ADDING MANY ω-SEQUENCES TO A SINGULAR CARDINAL

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1. Introduction

These notes are based on a lecture given by Moti Gitik at the Appalachian Set Theory workshop on April 3, 2010. Spencer Unger was the official note-taker. During the lecture Gitik presented many forcings for adding ω-sequences to a singular cardinal of cofinality ω. The goal of these notes is to provide the reader with an introduction to the main ideas of a result due to Gitik.*

Theorem 1. *Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence with each κ_n κ_n^{+n+2}-strong, and $\kappa =_{def} \sup_{n<\omega} \kappa_n$. There is a cardinal preserving forcing extension in which no bounded subsets of κ are added and $\kappa^\omega = \kappa^{++}$.*

In order to present this result, we approach it by proving some preliminary theorems about different forcings which capture the main ideas in a simpler setting. For the entirety of the notes we work with an increasing sequence of large cardinals $\langle \kappa_n \mid n < \omega \rangle$ with $\kappa =_{def} \sup_{n<\omega} \kappa_n$. The large cardinal hypothesis that we use varies with the forcing. A recurring theme is the idea of a cell. A cell is a simple poset which is designed to be used together with other cells to form a large poset. Each of the forcings that we present has ω-many cells which are put together in a canonical way to make the forcing.

First, we will present Diagonal Prikry Forcing which adds a single cofinal ω-sequence to κ. The key property that we wish to present with this forcing is the *Prikry condition*. The Prikry condition is the property of the forcing that allows us to show that no bounded subsets of κ are added. All subsequent forcings share this property. We also show that this forcing preserves cardinals and cofinalities above κ using a chain condition argument.

Second, we present a forcing for adding λ-many ω-sequences to κ using long extenders. This forcing can be seen as both a more complicated version of Diagonal Prikry forcing and an approximation of the forcing used to prove Theorem 1. We want to repeat many of the arguments from the first poset. Each argument becomes more difficult. We sketch a proof of the Prikry condition and mention a strengthening needed to show that κ^+ is preserved. The chain condition argument is also more difficult and we sketch a proof of this too.

Third, we present the forcing from Theorem 1. To do this we present an attempt at a definition, which we ultimately refine to obtain the actual forcing. This is instructive because our attempt at a definition is similar to the forcing with long

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*During the workshop, Gitik also discussed the preparation forcing needed to add κ^{+++} ω-sequences to a singular cardinal κ, but we omit that discussion here. The original account of this forcing can be found in [3] and a revised version that is closer to the presentation given at the workshop can be found in [1].

extenders and it is straightforward to see that they share many properties. In particular this attempted definition satisfies the Prikry condition. However there is a problem with the chain condition. We explain where the argument goes awry and present a revised definition which allows us to recover the chain condition.

Diagonal Prikry forcing first appeared in [7]. The second forcing originally appeared in [5]. A full presentation of the first and the second forcing as they appear in these notes can be found in [4]. Theorem 1 originally appeared in [2], but in a slightly different form. The presentation of the third forcing in these notes is closer to the presentation found in [6].

2. Adding a single ω-sequence

For this section we assume that each κ_n is measurable and let U_n be a nonprincipal κ_n-complete ultrafilter on κ_n. As mentioned in the introduction our forcing is made up of ω-many cells Q_n for $n < \omega$. Each cell has two parts Q_{n0} and Q_{n1}. Fixing $n < \omega$ we set $Q_{n1} =_{def} \kappa_n$ and $Q_{n0} =_{def} U_n$. While Q_{n1} can be ordered by ordinal less than, we will not use this ordering. More importantly, we define an order on Q_{n0}, $\leq_{Q_{n0}}$ by $A \geq_{Q_{n0}} B$ if and only if $A \subseteq B$. Our forcing convention is $A \geq B$ means that A is stronger than B.

Now we are ready to define the n^{th} cell Q_n. We set $Q_n =_{def} Q_{n0} \cup Q_{n1}$ and define two orderings $\leq_n$ and $\leq_n^*$. $\leq_n$ is the ordering that we force with. $\leq_n^*$ will be a notion of *direct extension*, which is an important subordering of $\leq_n$. We define direct extension first. For $p, q \in Q_n$, let $p \leq_n^* q$ if and only if either $p = q$ or $p, q \in Q_{n0}$ and $p \leq_{Q_{n0}} q$. Now we define $p \leq_n q$ if and only if either $p \leq_n^* q$ or $p \in Q_{n0}, q \in Q_{n1}$ and $q \in p$. There are two ways to extend in the $\leq_n$ relation. Starting with a measure one set we are allowed to pick another measure one set contained in the first, but then eventually we must pick an ordinal from the current measure one set. Once we pick an ordinal no further extensions are possible. Next we define the Prikry condition, which shows the purpose of the direct extension relation.

Definition 2. *A forcing $\langle P, \leq, \leq^* \rangle$ where $\leq^* \subseteq \leq$ satisfies the Prikry condition if and only if for all $p \in P$ and for all statements σ in the forcing language, there is $q \,{}^*\!\geq p$, $q \parallel \sigma$.*

Proposition 3. *$\langle Q_n, \leq_n, \leq_n^* \rangle$ satisfies the Prikry condition.*

Proof. Let $p \in Q_n$ and σ be in the forcing language. If $p \in Q_{n1}$ then we are done since p decides all statements in the forcing language. Suppose that $p \in Q_{n0}$. We have that $p \in U_n$. Let $A_0 = \{\nu \in p \mid \nu \Vdash \sigma\}$ and $A_1 = \{\nu \in p \mid \nu \Vdash \neg\sigma\}$. A_0, A_1 partition p. So one of them must be in U_n. Without loss of generality assume that $A_0 \in U_n$. Then $A_0 \,{}^*\!\geq p$ and $A_0 \Vdash \sigma$. □

Note that the forcing Q_n is equivalent to the trivial forcing. We now put the Q_n's together to define a forcing notion P.

Definition 4. *$p \in P$ if and only if $p = \langle p_n \mid n < \omega \rangle$ such that for all $n < \omega$, $p_n \in Q_n$ and there is $\ell(p) < \omega$ such that for all $n \geq \ell(p)$, $p_n \in Q_{n0}$ and for all $n < \ell(p)$, $p_n \in Q_{n1}$. Let $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle$ be members of P. Set $p \geq q$ if and only if for all $n < \omega$, $p_n \geq_n q_n$. And set $p \,{}^*\!\geq q$ if and only if for all $n < \omega$, $p_n \,{}^*\!\geq_n q_n$.*

Lemma 5. *$\langle P, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. Let $p \in P$ with $p = \langle p_n \mid n < \omega\rangle$. Let σ be a statement in the forcing language. We need $p^* \,{}^*\!\geq p$ such that $p^* \parallel \sigma$. Suppose that there is no such p^*. We will construct a direct extension of p so that no extension decides σ, a contradiction. Assume for simplicity that $\ell(p) = 0$, ie that for all $n < \omega$, $p_n \in U_n$. By induction we construct an $\leq^*$-increasing sequence of conditions $\langle t(n) \mid n < \omega\rangle$ above p with the property that for all n and $s \geq t$ with $\ell(s) = n$, s does not decide σ.

