Non-homogeneity of Quotients of Prikry Forcings

Moti Gitik[1] and Eyal Kaplan

Tel Aviv University

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Abstract

We study non-homogeneity of quotients of Prikry and tree Prikry forcings with non-normal ultrafilters over some natural distributive forcing notions.

Introduction

Let $\langle Q, \leq_Q \rangle$ be a $\kappa-$distributive forcing notion of cardinality κ. For $q \in Q$ let $Q/q = \{p \in Q \mid p >_Q q\}$. Consider $F_{Q/q} = \{D \subseteq Q/q \mid D$ is a dense open $\}$, for every $q \in Q$. It is a $\kappa-$complete filter over a set of cardinality κ. Assuming large cardinals, for example, if κ is a $\kappa-$compact cardinal, then every $F_{Q/q}$ extends to a $\kappa-$complete ultrafilter $F^*_{Q/q}$. Let $\vec{F}^* = \langle F^*_{Q/q} \mid q \in Q \rangle$.
Force with the corresponding tree Prikry forcing $P_{\vec{F}^*}$. There will be a $V-$generic subset of Q in the extension.
We will study the resulting quotient forcing.
Our goal will be to prove the consistency of a strong occurrence of non-homogeneity of this forcing:

Theorem. *Consistently from κ^+-supercompactness of κ, for every non-trivial, κ-distributive forcing notion Q with $|Q| = \kappa$, there exists a choice of measures $\vec{F}^*$, such that the following property holds: Given two generic Prikry sequences $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$ for $P_{\vec{F}^*}$ such that $\langle q_n : n < \omega \rangle \in V[\langle p_n : n < \omega \rangle]$, it follows that $\langle p_n : n < \omega \rangle = \langle q_n : n < \omega \rangle$.*

This extends the main result of Koepke, Rasch and Schlicht [5] which deals with normal measures only.

In the second chapter, we force with the standard Prikry forcing P_{F^*}, where $F^* = F^*_Q$ is a $\kappa-$complete ultrafilter which extends the filter of dense open subsets of Q. We will study the possible consequences of having–

$$\langle q_n : n < \omega \rangle \in V[\langle p_n : n < \omega \rangle]$$

for two disjoint generic Prikry sequences $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$. We will prove that this induces a non-trivial projection of ${F^*}^n$ onto F^*, for some $n < \omega$.

Notations

1. **Forcing:** We force over the ground model V. Given a forcing notion $\langle Q, <_Q \rangle$ and elements $p, q \in Q$, $p >_Q q$ means "p extends q". Let $Q/q = \{p \in Q \colon p \geq_Q q\}$ be the cone of Q above q. If $G \subseteq Q$ is generic over V, then, for every P-name $\underset{\sim}{\sigma}$, $\left(\underset{\sim}{\sigma}\right)_G$ is the interpretation of $\underset{\sim}{\sigma}$ in $V[G]$.

2. **Sequences:** The set of finite increasing sequences of ordinals below a cardinal κ in denoted by $[\kappa]^{<\omega}$.

   We extend this notation to strictly increasing finite sequences of elements in a forcing notion $\langle Q, <_Q \rangle$: $[Q]^{<\omega}$ is the set of sequences $\langle q_0, \ldots, q_n \rangle$, where $q_{i+1} >_Q q_i$ for every $0 \leq i \leq n-1$. The set $[Q]^n$ of increasing sequences of length n is defined similarly. In the case where $n = 0$, $[Q]^0 = \{\langle\rangle\}$, i.e., the set which includes only the empty sequence.

   Given a sequence $\vec{a} = \langle a_0, \ldots, a_n \rangle$, we denote it's length by $\mathrm{lh}(\vec{a}) = n+1$. The length of the empty sequence is 0. If $\langle Q, <_Q \rangle$ is a forcing notion and $\vec{a} \in [Q]^n$, the maximal coordinate of $\vec{a}$ is denoted by $\mathrm{mc}(\vec{a}) = a_n$. If $\vec{a}$ is the empty sequence, we set artificially $\mathrm{mc}(\vec{a}) = 0_Q$.

   We use the notation $\frown$ for concatenation of sequences: Given sequences $t = \langle \alpha_0, \ldots, \alpha_n \rangle, s = \langle \beta_0, \ldots, \beta_n \rangle$, let $t^\frown s$ be the sequence –

   $$\langle \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \rangle$$

   (of course, $[\kappa]^{<\omega}$ and $[Q]^{<\omega}$ are not closed under concatenations).

3. **Trees:** If $\langle T, <_T \rangle$ is a tree and $t \in T$, then $\mathrm{Succ}_T(t)$ is the set of immediate successors of t in T. We mostly work with (sub-trees of) trees of the form $[Q]^{<\omega}$, where Q is a forcing notion, ordered by $\lhd$,

   $$\langle a_0, \ldots, a_n \rangle \lhd \langle b_0, \ldots, b_m \rangle$$

if and only if $m \geq n$, and for every $0 \leq i \leq n, a_i = b_i$.

Under these settings, for every $t \in T$, we denote $T_t = \{s \in T \colon s \lhd t\}$.

4. **Ultrafilters:** Given an ultrafilter V on κ and a function $f \colon \kappa \to \kappa$, denote $f_* V = \{A \subseteq \kappa \colon f^{-1} A \in V\}$. Also, for ultrafilters V, W, denote $V \leq_{RK} W$ if for some function $f \colon \kappa \to \kappa$, $V = f_* W$ (this is the Rudin-Keisler order). If $V \leq_{RK} W \leq_{RK} V$, denote $V \equiv_{RK} W$.

   Given a measure U on a cardinal κ and a function $f \colon \kappa \to V$, $[f]_U$ is the standard equivalence class of f in the ultrapower construction.

Preliminaries

We assume familiarity with forcing and large cardinals. We will use some standard arguments about distributive forcing notions, quotient forcings and limits of ultrafilters. For sake of completeness, we provide the relevant details in this section.

