \mathrm{Succ}(I)$. Since $p\in D$, $\gamma(t_{i_{j-1}},p)$ is the predecessor of $\gamma(t_{i_j},p)$ in I. Use the induction hypothesis to see that

$$I(t'_{i_j},\pi_I(p))=\min(\beta\in I\setminus \gamma(t_{i_{j-1}},p)+1\mid o(\beta)=o^{\vec{U}}(t_{i_j}))=\gamma(t_{i_j},p)$$

For $\gamma(t_{i_j},p)\in \mathrm{Lim}(I)$, use condition (1) of the definition of D to see that $\gamma(t_{i_{j-1}},p)+\omega^{o^{\vec{U}}(t_{i_j})}=\gamma(t_{i_j},p)$. Thus

$$\forall r\in I\cap(\gamma(t_{i_{j-1}},p),\gamma(t_{i_j},p))\ (o(r)<o^{\vec{U}}(t_{i_j}))$$

In Particular,

$$I(t'_{i_j}, \pi_I(p)) = \min(\beta \in I \setminus \gamma(t_{i_{j-1}}, p) + 1 \mid o(\beta) = o^{\vec{U}}(t_{i_j})) = \gamma(t_{i_j}, p)$$

$\blacksquare_{of\ claim}$

Checking definition 4.2, show that $\pi_I(p) \in \mathbb{M}_I[\vec{U}]$: (1), (2.a.i), (2.b.i), (2.b.iii) are immediate from the definition of π_I. Use the claim to verify that (2.a.ii), (2.b.ii) follows from (1),(2) in D respectively. For (2.a.iii), let $1 \leq j \leq n'$, write

$$\gamma(t_{i_{j-1}}, p) + \sum_{i_{j-1} < l \leq i_j} \omega^{o^{\vec{U}}(t_l)} = \gamma(t_{i_j}, p)$$

This equation induces a C.N.F equation

$$\mathrm{I}(\mathrm{t}_{i_{j-1}}, \pi_I(p)) + \textstyle\sum_{k=1}^{n_0} \omega^{o^{\vec{U}}(t_{l_k})} = I(t_{i_j}, \pi_I(p)) \quad \text{(C.N.F)}$$

Thus

$$\langle \kappa(t_{l_1}), ..., \kappa(t_{l_{n_0 - 1}}) \rangle \in Y(o^{\vec{U}}(t_{l_1})) \times ... \times Y(o^{\vec{U}}(t_{l_{n_0-1}})) \bigcap [(\kappa(t_{i_{j-1}}), \kappa(t_{i_j}))]^{<\omega}$$

(1)- Let $q = \langle t'_1, ..., t'_{n+1} \rangle \in \mathbb{M}_I[\vec{U}]$. For every t'_j such that $I(t'_j, q) \in \mathrm{Succ}(I)$, use definition 4.2 (2.a.iii) to find $\vec{s_j} = \langle s_{j,1}, ..., s_{j,m_j} \rangle$ such that

$$\langle \kappa(s_{j,1}), ..., \kappa(s_j r, m_j) \rangle \in Y(\gamma_1) \times ... \times Y(\gamma_{m-1}) \bigcap [(\kappa(t'_{i_r - 1}), \kappa(t'_{i_r}))]^{<\omega}$$

where $I(t'_{i_r-1}, q) + \sum_{i=1}^{m} \omega^{\gamma_i} = I(t'_{i_r}, q)$ (C.N.F).
For each $i = 1, ..., n$ such that $o^{\vec{U}}(t'_i) > 0$ and $\kappa(t'_i) \in \mathrm{Succ}(I)$ pick some $B(t'_i) \in \bigcap_{\xi < o^{\vec{U}(t'_i)}} U(t_i, \xi)$.
Define $p = \langle t_1, ..., t_{n+1} \rangle^\frown \langle \vec{s_r} \mid I(t_r, q) \in \mathrm{Succ}(I) \rangle$

$$t_i = \begin{cases} \langle \kappa(t'_i), B(t'_i) \setminus \kappa(s_{i,m_i}) + 1 \rangle & o^{\vec{U}}(t'_i) > 0 \\ \kappa(t'_i) & otherwise \end{cases}$$

Once we prove that $\gamma(s_{r,j}, p) \notin I$ and that p computes I correctly i.e. $\gamma(t_i, p) = I(t'_i, q)$, it will follow that $\pi_I(p) = \langle t'_i \mid \gamma(t_i, p) \in I \rangle = q$. By induction on i, for $i = 0$ it is trivial. Let $0 < i$ and assume the statement holds for i. If $I(t'_{i+1}, q) \in \mathrm{Lim}(I)$, then by 4.2 (b.ii)

$$I(t'_{i+1},q)=I(t'_i,q)+\omega^{o^{\vec{U}}(t'_{i+1})}=\gamma(t_i,p)+\omega^{o^{\vec{U}}(t_{i+1})}=\gamma(t_{i+1},p)$$

If $I(t'_{i+1},q)\in \mathrm{Succ}(I)$, then from 4.2 (a.ii) it follows that $I(t'_i,q)$ is the predecessor of $I(t'_{i+1},q)$. By the choice of $\vec{s_{i+1}}$,

$$\gamma(t_{i+1},p)=\gamma(t_i,p)+\sum_{i=1}^{m-1}\omega^{\gamma_1}n_i+\omega^{\gamma_m}(n_m-1)+\omega^{o^{\vec{U}}(t_{i+1})}=$$
$$=I(t'_i,q)+\sum_{i=1}^{m-1}\omega^{\gamma_1}n_i+\omega^{m_1}(n_{m_1}-1)+\omega^{o^{\vec{U}}(t'_{i+1})}=I(t'_{i+1},q)$$

Also, for all $1\leq r\leq m_{i+1}$, $\gamma(s_{i+1,r},p)$ is between two successor ordinals in I, hence $\gamma(s_{i+1,r},p)\notin I$. Finally, $p\in D$ follows from 4.3 (a.ii) and condition (1) and if $\gamma(t_i,p)\in \mathrm{Lim}(I)$ we did not add $\vec{s_i}$. Thus $i_{j-1}=i_j-1$.

(2)- Assume that $p,q\in D$, $p\leq q$. Using the claim, the verification of definition 4.3 it similar to (1).

(3)- We shall proof something weaker to ease notation. Nevertheless, the general statement if very similar. Let $p=\langle t_1,...,t_{n+1}\rangle\in\mathbb{M}[\vec{U}]$. Assume that

$$\pi_I(p)=\langle t'_{i_1},...,t'_{i_{n'}}\rangle\leq_I\langle t'_{i_1},...,t'_{i_{j-1}},s_1,..,s_m,t'_{i_j},...,t'_{i_n}\rangle=q'\in\mathbb{M}_I[\vec{U}]$$

For every $l=1,...,m$ such that $I(s_l,\pi_I(p))\in\mathrm{Succ}(I)$ use definition 4.3 (2b) to find $\vec{s_l}=\langle s_{l,1},...,s_{l,m_l}\rangle$ such that

$$\langle\kappa(s_{l,1}),...,\kappa(s_{l,m_l})\rangle\in B(t_{i_j},\gamma_1)\times...\times B(t_{i_j},\gamma_{m-1})\bigcap[(\kappa(s_{l-1}),\kappa(s_l))]^{<\omega}$$

where $I(s_{l-1},\pi_I(p))+\sum_{i=1}^{m}\omega^{\gamma_i}=I(s_l,\pi_I(p))$ (C.N.F). Define $p\leq p'$ to be the extension $p'=p^\frown\langle s'_1,..,,s'_m\rangle^\frown\langle\vec{s_l}\mid I(s_l,\pi_I(p))\in\mathrm{Succ}(I)\rangle$ where

$$s'_i=\begin{cases}\langle\kappa(s_i),B_i\setminus\kappa(s_{i,m_i})+1\rangle & o^{\vec{U}}(s_i)>0\\ s_i & otherwise\end{cases}$$

As in (1), $\pi_I(p')=\langle t'_{i_1},...,t'_{i_{j-1}},(s'_1)',...,(s'_m)',...t_{i_{n'}}\rangle$. Notice that since we only change s_l such that $I(s_l,\pi_I(p))\in\mathrm{Succ}(I)$, $(s'_l)'=s_l$. Thus $\pi_I(p')=q$ and $p'\in D$ follows.

■

Definition 4.9 *Let G_I be $\mathbb{M}_I[\vec{U}]$ generic, the quotient forcing is*

$$\mathbb{M}[\vec{U}]/G_I = \pi^{-1}{}_I'' G_I = \{p \in \mathbb{M}[\vec{U}] \mid \pi_I(p) \in G_I\}$$

■

The forcing $\mathbb{M}[\vec{U}]/G_I$ completes $V[G_I]$ to $V[G]$ in the sense that if $G \subseteq \mathbb{M}[\vec{U}]$ is generic such that $\pi_I^*(G) = G_I$ then G is also $\mathbb{M}[\vec{U}]/G_I$-generic.