For the base case we set $t(0) = p$ and note that our assumption for a contradiction shows that $t(0)$ has the desired property. Assume that we have constructed $t(n)$ as claimed. We work to construct $t(n+1)$. Let $\vec{\nu}$ be a sequence of length n such that $t(n)^\frown\vec{\nu} =_{def} \langle \nu_0, \nu_1, \ldots \nu_{n-1}, t(n)_n, t(n)_{n+1}, \ldots\rangle \geq t(n)$ (If $n = 0$, then the empty sequence is the only such $\vec{\nu}$.) For each one point extension of $t(n)^\frown\vec{\nu}$, we wish to capture a direct extension which decides σ, if one exists.

Let $\langle \nu_\eta \mid \eta < \kappa_n\rangle$ enumerate $t(n)_n$ in increasing order. We inductively construct an increasing sequence of direct extensions of $t(n)$, $\langle p_{\vec{\nu}}(\eta) \mid \eta < \kappa_n\rangle$. We set $p_{\vec{\nu}}(0) =_{def} t(n)$. Suppose that we have constructed $p_{\vec{\nu}}(\eta)$ for some η. If there is a direct extension of $p_{\vec{\nu}}(\eta)^\frown\vec{\nu}^\frown\nu_\eta$ that decides σ, then let q be such an extension with $q = \langle q_k \mid k < \omega\rangle$. We set $p_{\vec{\nu}}(\eta+1) =_{def} \langle t(n)_0, t(n)_1, \ldots t(n)_n, q_{n+1}, q_{n+2}, \ldots\rangle$. If $\gamma < \kappa_n$ is limit we let $p_{\vec{\nu}}(\gamma)_k =_{def} t(n)_k$ for $k \leq n$ and $p_{\vec{\nu}}(\gamma)_k =_{def} \cap_{\eta<\gamma} p_{\vec{\nu}}(\eta)_k$ for $k > n$. Note that for $k > n$, U_k is κ_k-complete and $\kappa_k > \kappa_n > \gamma$. So $p(\gamma)\,{}^*\!\geq p(\eta)$ for all $\eta < \gamma$. This completes the inductive construction of the sequence $\langle p_{\vec{\nu}}(\eta) \mid \eta < \kappa_n\rangle$. Again using completeness of the relevant measures we can find $r(\vec{\nu}) \,{}^*\!\geq p_{\vec{\nu}}(\eta)$ for all $\eta < \kappa_n$.

We need to refine $t(n)_n$ and we do so as follows. For each $\vec{\nu}$ we can partition $t(n)_n$ into three sets

$$
\begin{aligned}
A_0^{\vec{\nu}} &= \{\rho \in t(n)_n \mid r^\frown\vec{\nu}^\frown\rho \Vdash \sigma\},\\
A_1^{\vec{\nu}} &= \{\rho \in t(n)_n \mid r^\frown\vec{\nu}^\frown\rho \Vdash \neg\sigma\}, \text{ and}\\
A_2^{\vec{\nu}} &= \{\rho \in t(n)_n \mid r^\frown\vec{\nu}^\frown\rho \not\parallel \sigma\}.
\end{aligned}
$$

We claim that for all $\vec{\nu}$, $A_2^{\vec{\nu}} \in U_n$. Assume otherwise. Then without loss of generality there is $\vec{\nu}$ such that $A_0^{\vec{\nu}} \in U_n$. Then there is a direct extension of $t(n)^\frown\vec{\nu}$ that decides σ namely $\langle \nu_0, \nu_1, \ldots \nu_{n-1}, A_0^{\vec{\nu}}, r(\vec{\nu})_{n+1}, r(\vec{\nu})_{n+2}, \ldots\rangle$. This contradicts our inductive assumption in the construction of $\langle t(n) \mid n < \omega\rangle$. So for all $\vec{\nu}$, $A_2^{\vec{\nu}} \in U_n$. We are now ready to define $t(n+1)$. For $k < n$, we let $t(n+1)_k =_{def} t(n)_k$. For the other coordinates we wish to take the intersection of the relevant measure one sets. In particular, we let

$$
\begin{aligned}
t(n+1)_n &=_{def} \bigcap_{\vec{\nu}} A_2^{\vec{\nu}}, \text{ and}\\
t(n+1)_k &=_{def} \bigcap_{\vec{\nu}} r(\vec{\nu})_k \text{ for all } k > n
\end{aligned}
$$

There are at most κ_{n-1} many possible $\vec{\nu}$. So by the completeness of each U_k for $k \geq n$, each of the above sets is measure one for the appropriate measure. It follows that $t(n+1) \in P$ and $t(n+1) \,{}^*\!\geq t(n)$. Moreover, if $q =_{def} t(n+1)^\frown\vec{\nu}^\frown\nu$ is an $n+1$ step extension of $t(n+1)$, then $\nu \in A_2^{\vec{\nu}}$ and hence q does not decide σ. This completes the construction of $\langle t(n) \mid n < \omega\rangle$.

We are now ready to conclude the proof that P satisfies the Prikry condition. Let $p^* \ {}^*\!\geq t(n)$ for all $n < \omega$. It is easy to see that for all $n < \omega$, there is no $s \geq p^*$ with $\ell(s) = n$ such that s decides σ. So in fact no extension of p^* decides σ, a contradiction. $\square$

Using the Prikry condition we can show that forcing with P does not add bounded subsets of κ.