0.1 Distributivity

Definition 0.1.1. *Given an uncountable cardinal κ, we say that a forcing notion $\langle Q, <_Q \rangle$ is κ-distributive if forcing with Q adds no new $< \kappa$ sequences of ordinals.*

The following is a well known (See, for example, [4]):

Proposition 0.1.2. *Let κ be an uncountable cardinal, and $\langle Q, <_Q \rangle$ a separative forcing notion. The following are equivalent:*

1. *Q is κ-distributive.*
2. *For every $\xi < \kappa$ and a sequence $\langle D_\alpha \colon \alpha < \xi \rangle$ of dense open subsets of Q,*
$$\bigcap_{\alpha<\xi} D_\alpha$$
is dense and open.
3. *For every $\xi < \kappa$, $q \in Q$ and for every sequence $\langle D_\alpha \colon \alpha < \xi \rangle$ of dense open subsets of Q above q,*
$$\bigcap_{\alpha<\xi} D_\alpha$$
is dense and open above q.

Remark 0.1.3. *The last proposition will be applied as follows: Given a separative, κ-distributive forcing notion, $\langle Q, <_Q \rangle$, and $q \in Q$, let –*

$$F_q = \{E \subseteq Q/q \colon E \text{ contains a dense open subset } D \text{ of } Q/q\}$$

where $Q/q = \{p \in Q \colon p >_Q q\}$. Then F_q is a κ-complete filter on Q/q. Under a suitable large cardinal assumption, F_q can be extended to a κ-complete ultrafilter, F_q^*.

The following lemma will be useful later:

Lemma 0.1.4. *Let Q be a separative, κ-distributive notion of forcing of cardinality κ. Then Q can be partitioned to κ-many disjoint dense subsets.*

Proof. Assume that $Q = \{q_\alpha \colon \alpha < \kappa\}$. For every $A \subseteq Q$ with $|A| < \kappa$, let –

$$E(A) = \bigcap_{q \in A} \{p \in Q \colon p >_Q q \text{ or } p, q \text{ are incompatible}\}$$

Then $E(A)$ is a dense and open subset of Q, disjoint from A.

Let $G \colon \kappa \to \kappa \times \kappa$ be Godel's Pairing function. We define a sequence $\langle p_\xi \colon \xi < \kappa\rangle$ as follows: Assume that $\eta < \kappa$ and $\langle p_\xi \colon \xi < \eta\rangle$ were defined. Let us define p_η. Assume that $G(\eta) = (\alpha, \beta)$. Choose $p_\eta \in E(\{p_\xi \colon \xi < \eta\})$ such that p_η extends q_β. This finishes the construction.

Set, for every $\alpha < \kappa$, $D_\alpha = \{p_\xi \colon \ \exists \beta < \kappa \ G(\xi) = (\alpha, \beta)\}$. We claim that D_α is dense for every $\alpha < \kappa$. Indeed, given $q_\beta \in Q$, let $\xi = G^{-1}(\alpha, \beta)$. Then $p_\xi \in D_\alpha$ and extends q_β.

By our construction, the dense sets $\langle D_\alpha \colon \alpha < \kappa\rangle$ are pairwise disjoint.

0.2 Quotient Forcings

Suppose that P, Q are two separative forcing notions, such that every generic extension $V[G]$ for P, contains a generic set $H \in V[G]$ for Q over V. Under these settings, we describe a forcing notion in $V[H]$ whose generic extensions could be obtained by forcing directly with P over V.

We assume here that Q is a complete boolean algebra. This will not be the case in further applications, but we can always replace Q with it's completion, $\mathrm{RO}(Q)$ (i.e., the complete boolean algebra in which Q densely embeds. To be precise, we should remove from $\mathrm{RO}(Q)$ the strongest element, $0_{\mathrm{RO}(Q)}$).

Definition 0.2.1. *A projection $\pi \colon P \to Q$ is a function which satisfies:*

1. *If p' extends p, then $\pi(p')$ extends $\pi(p)$.*

2. *For every $p \in P$, $\pi''(P/p)$ is dense above $\pi(p)$ in Q.*

We state some standard properties, which are presented with more details in [4], for example.

Proposition 0.2.2. *Assume that P, Q are separative forcing notions. Suppose that $\underset{\sim}{H}$ is a P-name for a generic set for Q, and this is forced by the weakest condition in P. Define a function $\pi\colon P \to Q$ as follows: for every $p \in P$,*

$$\pi(p) = \prod\{q \in Q\colon p \Vdash \check{q} \in \underset{\sim}{H}\}$$

Then π is a projection.

Definition 0.2.3. *Suppose that P, Q are separative forcing notions, and $\pi\colon P \to Q$ is a projection. Assume that H is Q-generic over V. Define, in $V[H]$, the quotient forcing, $P/H = \{p \in P\colon \pi(p) \in H\}$, ordered by the order induced from P.*

Proposition 0.2.4. *Let P, Q be as in the last definition. Then every generic set G for P/H is generic for P over V as well. Also, $V[H][G] = V[G]$.*

Claim 0.2.5. *Let P, Q be as above. Assume that $\underset{\sim}{H}$ is a P-name, forced by the weakest condition in P to be Q-generic over V. Let $\pi\colon P \to Q$ be the induced projection. Let G be P-generic over V, and $\left(\underset{\sim}{H}\right)_G = H$. Then for every generic set G' for the quotient forcing P/H over $V[H]$, $\left(\underset{\sim}{H}\right)_{G'} = H$.*

Proof. Assume first that in $V[G']$, $h \in \left(\underset{\sim}{H}\right)_{G'}$. Then for some $p \in G'$, $p \Vdash \check{h} \in \underset{\sim}{H}$. Therefore $\pi(p) \in H$, and $\pi(p)$ extends h; Thus, $h \in H$. So in $V[G']$, $\left(\underset{\sim}{H}\right)_{G'} \subseteq H$. But $\left(\underset{\sim}{H}\right)_{G'}, H$ are Q-generic over V, so $\left(\underset{\sim}{H}\right)_{G'} = H$. □