Proposition 4.10 *Let $x, p \in \mathbb{M}[\vec{U}]$ and $q \in \mathbb{M}_I[\vec{U}]$, then*

1. $\pi_I(p) \leq_I q \Rightarrow q \Vdash_{\mathbb{M}_I[\vec{U}]} \check{p} \in \mathbb{M}[\vec{U}]/\underset{\sim}{G}_I$
2. $q \Vdash_{\mathbb{M}_I[\vec{U}]} \check{p} \in \mathbb{M}[\vec{U}]/\underset{\sim}{G}_I \Rightarrow \pi_I(p), q$ *are compatible*
3. $x \Vdash_{\mathbb{M}[\vec{U}]} \check{p} \in \mathbb{M}[\vec{U}]/\underset{\sim}{G}_I \Rightarrow \pi_I(p), \pi_I(x)$ *are compatible*

■

Lemma 4.11 *Let G_I be $\mathbb{M}_I[\vec{U}]$-generic. Then the forcing $\mathbb{M}[\vec{U}]/G_I$ satisfies $\kappa^+ -$ c.c. in $V[G_I]$.*

Proof: Fix $\{p_\alpha \mid \alpha < \kappa^+\} \subseteq \mathbb{M}[\vec{U}]/G_I$ and let

$$r \in G_I,\ r \Vdash_{\mathbb{M}_I[\vec{U}]} \forall \alpha < \kappa^+ \ \ \underset{\sim}{p_\alpha} \in \mathbb{M}[\vec{U}]/\underset{\sim}{G}_I$$

Next we shall show that

$$E = \{q \in \mathbb{M}_I[\vec{U}] \mid (q \perp r) \bigvee (q \Vdash_{\mathbb{M}_I[\vec{U}]} \exists \alpha, \beta < \kappa^+ \ (\underset{\sim}{p_\alpha}, \underset{\sim}{p_\beta} \text{ are compatible})\}$$

is a dense subset of $\mathbb{M}_I[\vec{U}]$. Assume $r \leq_I r'$, for every $\alpha < \kappa^+$ pick some $r' \leq_I q^*_\alpha \in \mathbb{M}_I[\vec{U}]$, $p^*_\alpha \in \mathbb{M}[\vec{U}]$ such that

- $\pi_I(p^*_\alpha) = q^*_\alpha$

- $q^*_\alpha \Vdash \underset{\sim}{p_\alpha} \leq \overset{\vee}{p^*_\alpha} \in \mathbb{M}[\vec{U}]/\underset{\sim}{G_I}$

There exists such q^*_α ,p^*_α: Find $r' \leq_I q'_\alpha$ and p'_α such that $q'_\alpha \Vdash \overset{\vee}{p'_\alpha} = \underset{\sim}{p_\alpha}$ then by the proposition 4.10 (2), there is $q^*_\alpha \geq_I \pi_I(p'_\alpha), q'_\alpha$. By lemma 4.8 (3) there is $p^*_\alpha \geq p'_\alpha$ such that $q^*_\alpha := \pi_I(p^*_\alpha)$. It follows from proposition 4.10 (1) that

$$q^*_\alpha \Vdash \underset{\sim}{p_\alpha} \leq \overset{\vee}{p^*_\alpha} \in \mathbb{M}[\vec{U}]/\underset{\sim}{G_I}$$

Denote $p^*_\alpha = \langle t_{1,\alpha}, ..., t_{n_\alpha,\alpha}, t_{n_\alpha+1,\alpha}\rangle$, $q^*_\alpha = \langle t_{i_1,\alpha}, ..., t_{i_{m_\alpha},\alpha}, t_{n_\alpha+1,\alpha}\rangle$. Find $S \subseteq \kappa^+$, $n < \omega$ and $\langle \kappa_1, ..., \kappa_n\rangle$ such that $|S| = \kappa^+$ and for any $\alpha \in S$, $n_\alpha = n$ and

$$\langle \kappa(t_{1,\alpha}), ..., \kappa(t_{n_\alpha,\alpha})\rangle = \langle \kappa_1, ..., \kappa_n\rangle.$$

Since $\pi_I(p^*_\alpha) = q^*_\alpha$ it follows that

$$\langle \kappa(t_{i_1,\alpha}), ..., \kappa(t_{i_{m_\alpha},\alpha})\rangle = \langle \kappa_{i_1}, ..., \kappa_{i_m}\rangle$$

for some $m < \omega$ and $1 \leq i_1 < ... < i_m \leq n$.
Fix any $\alpha, \beta \in S$ and let $p^* = \langle t_1, ..., t_n, t_{n+1}\rangle$ where

$$t_i = \begin{cases} \langle \kappa_i, B(t_{i,\alpha}) \cap B(t_{i,\beta})\rangle & o^{\vec{U}}(t_{i,\alpha}) > 0 \\ \kappa_i & otherwise \end{cases}$$

Inspired by the boolean algebras we shell denote $p^*_\alpha \cap p^*_\beta = p^*$. Set

$$q^* = \pi_I(p^*) = \langle t'_{i_1}, ..., t'_{i_m}\rangle$$

Then $r' \leq_I q^*_\alpha \cap q^*_\beta = \pi_I(p^*_\alpha) \cap \pi_I(p^*_\beta) = \pi_I(p^*_\alpha \cap p^*_\beta) = \pi_I(p^*) = q^*$. It follows that $q^* \in E$ since by proposition 4.10 (1) $q^* \Vdash_{\mathbb{M}_I[\vec{U}]} \overset{\vee}{p^*} \in \mathbb{M}[\vec{U}]/\underset{\sim}{G_I}$ and

$$q^* \Vdash_{\mathbb{M}_I[\vec{U}]} \underset{\sim}{p_\alpha} \leq \overset{\vee}{p^*_\alpha} \leq^* \overset{\vee}{p^*} \ \wedge\ \underset{\sim}{p_\beta} \leq \overset{\vee}{p^*_\beta} \leq^* \overset{\vee}{p^*}$$

The rest is routine.

■

Lemma 4.12 *Let G be $\mathbb{M}[\vec{U}]$-generic. Then the forcing $\mathbb{M}[\vec{U}]/G_I$ satisfies $\kappa^+ - c.c.$ in $V[G]$.*

Proof: Fix $\{p_\alpha \mid \alpha < \kappa^+\} \subseteq \mathbb{M}[\vec{U}]/G_I$ in $V[G]$ and let

$$r \in G,\ r \Vdash_{\mathbb{M}[\vec{U}]} \forall\alpha < \kappa^+ \ \ \underset{\sim}{p}_\alpha \in \mathbb{M}[\vec{U}]/\underset{\sim}{G}_I$$

Similar to lemma 4.11 we shall show that

$$E = \{x \in \mathbb{M}[\vec{U}] \mid (q \perp r) \bigvee (q \Vdash_{\mathbb{M}[\vec{U}]} \exists \alpha, \beta < \kappa^+ (\underset{\sim}{p}_\alpha, \underset{\sim}{p}_\beta) \textit{ are compatible})\}$$

is a dense subset of $\mathbb{M}[\vec{U}]$. Assume $r \leq r'$, for every $\alpha < \kappa^+$ pick some $r' \leq x'_\alpha \in \mathbb{M}[\vec{U}]$, $p'_\alpha \in \mathbb{M}[\vec{U}]$ such that $x'_\alpha \Vdash_{\mathbb{M}[\vec{U}]} \underset{\sim}{p}_\alpha = \check{p}'_\alpha$. By proposition 4.10 (3), we can find $\pi_I(x'_\alpha), \pi_I(p'_\alpha) \leq_I y_\alpha$. By lemma 4.8 (3), There is $x'_\alpha \leq x^*_\alpha$,$p'_\alpha \leq p^*_\alpha$ such that

$$\pi_I(x'_\alpha), \pi_I(p''_\alpha) \leq_I y_\alpha = \pi_I(p^*_\alpha) = \pi_I(x^*_\alpha)$$

Denote

$$x^*_\alpha = \langle s_{1_\alpha}, ..., s_{k_\alpha,\alpha}, s_{k_\alpha+1,\alpha} \rangle \ , \ p^*_\alpha = \langle t_{1,\alpha}, ..., t_{n_\alpha,\alpha}, t_{n_\alpha+1,\alpha} \rangle$$
$$\pi_I(x^*_\alpha) = \langle t'_{i_1,\alpha}, ..., t'_{i_{k'_\alpha},\alpha} t'_{k_\alpha+1} \rangle = \pi_I(p_\alpha)$$

Find $S \subseteq \kappa^+$ $|S| = \kappa^+$ and $\langle \kappa_1, ..., \kappa_n \rangle, \langle \nu_1, ..., \nu_k \rangle$ such that for any $\alpha \in S$

$$\langle \kappa(t_{1,\alpha}), ..., \kappa(t_{n_\alpha,\alpha}) \rangle = \langle \kappa_1, ..., \kappa_n \rangle, \ \ \langle \kappa(s_{1,\alpha}), ..., \kappa(s_{k,\alpha}) \rangle = \langle \nu_1, ..., \nu_k \rangle$$

Fix any $\alpha, \beta \in S$ and let $p^* = p^*_\alpha \cap p^*_\beta$, $x^* = x^*_\alpha \cap x^*_\beta$. Then $p'_\alpha, p'_\beta \leq^* p^*$ and $x_\alpha, x_\beta \leq^*_I x^*$. Finally claim that $x^* \in E$:

$$\pi_I(p^*) = \pi_I(p^*_\alpha) \cap \pi_I(p^*_\beta) = \pi_I(x^*_\alpha) \cap \pi_I(x^*_\beta) = \pi_I(x^*)$$

thus $x^* \Vdash_{\mathbb{M}[\vec{U}]} \check{p}^* \in \mathbb{M}[\vec{U}]/\underset{\sim}{G}_I$. Moreover, $x_\alpha \leq^* x^*$ which implies that $x^* \Vdash_{\mathbb{M}[\vec{U}]} \check{p}^* \geq \underset{\sim}{p}_\alpha, \underset{\sim}{p}_\beta$.