Lemma 6. *$\langle P, \leq \rangle$ does not add bounded subsets of κ*

Proof. Let $\underset{\sim}{a}$ be a P-name for a bounded subset of κ. We may assume that $\Vdash_P \underset{\sim}{a} \subseteq \underset{\sim}{\lambda}$ for some $\underset{\sim}{\lambda} < \kappa$. We choose $p \in P$ that decides the value of $\underset{\sim}{\lambda}$ to be $\lambda < \kappa$. There is $i < \omega$ such that $\kappa_i > \lambda$ and we choose $p' \geq p$ with $\ell(p') = i$. Now let $\sigma_\alpha \equiv$ "$\alpha \in a$." We inductively build an increasing sequence of direct extensions of p' $\langle p(\alpha) \mid \alpha < \lambda \rangle$. Let $p(0) =_{def} p'$. At stage $\alpha + 1$ we use the Prikry condition to find $p(\alpha+1) \ {}^*\!\geq p(\alpha)$ which decides σ_α. Since the direct extension relation is κ_i closed for conditions of length i, we can take upper bounds at both limit stages and for the whole construction. We let p'' be an upper bound for the sequence $\langle p(\alpha) \mid \alpha < \lambda \rangle$. Now p'' forces that $\underset{\sim}{a} = a$ for some $a \subseteq \lambda$ with $a \in V$. So there is a dense set of conditions which force $\underset{\sim}{a} \in V$. $\square$

Let $G \subseteq P$ be V-generic over P. Define $t : \omega \to \kappa$ by $t(n) = \alpha$ if and only if there is $p \in G$ such that $\ell(p) > n$ and $p_n = \alpha$. Then we have that $t \in \prod_{n<\omega} \kappa_n$. Let $\underset{\sim}{t}$ be the canonical name for t. Using an easy density argument we can see that t is bigger than any $s \in (\prod_{n<\omega} \kappa_n)^V$ mod finite, in particular $t \notin V$.

Lemma 7. *For each $s \in \prod_{n<\omega} \kappa_n$ with $s \in V$, we have $s <^* t$.*

Proof. Let $p \in P$ and $s \in \prod_{n<\omega} \kappa_n \cap V$. For each $n \geq \ell(p)$ replace p_n by $p_n \setminus (s(n)+1)$. Let q be the result. Then $q \Vdash \forall n \geq \ell(q),\ \underset{\sim}{t}(n) > s(n)$ $\square$

Finally we show that cardinals and cofinalities above κ are preserved.

Lemma 8. *$\langle P, \leq \rangle$ has κ^+-cc*

Proof. Assume that $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$ is a sequence of pairwise incompatible members of P. Note that if two conditions $p, q \in P$ begin with the same sequence of ordinals, then they are compatible. To find an upper bound we just take the sequence of ordinals followed by $p_n \cap q_n$ for all $n \geq \ell(p) = \ell(q)$. There are only κ-many such finite sequences of ordinals. It follows that there are $\alpha, \beta < \kappa^+$ such that p_α is compatible with p_β, a contradiction. $\square$

This completes the presentation of the forcing for adding a single ω-sequence. The combination of the facts that P adds no bounded subsets of κ and that P has the κ^+-cc shows that cardinals are preserved in the extension.

3. Adding many ω-sequences with long extenders

In this section we present a forcing to add λ-many ω-sequences to κ and for this we use a stronger large cardinal assumption. This time we assume GCH in V and let $\lambda > \kappa^+$ be regular. For this forcing we assume that for each $n < \omega$ there is a $(\kappa_n, \lambda+1)$-extender E_n over κ_n.

In particular we assume that for each $n < \omega$, there is $j_n : V \to M_n$ with M_n transitive, ${}^{\kappa_n}M_n \subseteq M_n$, $M_n \supseteq V_{\lambda+1}$, $\mathrm{crit}(j_n) = \kappa_n$ and $j_n(\kappa_n) > \lambda$. Now for all α

with $\kappa_n \leq \alpha < \lambda$, we define $E_{n\alpha} = \{X \subseteq \kappa_n \mid \alpha \in j_n(X)\}$, which is a nonprincipal, κ_n-complete ultrafilter on κ_n. Note that $E_{n\kappa_n}$ is normal.

Next we define an ordering that plays a key role in the definition of the ordering of the poset for this section. For $\alpha, \beta < \lambda$ define $\alpha \leq_{E_n} \beta$ if and only if there is $f : \kappa_n \to \kappa_n$ such that $j_n(f)(\beta) = \alpha$. It turns out that $\leq_{E_n}$ is κ_n-directed and this fact will play an important role in the proofs below.

Lemma 9. *Fix $n < \omega$ and $\tau < \kappa_n$. Assume that $\langle \alpha_\nu \mid \nu < \tau \rangle$ is a sequence of ordinals less than λ. There are unboundedly many $\alpha < \lambda$ such that for all $\nu < \tau$, $\alpha_\nu \leq_{E_n} \alpha$.*

Proof. To proceed with the proof we need a particular enumeration of small subsets of κ_n. Using GCH, we can construct $\langle a_\beta \mid \beta < \kappa_n \rangle$, an enumeration of $\kappa_n^{<\kappa_n}$ with the following property. For every regular $\delta < \kappa_n$, the sequence $\langle a_\beta \mid \beta < \delta \rangle$ enumerates $\delta^{<\delta}$ so that each $x \in \delta^{<\delta}$ appears unboundedly often in the sequence below δ. We will be interested in the object $j_n(\langle a_\beta \mid \beta < \kappa_n \rangle)$, which when restricted to λ enumerates $\lambda^{<\lambda}$ with the property given to δ above. We call this enumeration $\langle a_\beta \mid \beta < \lambda \rangle$. Fix $\alpha < \lambda$ such that $a_\alpha = \langle \alpha_\nu : \nu < \tau \rangle$. We claim that for all $\nu < \tau$, $\alpha_\nu \leq_{E_n} \alpha$. Recall that by the property of the enumeration there are λ many such α, so this will finish the proof of the lemma.

By the general theory of ultrapowers we have the following diagram.

$$\begin{tikzcd}
 & M_n & \\
V \arrow[ur, "j_n"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"] & \\
 & N_{\alpha_\nu} \arrow[uu, bend right=60, "k_{\alpha_\nu}"'] &
\end{tikzcd}$$

In the diagram $N_\alpha \simeq \mathrm{Ult}(V, E_{n\alpha})$ and $k_\alpha([f]_{E_{n\alpha}}) = j_n(f)(\alpha)$, and we have the same for α_ν in place of α. Each of the maps in the diagram is an elementary embedding. The diagram commutes, so in particular we have $j_n(\langle a_\beta \mid \beta < \kappa_n \rangle) = k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa_n \rangle)$. The α^{th} member of this sequence is a_α. Moreover we can write a_α as the image of a τ-sequence of ordinals in N_α.

$$a_\alpha = j_n(\langle a_\beta \mid \beta < \kappa_n \rangle)(\alpha) = k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa_n \rangle)([id]_{E_{n\alpha}}))$$

Since $\mathrm{crit}\, k_\alpha \geq \kappa_n = \mathrm{crit}(j_n)$, we have that $i_\alpha(\langle a_\beta \mid \beta < \kappa_n \rangle)([id]_{E_{n\alpha}})$ is a τ-sequence of ordinals in N_α by the elementarity of the map k_α. We let α^*_ν be the ν^{th} member of the τ-sequence from N_α. By elementarity $k_\alpha(\alpha^*_\nu) = \alpha_\nu$. This allows us to define a elementary embedding $k_{\alpha_\nu\alpha} : N_{\alpha_\nu} \to N_\alpha$ by the formula $k_{\alpha_\nu\alpha}([f]_{E_{n\alpha_\nu}}) = i_\alpha(f)(\alpha^*_\nu)$. The proofs that $k_{\alpha_\nu\alpha}$ is elementary and that the following diagram commutes are easy and will be omitted.