0.3 Limits of Ultrafilters

Definition 0.3.1. *Assume that U, W and V_α, for every $\alpha < \kappa$, are ultrafilters on κ. Then $U = W\text{-}lim\langle V_\alpha\colon \alpha < \kappa\rangle$ means that, for every $X \subseteq \kappa$,*

$$X \in U \iff \{\alpha < \kappa\colon X \in V_\alpha\} \in W$$

Definition 0.3.2. *A sequence $\langle V_\alpha\colon \alpha < \kappa\rangle$ of ultrafilters on κ is called discrete, if there exists a partition $\langle A_\alpha\colon \alpha < \kappa\rangle$ of κ such that $A_\alpha \in V_\alpha$ for every $\alpha < \kappa$.*

The next lemma is well known:

Lemma 0.3.3. *Every sequence of pairwise distinct normal ultrafilters is discrete.*

Definition 0.3.4. *Let U, W be ultrafilters on κ. We say that $W \leq_{RF} U$ (Rudin-Frolik order) if there exists a discrete sequence $\langle V_\alpha \colon \alpha < \kappa \rangle$ of ultrafilters on κ, such that $U = W\text{-}lim\langle V_\alpha \colon \alpha < \kappa\rangle$.*

The following lemmas are well known as well; For sake of completeness, we provide the proof here.

Lemma 0.3.5. *$W \leq_{RF} U \to W \leq_{RK} U$.*

Proof. Suppose that $\langle V_\alpha \colon \alpha < \kappa \rangle$ is a discrete sequence of ultrafilters such that $U = W\text{-}\lim\langle V_\alpha \colon \alpha < \kappa\rangle$. Let $\langle A_\alpha \colon \alpha < \kappa\rangle$ be a partition of κ, such that $A_\alpha \in V_\alpha$ for every $\alpha < \kappa$.

Let $h \colon \kappa \to \kappa$ be the function $h(x) = \alpha \iff x \in A_\alpha$, i.e., $h(x)$ is the unique index α such that $x \in A_\alpha$. Then $X \in W \iff h^{-1}X \in U$, since $h^{-1}(X) = \bigcup\limits_{\alpha \in X} A_\alpha$. In particular, $W \leq_{RK} U$. □

Proposition 0.3.6. *Suppose that $U = W\text{-}lim\langle V_\alpha \colon \alpha < \kappa\rangle$, where $\langle V_\alpha \colon \alpha < \kappa\rangle$ is a discrete sequence of ultrafilters measures on κ.*

Let $M_U \simeq Ult(V, U), M_W \simeq Ult(V, W)$ be the ultrapowers of U, W, with corresponding elementary embeddings j_U, j_W. Define $V' \in M_W$ as follows –

$$V' = j_W(V_\alpha \colon \alpha < \kappa)([Id]_W)$$

Then V' is a measure on $j_W(\kappa)$ and $M_U \simeq Ult(M_W, V')$. Moreover, if j' is the ultrapower embedding of V', then $j' \circ j_W = j_U$.

$$\begin{tikzcd}
 & M_U \simeq \mathrm{Ult}(M_W, V') \\
V \arrow[ur, "j_U"] \arrow[r, "j_W"'] & M_W \arrow[u, "j'"']
\end{tikzcd}$$

Proof. By elementarity, V' is a measure on $j_W(\kappa)$. It suffices to prove that $\mathrm{Ult}(V, U)$ and $\mathrm{Ult}(M_W, V')$ are isomorphic, and thus have the same transitive collapse. We define an isomorphism, definable in V, $\phi \colon \mathrm{Ult}(V, U) \to \mathrm{Ult}(M_W, V')$, as follows:

$$\phi\left([f]_U\right) = [j_W(f)]_{V'}$$

for every function f with domain κ. Let us prove that ϕ is well-defined, and an isomorphism. Suppose that $[f]_U = [g]_U$. Then $\{\alpha < \kappa\colon f(\alpha) = g(\alpha)\} \in U$, and in the ultrapower by W,

$$\{\alpha < j_W(\kappa)\colon j_W(f)(\alpha) = j_W(g)(\alpha)\} \in V'$$

thus, $[j_W(f)]_{V'} = [j_W(g)]_{V'}$.

Proving elementarity is similar. Let us prove that ϕ is onto. Assume that $[f]_{V'} \in \mathrm{Ult}\,(M_W, V')$. For some $g\colon \kappa \to V$, $f = [g]_W$. Since $f\colon j_W(\kappa) \to M_W$ is a function, we can assume without loss of generality that, for every $\beta < \kappa$, $g(\beta)$ is a function from κ to V. Now, define $f'\colon \kappa \to \kappa$ as follows: For every $\alpha < \kappa$, set $f'(\alpha) = g(\beta_\alpha)(\alpha)$, where β_α is the unique index β such that $\alpha \in A_\beta$. We claim that $\phi([f']_U) = [f]_{V'}$. It suffices to prove that –

$$\{\alpha < j_W(\kappa)\colon j_W(f')(\alpha) = [g]_W\,(\alpha)\} \in V'$$

or –

$$\{\beta < \kappa\colon \{\alpha < \kappa\colon f'(\alpha) = g(\beta)(\alpha)\} \in V_\beta\} \in W$$

This holds: Indeed, fix $\beta < \kappa$. Then $\{\alpha < \kappa\colon f'(\alpha) = g(\beta)(\alpha)\} \supseteq A_\beta \in V_\beta$.