■

Lemma 4.13 *If $A \in V[G]$, $A \subseteq \kappa^+$ then there exists $C^* \subseteq C_G$ such that $V[A] = V[C^*]$.*

Proof: Work in $V[G]$, for every $\alpha < \kappa^+$ find subsequences $C_\alpha \subseteq C_G$ such that $V[C_\alpha] = V[A \cap \alpha]$ using the induction hypothesis. The function $\alpha \mapsto C_\alpha$ has range $P(C_G)$ and domain κ^+ which is regular in $V[G]$. Therefore there exist $E \subseteq \kappa^+$ unbounded in κ^+ and $\alpha^* < \kappa^+$ such that for every $\alpha \in E$, $C_\alpha = C_{\alpha^*}$. Set $C^* = C_{\alpha^*}$, then

1. $C^* \subseteq C_G$
2. $C^* \in V[A \cap \alpha^*] \subseteq V[A]$
3. $\forall \alpha < \kappa^+ \ A \cap \alpha \in V[C^*]$

Since C_G is a club, it can be assumed that C^* is a club by adding the limit points of C^* to C^*, clearly it will still satisfy (1)-(3). Unlike A's that were subsets of κ, for which we added another piece of C_G to C^* to obtain C' such that $V[A] = V[C']$, here we claim that $V[A] = V[C^*]$:
By (2), $C^* \in V[A]$. For the other direction, denote by I the indexes of C^* in C and consider the forcings $\mathbb{M}_I[\vec{U}], \mathbb{M}[\vec{U}]/G_I$. Assume that $A \notin V[C^*]$, we shall reach a contradiction: Let $\underset{\sim}{A}$ be a name for A in $\mathbb{M}[\vec{U}]/G_I$ where $\pi''_I G = G_I$. Work in $V[G_I]$, by lemma 4.6 (2), $V[G_I] = V[C^*]$. For every $\alpha < \kappa^+$ define

$$X_\alpha = \{B \subseteq \alpha \mid ||\underset{\sim}{A} \cap \alpha = B|| \neq 0\}$$

where the truth value is taken in $RO(\mathbb{M}[\vec{U}]/G_I)$- the complete boolean algebra of regular open sets for $\mathbb{M}[\vec{U}]/G_I$. By lemma 4.11

$$\forall \alpha < \kappa^+ \ |X_\alpha| \leq \kappa.$$

For every $B \in X_\alpha$ define $b(B) = ||\underset{\sim}{A} \cap \alpha||$. Assume that $B' \in X_\beta$ and $\alpha \leq \beta$ then $B = B' \cap \alpha \in X_\alpha$. Switching to boolean algebra notation ($p \leq_B q$ means p extends q) $b(B') \leq_B b(B)$. Note that for such B, B' if $b(B') <_B b(B)$, then there is

$$0 < p \leq_B (b(B) \setminus b(B')) \leq_B b(B)$$

Therefore

$$p \cap b(B') \leq_B (b(B) \setminus b(B')) \cap b(B') = 0$$

Hence $p \perp b(B')$. Work in V[G], denote $A_\alpha = A \cap \alpha$. Recall that

$$\forall \alpha < \kappa^+ \; A_\alpha \in V[C^*]$$

thus $A_\alpha \in X_\alpha$. Consider the $\leq_B$-non-increasing sequence $\langle b(A_\alpha) \mid \alpha < \kappa^+ \rangle$. If there exists some $\gamma^* < \kappa^+$ on which the sequence stabilizes, define

$$A' = \bigcup \{ B \subseteq \kappa^+ \mid \exists \alpha \; b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha = B \} \in V[C^*]$$

To see that $A' = A$, notice that if B, B', α, α' are such that

$$b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha = B, \; b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha' = B'$$

if $\alpha \leq \alpha'$ then we must have $B' \cap \alpha = B$ otherwise, the non zero condition $b(A_{\gamma^*})$ would force contradictory information. Consequently, for every $\xi < \kappa^+$ there exists $\xi < \gamma < \kappa^+$ such that $b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \gamma = A \cap \gamma$, hence $A' \cap \gamma = A \cap \gamma$. This is a contradiction to $A \notin V[C^*]$. Therefore, the sequence $\langle b(A_\alpha) \mid \alpha < \kappa^+ \rangle$ does not stabilize. By regularity of κ^+, there exists a subsequence $\langle b(A_{i_\alpha}) \mid \alpha < \kappa^+ \rangle$ which is strictly decreasing. Use the observation we made to find $p_\alpha \leq_B b(A_{i_\alpha})$ such that $p_\alpha \perp b(A_{i_{\alpha+1}})$. Since $b(A_{i_\alpha})$ are decreasing, for any $\beta > \alpha$ $p_\alpha \perp b(A_{i_\beta})$ thus $p_\alpha \perp p_\beta$. This shows that $\langle p_\alpha \mid \alpha < \kappa^+ \rangle \in V[G]$ is an antichain of size κ^+ which contradicts Lemma 4.12. Thus $V[A] = V[C^*]$.

■

End of the proof of Theorem 3.3: By induction on $\sup(A) = \lambda > \kappa^+$. It suffices to assume that λ is a cardinal.

case1: $(cf^{V[G]}(\lambda) > \kappa)$ the arguments of lemma 4.13 works.

case2: $(cf^{V[G]}(\lambda) \leq \kappa$ $)$ Since $\mathbb{M}[\vec{U}]$ satisfies $\kappa^+ - c.c.$ we must have that $\nu := cf^V(\lambda) \leq \kappa$. Fix $\langle \gamma_i | \; i < \nu \rangle \in V$ cofinal in λ. Work in $V[A]$, for every $i < \nu$ find $d_i \subseteq \kappa$ such that $V[d_i] = V[A \cap \gamma_i]$. By induction, there exists $C^* \subseteq C_G$ such that $V[\langle d_i \mid i < \nu \rangle] = V[C^*]$, therefore

1. $\forall i < \nu \; A \cap \gamma_i \in V[C^*]$
2. $C^* \in V[A]$

Work in $V[C^*]$, for $i < \nu$ define $X_i = \{B \subseteq \alpha \mid \ ||\underset{\sim}{A} \cap \gamma_i = B|| \neq 0\}$. By lemma 4.11, $|X_i| \leq \kappa$. For every $i < \nu$ fix an enumeration

$$X_i = \langle X(i, \xi) \mid \ \xi < \kappa \rangle \in V[C^*]$$

There exists $\xi_i < \kappa$ such that $A \cap \gamma_i = X(i, \xi_i)$. Moreover, since $\nu \leq \kappa$ the sequence $\langle A \cap \gamma_i \mid i < \nu \rangle = \langle X(i, \xi_i) \mid i < \nu \rangle$ can be coded in $V[C^*]$ as a sequence of ordinals below κ . By induction there exists $C'' \subseteq C_G$ such that $V[C''] = V[\langle \xi_i \mid i < \nu \rangle]$. It follows

$$V[C'', C^*] = (V[C^*])[\langle \xi_i \mid i < \nu \rangle] = V[A]$$

Finally, we can take for example, $C' = C'' \cup C^* \subseteq C_G$ to obtain $V[A] = V[C']$

$\blacksquare_{theorem\ 3.3}$

5 Classification of subforcing of Magidor

Definition 5.1 *Let $\vec{U}$ be a coherent sequence and κ a measurable cardinal with $0 < o^{\vec{U}}(\kappa) < \min(\nu \mid o^{\vec{U}}(\nu) > 0)$. Let $I \subseteq \omega^{o^{\vec{U}}(\kappa)}$ be a closed subset. Define:*

1. $0_{\mathbb{M}_I[\vec{U}]} = \big\langle \langle\rangle, \langle \kappa, B^* \rangle \big\rangle$ *where B^* has the following properties*

   - $B^* \in \bigcap_{\xi < o^{\vec{U}}(\kappa)} U(\kappa, \xi)$
   - *For every $\beta \in B^*$ $o^{\vec{U}}(\beta) < o^{\vec{U}}(\kappa)$*
   - *For every $\beta \in B^*$ $B \cap \beta \in \bigcap_{\xi < o^{\vec{U}}(\beta)} U(\beta, \xi)$*

2. *For every $p = \big\langle t_1, ..., t_n, \langle \kappa, B' \rangle \big\rangle$ such that each t_r is an ordinal or a pair, define $\gamma_I(t_0, p) = 0$ and*

$$\gamma_I(t_r, p) = \min(i \in I \setminus \gamma_I(t_{r-1}, p) + 1 \mid o(i) = o^{\vec{U}}(t_r))$$

   If for some $1 \leq r \leq n$, $\{i \in I \setminus \gamma_I(t_{r-1}, p) + 1 \mid o(i) = o^{\vec{U}}(t_r)\} = \emptyset$ then for every $1 \leq j \leq n$ let $\gamma_I(t_j, p) = N/A$.

