A note on blowing up powers of measurable cardinals

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Abstract

We examine the known constructions for blowing up the power of a measurable cardinal and exploit differences in order to answer questions of P. Lücke and S. Müller from [5] and of P. Lücke.

1 Introduction

The purpose of this note is to answer the following questions:

1. (P. Lücke and S. Müller [5]) *Is it consistent that there exist normal ultrafilters* U_0 *and* U_1 *on a measurable cardinal* κ *such that there is a limit ordinal* λ *with the property that no unbounded subset of* λ *is fresh over* $\mathrm{Ult}(V, U_0)$ *and there exists an unbounded subset of* λ *that is fresh over* $\mathrm{Ult}(V, U_1)$*?*

2. (P. Lücke) *Let* U_0, U_1 *be two* $\kappa-$*complete non-principal ultrafilters over a measurable cardinal* κ*. Let* $j_{U_0} : V \to M_{U_0}, j_{U_1} : V \to M_{U_1}$ *be the corresponding elementary embeddings. Is it possible to have a limit ordinal* α *such that* $\mathrm{cof}(j_{U_0}(\alpha)) \neq \mathrm{cof}(j_{U_1}(\alpha))$*?*

We will use small modifications of well known methods for blowing powers of measurable cardinals. An excellent exposition of the subject can be found in J. Cummings handbook article [2].

2 On fresh sets in the ultrapower

P. Lücke and S. Müller asked the following question in [5]:

Is it consistent that there exist normal ultrafilters U_0 *and* U_1 *on a measurable cardinal* κ *such that there is a limit ordinal* λ *with the property that no unbounded subset of* λ *is fresh over* $\mathrm{Ult}(V, U_0)$ *and there exists an unbounded subset of* λ *that is fresh over* $\mathrm{Ult}(V, U_1)$*?*

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Our aim here will be to present a construction that gives an affirmative answer to the question even with $\lambda = \kappa^+$.

Start with a GCH model with a $(\kappa, \kappa^{++})-$extender E. Force over it Cohen function for each inaccessible $\alpha \leq \kappa$.
Then in the extension, which we denote still by V, there will be two (actually many) $(\kappa, \kappa^{++})-$extenders E_0, E_1 which extend E and such that, for every $i < 2$,

1. $E_0(\kappa) \neq E_1(\kappa)$, where $E_i(\alpha) = \{X \subseteq \kappa \mid \alpha \in j_{E_i}(X)\}$, $\alpha < \kappa^{++}$,
2. $M_i = M_{E_i} = \text{Ult}(V, E_i)$ is closed under $\kappa-$sequences of its elements,
3. $V_{\kappa+2} \subseteq M_i$.

It is easy to obtain such situation even starting from $o(\kappa) = \kappa^{++} + 1$.

Pick some $A \in E_0(\kappa) \setminus E_1(\kappa)$ consisting of inaccessible cardinals.

Define by induction an iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha < \kappa + 2, \beta < \kappa + 1\rangle$.

In V^{P_α}, let Q_α will be trivial unless α is an inaccessible.
Suppose that α is an inaccessible.
If $\alpha \notin A$, then let Q_α be the Cohen forcing $Cohen(\alpha, \alpha^{++})$ for blowing up the power of α to α^{++}.
If $\alpha \in A$, then let Q_α be $(Cohen(\alpha, \alpha^+) * Cohen(\alpha^+, \alpha^{++})) \times Cohen(\alpha, \alpha^{++})$.
Finally set $Q_\kappa = (Cohen(\kappa, \kappa^+) * Cohen(\kappa^+, \kappa^{++})) \times Cohen(\kappa, \kappa^{++})$.

Let G be a generic subset of $P_{\kappa+1}$. $V[G]$ will be a desired model.
Denote $G \cap P_\alpha$ by G_α.