Let us prove the equality $j' \circ j_W = j_U$. Denote, for every $x \in V$, the function $c_x\colon \kappa \to V$, defined as follows: $\forall \alpha < \kappa,\ c_x(\alpha) = x$. Now, for every $x \in V$,

$$\phi([c_x]_U) = [j_W(c_x)]_{V'} = j' \circ j_W(x)$$

where the last equality can be easily checked (we slightly abused the notation and identified elements in $\mathrm{Ult}(M_W, V')$ with their image under the transitive collapse). $\square$

Chapter 1

Tree Prikry Forcing

1.1 Definitions and Basic Properties

Definition 1.1.1. *κ is a κ-compact cardinal if every κ-complete filter on κ can be extended to a κ-complete ultrafilter on κ.*

Let κ be a κ-compact cardinal. Consider a κ-distributive forcing notion $\langle Q, <_Q \rangle$ of cardinality κ. Let $[Q]^{<\omega}$ be the full tree of finite $<_Q$-increasing sequences of elements of Q, ordered by end-extensions, i.e.,

$$[Q]^{<\omega} = \{\langle \nu_1, \ldots, \nu_n \rangle \colon n < \omega, \nu_i \in Q \text{ and } \nu_1 <_Q \nu_2 <_Q \ldots <_Q \nu_n\}$$

For $t = \langle a_1, \ldots, a_n \rangle, s = \langle b_1, \ldots, b_m \rangle \in [Q]^{<\omega}$, denote $t \lhd s$ if $n \leq m$ and for every $i = 1, \ldots, n$, $a_i = b_i$. For every non-empty sequence $t = \langle a_1, \ldots, a_n \rangle \in [Q]^{<\omega}$, set $\mathrm{mc}(t) = a_n$. If $t = \langle \rangle$, set artificially $\mathrm{mc}(t) = 0_Q$, where 0_Q is the weakest condition of Q.

For every $q \in Q$, denote $Q/q = \{p \in Q \colon p \geq_Q q\}$. This is the cone of Q above q.

Remark 1.1.2. *If Q is separative, then, for every $q \in Q$, $|Q/q| = \kappa$. Indeed, else, if $Q/q = \{p_\alpha \colon \alpha < \xi\}$ for some $\xi < \kappa$, define $D_\alpha = \{p \in Q \colon p >_Q p_\alpha \text{ or } p \perp p_\alpha\}$. Since Q is separative, D_α is dense and open for every $\alpha < \xi$. Then $(Q/q) \cap \bigcap_{\alpha<\xi} D_\alpha = \emptyset$, a contradiction.*

For every $t \in [Q]^{<\omega}$, let F_t be the κ-complete filter generated by the subsets of Q, which are dense and open above $\mathrm{mc}(t)$ –

$$F_t = \{E \subseteq Q/\mathrm{mc}(t) \ : \ D \subseteq E \text{ for some dense open subset } D \text{ of } Q/\mathrm{mc}(t)\}$$

By κ-compactness of κ, for every $t \in [Q]^{<\omega}$, there exists a κ-complete ultrafilter, F_t^*, which extends F_t. Denote $\vec{F}^* = \langle F_t^* \colon t \in [Q]^{<\omega} \rangle$ (in the next sections, we will assume κ^+-supercompactness of κ, and choose F_t^* more carefully).

Let us present a Prikry type forcing $P_{\vec{F}^*}$. We follow the presentation and notations from Gitik's handbook article [2].

Definition 1.1.3. *Let $t \in [Q]^{<\omega}$. A tree $T \subseteq [Q]^{<\omega}$ is a $\langle F_s^* \colon s \in [Q]^{<\omega} \rangle$-tree with trunk t if –*

1. *$T \subseteq [Q]^{<\omega}$, ordered by end-extensions.*
2. *t is the trunk of T, i.e., for every $s \in T$, $s \lhd t$ or $t \lhd s$.*
3. *For every $s \in T$ such that $t \lhd s$, $Succ_T(s) = \{q \in Q \colon s^\frown \langle q \rangle \in T\} \in F_s^*$.*

Let $\langle P_{\vec{F}^*}, \leq, \leq^* \rangle$, consist of elements of the form $\langle t, T \rangle$, where $t \in [Q]^{<\omega}$ and $T \subseteq [Q]^{<\omega}$ is a $\langle F_s^* \colon s \in [Q]^{<\omega} \rangle$-tree with trunk t. We say that $\langle t, T \rangle$ extends $\langle s, S \rangle$ if $T \subseteq S$ (in particular, $t \rhd s$). If, in addition, $t = s$, we say that $\langle t, T \rangle$ is a Direct Extension of $\langle s, S \rangle$, and denote it by $\langle t, T \rangle \geq^* \langle s, S \rangle$.

We will show some Prikry-type properties of $\langle P_{\vec{F}^*}, \leq, \leq^* \rangle$. First, we define the Prikry sequence corresponding to a generic set $G \subseteq P$.

Lemma 1.1.4. *Let G be a $P_{\vec{F}^*}$-generic set. Then –*

$$C = \cup \{ t \in [Q]^{<\omega} : \exists T \ \langle t, T \rangle \in G \}$$

is a $<_Q$-increasing ω-sequence (we refer to it as the Prikry sequence corresponding to G). Moreover, $V[G] = V[C]$, and $V[G]$ contains an ω-sequence of ordinals, which is cofinal in κ.

Proof. First, we show that the set C is, indeed, an ω-sequence. For every pair of compatible elements $\langle t, T \rangle, \langle s, S \rangle$, $t \rhd s$ or $t \lhd s$. Therefore, the Prikry sequence $\cup \{ t \in [Q]^{<\omega} : \exists T \ \langle t, T \rangle \in G \}$ is well defined. We also note that, given $n < \omega$, the set $D_n = \{ \langle t, T \rangle \colon \mathrm{lh}(t) \geq n \}$ is dense in $P_{\vec{F}^*}$. Therefore, the length of the Prikry sequence is ω. Clearly, it's $<_Q$-increasing, as a union of such sequences. It's not hard to see that $G \in V[C]$, since $G = \{ \langle t, T \rangle \in P_{\vec{F}^*} \colon \forall n < \omega \ C \restriction n \in T \}$.

It remains to show that κ changes it's cofinality in $V[G]$. In V, fix a bijection $f \colon Q \to \kappa$. Let $\langle p_n \colon n < \omega \rangle \in V[G]$ be the Prikry sequence corresponding to G. We show that $\{ f(p_n) \colon n < \omega \}$ is cofinal in κ in $V[G]$. Let $\alpha < \kappa$. Define, in V, the set $D_\alpha = \{\langle t, T\rangle\colon\ f(\mathrm{mc}(t)) \geq \alpha\}$. It suffices to prove that D_α is dense in $P_{\vec{F}^*}$. Indeed, Take arbitrary $\langle t, T\rangle \in P_{\vec{F}^*}$. We note that $f^{-1\prime\prime}\alpha$ is of cardinality $< \kappa$, so $f^{-1\prime\prime}\alpha \notin F^*_t$. Now, choose $q \in \mathrm{Succ}_T(t)$ with $f(q) \geq \alpha$. Then $\langle t^\frown\langle q\rangle, T\rangle \in D$.