3. *The elements of $\mathbb{M}_I[\vec{U}]$ are of the form $p = \big\langle t_1, ..., t_n, \langle \kappa, B \rangle \big\rangle$ such that each t_r is an ordinal or a pair and $\gamma_I(t_{r1}, p) \neq N/A$ for every $1 \leq r \leq n$, such that:*

   (a) $\kappa(t_1) < ... < \kappa(t_n) < \kappa$

   (b) $B \subseteq B^*$, $B \in \bigcap_{\xi < o^{\vec{U}}(\kappa)} U(\kappa, \xi)$

   (c) *For every $1 \leq r \leq n$*

      i. *If $\gamma_I(t_r, p) \in \mathrm{Succ}(I)$ then*

         A. $t_r = \kappa(t_r) \in B^*$

         B. *$\gamma_I(t_{r-1}, p)$ is the predecessor in I of $\gamma_I(t_r, p)$*

      ii. *If $\gamma_I(t_r, p) \in \mathrm{Lim}(I)$*

         A. $t_r = \langle \kappa(t_r), B(t_r) \rangle \in B^* \times P(B^*)$, $B(t_i) \in \bigcap_{\xi < o^{\vec{U}}(t_r)} U(t_r, \xi)$

B. $\gamma_I(t_{r-1}, p) + \omega^{o^{\vec{U}}(t_r)} = \gamma_I(t_r, p)$
C. $\min(B(t_r)) > \kappa(t_{r-1})$, *where* $\kappa(t_0) = 0$

4. *Let* $p = \langle t_1, ..., t_n, t_{n+1}\rangle, q = \langle s_1, ..., s_m, s_{m+1}\rangle \in \mathbb{M}_I[\vec{U}]$. *Define* $\langle t_1, ..., t_n, t_{n+1}\rangle \leq_I \langle s_1, ..., s_m, s_{m+1}\rangle$ *iff* $\exists 1 \leq i_1 < ... < i_n \leq m < i_{n+1} = m + 1$ *such that*

   (a) $\kappa(t_r) = \kappa(s_{i_r})$ *and* $B(s_{i_r}) \subseteq B(t_r)$

   (b) *If* $i_k < j < i_{k+1}$

      i. $\kappa(s_j) \in B(t_{k+1})$
      ii. $I(s_j, q) \in \mathrm{Lim}(I) \to B(s_j) \subseteq B(t_{k+1}) \cap \kappa(s_j)$

■

Definition 5.2 *The forcings* $\{\mathbb{M}_I[\vec{U}] \mid I \in P(\omega^{o^{\vec{U}}(\kappa)})\}$ *is the family of Magidor-type forcing with the coherent sequence* $\vec{U}$.

In practice, Magidor-type forcings are just Magidor forcing with a subsequence of $\vec{U}$; If I is any closed subset of indexes, we can read the measures of $\vec{U}$ from which the elements of the final sequence are chosen using the map $I \mapsto \langle o(i) \mid i \in I\rangle$ (recall that $o(i) = \gamma_n$ where $i = \omega^{\gamma_1} + ... + \omega^{\gamma_n}$ *C.N.F*).

Example: Assume that $o^{\vec{U}}(\kappa) = 2$ and let a

$$I = \{1, \omega, \omega + 1\} \cup (\omega \cdot 3 \setminus \omega \cdot 2) \cup \{\omega \cdot 3, \omega \cdot 4, ...\} \in P(\omega^2)$$

Then $\langle o(i) \mid i \in I\rangle = \langle 0, 1, \underbrace{0, 0, 0...}_{\omega}, \underbrace{1, 1, 1...}_{\omega}\rangle$. Therefore $\mathbb{M}_I[\vec{U}]$ is just Prikry foricing with $U(\kappa_1, 0)$ for some measurable $\kappa_1 < \kappa$ followed by Prikry forcing with $U(\kappa, 1)$.
Although in this example the noise at the beginning is neglectable, there are I's for which we do not get "pure" Magidor forcing which uses one measure at a time and combine several measure. The next theorem is a Mathias characterization for Magidor-type forcing and is proven in [**?**].

Theorem 5.3 *Let* $\mathbb{M}_I[\vec{U}]$ *be a Magidor-type forcing,* $C = \langle C(i) \mid i \in I\rangle$ *be any increasing continues sequence. Then*

$$G_C = \{p \in \mathbb{M}_I[\vec{U}] \mid \kappa(p) \subseteq C,\ C \setminus \kappa(p) \subseteq B(p)\}$$

is a generic for $\mathbb{M}_I[\vec{U}]$ *iff:*

1. *For every* $i \in I$ $o^{\vec{U}}(C(i)) = o(i)$
2. *For every* $\langle c_1, ..., c_n\rangle \in [\mathrm{Lim}(C)]^{<\omega}$ *and every* $A_r \in \bigcap_{j<o^{\vec{U}}(c_r)} U(c_r, j)$ *for* $1 \leq r \leq n$*, there exists* $\alpha_1 < c_1 \leq \alpha_2 < c_2 \leq ... \leq \alpha_n < c_n$ *such that* $C \cap (\alpha_r, c_r) \subseteq A_r$

We restate Theorem 3.3 in terms of complete subforcing [?].

Theorem 5.4 *Let* $\mathbb{P} \subseteq \mathbb{M}[\vec{U}]$ *be a complete subforcing of* $\mathbb{M}[\vec{U}]$ *then there exists a maximal antichain* $Z \subseteq \mathbb{P}$ *and* I_p*,* $p \in Z$ *such that* $\mathbb{P}_{\geq p}$ *(the forcing* $\mathbb{P}$ *above* p*) is equivalent to the Magidor-type forcing* $\mathbb{M}_{I_p}[\vec{U}]_{\geq q_p}$*.*

Proof: Let $H \subseteq \mathbb{P}$ be generic, then there exists $G \subseteq \mathbb{M}[\vec{U}]$ generic such that $H = G \cap \mathbb{P}$, in particular $V \subseteq V[H] \subseteq V[G]$. By Theorem 3.3, there is a closed $C' \subseteq C_G$ such that $V[C'] = V[H]$. Let $\underset{\sim}{C}'$ be a $\mathbb{P}$-name of C' and I it's set of indexes in C_G. The assumption $o^{\vec{U}}(\kappa)$ is crucial to claim that $I \in V$. By the Mathias characterization (see theorem 5.4), C' is generic for $\mathbb{M}_I[\vec{U}]$. Let $p \in \mathbb{P}$ such that

$$p \Vdash \underset{\sim}{C}' \text{ is generic for } I = I_p \text{ and } V[\underset{\sim}{H}] = V[\underset{\sim}{C}']$$

This is indeed a formula in the forcing language since for any set A, $V[A] = \bigcup_{z \subseteq ord, z \in V} L[z, A]$ where $L[z, A]$ is the class of all constructable sets relative to z, A. Redefine $\underset{\sim}{C}', \underset{\sim}{H}$ to be $\mathbb{M}_{I_p}[\vec{U}]$-names for C', H and let $q_p \in RO(\mathbb{M}_{I_p}[\vec{U}])$ be

$$q_p = ||\underset{\sim}{H} \text{ is generic for } \mathbb{P},\, p \in \underset{\sim}{H} \text{ and } V[\underset{\sim}{H}] = V[\underset{\sim}{C}']||$$

Clearly $\mathbb{M}_{I_p}[\vec{U}]_{\geq q_p}$ and $\mathbb{P}_{\geq p}$ have the same generic extensions

■

6 Prikry forcings with non-normal ultrafilters.

Let κ be a measurable cardinal and let $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega}\rangle$ be a tree consisting of $\kappa-$complete non-trivial ultrafilter over κ.

Recall the definition due to Prikry of the tree Prikry forcing with $\mathbb{U}$.

Definition 6.1 *$P(\mathbb{U})$ is the set of all pairs $\langle p, T\rangle$ such that*

1. *p is a finite sequence of ordinals below κ,*
2. *$T \subseteq [\kappa]^{<\omega}$ is a tree with trunk p such that for every $q \in T$ with $q \geq_T p$, the set of the immediate successors of q in T, i.e. $Suc_T(q)$ is in U_q.*

The orders $\leq, \leq^*$ are defined in the usual fashion.

■

For every $a \in [\kappa]^{<\omega}$, let π_a be a projection of U_a to a normal ultrafilter. Namely, let $\pi_a : \kappa \to \kappa$ be a function which represents κ in the ultrapower by U_a, i.e. $[\pi]_{U_a} = \kappa$. Once U_a is a normal ultrafilter, then let π_a be the identity.

By passing to a dense subset of $P(\mathbb{U})$, we can assume that for each $\langle p, T\rangle \in P(\mathbb{U})$, for every $\langle \nu_1, ..., \nu_n\rangle \in T$ we have

$$\nu_1 < \pi_{\langle \nu_1\rangle}(\nu_2) \leq \nu_2 < ... \leq \nu_{n-1} < \pi_{\langle \nu_1,...,\nu_{n-1}\rangle}(\nu_n)$$

and for every $\nu \in Suc_T(\langle \nu_1, ..., \nu_n\rangle), \pi_{\langle \nu_1,...,\nu_n\rangle}(\nu) > \nu_n$.