$$\begin{tikzcd}
 & M_n \\
V \arrow[ur, "j_n"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"] \\
 & N_{\alpha_\nu} \arrow[u, "k_{\alpha_\nu\alpha}"] \arrow[uu, bend right=60, "k_{\alpha_\nu}"']
\end{tikzcd}$$

Using $k_{\alpha_\nu\alpha}$, we can define a map, $\pi_{\alpha\alpha_\nu}$ which witnesses that $\alpha_\nu \leq_{E_n} \alpha$. Let $\pi_{\alpha\alpha_\nu} : \kappa_n \to \kappa_n$ such that $[\pi_{\alpha\alpha_\nu}]_{E_{n\alpha}} = \alpha_\nu^*$. Then $j_n(\pi_{\alpha\alpha_\nu})(\alpha) = k_\alpha([\pi_{\alpha\alpha_\nu}]_{E_{n\alpha}}) = k_\alpha(\alpha_\nu^*) = \alpha_\nu$. So $\alpha_\nu \leq_{E_n} \alpha$. $\square$

For all $\alpha, \beta < \lambda$ such that $\beta \leq_{E_n} \alpha$, we let $\pi_{\alpha\beta}$ be the projection as defined in the previous lemma. Also we let $\pi_{\alpha\alpha}$ be the identity map. We will use these projections to relate the Prikry sequences that we add.

We define a forcing that is similar in form to the forcing for adding a single ω-sequence. First we define the n^{th} cell Q_n and its associated orderings $\leq_n$ and $\leq_n^*$. As before Q_n has two parts Q_{n1} and Q_{n0}. For this forcing Q_{n1} is a kind of Cohen forcing that is restricted by Q_{n0} and the definition of Q_{n0} is significantly more complex than before.

Definition 10. *We define $Q_{n1} =_{def} \{f \mid f$ is a partial function from λ to κ_n with $|f| \leq \kappa\}$. We order Q_{n1} by containment and call this ordering $\leq_{Q_{n1}}$. A triple $\langle a, A, f\rangle \in Q_{n0}$ if and only if all of the following conditions hold:*

1. *$f \in Q_{n1}$,*
2. *$a \subseteq \lambda$ such that*
   (a) *$|a| < \kappa_n$,*
   (b) *a has a maximal element in the ordinal sense and $\max(a) \geq_{E_n} \beta$ for all $\beta \in a$ and*
   (c) *$a \cap \mathrm{dom}(f) = \emptyset$,*
3. *$A \in E_{n\max(a)}$,*
4. *If $\alpha > \beta$ and $\alpha, \beta \in a$, then for all $\nu \in A$, $\pi_{\max(a)\alpha}(\nu) > \pi_{\max(a)\beta}(\nu)$,*
5. *If $\alpha, \beta, \gamma \in a$ with $\alpha \geq_{E_n} \beta \geq_{E_n} \gamma$, then for all $\rho \in \pi_{\max(a)\alpha}$"$A$, $\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$.*

Next we define the ordering $\leq_{Q_{n0}}$. $\langle a, A, f\rangle \geq_{Q_{n0}} \langle b, B, g\rangle$ if and only if

1. *$f \supseteq g$,*
2. *$a \supseteq b$ and*
3. *$\pi_{\max(a)\max(b)}$"$A \subseteq B$.*

We define $Q_n =_{def} Q_{n0} \cup Q_{n1}$ as before. We let $\leq_n^* =_{def} \leq_{Q_{n0}} \cup \leq_{Q_{n1}}$. For $p, q \in Q_n$, $p \leq_n q$ if and only if either $p \leq_n^* q$ or $p \in Q_{n0}$ with $p = \langle a, A, f\rangle$, $q \in Q_{n1}$ and

1. *$q \supseteq f$,*
2. *$\mathrm{dom}(q) \supseteq a$,*
3. *$q(\max(a)) \in A$ and*
4. *$\forall \beta \in a\ q(\beta) = \pi_{\max(a)\beta}(q(\max(a)))$.*

Note 11. *It is not hard to show that $\leq_n$ is transitive.*

Remark 12.
- We call f *the Cohen part* of $\langle a, A, f\rangle$.
- There are two technical lemmas here which show that (4) and (5) in the definition of definition of Q_{n0} are possible. We omit them and refer the interested reader to [4] for their statements and proofs.
- The forcing Q_n is equivalent to Cohen forcing.
- In contrast to the first forcing, a single condition can have many incompatible direct extensions.

Again we show that the n^{th} cell satisfies the Prikry condition.

Lemma 13. *$\langle Q_n, \leq_n, \leq_n^*\rangle$ satisfies the Prikry condition.*

Proof. It is enough to consider conditions of the form $\langle a, A, f\rangle$, since conditions in Q_{n1} can only be extended to stronger conditions in Q_{n1} and this is a direct extension. Let σ be a statement in the forcing language. For $\nu \in A$ let $\langle a, A, f\rangle^\frown\nu =_{def} f \cup \{\langle \beta, \pi_{\max(a)\beta}(\nu) \mid \beta \in A\}$. We construct a $\leq_n^*$-increasing sequence of conditions in Q_{n0} of length κ_n. At the induction step, we find a nondirect extension that decides σ and then we extend the Cohen part of our condition to take this nondirect extension into account.

Let $\langle \nu_\eta \mid \eta < \kappa_n\rangle$ be an increasing enumeration of A and let $q(0) =_{def} \langle a, A, f\rangle$. Assume that we have defined $q(\eta) =_{def} \langle a_\eta, A_\eta, f_\eta\rangle$ for some $\eta < \kappa_n$. We choose $p(\eta) \geq_{Q_{n1}} q(\eta)^\frown\nu_\eta$ such that $p(\eta) \parallel \sigma$. We want a direct extension of $\langle a_\eta, A_\eta, f_\eta\rangle$ that takes $p(\eta)$ into account. We are going to fix the first two coordinates of $q(\eta)$ and extend the third(Cohen) part. Let $f_{\eta+1} = p(\eta) \restriction (\mathrm{dom}(p(\nu_\eta)) \setminus a)$ and set $q(\eta+1) =_{def} \langle a, A, f_{\eta+1}\rangle$. Note that $q(\eta)^\frown\nu_\eta = p(\eta)$. This finishes the successor step. Assume that $\gamma < \kappa_n$ is a limit ordinal and that for all $\eta < \gamma$ we have constructed $q(\eta)$. By induction we can assume that the first two coordinates of each $q(\eta)$ are fixed as a, A. On the third coordinate we just take the union of our increasing sequence of functions. Let $f_\gamma =_{def} \cup_{\eta<\gamma} f_{\eta+1}$ and define $q(\gamma) =_{def} \langle a, A, f_\gamma\rangle$. This process at limits gives a condition since the size of the union is less than or equal to κ which is the allowed size of a condition in the Cohen part. Similar reasoning allows us to take an upper bound $\langle a, A, g\rangle$ for the whole sequence that we built. Also notice that for all $\nu \in A$, $\langle a, A, g\rangle^\frown\nu \geq_{Q_n1} p(\nu)$. So for each ν, $\langle a, A, g\rangle^\frown\nu \parallel \sigma$.