$\square$

Lemma 1.1.5. *Let $T \subseteq [Q]^{<\omega}$ be a a $\langle F^*_s\colon s \in [Q]^{<\omega}\rangle$-tree with trunk t. Assume that $\alpha < \kappa$, and $f\colon T \to \alpha$ is some function. Then there exists a $\langle F^*_s\colon s \in [Q]^{<\omega}\rangle$-tree $S \subseteq T$ with trunk t, such that, for every $n < \omega$, $f \restriction Lev_n(S)$ is constant.*

Proof. Assume for simplicity that $t = \langle\rangle$. First, let us prove the claim for each $n < \omega$ separately. This is clear for $n = 0$. We proceed by induction on $n < \omega$. Assume the claim holds for n, and let us prove it for $n+1$. For every $q \in \mathrm{Lev}_1(T)$, let $T_q = \{\langle q_1, \ldots, q_m\rangle \in [Q]^{<\omega}\colon \langle q, q_1, \ldots, q_m\rangle \in T\}$. Let $f_q\colon T_q \to \alpha$ be defined as follows: $f_q\left(\langle q_1, \ldots, q_m\rangle\right) = f\left(\langle q, q_1, \ldots, q_m\rangle\right)$. Then there exists $S_q \subseteq T_q$ and $\alpha_q < \alpha$ such that $f_q\left(\langle q_1, \ldots, q_n\rangle\right) = \alpha_q$ for every $\langle q_1, \ldots, q_n\rangle \in S_q$. Now, take $A \in F^*_{\langle\rangle}$, $A \subseteq \mathrm{Lev}_1(T)$ such that, for some $\beta < \kappa$, $\alpha_q = \beta$ for every $q \in A$.

Define $S = \{\langle q, q_1, \ldots, q_m\rangle\colon q \in A \text{ and } \langle q_1, \ldots, q_m\rangle \in S_q\}$. Let us claim that $f \restriction \mathrm{Lev}_{n+1}(S)$ is constant. Let $\langle q, q_1, \ldots, q_n\rangle \in S$. Then $\langle q_1, \ldots, q_n\rangle \in S_q$, so $f\left(\langle q, q_1, \ldots, q_n\rangle\right) = \alpha_q = \beta$.

Now, assume that for every $n < \omega$ there exists a $\langle F^*_s\colon s \in [Q]^{<\omega}\rangle$-tree, $S_n \subseteq T$, such that $f \restriction \mathrm{Lev}_n(S_n)$ is constant. Let $S = \bigcap\limits_{n<\omega} S_n$. Then S is a $\langle F^*_s\colon s \in [Q]^{<\omega}\rangle$-tree as desired. $\square$

Now, in a standard fashion, we conclude the following:

Lemma 1.1.6. *(The Prikry Condition) Let $\langle t, T\rangle \in P_{\vec{F}^*}$ and σ be a statement in the forcing language. Then there exists a direct extension $\langle t, S\rangle \geq^* \langle t, T\rangle$ such that $\langle t, S\rangle \parallel \sigma$.*

Corollary 1.1.7. *$P_{\vec{F}^*}$ preserves all cardinals.*

The next lemmas will be applied in the next section.

Lemma 1.1.8. *Assume that $A_s \in F^*_s$ for every $s \in [Q]^{<\omega}$, and $\langle p_n\colon n < \omega\rangle$ is a Prikry sequence for $P_{\vec{F}^*}$. Then for some $n_0 < \omega$, and for every $n \geq n_0$, $p_{n+1} \in A_{\langle p_0, \ldots, p_n\rangle}$.*

Proof. Assume that $G \subseteq P_{\vec{F}^*}$ is the generic set corresponding to $\langle p_n \colon n < \omega \rangle$. Define a dense set as follows:

$$D = \{\langle t, T \rangle \in P_{\vec{F}^*} \colon \ \forall s \in T, s \rhd t \to \mathrm{Succ}_T(s) \subseteq A_s\}$$

D is dense in $P_{\vec{F}^*}$. Indeed, given a condition $\langle t, T \rangle \in P_{\vec{F}^*}$, define a $\langle F^*_s \colon s \in [Q]^{<\omega} \rangle$-tree, T', such that for every $s \rhd t$, $\mathrm{Succ}_{T'}(s) \subseteq \mathrm{Succ}_T(s) \cap A_s$ (Applying the intersections inductively, shrinking T level-by-level). Then $\langle t, T' \rangle \in D$ extends $\langle t, T \rangle$.

Now, take $\langle s, S \rangle \in G \cap D$. Then for every $n \geq \mathrm{lh}(s)$, $p_{n+1} \in \mathrm{Succ}_S(p_n) \subseteq A_{\langle p_0, \ldots, p_n \rangle}$. □

Lemma 1.1.9. *Assume that there exists a partition of Q, $\langle A_s \colon s \in [Q]^{<\omega} \rangle$, such that $A_s \in F^*_s$, for every $s \in [Q]^{<\omega}$. Let $\langle p_n \colon n < \omega \rangle$, $\langle q_n \colon n < \omega \rangle$ be a pair of different Prikry sequences for P_{F^*} such that $\langle q_n \colon n < \omega \rangle \in V[\langle p_n \colon n < \omega \rangle]$. Then, for every $i < \omega$ there exists some $k < \omega$, $k \geq i$, such that $\{p_n \colon k < n < \omega\}, \{q_n \colon k < n < \omega\}$ are disjoint.*

Proof. First, apply the last lemma: Let $i_0 < \omega$ be such that, for every $k \geq i_0$, $p_{k+1} \in A_{\langle p_0, \ldots, p_k \rangle}$, and $q_{k+1} \in A_{\langle q_0, \ldots, q_k \rangle}$. We can assume that $i_0 \geq i$, or else, enlarge i_0.