Note that once the measures over a certain level (or certain levels) are the same - say for some $n < \omega$ and U, for every $a \in [\kappa]^n$, $U_a = U$, then a modified diagonal intersection

$$\Delta^*_{\alpha<\kappa} A_\alpha := \{\nu < \kappa \mid \forall \alpha < \pi_k(\nu)(\nu \in A_\alpha)\} \in U,$$

once $\{A_\alpha \mid \alpha < \kappa\} \subseteq U$, can be used to avoid or to simplify the tree structure.

For example, if $\langle \mathcal{V}_n \mid n < \omega\rangle$ is a sequence of $\kappa-$complete ultrafilters over κ, then the Prikry forcing with it $P(\langle \mathcal{V}_n \mid n < \omega\rangle)$ is defined as follows:

Definition 6.2 *$P(\langle \mathcal{V}_n \mid n < \omega\rangle)$ is the set of all pairs $\langle p, \langle A_n \mid |p| < n < \omega\rangle\rangle$ such that*

1. $p = \langle \nu_1, ..., \nu_k \rangle$ *is a finite sequence of ordinals below* κ*, such that* $\nu_j < \pi_i(\nu_i)$*, whenever* $1 \le j < i \le k$*,*

2. $A_n \in \mathcal{V}_n$*, for every* $n, |p| < n < \omega$*, and*

3. $\pi_{k+1}(\min(A_{k+1})) > \max(p)$*, where* $\pi_n : \kappa \to \kappa$ *is a projection of* $\mathcal{V}_n$ *to a normal ultrafilter, i.e.* π_n *is a function which represents* κ *in the ultrapower by* $\mathcal{V}_n$*,* $[\pi]_{\mathcal{V}_n} = \kappa$*.*

■

A simpler case is once all $\mathcal{V}_n$ are the same, say all of them are U. Then we will have the Prikry forcing with U:

Definition 6.3 $P(U)$ *is the set of all pairs* $\langle p, A \rangle$ *such that*

1. $p = \langle \nu_1, ..., \nu_k \rangle$ *is a finite sequence of ordinals below* κ*, such that* $\nu_j < \pi(\nu_i)$*, whenever* $1 \le j < i \le k$*,*

2. $A \in U$*, and*

3. $\pi(\min(A)) > \max(p)$*, where* π *is a projection of* U *to a normal ultrafilter.*

■

Let G be a generic for $\langle P(\mathbb{U}), \le \rangle$. Set

$$C = \bigcup \{p \mid \exists T \quad \langle p, T \rangle \in G\}.$$

It is called a Prikry sequence for $\mathbb{U}$.

For every natural $n \ge 1$ we would like to define a $\kappa-$complete ultrafilter U_n over $[\kappa]^n$ which correspond to the first $n-$levels of trees in $P(\mathbb{U})$.
If $n = 1$, set $U_1 = U_{\langle\rangle}$.
Deal with the next step $n = 2$. Here for each $\nu < \kappa$ we have U_ν.
Consider the ultrapower by $U_{\langle\rangle}$:

$$i_{\langle\rangle} : V \to M_{\langle\rangle}.$$

Then the sequence $i_{\langle\rangle}(\langle U_{\langle\nu\rangle} \mid \nu < \kappa\rangle)$ will have the length $i_{\langle\rangle}(\kappa)$.
Let $U_{\langle [id]_{U_{\langle\rangle}}\rangle}$ be its $[id]_{U_{\langle\rangle}}$ ultrafilter in $M_{\langle\rangle}$ over $i_{\langle\rangle}(\kappa)$. Consider its ultrapower

$$i_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}} : M_{\langle\rangle} \to M_{\langle [id]_{U_{\langle\rangle}}\rangle}$$

Set

$$i_2 = i_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}} \circ i_{\langle\rangle}.$$

Then

$$i_2 : V \to M_{\langle [id]_{U_{\langle\rangle}}\rangle}.$$

Note that if all of $U_{\langle\nu\rangle}$'s are the same or just for a set of ν's in $U_{\langle\rangle}$ they are the same, then this is just an ultrapower by the product of $U_{\langle\rangle}$ with this ultrafilter. In general it is an ultrapower by

$$U_{\langle\rangle} - Lim\langle U_{\langle\nu\rangle} \mid \nu < \kappa\rangle,$$

where

$$X \in U_{\langle\rangle} - Lim\langle U_{\langle\nu\rangle} \mid \nu < \kappa\rangle \text{ iff } [id]_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}} \in i_2(X).$$

Note that once most of $U_{\langle\nu\rangle}$'s are normal, then $U_{\langle [id]_{U_{\langle\rangle}}\rangle}$ is normal as well, and so, $[id]_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}} = i_{\langle\rangle}(\kappa)$.

Define an ultrafilter U_2 on $[\kappa]^2$ as follows:

$$X \in U_2 \text{ iff } \langle [id]_{U_{\langle\rangle}}, [id]_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}}\rangle \in i_2(X).$$

Define also for $k = 1, 2$, ultrafilters U_2^k over κ as follows:

$$X \in U_2^1 \text{ iff } [id]_{U_{\langle\rangle}} \in i_2(X),$$

$$X \in U_2^1 \text{ iff } [id]_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}} \in i_2(X).$$

Clearly, then $U_2^1 = U_1$ and $U_2^2 = U_{\langle\rangle} - Lim\langle U_{\langle\nu\rangle} \mid \nu < \kappa\rangle$. Also U_2^1 is the projection of U^2 to the first coordinate and U_2^2 to the second.

Let $\langle\langle\rangle, T\rangle \in P(\mathbb{U})$. It is not hard to see that $T \restriction 2 \in U_2$.

Continue and define in the similar fashion the ultrafilter U_n over $[\kappa]^n$ and its projections to the coordinates U_n^k for every $n > 2, 1 \leq k \leq n$. We will have that for any $\langle\langle\rangle, T\rangle \in P(\mathbb{U})$, $T \restriction n \in U_n$. Also, if $1 \leq n \leq m < \omega$, then the natural projection of U_m to $[\kappa]^n$ will be U_n.

It is easy to see that C is a Prikry sequence for $\langle U_n^n \mid 1 \leq n < \omega\rangle$, in a sense that for every sequence $\langle A_n \mid n < \omega\rangle \in V$, with $A_n \in U_n^n$, there is $n_0 < \omega$ such that for every $n > n_0$, $C(n) \in U_n^n$.
However, it does not mean that C is generic for the forcing $P(\langle U_n^n \mid 1 \leq n < \omega\rangle)$ defined above (Definition ??). The problem is with projection to normal. All U_n^n's have the same normal U_1.

Suppose now that we have an ultrafilter W over $[\kappa]^\ell$ which is Rudin-Keisler below some $\mathfrak{V}$ over $[\kappa]^k$ ($W \leq_{RK} \mathfrak{V}$), for some $k, \ell, 1 \leq \ell, k < \omega$. This means that there is a function $F : [\kappa]^k \to [\kappa]^\ell$ such that

$$X \in W \text{ iff } F^{-1\prime\prime}X \in \mathfrak{V}.$$

So F projects $\mathfrak{V}$ to W. Let us denote this by $W = F_*\mathfrak{V}$.

The next statement characterizes $\omega-$sequences in $V[C]$.

Theorem 6.4 *Let $\langle \alpha_k \mid k < \omega\rangle \in V[C]$ be an increasing cofinal in κ sequence. Then $\langle \alpha_k \mid k < \omega\rangle$ is a Prikry sequence for a sequence in V of $\kappa-$complete ultrafilters which are Rudin -Keisler below $\langle U_n \mid n < \omega\rangle$.*[5]
Moreover, there exist a non-decreasing sequence of natural numbers $\langle n_k \mid k < \omega\rangle$ and a sequence of functions $\langle F_k \mid k < \omega\rangle$ in V, $F_k : [\kappa]^{n_k} \to \kappa$, $(k < \omega)$, such that

1. *$\alpha_k = F_k(C \restriction n_k)$, for every $k < \omega$.*

2. *Let $\langle n_{k_i} \mid i < \omega\rangle$ be the increasing subsequence of $\langle n_k \mid k < \omega\rangle$ such that*

   (a) *$\{n_{k_i} \mid i < \omega\} = \{n_k \mid k < \omega\}$, and*

   (b) *$k_i = \min\{k \mid n_k = n_{k_i}\}$.*

Set $\ell_i = |\{k \mid n_k = n_{k_i}\}|$. Then $\langle F_k(C \restriction n_{k_i}) \mid i < \omega, n_k = n_{k_i}\rangle$ will be a Prikry sequence for $\langle W_i \mid i < \omega\rangle$, i.e. for every sequence $\langle A_i \mid i < \omega\rangle \in V$, with $A_i \in W_i$, there is $i_0 < \omega$ such that for every $i > i_0$, $\langle F_k(C \restriction n_{k_i}) \mid i < \omega, n_k = n_{k_i}\rangle \in A_i$, where each W_i is an ultrafilter over $[\kappa]^{\ell_i}$ which is the projection of $U_{n_{k_i}}$ by $\langle F_{k_i}, ..., F_{k_i+\ell_i-1}\rangle$.

Proof. Work in V. Given a condition $\langle q, S\rangle$, we will construct by induction, using the Prikry property of the forcing $P(\mathbb{U}$, a stronger condition $\langle p, T\rangle$ which decides $\underset{\sim}{\alpha}_k$ once going up to a certain level n_k of T. Let us assume for simplicity that q is the empty sequence.