For the next stage we want to shrink A so that each ν gives the same decision. We define a partition of A into

$$A_0 = \{\nu \in A \mid \langle a, A, g\rangle^\frown\nu \Vdash \sigma\} \text{and} \tag{1}$$

$$A_1 = \{\nu \in A \mid \langle a, A, g\rangle^\frown\nu \Vdash \neg\sigma\}. \tag{2}$$

Since $A \in E_{n\max(a)}$, we must have $A_0 \in E_{n\max(a)}$ or $A_1 \in E_{n\max(a)}$. Without loss of generality assume that $A_0 \in E_{n\max(a)}$. Then $\langle a, A_0, g\rangle \Vdash \sigma$, since every nondirect extension is below $p(\nu)$ for some $\nu \in A_0$ and each such $p(\nu)$ forces σ. $\square$

We are now ready to define $\langle P, \leq, \leq^*\rangle$.

Definition 14. *$p \in P$ if and only if $p = \langle p_n \mid n < \omega\rangle$ with the following properties.*
1. *For all $n < \omega$, $p_n \in Q_n$.*
2. *There is $\ell(p) < \omega$ such that for all $n \geq \ell(p)$ $p_n \in Q_{n0}$ and for all $n < \ell(p)$, $p_n \in Q_{n1}$.*
3. *If $n \geq \ell(p)$ and $p_n = \langle a_n, A_n, f_n\rangle$, then $m \geq n$ implies $a_m \supseteq a_n$.*

Suppose that $p = \langle p_n \mid n < \omega\rangle$ *and* $q = \langle q_n \mid n < \omega\rangle$ *are in* P. *Then* $p \geq q$ *if and only if for all* n, $p_n \geq_n q_n$ *and* $p \,{}^*\!\geq q$ *if and only if for all* n, $p_n \,{}^*\!\geq_n q_n$

This forcing is considerably more complex than the previous one. One way that we see this is that our new forcing only satisfies the κ^{++}-cc.

Lemma 15. $\langle P, \leq\rangle$ *satisfies* κ^{++}*-cc*

We will only sketch the proof here. Let $\langle p(\alpha) \mid \alpha < \kappa^{++}\rangle$ be a sequence of conditions from P. Without loss of generality we can assume that $\ell(p(\alpha))$ is constant for all α with value ℓ. For $n < \ell$, the forcing is essentially Cohen forcing, so we can assume that the sets $\mathrm{dom}(p(\alpha)_n)$ form a delta system and that the conditions agree on the root. For $n \geq \ell$, we can form a delta system out of $\{a(\alpha)_n \cup f(\alpha)_n \mid \alpha < \kappa^{++}\}$. With a little more work, we can refine further to find a κ^{++}-sequence of conditions with the property that any two have a common refinement. In this final argument, we use the fact that $\leq_{E_n}$ is κ_n-directed and the two lemmas that we omitted that show that certain conditions in the forcing are possible.

We can also show that $\langle P, \leq, \leq^*\rangle$ satisfies the Prikry condition. In fact it satisfies a stronger condition, which allows us to show that κ^+ is preserved. We omit the statement and proof of the stronger condition and give the basic idea of the proof that P satisfies the Prikry condition.

Lemma 16. $\langle P, \leq, \leq^*\rangle$ *satisfies the Prikry condition.*

The argument goes by combining the ideas from two of the proofs that we have already given. We take ideas from the proof that the first forcing satisfies the Prikry condition and the proof that the n^{th} cell Q_n of the current forcing satisfies the Prikry condition. The proof that the first forcing satisfies the Prikry condition shows us that we need to diagonalize over all possible non-direct extensions by using a direct extension that captures the information from the non-direct extension. The proof that Q_n from the current forcing satisfies the Prikry condition shows us how to capture the information from a non-direct extension without increasing the length of the condition. The proof of the Prikry lemma for this forcing uses the closure of the Cohen conditions and the completeness of the measures heavily.

We now argue that we added λ-many new ω-sequences. Let $G \subseteq P$ be V-generic. Let $n < \omega$ and define a function $F_n : \lambda \to \kappa_n$ by $F_n(\alpha) = \nu$ if and only if there is $p \in G$ such that $\ell(p) > n$, $\alpha \in \mathrm{dom}(p_n)$ and $p_n(\alpha) = \nu$. Note that F_n is a function since G is a filter and F_n is defined on all of λ by genericity. Let $t_\alpha = \langle F_n(\alpha) \mid n < \omega\rangle$ for all $\alpha < \lambda$. Then we have $t_\alpha \in \prod \kappa_n$ for all $\alpha < \lambda$. It is possible that for some α, $t_\alpha \in V$, since it is possible that t_α is completely determined by a single condition. However the following lemma shows that the set $\{t_\alpha \mid \alpha < \lambda\}$ has size λ, which is enough to see that we added λ-many ω-sequences to κ.

Lemma 17. *Let* $\beta < \lambda$. *Then there is* α *with* $\beta < \alpha < \lambda$ *such that for all* $\gamma < \alpha$, $t_\gamma <^* t_\alpha$