Assume for contradiction, that for every $k \geq i_0$, $\{p_n \colon k < n < \omega\}$, $\{q_n \colon k < n < \omega\}$ are not disjoint; So $p_n = q_m$ for some $n, m \geq k$. In particular, $A_{\langle p_0, \ldots, p_{m-1} \rangle}$ is not disjoint from $A_{\langle q_0, \ldots, q_{n-1} \rangle}$, so $\langle p_0, \ldots, p_{m-1} \rangle = \langle q_0, \ldots, q_{n-1} \rangle$. This could be done for every $k \geq i_0$; Therefore, $\langle p_n \colon n < \omega \rangle = \langle q_n \colon n < \omega \rangle$. □

Our next observation is that, given $G \subseteq P_{\vec{F}^*}$ generic over V, there exists $H \in V[G]$ such that H is Q-generic over V.

Lemma 1.1.10. *Given a generic Prikry sequence $\langle p_n \colon n < \omega \rangle$ for $P_{\vec{F}^*}$, define $H \in V[\langle p_n \colon n < \omega \rangle]$, $H = \{q \in Q \colon \exists n < \omega \ q \leq p_n\}$. Then H is Q generic over V. In particular, if $\langle Q, <_Q \rangle$ is a separative forcing notion, then $P_{\vec{F}^*}$ is not minimal, i.e., every generic extension, obtained by forcing with $P_{\vec{F}^*}$ over V, has a non-trivial intermediate model.*

Proof. First, we prove that H is Q-generic over V. The only non-trivial property is that, for every $D \subseteq Q$ dense and open, $D \cap H \neq \emptyset$. Indeed, given such D, define, for every $s \in [Q]^{<\omega}$, $A_s = D \cap (Q/\text{mc}(s))$. Then $A_s \in F^*_s$, and thus, for some n_0, and for every $n \geq n_0$, $p_n \in A_{\langle p_0,\dots,p_{n-1}\rangle}$. In particular, $p_n \in D$.

Assuming that Q is separative, it follows that $H \notin V$. Moreover, $V[H] \subsetneq V[\langle p_n : n < \omega\rangle]$, since, in $V[H]$, κ is still regular. Thus, $P_{\vec{F}^*}$ is not minimal. $\square$

Remark 1.1.11. *By [5], $P_{\vec{F}^*}$ might be minimal. Consider $\langle Q, <_Q\rangle = \langle \kappa, \in\rangle$. Assume that $\langle U_\alpha : \alpha < \kappa\rangle$ is a sequence of pairwise distinct normal ultrafilters. Set, for every $t \in [\kappa]^{<\omega}$, $F^*_t = U_{mc(t)}$ (more precisely, $F^*_t = \{A \cap (\kappa \setminus mc(t)) : A \in U_{mc(t)}\}$). Under these conditions, it is proved in [5] that every generic extension, which is obtained by forcing with $P_{\vec{F}^*}$ over V, doesn't have non-trivial intermediate models, i.e., $P_{\vec{F}^*}$ is minimal. In particular, if $\langle p_n : n < \omega\rangle$ and $\langle q_n : n < \omega\rangle$ are generic Prikry sequences for $P_{\vec{F}^*}$, such that –*

$$\langle q_n : n < \omega\rangle \in V[\langle p_n : n < \omega\rangle]$$

then–

$$V[\langle q_n : n < \omega\rangle] = V[\langle p_n : n < \omega\rangle]$$

By lemma 1.1.10, there exists a projection $\pi : P_{\vec{F}^*} \to \text{RO}(Q)$. Given an arbitrary generic set $H \subseteq \text{RO}(Q)$ over V, the quotient forcing $P_{\vec{F}^*}/H$ is non-trivial, since κ is still regular in $V[H]$.

Definition 1.1.12. *We say that a forcing notion $\langle P, <_P\rangle$ is cone homogeneous, if for every $a, b \in P$ there are extensions $a' >_P a$, $b' >_P b$ such that P/a' and P/b' are isomorphic.*

Proposition 1.1.13. *Let $H \subseteq RO(Q)$ be generic over V. Suppose that $P_{\vec{F}^*}/H$ is cone homogeneous. Then there are two different Prikry sequences for $P_{\vec{F}^*}$, $\langle p_n : n < \omega\rangle$, $\langle q_n : n < \omega\rangle$, such that $\langle q_n : n < \omega\rangle \in V[\langle p_n : n < \omega\rangle]$.*

Proof. Since $P_{\vec{F}^*}/H$ is non-trivial, there are incompatible elements $\langle p_0, \dots p_n, T\rangle$, $\langle q_0, \dots q_m, S\rangle$ in $P_{\vec{F}^*}/H$. By extending those elements, we can assume that, for some $i < \omega$, $p_i \neq q_i$. By homogeneity, there exists an automorphism $\sigma \in V[H]$, mapping the cone of of $P_{\vec{F}^*}/H$ above an extension of $\langle p_0, \dots p_n, T\rangle$, to the cone above some extension of $\langle q_0, \dots q_m, S\rangle$. Thus, there are pair of Prikry sequences for $P_{\vec{F}^*}/H$, $\langle p_n : n < \omega\rangle$, $\langle q_n : n < \omega\rangle$, such that σ maps the generic set (of $P_{\vec{F}^*}/H$, over $V[H]$) corresponding to $\langle p_n : n < \omega \rangle$ into the generic set corresponding to $\langle q_n : n < \omega \rangle$ (this follows by extending one sequence to a generic Prikry sequence for the quotient forcing, and then applying the point-wise image under σ). Since $\sigma \in V[H]$, $\langle q_n : n < \omega \rangle \in V[H][\langle p_n : n < \omega \rangle] = V[\langle p_n : n < \omega \rangle]$. It's clear that those sequences are different (because they have different initial segments). $\square$

Remark 1.1.14. *The same argument as in the last proposition proves that if $P_{\vec{F}^*}$ itself is cone homogeneous, then there are two different Prikry sequences for $P_{\vec{F}^*}$, $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$, such that $\langle q_n : n < \omega \rangle \in V[\langle p_n : n < \omega \rangle]$.*

1.2 Prikry Sequences Inside Generic Extensions

Assume that $\langle p_n : n < \omega \rangle$ is P-generic over V. It's natural to ask if $V[\langle p_n : n < \omega \rangle]$ contains another Prikry sequence for $P_{\vec{F}^*}$, $\langle q_n : n < \omega \rangle$. If it does, could $\langle p_n : n < \omega \rangle$ and $\langle q_n : n < \omega \rangle$ be disjoint, or "far" from each other in any other way?