[5]Let $\langle \mathcal{V}_k \mid k < \omega\rangle$ be such sequence of ultrafilters over κ. We do not claim that $\langle \alpha_k \mid k < \omega\rangle$ is Prikry generic for the forcing $P(\langle \mathcal{V}_k \mid k < \omega\rangle)$, but rather that for every sequence $\langle A_k \mid k < \omega\rangle \in V$, with $A_k \in \mathcal{V}_k$, there is $k_0 < \omega$ such that for every $k > k_0$, $\alpha_k \in \mathcal{V}_k$.

Build by induction $\langle\langle\rangle, T\rangle \geq^* \langle\langle\rangle, S\rangle$ and a non-decreasing sequence of natural numbers $\langle n_k \mid k < \omega\rangle$ such that for every $k < \omega$

1. for every $\langle \eta_1, ..., \eta_{n_k}\rangle \in T$ there is $\rho_{\langle \eta_1,...,\eta_{n_k}\rangle} < \kappa$ such that
   (a) the condition $\langle\langle \eta_1, ..., \eta_{n_k}\rangle, T_{\langle \eta_1,...,\eta_{n_k}\rangle}$ forces "$\underset{\sim}{\alpha}_k = \rho_{\langle \eta_1,...,\eta_{n_k}\rangle}$",
   (b) $\rho_{\langle \eta_1,...,\eta_{n_k}\rangle} \geq \pi_{\langle \eta_1,...,\eta_{n_{k-1}}\rangle}(\eta_{n_k})$,
2. there is no $n, n_k \leq n < n_{k+1}$ such that for some $\langle \eta_1, ..., \eta_n\rangle \in T$ and E the condition $\langle\langle \eta_1, ..., \eta_n\rangle, E\rangle$ decides the value of $\underset{\sim}{\alpha}_{k+1}$,

Now, using the density argument and making finitely many changes, if necessary, we can assume that such $\langle\langle\rangle, T\rangle$ in the generic set.

For every $k < \omega$, define a function $F_k : Lev_{n_k}(T) \to \kappa$ by setting

$$F_k(\eta_1, ..., \eta_{n_k}) = \nu \text{ if } \langle\langle \eta_1, ..., \eta_{n_k}\rangle, T_{\langle \eta_1,...,\eta_{n_k}\rangle}\rangle \Vdash \underset{\sim}{\alpha}_k = \nu.$$

■

We restrict now our attention to ultrafilters U which are P-points. This will allow us to deal with arbitrary sets of ordinals in $V[C]$.
Recall the definition.

Definition 6.5 *U is called a P-point iff every non-constant (mod U) function $f : \kappa \to \kappa$ is almost one to one (mod U), i.e. there is $A \in U$ such that for every $\delta < \kappa$,*

$$|\{\nu \in A \mid f(\nu) = \delta\}| < \kappa.$$

■

Note that, in particular, the projection to the normal ultrafilter π is almost one to one. Namely,

$$|\{\nu < \kappa \mid \pi(\nu) = \alpha\}| < \kappa,$$

for any $\alpha < \kappa$.
Denote by U^{nor} the projection of U to the normal ultrafilter.

Lemma 6.6 *Assume that $\mathbb{U} = \langle U_a \mid 1 \leq a \in [\kappa]^{<\omega}\rangle$ consists of P-point ultrafilters. Suppose that $A \in V[C] \setminus V$ is an unbounded subset of κ. Then κ has cofinality ω in $V[A]$.*

Proof. Work in V. Let $\underset{\sim}{A}$ be a name of A and $\langle s, S\rangle \in P(\mathbb{U})$. Suppose for simplicity that s is the empty sequence. Define by induction a subtree T of S. For each $\nu \in Lev_1(S)$ pick some subtree S'_ν of $S_{\langle\nu\rangle}$ and $a_\nu \subseteq \pi_{\langle\rangle}(\nu)$ such that

$$\langle\langle\nu\rangle, S'_\nu\rangle \Vdash \underset{\sim}{A} \cap \pi_{\langle\rangle}(\nu) = a_\nu.$$

Let $S(0)'$ be a subtree of S obtained be replacing $S_{\langle\nu\rangle}$ by S'_ν, for every $\nu \in Lev_1(S)$.
Consider the function $\nu \to a_\nu, (\nu \in Lev_1(S))$. By normality of $\pi_{\langle\rangle *}U_{\langle\rangle}$ it is easy to find $A(0) \subseteq \kappa$ and $T(0) \subseteq Lev_1(S(0)'), T(0) \in U_{\langle\rangle}$ such that $A(0) \cap \pi_{\langle\rangle}(\nu) = a_\nu$, for every $\nu \in T(0)$. Set the first level of T to be $T(0)$. Set $S(0)$ to be a subtree of $S(0)'$ obtained by shrinking the first level to $T(0)$.
Let now $\langle\nu_1, \nu_2\rangle \in Lev_2(S(0))$. So, $\pi_{\langle\nu_1\rangle}(\nu_2) > \nu_1$. Find a subtree S'_{ν_1,ν_2} of $(S(1)_{\langle\nu_1,\nu_2\rangle})$, and $a_{\nu_0,\nu_1} \subseteq \pi_{\langle\nu_1\rangle}(\nu_2)$ such that

$$\langle\langle\nu_1, \nu_2\rangle, \vec{S}'_{\nu_0,\nu_1}\rangle \Vdash \underset{\sim}{A} \cap \pi_{\langle\nu_1\rangle}(\nu_2) = a_{\nu_1,\nu_2}.$$

Let $S(1)'$ be a subtree of $S(0)$ obtained be replacing $S_{\langle\nu_1,\nu_2\rangle}$ by S'_{ν_1,ν_2}, for every $\langle\nu_1, \nu_2\rangle \in Lev_2(S(0))$.
Again, we consider the function $\nu \to a_\nu, (\nu \in S(1)'_{\nu_1})$. By normality of $\pi_{\langle\nu_1\rangle *}U_{\langle\nu_1\rangle}$ it is easy to find $A(\nu_1) \subseteq \kappa$ and $T(\nu_1) \subseteq (S(1)'_{\langle\nu_1\rangle}), T(\nu_1) \in U_{\langle\nu_1\rangle}$ such that $A(\nu_1) \cap \pi_{\langle\nu_1\rangle}(\nu) = a_{\nu_1,\nu}$, for every $\nu \in T(\nu_1)$.
Define the set of the immediate successors of ν_1 to be $T(\nu_1)$, i.e. $Suc_T(\nu_1) = T(\nu_1)$. Let $S(1)$ be a subtree of $S(1)'$ obtained this way.
This defines the second level of T. Continue similar to define further levels of T.
We will have the following property:

(*) for every $\langle\eta_1, ..., \eta_n\rangle \in T$,

$$\langle\langle\eta_1, ..., \eta_n\rangle, T_{\langle\eta_1,...,\eta_n\rangle}\rangle \Vdash \underset{\sim}{A} \cap \pi_{\langle\eta_1,...,\eta_{n-1}\rangle}(\eta_n) = A(\eta_1, ..., \eta_{n-1}) \cap \pi_{\langle\eta_1,...,\eta_{n-1}\rangle}(\eta_n).$$

A simple density argument implies that there is a condition which satisfies (*) in the generic set. Assume for simplicity that already $\langle\langle\rangle, T\rangle$ is such a condition. Then, $C \subseteq T^*$. Let $\langle\kappa_n \mid n < \omega\rangle = C$. So, for every $n < \omega$,

$$A \cap \pi_{\langle\kappa_0,...,\kappa_{n-1}\rangle}(\kappa_n) = A(\kappa_0, ..., \kappa_{n-1}) \cap \pi_{\langle\kappa_0,...,\kappa_{n-1}\rangle}(\kappa_n).$$

Let us work now in $V[A]$ and define by induction a sequence $\langle\eta_n \mid n < \omega\rangle$ as follows. Consider $A(0)$. It is a set in V, hence $A(0) \neq A$. So there is η such that for every $\nu \in Lev_1(T)$ with $\pi_{\langle\rangle}(\nu) \geq \eta$ we have $A \cap \pi_{\langle\rangle}(\nu) \neq A(0) \cap \pi_{\langle\rangle}(\nu)$. Set η_0 to be the least such η.
Turn to η_1. Let $\xi \in Lev_1(T)$ be such that $\pi_{\langle\rangle}(\xi) < \eta_0$. Consider $A(\xi)$. It is a set in V, hence $A(\xi) \neq A$. So there is η such that for every $\nu \in Lev_2(T_{\langle\xi\rangle})$ with $\pi_{\langle\xi\rangle}(\nu) \geq \eta$ we have $A \cap \pi_{\langle\xi\rangle}(\nu) \neq A(\xi) \cap \pi_{\langle\xi\rangle}(\nu)$. Set $\eta(\xi)$ to be the least such η. Now define η_1 to be $\sup(\{\eta(\xi) \mid \pi_1(\xi) < \eta_0\})$. The crucial point now is that the number of ξ's with $\pi_{\langle\rangle}(\xi) < \eta_0$ is less than κ, since $U_{\langle\rangle}$ is a P-point.