Proof. Work in V and let $p = \langle p_n \mid n < \omega\rangle$ be a condition with $p_n = \langle a_n, A_n, f_n\rangle$ for $n \geq \ell(p)$. We work to define an extension of p which forces the conclusion for a particular choice of α. By the definition of conditions in our poset, we have $|\cup_{n<\omega} \mathrm{dom}(f_n)| \leq \kappa$. We choose $\alpha \in \lambda \setminus (\sup(\cup_{n<\omega} \mathrm{dom}(f_n)) \cup (\cup_{n<\omega} a_n))$. Now by Lemma 9, for each n there is α_n^* such that $a_n \cup \{\alpha\} \leq_{E_n} \alpha_n^*$. Let $q \geq p$ be the condition given by taking p and replacing each a_n with $a_n \cup \{\alpha\} \cup \{\alpha^*_n\}^\dagger$ and let $q_n =_{def} \langle b_n, B_n, g_n\rangle$ for $n \geq \ell(q) = \ell(p)$. We claim that q forces that $t_\gamma <^* t_\alpha$ for all $\gamma < \alpha$. We break the proof into two cases. In the first case we assume that $t_\gamma \in V$. We show that there is a dense set above q which forces $t_\gamma <^* t_\alpha$. Let $r \geq q$ with $r_n = \langle c_n, C_n, h_n\rangle$ for $n \geq \ell(r)$. Define an extension of r by finding a measure one set $D_n \subseteq C_n$ for each $n \geq \ell(r)$ so that for all $\zeta \in \pi_{\max(c_n)\alpha}``D_n$, $\zeta > t_\gamma(n) + 1$. Using the definition of the ordering, this extension forces $t_\gamma <^* t_\alpha$. Otherwise, we assume that $t_\gamma \notin V$. Then there is a dense set of $r \geq q$ such that $\gamma \in c_n$ for all $n \geq \ell(r)$ where $r_n = \langle c_n, C_n, h_n\rangle$. Then by condition 4 in the definition of the triple $\langle a, A, f\rangle$ and condition 4 in the definition of $\leq^*_n$ such a condition r forces that $t_\gamma <^* t_\alpha$. $\square$

4. Adding many ω-sequences with short extenders

It is natural to ask whether the long extenders used in the last section are required. In the long extender forcing each ω-sequence is controlled by ω-many measure one sets, one from each extender. It seems natural to use extenders whose length is the number of ω-sequences that we wish to add. In this way we only need to use each measure once. In this section we introduce a forcing that uses shorter extenders. The upshot is that we are required to use one measure from a given extender to control the values of more than one ω-sequence.

We begin by describing a naive attempt to define a forcing that adds λ-many ω-sequences using short extenders. The forcing resembles the long extender version with a few changes prompted by the discussion in the previous paragraph. It satisfies the strengthening of the Prikry property needed to see that κ^+ is preserved, however it collapses λ to have size κ^+. More specifically the chain condition argument from before no longer works. We show that one can modify the definition and argue that a subforcing satisfies the κ^{++}-cc in the case when $\lambda = \kappa^{++}$. The subforcing that we identify is the forcing needed to prove Theorem 1. The case when $\lambda > \kappa^{++}$ requires a preparation forcing and is significantly more difficult.

4.1. A naive attempt.

We continue to work with κ singular of cofinality ω with $\langle \kappa_n \mid n < \omega\rangle$ increasing and cofinal in κ. This time we assume that for each κ_n we have an extender of length κ_n^{+n+2}. In particular we assume that for each n there is an elementary embedding $j_n : V \to M_n$ with $\mathrm{crit}(j_n) = \kappa_n$, ${}^{\kappa_n}M_n \subseteq M_n$, $j_n(\kappa_n) > \kappa_n^{+n+2}$ and $V_{\kappa_n^{+n+2}} \subseteq M_n$. Then we derive our extender as usual $E_{n\alpha} =_{def} \{X \subseteq \kappa_n \mid \alpha \in j_n(X)\}$. We let $\lambda > \kappa^+$ be regular and attempt to define a forcing to add λ-many cofinal ω-sequences to κ.

We proceed as before by defining a cell for each n and then gluing them together. The definition of Q_{n1} will be the same and the definition of Q_{n0} will be slightly different.

Definition 18. *Fixing $n < \omega$, we define $Q_{n1} =_{def} \{f \mid f$ is a partial function from λ to κ_n with $|f| \leq \kappa\}$ and order it by extension, which we call $\leq_{Q_{n1}}$. Let $\langle a, A, f\rangle \in Q_{n0}$ if and only if*

1. *$f \in Q_{n1}$,*
2. *a is a partial order preserving function from λ to κ_n^{+n+2} with $|a| < \kappa_n$ such that*
   - (a) *$\mathrm{dom}(a)$ has a maximal element in the ordinal sense,*

[†] In order to obtain this condition we use the Lemmas mentioned in Remark 12.

(b) $a(\max(\mathrm{dom}(a))) = \max(\mathrm{rng}(a))$ *and*
(c) *for all* $\beta \in rng(a)$, $\beta \leq_{E_n} a(\max(\mathrm{dom}(a)))$, *and*

(3) $A \in E_{na(\max(\mathrm{dom}(a)))}$,
(4) *If* $\alpha > \beta$ *and* $\alpha, \beta \in \mathrm{dom}(a)$, *then for all* $\nu \in A$, $\pi_{a(\max(\mathrm{dom}(a)))a(\alpha)}(\nu) > \pi_{a(\max(\mathrm{dom}(a)))a(\beta)}(\nu)$ *and*
(5) *if* $\alpha, \beta, \gamma \in \mathrm{dom}(a)$ *with* $a(\alpha) \geq_{E_n} a(\beta) \geq_{E_n} a(\gamma)$, *then for all* $\rho \in \pi_{a(\max(\mathrm{dom}(a)))a(\alpha)}$"$A$, $\pi_{a(\alpha)a(\gamma)}(\rho) = \pi_{a(\beta)a(\gamma)}(\pi_{a(\alpha)a(\beta)}(\rho))$.

We define $\langle b, B, g\rangle \leq_{Q_{n0}} \langle a, A, f\rangle$ *if and only if*

(1) $g \leq_{Q_{n1}} f$,
(2) $b \subseteq a$ *and*
(3) $\pi_{a(\max(\mathrm{dom}(a)))b(\max(\mathrm{dom}(b)))}$"$A \subseteq B$.

Let $\leq^*_{Q_n} =_{def} \leq_{Q_{n0}} \cup \leq_{Q_{n1}}$. *Define* $p \leq_{Q_n} q$ *if and only if*

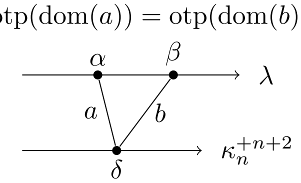
(1) $p \leq^*_{Q_n} q$ *or*
(2) $p = \langle a, A, f\rangle \in Q_{n0}, q \in Q_{n1}$ *and*
  (a) $f \leq_{Q_{n1}} q$,
  (b) $dom(a) \subseteq \mathrm{dom}(q)$,
  (c) $q(\max(\mathrm{dom}(a))) \in A$ *and*
  (d) *for all* $\beta \in \mathrm{dom}(a)$, $q(\beta) = \pi_{a(\max(\mathrm{dom}(a)))a(\beta)}(q(\max(\mathrm{dom}(a))))$.

Remark 19. As in the long extender forcing, we omit Lemmas which show that (4) and (5) in the definition of Q_{n0} are possible.