By arguments of H. Woodin (see Cummings handbook article [2] or [3]), the ultrapower embeddings $j_{E_i} : V \to M_{E_i}, i < 2$ extend.
More specifically, j_{E_0} extends to $j_0^* : V[G] \to M_0[G, G_{(\kappa^{++}, j_{E_0}(\kappa)+1]}] = M_0^*$.
It is not hard to see that $M_0^* \supseteq \mathcal{P}(\kappa^+)$ of $V[G]$ and that it is an ultrapower by a normal measure over κ which extends $E_0(\kappa)$[1].

Now, j_{E_1} extends to $j_1^* : V[G] \to M_1[G_\kappa, G \cap (Cohen(\kappa, \kappa^+) \times Cohen(\kappa, \kappa^{++})), H] = M_1^*$, where H is $M_1[G_\kappa, G \cap (Cohen(\kappa, \kappa^+) \times Cohen(\kappa, \kappa^{++}))]$- generic for the continuation of $j_{E_1}(P)$ above κ^{++}. Note that this part does not add no new subsets to κ^{++} (over $M_1[G_\kappa, G \cap$

[1] It is possible just to change values of Cohen functions $f_{j_{E_0}(\kappa) j_{E_0}(\alpha)}$ at κ to α, for every $\alpha < \kappa^{++}$ in order to "capture" all the generators of E_0.
Omer Ben Neria pointed out that actually there is no need in this change. Thus, for every $\alpha < \kappa^{++}$, we can consider a function $h_\alpha : \kappa \to \kappa$ defined by setting $h_\alpha(\nu) = \min(\{\gamma < \nu^{++} \mid f_{\kappa\alpha} \restriction \nu = f_{\nu\gamma}\})$, if exists and 0 otherwise. Then, $j^*_{E_1}(h_\alpha)(\kappa) = \alpha$, since $f_{j(\kappa) j(\alpha)} \restriction \kappa = f_{\kappa\alpha}$.

$(Cohen(\kappa, \kappa^+) \times Cohen(\kappa, \kappa^{++}))]$).
Again it is possible to argue that M_1^* is an ultrapower by a normal measure over κ which extends $E_1(\kappa)$.
However, the Cohen subsets of κ^+ which were added by $G \cap Cohen(\kappa^+, \kappa^{++})$ are missing there.

The consistency strength that was used for the construction above is $o(\kappa) = \kappa^{++} + 1$. Let us argue that it is optimal.

Proposition 2.1 *Assume* $\neg o^\P$. *Suppose that* κ *is a measurable cardinal,* $2^\kappa > \kappa^+$ *and for some normal ultrafilter* U *over* κ, $M_U \supseteq \mathcal{P}(\kappa^+)$.
Then $o(\kappa) \geq \kappa^{++} + 1$.

Proof. Suppose otherwise. Then, necessarily, $o(\kappa) \geq \kappa^{++}$. Let $\mathcal{K}$ be the core model. By W. Mitchell [6] (or in a more general setting, by R. Schindler [7]), $j_U \restriction \mathcal{K} : \mathcal{K} \to \mathcal{K}^{M_U}$ is an iterate ultrapower of $\mathcal{K}$ by its measures. Let $U(\kappa, \alpha)$ be the first measure used in this ultrapower. Then $U(\kappa, \alpha)$ cannot be in M_U, since the core model of M_U is $\mathcal{K}^{M_U}$ and $U(\kappa, \alpha)$ is not on the sequence there.
$\square$

3 On cofinality in ultrapowers

It was shown in [4], answering a question of D. Fremlin, that:

If U_0, U_1 *are two* $\kappa-$*complete ultrafilters over a measurable cardinal* κ *then for every ordinal* α, $|j_{U_0}(\alpha)| = |j_{U_1}(\alpha)|$.