If $\eta_1 = \kappa$, then the cofinality of κ (in $V[A]$) is at most η_0. So it must be ω since the Prikry forcing used does not add new bounded subsets to κ, and we are done.
Let us argue however that this cannot happen and always $\eta_1 < \kappa$.

Claim 1 $\eta_1 < \kappa$.

Proof. Suppose otherwise. Then

$$\sup(\{\eta(\xi) \mid \pi_{\langle\rangle}(\xi) < \eta_0\}) = \kappa.$$

Hence for every $\alpha < \kappa$ there will be ξ with $\pi_{\langle\rangle}(\xi) < \eta_0$ such that

$$A \cap \alpha = A(\xi) \cap \alpha.$$

Then, for every $\alpha < \kappa$ there will be ξ, ξ' with $\pi_{\langle\rangle}(\xi), \pi_{\langle\rangle}(\xi') < \eta_0$ such that

$$A(\xi) \cap \alpha = A(\xi') \cap \alpha.$$

Now, in V, set $\rho_{\xi,\xi'}$ to be the least $\rho < \kappa$ such that

$$A(\xi) \cap \rho \neq A(\xi') \cap \rho,$$

if it exists and 0 otherwise, i.e. if $A(\xi) = A(\xi')$. Let

$$Z = \{\rho_{\xi,\xi'} \mid \pi_{\langle\rangle}(\xi), \pi_{\langle\rangle}(\xi') < \eta_0\}.$$

Then $|Z|^V < \kappa$, since the number of possible ξ, ξ' is less than κ. But Z should be unbounded in κ due to the fact that for every $\alpha < \kappa$ there will be ξ with $\pi_{\langle\rangle}(\xi) < \eta_0$ such that $A \cap \alpha = A(\xi) \cap \alpha$ and $A \neq A(\xi)$. Contradiction.

$\blacksquare_{of\ the\ claim}$

Suppose that $\eta_0, ..., \eta_n < \kappa$ are defined. Define η_{n+1}. Let $\langle \xi_0, ..., \xi_n \rangle$ be in T. Consider $A(\xi_0, ..., \xi_n)$. It is a set in V, hence $A(\xi_0, ..., \xi_n) \neq A$. So there is η such that for every $\nu \in Lev_{n+2}(T_{\langle \xi_0,...,\xi_n \rangle})$ with $\pi_{\langle \xi_0,...,\xi_n \rangle}(\nu) \geq \eta$ we have $A \cap \pi_{\langle \xi_0,...,\xi_n \rangle}(\nu) \neq A(\xi_0, ...\xi_n) \cap \pi_{\langle \xi_0,...,\xi_n \rangle}(\nu)$. Set $\eta(\xi_0, ...\xi_n)$ to be the least such η. Now define η_{n+1} to be $\sup(\{\eta(\xi_0, ...\xi_n) \mid \pi_{\langle\rangle}(\xi_0) < \eta_0, ..., \pi_{\langle \xi_0,...,\xi_{n-1} \rangle}(\xi_n) < \eta_n\})$.
Each relevant ultrafilter is a P-point, and so, the number of relevant $\xi_0, ...\xi_n$ is bounded in κ. So, $\eta_{n+1} < \kappa$, as in the claim above.

This completes the definition of the sequence $\langle \eta_n \mid n < \omega \rangle$.
Let us argue that it is cofinal in κ.
Suppose otherwise.
Note that the sequence $\langle \pi_{\langle \kappa_0,...,\kappa_{n-1} \rangle}(\kappa_n) \mid n < \omega \rangle$ is unbounded in κ.
Let k be the least such that $\pi_{\langle \kappa_0,...,\kappa_{k-1} \rangle}(\kappa_k) > \sup(\{\eta_n \mid n < \omega\})$. Then

$$A \cap \pi_{\langle \kappa_0,...,\kappa_{k-1} \rangle}(\kappa_k) = A(\kappa_0, ..., \kappa_{k-1}) \cap \pi_{\langle \kappa_0,...,\kappa_{k-1} \rangle}(\kappa_k).$$

This is impossible, since $\eta_k < \pi_{\langle \kappa_0,...,\kappa_{k-1} \rangle}(\kappa_k)$.

■

Theorem 6.7 *Let $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega}\rangle$ consists of P-point ultrafilters over κ. Then for every new set of ordinals A in $V^{P(\mathbb{U})}$, κ has cofinality ω in $V[A]$.*

Proof. Let A be a new set of ordinals in $V[G]$, where $G \subseteq P(\mathbb{U})$ is generic. By Lemma **??**, it is enough to find a new subset of A of size κ.
Suppose that every subset of A of size κ is in V. Let us argue that then A is in V as well. Let $\lambda = \sup(A)$.
The argument is similar to [**?**](Lemma 0.7).
Note that $(\mathcal{P}_{\kappa^+}(\lambda))^V$ remains stationary in $V[G]$, since $P(\mathbb{U})$ satisfies κ^+-c.c. For each $x \in (\mathcal{P}_{\kappa^+}(\lambda))^V$ pick $\langle s_x, S_x\rangle \in G$ such that

$$\langle s_x, S_x\rangle \Vdash \underset{\sim}{A} \cap x = A \cap x.$$

There are a stationary $E \subseteq (\mathcal{P}_{\kappa^+}(\lambda))^V$ and $s \in [\kappa]^{<\omega}$ such that for each $x \in E$ we have $s = s_x$. Now, in V, we consider

$$H = \{\langle s, T\rangle \in P(U) \mid \exists x \in \mathcal{P}_{\kappa^+}(\lambda) \exists a \subseteq x \quad \langle s, T\rangle \Vdash \underset{\sim}{A} \cap x = a\}.$$

Note that if $\langle s, T\rangle, \langle s, T'\rangle \in P(U)$ and for some $x \subseteq y$ in $\mathcal{P}_{\kappa^+}(\lambda)$, $a \subseteq x, b \subseteq y$ we have

$$\langle s, T\rangle \Vdash \underset{\sim}{A} \cap x = a \text{ and } \langle s, T'\rangle \Vdash \underset{\sim}{A} \cap y = b,$$

then $b \cap x = a$. Just conditions of this form are compatible, and so they cannot force contradictory information.
Apply this observation to H. Let

$$X = \{a \subseteq \lambda \mid \exists \langle s, S\rangle \in H \quad \exists x \in \mathcal{P}_{\kappa^+}(\lambda) \langle s, T\rangle \Vdash \underset{\sim}{A} \cap x = a\}.$$

Then necessarily, $\bigcup X = A$.

■*of the claim*

We do not know wether $V[A]$ for $A \in V[C] \setminus V$ is equivalent to a single $\omega-$sequence even for $A \subseteq \kappa^+$. The problematic case is once U_n's have κ^+-many different ultrafilters below in the Rudin-Keisler order.

Theorem 6.8 *Assume that there is no inner model with $o(\alpha) = \alpha^{++}$. Let U be $\kappa-$complete ultrafilter over κ and $V = L[\vec{E}]$, for a coherent sequence of measures $\vec{E}$. Force with the Prikry forcing with U. Suppose that A is a new set of ordinals in a generic extension. Then the cofinality of κ is ω in $V[A]$.*

Proof. Consider

$$i_U : V \to M \simeq V^\kappa/U.$$

By Mitchell [?], i_U is an iterated ultrapower using measures from $\vec{E}$ and images of $\vec{E}$. In addition we have that ${}^\kappa M \subseteq M$. Hence it should be a finite iteration using.
κ is the critical point, hence no measures below κ are involved and the first one applied is a measure on κ in $\vec{E}$. Denote it by E_0 and let

$$i_0 : V \to M_1$$

be the corresponding embedding. Let $\kappa_1 = i_0(\kappa)$. Rearranging, if necessary, we can assume that the next step was to use a measure E_1 over κ_1 from $i_0(\vec{E})$. So, it is either the image of one of the measures of $\vec{E}$ or $E_0 - Lim\langle E^\xi \mid \xi < \kappa\rangle$, where $\langle E^\xi \mid \xi < \kappa\rangle$ is a sequence of measures over κ from $\vec{E}$ which represents in M_1 the measure used over κ_1.
Let

$$i_1 : M_1 \to M_2$$

be the corresponding embedding and $\kappa_2 = i_1(\kappa_1)$.
κ_2 can be moved further in our iteration, but only finitely many times. Suppose for simplicity that it does not move.
If nothing else is moved then U is equivalent to $E_0 - Lim\langle E^\xi \mid \xi < \kappa\rangle$ and **??** easily provides the desired conclusion.
Suppose $i_1 \circ i_0$ is not i_U. Then some measures from $i_1 \circ i_0(\vec{E})$ with critical points in the intervals $(\kappa, \kappa_1), (\kappa_1, \kappa_2)$ are applied. Again, only finitely many can be used.
Thus suppose for simplicity that only one is used in each interval. The treatment of a general case is more complicated only due to notation.
So suppose that a measure E_2 with a critical point $\delta \in (\kappa, \kappa_1)$ is used on the third step of the iteration.
Let

$$i_2 : M_2 \to M_3$$

be the corresponding embedding. Note that the ultrafilter $\mathcal{V}$ defined by

$$X \in \mathcal{V} \text{ iff } i_2(\delta) \in i_2 \circ i_1 \circ i_0(X)$$

is $P-$point. Thus, a function $f : \kappa \to \kappa$ which represents δ in M_1, i.e. $\delta = i_0(f)(\kappa)$, will witness this.
Similar an ultrafilter used in the interval (κ_1, κ_2) will be $P-$point in M_1, and so, in V, it will be equivalent to a limit of $P-$points.
So such situation is covered by **??**.