By [5], there exists a variation of $P_{\vec{F}^*}$ which is minimal, i.e., every generic extension has no non-trivial intermediate models. We would like to consider variations of $P_{\vec{F}^*}$ which are not necessarily minimal, but still have the following property: If $\langle p_n : n < \omega \rangle, \langle q_n : n < \omega \rangle$ are Prikry sequences for $P_{\vec{F}^*}$, and–

$$\langle q_n : n < \omega \rangle \in V[\langle p_n : n < \omega \rangle]$$

then–

$$\langle p_n : n < \omega \rangle = \langle q_n : n < \omega \rangle$$

In particular, every generic extension, obtained by forcing with $P_{\vec{F}^*}$, doesn't have non-trivial intermediate models which are themselves generic extensions, obtained by forcing with $P_{\vec{F}^*}$ over V.

As a first example, we consider the case where the measures $\vec{F}^*$ are pairwise distinct and normal. Then, we will consider the general case.

1.2.1 Trees With Pairwise Distinct Normal Measures

Suppose that $\langle Q, <_Q \rangle = \langle \kappa, \in \rangle$, and $\langle F^*_t : t \in [\kappa]^{<\omega} \rangle$ is a sequence of pairwise distinct normal ultrafilters. We note that, for every $t \in [\kappa]^{<\omega}$, any dense open set of $Q/\mathrm{mc}(t)$ is an interval of ordinals of the form $[\alpha, \kappa)$ where $\alpha > \mathrm{mc}(t)$. Thus, any normal ultrafilter on κ will extend the κ-complete filter of dense and open sets above $\mathrm{mc}(t)$. Under these settings, we have the following property:

Theorem 1.2.1. *Suppose that $\langle Q, <_Q \rangle = \langle \kappa, \in \rangle$, and $\langle U_t \colon t \in [\kappa]^{<\omega} \rangle$ is a sequence of pairwise distinct normal ultrafilters. Consider the forcing $P_{\vec{U}}$ (which is the forcing $P_{\vec{F}^*}$, where, for every $t \in [\kappa]^{<\omega}$, $F^*_t = \{A \cap (Q/mc(t)) : A \in U_t\}$). Then for every pair of Prikry sequences for $P_{\vec{U}}$, $\langle p_n \colon n < \omega \rangle$, $\langle q_n \colon n < \omega \rangle$,*

$$\langle q_n \colon n < \omega \rangle \in V\left[\langle p_n \colon n < \omega \rangle\right] \iff \langle p_n \colon n < \omega \rangle = \langle q_n \colon n < \omega \rangle$$

Proof. Since $\langle U_t \colon t \in [Q]^{<\omega} \rangle$ are pairwise distinct normal ultrafilters, there exists a partition $\langle A_s \colon s \in [Q]^{<\omega} \rangle$ of Q, such that $A_s \in U_s$. Assume by contradiction that $\langle p_n \colon n < \omega \rangle$, $\langle q_n \colon n < \omega \rangle$ are two Prikry sequences, such that $\langle q_n \colon n < \omega \rangle \in V\left[\langle p_n \colon n < \omega \rangle\right]$.

Apply lemma 1.1.8 to find $i_0 < \omega$ such that for every $i \geq i_0$, $q_{i+1} \in A_{\langle q_0, \ldots, q_i \rangle}$ and $p_{i+1} \in A_{\langle p_0, \ldots, p_i \rangle}$. Apply lemma 1.1.9 to find $k \geq i_0$ such that $\{p_n \colon k \leq n < \omega\}, \{q_n \colon k \leq n < \omega\}$ are disjoint. Denote by G the generic set over V which corresponds to $\langle p_n \colon n < \omega \rangle$. Let $\underset{\sim}{\sigma}$ be a P-name for the sequence $\langle q_n \colon n < \omega \rangle$. Let $\langle r, T \rangle \in G$ be an element which forces the following:

1. $\underset{\sim}{\sigma}$ is a name of a P-generic Prikry sequence.
2. $\langle \underset{\sim}{\sigma}(n) \colon k \leq n < \omega \rangle$ is disjoint from $\langle p_n \colon k \leq n < \omega \rangle$ (we use the canonical name of the generic set to express $\langle p_n \colon k \leq n < \omega \rangle$).
3. For every $k < i < \omega$, $\underset{\sim}{\sigma}(i) \in A_{\underset{\sim}{\sigma} \restriction i}$.

For notational simplicity, let us assume that $r = \langle \rangle$.