Philipp Lücke asked the following natural question:

Let U_0, U_1 *be two* $\kappa-$*complete non-principal ultrafilters over a measurable cardinal* κ. *Let* $j_{U_0} : V \to M_{U_0}, j_{U_1} : V \to M_{U_1}$ *be the corresponding elementary embeddings.*
Is it possible to have a limit ordinal α *such that* $\mathrm{cof}(j_{U_0}(\alpha)) \neq \mathrm{cof}(j_{U_1}(\alpha))$*?*

The following is immediate:

Proposition 3.1 *Let* U *be a* $\kappa-$*complete non-principal ultrafilter over a measurable cardinal* κ *and* $j_U : V \to M$ *the corresponding elementary embedding. Let* α *be a limit ordinal. Then the following hold:*

1. *if* $\mathrm{cof}(\alpha) < \kappa$, *then* $\mathrm{cof}(j_U(\alpha)) = \mathrm{cof}(\alpha)$;

2. *if* $\mathrm{cof}(\alpha) > \kappa$, *then* $j_U{}''\alpha$ *is cofinal in* $j_U(\alpha)$, *and so,* $\mathrm{cof}(j_U(\alpha)) = \mathrm{cof}(\alpha)$;

3. *if* $\mathrm{cof}(\alpha) = \kappa$, *then* $\mathrm{cof}(j_U(\alpha)) = \mathrm{cof}(j_U(\kappa)) \geq \kappa^+$.

In particular, the only possibility to have $\mathrm{cof}(j_{U_0}(\alpha)) \neq \mathrm{cof}(j_{U_1}(\alpha))$ is when $\mathrm{cof}(j_{U_0}(\kappa)) \neq \mathrm{cof}(j_{U_1}(\kappa))$.
Note that if $2^\kappa = \kappa^+$, then for any $\kappa-$complete non-principal ultrafilter U over κ, $|j_U(\kappa)| = \mathrm{cof}(j_U(\kappa)) = \kappa^+$, since M_U is closed under $\kappa-$sequences of its elements.

Our aim here will be to construct a model in which:

1. $2^\kappa = \kappa^{++}$,

2. there are U_0, U_1 two normal ultrafilters over a measurable cardinal κ such that $\mathrm{cof}(j_{U_0}(\kappa)) \neq \mathrm{cof}(j_{U_1}(\kappa))$.

Suppose V satisfies GCH and κ is a κ^+-supercompact cardinal.
Let W be a witnessing normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$ and $j_W : V \to M_W$ the corresponding elementary embedding.
It is easy to see, using GCH, that the following hold:

1. $(\kappa^{++})^{M_W} = \kappa^{++}$,

2. $(\kappa^{++})^{M_W} < j_W(\kappa) < \kappa^{+3}$,

3. $\mathrm{cof}(j_W(\kappa)) = \kappa^{++}$.

Let us derive two extenders from j_W - a $(\kappa, \kappa^{++})-$extender E_0 and a $(\kappa, j_W(\kappa))-$extender E_1. Namely, for every $a \in [\kappa^{++}]^{<\omega}$ and $X \subseteq [\kappa]^{|a|}$,

$$X \in E_0(a) \text{ iff } a \in j_W(X),$$

and, for every $a \in [j_W(\kappa)]^{<\omega}$ and $X \subseteq [\kappa]^{|a|}$,

$$X \in E_1(a) \text{ iff } a \in j_W(X).$$

Let $j_{E_i} : V \to M_{E_i}, i < 2$ be the corresponding elementary embeddings.
Define $k_i : M_{E_i} \to M_W$ by setting

$$k_i(j_{E_i}(f)(a)) = j_W(f)(a).$$

Lemma 3.2 $\mathrm{cof}((\kappa^{+3})^{M_{E_0}}) = \kappa^+$, *and so,* $(\kappa^{+3})^{M_{E_0}}$ *is a critical point of* k_0.