$\blacksquare_{\textit{of the claim}}$

7 Prikry forcing may add a Cohen subset.

Our aim here will be to show the following:

Theorem 7.1 *Suppose that V satisfies GCH and κ is a measurable cardinal. Then in a generic cofinality preserving extension there is a $\kappa-$complete ultrafilter U over κ such that the Prikry forcing with U adds a Cohen subset to κ over V. In particular, this forcing has a non-trivial subforcing which preserves regularity of κ.*

By [**?**] such F cannot by normal and by 6.6 F cannot be a P-point ultrafilter, since in any Cohen extension, κ stays regular.

Note that the above situation is impossible in $L[\mu]$. Just every $\kappa-$complete ultrafilter over the measurable κ is Rudin-Kiesler equivalent to μ^n, for some $n, 1 \leq n < \omega$, by [**?**]. But the Prikry forcing with μ^n is the same as the Prikry forcing with μ which is a normal measure.

We start with a GCH model with a measurable. Let κ be a measurable and U a normal measure on κ.
Denote by $j_U : V \to N \simeq Ult(V, U)$ the corresponding elementary embedding.

Define an iteration $\langle P_\alpha, Q_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ with Easton support as follows. Set $P_0 = 0$. Assume that P_α is defined. Set $\underset{\sim}{Q}_\alpha$ to be the trivial forcing unless α is an inaccessible cardinal.
If α is an inaccessible cardinal, then let $Q_\alpha = Q_{\alpha 0} * \underset{\sim}{Q}_{\alpha 1}$, where $Q_{\alpha 0}$ is an atomic forcing consisting of three elements $0_{Q_{\alpha 0}}, x_\alpha, y_\alpha$, such that $\widetilde{x_\alpha}, y_\alpha$ are two incompatible elements which are stronger than $0_{Q_{\alpha 0}}$.
Let $\underset{\sim}{Q}_{\alpha 1}$ be trivial once y_α is picked and let it be the Cohen forcing at α, i.e.

$$Cohen(\alpha, 2) = \{f : \alpha \to 2 \mid |f| < \alpha\}$$

once x_α was chosen.

Let $G_\kappa \subseteq P_\kappa$ be a generic. We extend now the embedding

$$j_U : V \to N,$$

in $V[G_\kappa]$, to

$$j^*_U : V[G_\kappa] \to N[G_\kappa * G_{[\kappa, j_U(\kappa))}],$$

for some $G_{[\kappa, j_U(\kappa))} \subseteq P_{[\kappa, j_U(\kappa))}$ which is $N[G_\kappa]-$generic for $P_{j_U(\kappa)}/G_\kappa$. This can be done easily, once over κ itself in $Q_{\kappa 0}$, we pick y_κ, which makes the forcing Q_κ a trivial one.
This shows, in particular, that κ is still a measurable in $V[G_\kappa]$, as witnessed by an extension of U.

Consider now the second ultrapower $N_2 \simeq \mathrm{Ult}(N, j_U(U))$.
Denote j_U by j_1, N by N_1. Let

$$j_{12} : N_1 \to N_2$$

denotes the ultrapower embedding of N_1 by $j_1(U)$. Let $j_2 = j_{12} \circ j_1$. Then

$$j_2 : V \to N_2.$$

Let us extend, in $V[G_\kappa]$, the embedding

$$j_{12} : N_1 \to N_2$$

to

$$j_{12}^* : N_1[G_\kappa * G_{[\kappa, j_1(\kappa))}] \to N_2[G_\kappa * G_{[\kappa, j_1(\kappa))} * G_{[j_1(\kappa), j_2(\kappa))}]$$

in a standard fashion, only this time we pick $x_{j_1(\kappa)}$ at stage $j_1(\kappa)$ of the iteration. Then a Cohen function should be constructed over $j_1(\kappa)$, which is not at all problematic to find in $V[G_\kappa]$.

Now we will have

$$j_2 \subseteq j_2^* : V[G_\kappa] \to N_2[G_\kappa * G_{[\kappa, j_1(\kappa))} * G_{[j_1(\kappa), j_2(\kappa))}]$$

which is the composition of j_1^* with j_{12}^*.

Define a $\kappa-$complete ultrafilter W over κ as follows:

$$X \in W \text{ iff } X \subseteq \kappa \text{ and } j_1(\kappa) \in j_2^*(X).$$

Proposition 7.1 *W has the following basic properties:*

1. $W \cap V = U$,
2. $\{\alpha < \kappa \mid x_\alpha$ *was picked at the stage* α *of the iteration* $\} \in W$,
3. *if* $C \subseteq \kappa$ *is a club, then* $C \in W$. *Moreover*

$$\{\nu \in C \mid \nu \textit{ is an inaccessible}\} \in W.$$

Proof:
(1) and (2) are standard. Let us show only (3). Let $C \subseteq \kappa$ be a club. Then, in N_2, $j_2(C)$ is a club at $j_2(\kappa)$. In addition, $j_2(C) \cap \kappa_1 = j_1(C)$. Now, $j_1(C)$ is a club in $j_1(\kappa)$. It follows that $j_1(\kappa) \in j_2(C)$.
In order to show that

$$\{\nu \in C \mid \nu \text{ is an inaccessible}\} \in W,$$

just note that $j_1(\kappa)$ is an inaccessible in N_2, and so W concentrates on inaccessibles.

■

Force with $Prikry(W)$ over $V[G_\kappa]$.
Let

$$C = \langle \eta_n \mid n < \omega \rangle$$

be a generic Prikry sequence.
By (2) in the previous proposition, there is $n^* < \omega$ such that for every $m \geq n^*$, at the stage η_m of the forcing P_κ, x_{η_m} was picked, and, hence, a Cohen function $f_{\eta_m} : \eta_m \to 2$ was added.

Define now $H : \kappa \to 2$ in $V[G_\kappa, C]$ as follows:

$$H = f_{\eta_{n^*}} \cup \bigcup_{n^* \leq m < \omega} f_{\eta_{m+1}} \restriction [\eta_m, \eta_{m+1}).$$

Proposition 7.2 *H is a Cohen generic function for κ over $V[G_\kappa]$.*

Proof Work in $V[G_\kappa]$. Let $D \in V[G_\kappa]$ be a dense open subset of $Cohen(\kappa)$. Consider a set

$C = \{\alpha < \kappa \mid$ if α is an inaccessible, then $D \cap V_\alpha[G_\alpha]$ is a dense open subset of $Cohen(\alpha)$ in $V[G_\alpha]\}$.

Claim 1 C is a club.

Proof. Suppose otherwise. Then $S = \kappa \setminus C$ is stationary. It consists of inaccessible cardinals by the definition of C.
Pick a cardinal χ large enough and consider an elementary submodel X of $\langle H_\chi, \in \rangle$ such that

1. $X \cap (V_\kappa)^{V[G_\kappa]} = (V_\delta)^{V[G_\kappa]}$, for some $\delta \in S$,
2. $\kappa, P_\kappa, D \in X$

Note that it is possible to find such X due to stationarity of S. Note also that $(V_\kappa)^{V[G_\kappa]} = V_\kappa[G_\kappa]$ and $(V_\delta)^{V[G_\kappa]} = V_\delta[G_\delta]$, since the iteration P_κ splits nicely at inaccessibles.

Let us argue that $D \cap V_\delta[G_\delta]$ is a dense open subset of $Cohen(\delta)$ in $V[G_\delta]$.
Just note that

$$D \cap X = D \cap X \cap (V_\kappa)^{V[G_\kappa]} = D \cap (V_\delta)^{V[G_\kappa]} = D \cap V_\delta[G_\delta].$$

So let $q \in (Cohen(\delta))^{V_\delta[G_\delta]}$. Then $q \in X$. Remember $X \preceq H_\chi$. So,

$$X \models D \text{ is dense open },$$

hence there is $p \geq q, p \in D \cap X$. But then, $p \in D \cap V_\delta[G_\delta]$, and we are done.
Contradiction.

$\blacksquare_{of\ claim}$

It follows now that $C \in W$. Hence there is $n^{**} \geq n^*$ such that for every $m, n^{**} \leq m < \omega$,

$$\eta_m \in C.$$

So, for every $m, n^{**} \leq m < \omega$,

$$f_{\eta_m} \in D,$$

since D is open.
It is almost what we need, however $H \restriction \eta_m$ need not be f_{η_m}, since an initial segment may was changed.
In order to overcome this, let us note the following basic property of the Cohen forcing:

Claim 2 Let E be a dense open subset of $Cohen(\kappa, 2)$, then there is a dense subset E^* of E such that for every $p \in E^*$ and every inaccessible cardinal $\tau \in \text{dom}(p)$ for every $q : \delta \to 2, p \restriction [\delta, \kappa) \cup q \in E^*$.

The proof is an easy use of $\kappa-$completeness of the forcing.

Now we can finish just replacing D by its dense subset which satisfies the conclusion of the claim. Then, $H \restriction \eta_m$ will belong to it as a bounded change of f_{η_m}.
So we are done.

$\blacksquare$

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