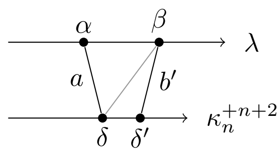
The definition the forcing P from the cells Q_n is exactly the same as in the long extender forcing. As we mentioned in the introduction to this section, our new forcing P satisfies the Prikry condition and therefore adds no new bounded subsets to κ. The proof that P satisfies the Prikry condition is similar to the proof for the long extender forcing. The key point is to deal with the fact that in the n^{th} cell we have replaced the less than κ_n sized subset of λ with a partial order preserving function from λ to κ_n^{+n+2}. In fact P also satisfies the strengthening of the Prikry condition need to see that κ^+ is preserved.

Theorem 20. *Forcing with* $(P, \leq)$ *preserves cardinals less than or equal to* κ^+.

The same chain condition argument from last time no longer works. It is instructive to see exactly what goes wrong. Let's say that we proceed as before and attempt to prove that the forcing is κ^{++}-cc. Given a κ^{++}-sequence of conditions, we can fix the length of the stem of all of the conditions. Again below the length of the stem we are essentially doing Cohen forcing, so this poses no problem. Working above the length of the stem, we form a Δ-system out of the domains of our partial functions, the objects corresponding to $\mathrm{dom}(a) \cup \mathrm{dom}(f)$ in each condition. We would like to be able to amalgamate two functions a and b with disjoint domains. However, $a \cup b$ need not be an order preserving function. In fact we may assume that $\mathrm{otp}(\mathrm{dom}(a)) = \mathrm{otp}(\mathrm{dom}(b))$ and for all $i < \mathrm{otp}(\mathrm{dom}(a))$, $a(i) = b(i)$. Pictorially,



So our argument cannot proceed. In order to recover the chain condition, we want to find a $\delta' > \delta$ that is similar in some sense to δ, so that we can instead map β to δ'. Note that our approach changes b to a similar function, say b'. Again pictorially we have



The similarity between b and b' induces an equivalence relation on conditions in a modified version of P. We will show that there is a projection from the modified version of P to the modified version of P modulo the equivalence relation.

4.2. **Gap 2: $\lambda = \kappa^{++}$.** In this section we work towards the final definition of the forcing to add κ^{++}-many cofinal ω-sequences to κ. We begin by clarifying the notion of similarity mentioned above, because it motivates the modified definition of the forcing and it will give us the projection map.

Let $n, k < \omega$ with $1 < k \leq n$ and define

$$\mathcal{A}_{n,k} =_{def} \langle H(\chi^{+k}), \in, <, \chi, E_n, \langle 0, 1, \ldots \tau, \cdots \mid \tau \leq \kappa_n^{+k} \rangle \rangle$$

where χ is a large regular cardinal, $<$ is a well-ordering of $H(\chi^{+k})$ and all of the other parameters mentioned are constants interpreted in the natural way. Let $\mathcal{L}_{n,k}$ be the language of $\mathcal{A}_{n,k}$. Let $\xi < \kappa_n^{+n+2}$ and define $\mathrm{tp}_{n,k}(\xi)$ be the type realized by ξ in the model $\mathcal{A}_{n,k}$.

We are actually interested in a slight expansion of the above language. We choose this presentation to highlight a specific constant. Let $\mathcal{L}^*_{n,k} =_{def} \mathcal{L}_{n,k} \cup \{c\}$ where c is a new constant. For a given ordinal δ, let $\mathcal{A}^\delta_{n,k}$ be the expansion of $\mathcal{A}_{n,k}$ to $\mathcal{L}^*_{n,k}$ where c is interpreted as δ. Let $\mathrm{tp}_{n,k}(\delta, \xi)$ be the type realized by ξ in $\mathcal{A}^\delta_{n,k}$.

Lemma 21. *For a given $n < \omega$, the set $C =_{def} \{\beta < \kappa_n^{+n+2} \mid \forall \gamma < \beta \ \mathrm{tp}_{n,n}(\gamma, \beta)$ is realized stationarily often below $\kappa_n^{+n+2}\}$ contains a club.*

Proof. Suppose for a contradiction that $S =_{def} \kappa_n^{+n+2} \setminus C = \{\beta < \kappa_n^{+n+2} \mid \exists \gamma < \beta \ \mathrm{tp}_{n,n}(\gamma, \beta)$ is not realized stationarily often below $\kappa_n^{+n+2}\}$ is stationary. By Fodor's Lemma there are a stationary set $S^* \subseteq S$ and an ordinal γ^* such that for all $\beta \in S^*$, $\mathrm{tp}_{n,n}(\gamma^*, \beta)$ is not realized stationarily often. A routine counting argument shows that there are only κ_n^{+n+1}-many possible types that each $\mathrm{tp}_{n,n}(\gamma^*, \beta)$ could be. It follows that $\beta \mapsto \mathrm{tp}_{n,n}(\gamma^*, \beta)$ is constant on a stationary set $S^{**} \subseteq S^*$. Let $\beta^* \in S^{**}$ be least. Then by the choice of S^{**}, $\mathrm{tp}_{n,n}(\gamma^*, (\beta^*)$ is realized stationarily often. This contradicts that $\beta^* \in S^*$. $\square$

Definition 22. *Let $n, k < \omega$ with $1 < k \leq n$ and $\beta < \kappa_n^{+n+2}$. β is* k-good if and only if

(1) $\mathrm{cf}(\beta) \geq \kappa_n^{++}$ and
(2) for all $\gamma < \beta$, $\mathrm{tp}_{n,k}(\gamma, \beta)$ is realized stationarily often below κ_n^{+n+2}.

We are now ready to modify our forcing P from above. In addition to previous properties demanded of a_n we require the following,

(1) $a_n : \kappa^{++} \to \kappa_n^{+n+2}$ and for all $\alpha \in \mathrm{dom}(a_n)$, $a_n(\alpha)$ is at least 2-good and
(2) if for some $p = \langle p_n \mid n < \omega \rangle$ and $\alpha < \kappa^{++}$ there is $i \geq \ell(p)$ such that $\alpha \in \mathrm{dom}(a_i)$ where $p_i = \langle a_i, A_i, f_i \rangle$, then there is a nondecreasing sequence $\langle k_m \mid i \leq m < \omega \rangle$ such that $k_m \to \infty$ as $m \to \infty$ and for every $m \geq i$, $a_m(\alpha)$ is k_m-good.

For ease of notation we will call this modified forcing P as well. For this modified forcing we have the same theorems that we had before. In particular P adds κ^{++} many ω-sequences to κ and preserves cardinals less than or equal to κ^+.

We work towards the definition of our equivalence relation. Note that if a_n is a function as above then we can code $\mathrm{rng}(a_n)$ as a single ordinal, namely its maximal element in the sense of $\leq_{E_n}$. This ordinal is one that we can take the type of in the sense discussed above.