Proof. The point is that the generators of E_0 are in the interval $[\kappa, \kappa^{++})$ only. Hence, if $j_{E_0(\kappa)} : V \to M_{E_0(\kappa)}$ is the ultrapower by the normal measure of E_0 and $k_{E_0(\kappa),E_0} : M_{E_0(\kappa)} \to M_{E_0}$ is defined by setting

$$k_{E_0(\kappa),E_0}(j_{E_0(\kappa)}(f)(\kappa)) = j_{E_0}(f)(\kappa),$$

then

$$k_{E_0(\kappa),E_0}{}''(\kappa^{+3})^{M_{E_0(\kappa)}} \text{ is unbounded in } (\kappa^{+3})^{M_{E_0}}.$$

□

The next lemma is similar:

Lemma 3.3 $\mathrm{cof}(j_{E_0}(\kappa)) = \kappa^+$.

Turn now to a longer extender E_1.

The following follows from the definition:

Lemma 3.4 *M_{E_1} agrees with M_W up to $j_W(\kappa)$.*

Lemma 3.5 $\mathrm{cof}(j_{E_1}(\kappa)) = \kappa^{++}$.

Proof. Just, $j_{E_1}(\kappa) = j_W(\kappa)$, and, by GCH, $\mathrm{cof}(j_W(\kappa)) = \kappa^{++}$.

□

Lemma 3.6 $\mathrm{cof}((j_{E_1}(\kappa)^+)^{M_{E_1}}) = \kappa^+$, *and so, $(j_{E_1}(\kappa)^+)^{M_{E_1}}$ is a critical point of k_1.*

Proof. Just note that

$$j_{E_1}{}''\kappa^+ \text{ is unbounded in } (j_{E_1}(\kappa)^+)^{M_{E_1}},$$

since every function from $V_\kappa \to \kappa^+$ is dominated by a constant function.

□

Now let us force $2^\kappa = \kappa^{++}$. Just iterate the Cohen forcing which adds $\eta^{++}-$Cohen functions $\langle f_{\eta\beta} \mid \beta < \eta^{++} \rangle$ from η to η for every inaccessible $\eta \leq \kappa$. Let G be a generic set.

By the Woodin argument, with an addition of Yoav Ben Shalom [1], j_{E_0} extends to elementary embedding $j_0^* : V[G] \to M_{E_0}[G^*]$ which is just an ultrapower embedding by a normal ultrafilter U_0 over κ extending $E_0(\kappa)$.

In particular, $\mathrm{cof}(j_0^*(\kappa)) = \kappa^+$, since $j_0^*(\kappa) = j_{E_0}(\kappa)$.

Turn now to W and E_1. Use the Silver method to extend j_W.
So we will have

$$j_W^* : V[G] \to M_W[G * H * \langle f_{j_W(\kappa)\beta} \mid \beta < (j_W(\kappa))^{++})^{M_W}\rangle],$$

where H is $M_W[G]-$generic for the iteration in the interval $(\kappa, j_W(\kappa))$ and $\langle f_{j_W(\kappa)\beta} \mid \beta < (j_W(\kappa))^{++})^{M_W}\rangle$ are Cohen functions for $j_W(\kappa)$. By the master condition,

$$f_{j_W(\kappa)j_W(\alpha)} \restriction \kappa = f_{\kappa\alpha},$$

for every $\alpha < \kappa^{++}$. Recall that $|j_W(\kappa)| = \kappa^{++}$. Pick, in V, an enumeration $\langle \tau_\nu \mid \nu < \kappa^{++}\rangle$ of $j_W(\kappa)$.
Next, we change $\langle f_{j_W(\kappa)\beta} \mid \beta < (j_W(\kappa))^{++})^{M_W}\rangle$ to $\langle f'_{j_W(\kappa)\beta} \mid \beta < (j_W(\kappa))^{++})^{M_W}\rangle$ as follows: set $f'_{j_W(\kappa)\beta} = f_{j_W(\kappa)\beta}$ unless β is not of the form $j_W(\alpha)$, for some $\alpha < \kappa^{++}$.
If $\beta = j_W(\alpha)$, then let $f'_{j_W(\kappa)\beta}(\xi) = f_{j_W(\kappa)\beta}(\xi)$, for every $\xi \neq \kappa$,
and set $f'_{j_W(\kappa)\beta}(\kappa) = \tau_\alpha$.[2]

Note that such defined sequence $\langle f'_{j_W(\kappa)\beta} \mid \beta < (j_W(\kappa))^{++})^{M_W}\rangle$
remains $M_W[G * H]-$generic sequence of Cohen functions, since $j_W{}''\kappa^{++}$ is unbounded in $j_W(\kappa^{++})$ and the Cohen forcing satisfies $j_W(\kappa^+)-$c.c. in $M_W[G * H]$.

Now deal with E_1. We would like to extend j_{E_1} to $j_{E_1}^* : V[G] \to M_{E_1}[R]$.
Let us use G, H to build $M_{E_1}[G * H]$.
The elementary embedding $k_1 : M_{E_1} \to M_W$ easily extends to $k_1^* : M_{E_1}[G, H] \to M_W[G, H]$.
Deal with the remaining part - the Cohen functions, as follows. Consider $k_1'' j_{E_1}(\kappa^{++})$. For every $\zeta < j_{E_1}(\kappa^{++})$, set $g_\zeta = f'_{j_W(\kappa)k_1(\zeta)}$.
Then $\langle g_\zeta \mid \zeta < j_{E_1}(\kappa^{++})\rangle$ will be the desired $M_{E_1}[G, H]-$generic sequence of Cohen functions.
Also k_1 extends to

$$k_1^{**} : M_{E_1}[G, H, \langle g_\zeta \mid \zeta < j_{E_1}(\kappa^{++})\rangle] \to M_W[G, H, \langle f'_{j_W(\kappa)\beta} \mid \beta < (j_W(\kappa))^{++})^{M_W}\rangle].$$

We have then $j_{E_1}^* : V[G] \to M_{E_1}[[G, H, \langle g_\zeta \mid \zeta < j_{E_1}(\kappa^{++})\rangle]]$.
Finally note this is just the ultrapower embedding by a normal measure U_1 over κ which extends $E_1(\kappa)$, since every generator of E_1 is now of the form $j_{E_1}^*(f_{\kappa\alpha})(\kappa)$, for some $\alpha < \kappa^{++}$.
In addition, we have $j_W(\kappa) = j_{E_1}(\kappa) = j_{U_1}(\kappa)$ has cofinality κ^{++}.

It is possible to obtain the above starting from $o(\kappa) = \kappa^{++}$. The argument of [3] allows to use the iterated ultrapower by all the measures over κ twice or one can stop after say κ^+ many steps. This way it is possible to insure that in the final generic extension we will have measures with $\mathrm{cof}(j(\kappa)) = \kappa^{++}$ and κ^+.

[2]Here this change is essential. Using [1], it is possible to argue that without it generators of E_1 above κ^{++} may be lost.

References

[1] Y. Ben Shalom, On the Woodin Construction of Failure of GCH at a Measurable Cardinal, Master thesis, Tel Aviv University, 2017, arXiv:1706.08143

[2] J. Cummings, Iterated forcing and elementary embeddings, in Handbook of Set Theory. Editors: Foreman, Matthew, Kanamori, Akihiro (Eds.), Springer 2010, 775-884.

[3] M. Gitik, On not SCH from $o(\kappa) = \kappa^{++}$. The negation of singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$, Ann. Pure Appl. Log. 43, 209–234. (1989)

[4] M. Gitik and S. Shelah, On elementary embeddings by complete ultrafilters, Ann. Pure Appl. Logic, 164(9), 855–865.

[5] P. Lücke and S. Müller,
Closure properties of measurable ultrapowers, arXiv:2009.09530.

[6] W. Mitchell, The covering lemma, in Handbook of Set Theory. Editors: Foreman, Matthew, Kanamori, Akihiro (Eds.), Springer 2010, 1497-1594.

[7] R. Schindler, Iterates of the core model, Journal of Symbolic Logic 71 (2006), 241-251.