Definition 23. *Let $n, k < \omega$ with $1 < k \leq n$ and $\langle a, A, f\rangle, \langle b, B, g\rangle \in Q_{n0}$. Define $\langle a, A, f\rangle \leftrightarrow_{n,k} \langle b, B, g\rangle$ if and only if*

1. *$f = g$,*
2. *$A = B$,*
3. *$dom(a) = dom(b)$ and*
4. *$\mathrm{rng}(a), \mathrm{rng}(b)$ (viewed as single ordinals) realize the same k-type.*

Using this definition we are ready to give the definition of the equivalence relation.

Definition 24. *Let $p = \langle p_n \mid n < \omega\rangle$ and $q = \langle q_n \mid n < \omega\rangle$ be members of P. Define $p \leftrightarrow q$ if and only if*

1. *$\ell(p) = \ell(q)$,*
2. *for all $n < \ell(p)$, $p_n = q_n$ and*
3. *there is a nondecreasing sequence $\langle k_m \mid \ell(p) \leq m < \omega\rangle$ such that $k_m \to \infty$ as $m \to \infty$ and for all $m \geq \ell(p)$, $p_m \leftrightarrow_{m,k_m} q_m$.*

It is easy to see that this is an equivalence relation. What is more interesting is that it works well with the definition of the ordering on P.

Lemma 25. *If $p, s, t \in P$ with $s \geq p \leftrightarrow t$, then there are $s' \geq s$, $t' \geq t$ such that $s' \leftrightarrow t'$.*

We can factor using this relation and get something nice, however we are going to explicitly describe the order that we will force with.

Definition 26. *Let $p, q \in P$. Define $p \to q$ if and only if there is an $m < \omega$ and a sequence of elements $\langle r_i \mid i < m\rangle$ such that*

1. *$r_0 = p$,*
2. *$r_{m-1} = q$ and*
3. *for each $i < m - 1$, either $r_i \leq r_{i+1}$ or $r_i \leftrightarrow r_{i+1}$.*

Since both $\leq$ and $\leftrightarrow$ are transitive we can assume that use of $\leq$ and $\leftrightarrow$ alternates along the sequence $\langle r_i \mid i < m\rangle$. Using Lemma 25 we can prove a nice fact about the interaction between $\leq$ and $\to$.

Lemma 27. *If $p \to q$, then there is $s \geq p$ such that $q \to s$*

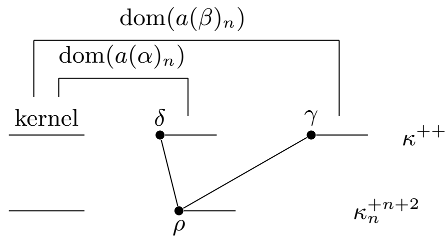
Lemma 27 is precisely what we need to show that the identity map from $\langle P, \leq\rangle$ to $\langle P, \to\rangle$ is a projection. We will now sketch the proof that $\langle P, \to\rangle$ has good chain condition.

Lemma 28. *$\langle P, \to\rangle$ satisfies κ^{++}-cc*

Sketch. Let $\langle p(\alpha) \mid \alpha < \kappa^{++}\rangle$ be a sequence of conditions in P. We sketch the construction of a pair of conditions $q(\alpha) \geq p(\alpha)$ and $q(\beta) \geq p(\beta)$ such that $q(\alpha) \leftrightarrow$

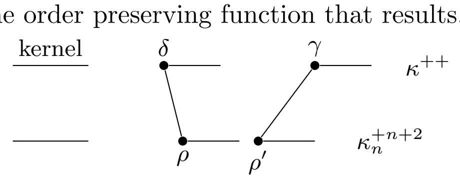
$q(\beta)$. This gives a contradiction since it follows that $p(\alpha), p(\beta)$ are compatible under $\rightarrow$. We may assume that for all $\alpha, \beta < \kappa^{++}$, $\ell(p(\alpha)) = \ell(p(\beta)) =_{def} l$ and for all $\alpha, \beta < \kappa^{++}$ and $n < l$ that $p(\alpha)_n \cup p(\beta)_n$ is a function.

What about above l? For $n \geq l$ let $p(\alpha)_n = \langle a(\alpha)_n, A(\alpha)_n, f(\alpha)_n \rangle$. We can assume that for all $n \geq l$, the sets $\mathrm{dom}(a(\alpha)_n) \cup \mathrm{dom}(f(\alpha)_n)$ form a Δ-system where each function takes the same values on the kernel. We assume that for every $\alpha, \beta < \kappa^{++}$, $\mathrm{rng}(a(\alpha)_n) = \mathrm{rng}(a(\beta)_n)$. Now we have the following picture.

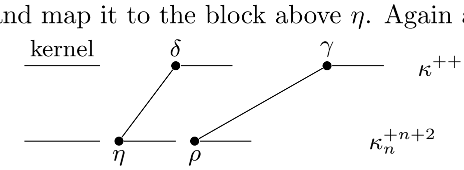


For simplicity we assume that $\gamma = \min(\cup_{m \geq l} \mathrm{dom}(a(\beta)_m) \setminus (kernel))$ and $\gamma \in \mathrm{dom}(a(\beta)_n)$. From the picture we have $(a(\alpha)_n)(\delta) = (a(\beta)_n)(\gamma) = \rho$. We assume that ρ is 5-good. Now we pick some $\rho' > \cup_{n \geq l} \mathrm{rng}(a(\alpha)_n)$ such that ρ' realizes the same type over the image of the kernel as ρ does. Here we are viewing the image of the kernel as coded by a single ordinal and taking types over it as discussed above.

We can now construct $q(\alpha)$. The key point is to construct the order preserving function part of each $q(\alpha)_n$. For this we take $a(\alpha)_n$ and add the off kernel part of $\mathrm{dom}(a(\beta)_n)$ to the domain and map this to a block that sits above ρ' and realizes the same 3-type as $a(\alpha)_n$ "$(\mathrm{dom}(a(\alpha)_n) \setminus (kernel))$. The following picture illustrates the order preserving function that results.



The construction of $q(\beta)$ is similar, but uses a fact that we have omitted. Namely if ξ is k-good, then there are unboundedly many $k-1$-good ordinals less than ξ. We use this to choose an $\eta < \rho$ (and above the image of the kernel) so that there is a block above η that realizes the same 3-type as the block beginning at ρ. We then add the off kernel part of $\mathrm{dom}(a(\alpha)_n)$ to the order preserving function $a(\beta)_n$ and map it to the block above η. Again a picture is helpful.



It follows that $q(\alpha)_n \leftrightarrow_{n,3} q(\beta)_n$. Working in a similar fashion with all $m \geq n$ we can obtain $q(\alpha) \leftrightarrow q(\beta)$. This finishes our sketch of the proof. $\square$

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