For every $i < \omega$, define a partial function f_i from some subset of T to Q, as follows: Given $t \in T$,

$$f_i(t) = q \iff \exists \langle t, T_i(t) \rangle \geq^* \langle t, T_t \rangle \text{ s.t. } \langle t, T_i(t) \rangle \Vdash \underset{\sim}{\sigma}(\check{i}) = \check{q} \tag{1.1}$$

where $T_t = \{s \in T \colon t \lhd s\}$. We note that $f_i(t)$ is well defined, since $\langle t, T_t \rangle$ can't have two direct extensions which force different values for $\underset{\sim}{\sigma}(\check{i})$ (because any two such direct extensions are compatible). We proceed with several lemmas:

Lemma 1.2.2. *The following properties hold:*

1. *Assume that $m < m'$ and $m' \geq k$. Then $dom(f_{m'}) \subseteq dom(f_m)$.*

2. *For every $t \in T$ the set $\{m < \omega : t \in dom(f_m)\}$ is finite.*

3. *Assume that $s = \langle f_0(t), \ldots, f_m(t) \rangle$, $m \geq k$ and $lh(t) \geq k$. Then s, t are $\lhd$-incompatible.*

Proof. 1. Take $t \in \mathrm{dom}(f_{m'})$. Then for some $T_i(t)$ as in equation 1.1, $\langle t, T_i(t) \rangle \Vdash \underset{\sim}{\sigma}(\check{m'}) = \widetilde{f_{m'}(t)}$. There exists a unique $s \in [Q]^{<\omega}$ such that $f_{m'}(t) \in A_s$. On the other hand, since $m' \geq k$, $f_{m'}(t) \in A_{\underset{\sim}{\sigma} \restriction m'}$. Therefore, $\langle t, T_i(t) \rangle \Vdash \underset{\sim}{\sigma} \restriction \check{m'} = \check{s}$. In particular, $\langle t, T_i(t) \rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \widetilde{s(m)}$.

2. Assume the contrary. Then, from property 1, $t \in \mathrm{dom}(f_m)$ for every $m < \omega$. Take $H \subseteq P$ generic over V, such that $\langle t, T_t \rangle \in H$. Then the Prikry sequence $(\underset{\sim}{\sigma})_H$ belongs to V, since $\underset{\sim}{\sigma}(m) = f_m(t)$ for every $m < \omega$, a contradiction.

3. This follows since the weakest condition forces that $\langle \underset{\sim}{\sigma}(n) : k \leq n < \omega \rangle$ is disjoint from the Prikry sequence derived from the canonical name of the generic set.

$\square$

Lemma 1.2.3. *There exists a $\langle F^*_t : t \in [Q]^{<\omega} \rangle$-tree $T^* \subseteq T$, such that, for every $m < \omega$ there exists $n < \omega$, for which $Lev_n(T^*) \subseteq dom(f_m)$, and, if $n \neq 0$, $Lev_{n-1}(T^*) \cap dom(f_m) = \emptyset$. Moreover, given $t \in Lev_n(T^*)$, $\langle t, T^*_t \rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \widetilde{f_m(t)}$.*

Proof. First, fix some $i < \omega$. By applying lemma 1.1.5, there exists a $\langle F^*_t : t \in [Q]^{<\omega} \rangle$-tree, $T_i \subseteq T$, with the following property: For every $n < \omega$, $\mathrm{Lev}_n(T_i)$ is entirely contained in $\mathrm{dom}(f_i)$, or disjoint from $\mathrm{dom}(f_i)$. Since all the trees T_i for $i < \omega$ have the same trunk, $T^* = \bigcap_{i<\omega} T_i$ is a $\langle F^*_t : t \in [Q]^{<\omega} \rangle$-tree.

Now, given $m < \omega$, there exists $n < \omega$, such that $\mathrm{Lev}_n(T_m) \subseteq \mathrm{dom}(f_m)$, since $\langle \langle\rangle, T_m \rangle$ has an extension which decides $\underset{\sim}{\sigma}(m)$. Take the first such n. Thus, $\mathrm{Lev}_n(T^*) \subseteq \mathrm{dom}(f_m)$. If $n \neq 0$, then $\mathrm{Lev}_{n-1}(T_m) \cap \mathrm{dom}(f_m) = \emptyset$; Thus, $\mathrm{Lev}_{n-1}(T^*) \cap \mathrm{dom}(f_m) = \emptyset$.

Given m and n as above, and $t \in \mathrm{Lev}_n(T^*)$, shrink T^* above t, such that every extension belongs to $T_m(t)$ (defined in equation 1.1). This ensures that $\langle t, T^*_t \rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \widetilde{f_m(t)}$. $\square$

For every $i<\omega$, let $n_i<\omega$ be the first level of T^* contained in $\operatorname{dom}(f_i)$. We note that $\langle n_i \colon i<\omega\rangle$ is unbounded, and, from the index k, weakly increasing (this follows from lemma 1.2.2). Thus, there are unboundedly many i's such that $n_i<n_{i+1}$.

For every $i<\omega$ such that $n_i<n_{i+1}$, and $\min\{i,n_i\}>k$, let us shrink $\mathrm{Lev}_{n_{i+1}}(T^*)$. Fix some $t\in \mathrm{Lev}_{n_{i+1}-1}(T^*)$. Then $t\in\operatorname{dom}(f_i)$ (since $n_{i+1}-1\geq n_i$). Denote $s=\langle f_0(t),\ldots,f_i(t)\rangle$. Since $\min\{n_i,i\}>k$, s,t are $\lhd$-incompatible, and thus $U_s\neq U_t$ (this follows from property 3 in lemma 1.2.2).

We note that $\mathrm{Succ}_{T^*}(t)\subseteq\operatorname{dom}(f_{i+1})$. Let $f^*_{i+1}\colon \mathrm{Succ}_{T^*}(t)\to Q$ be defined as follows:

$$\forall q\in \mathrm{Succ}_{T^*}(t)\quad f^*_{i+1}(q)=f_{i+1}\left(t^\frown\langle q\rangle\right)$$

Extend f^*_{i+1} arbitrarily to the domain $Q=\kappa$, and let us consider the ultrafilter $\left(f^*_{i+1}\right)_* U_t$. Then $U_s\neq\left(f^*_{i+1}\right)_* U_t$: Else, $U_s\leq_{RK} U_t$, and by normality, $U_s=U_t$, a contradiction. Thus, there are sets $B_t\in U_t$, $C_t\in U_s$ such that $f^*_{i+1}{}''B_t\cap C_t=\emptyset$.

Let $Z_s=\{t\in \mathrm{Lev}_{n_{i+1}-1}(T^*)\colon s=\langle f_0(t),\ldots,f_i(t)\rangle\}$. We define a set $E_s\in U_s$, and for every $t\in Z_s$, an ordinal δ_t, such that the following property holds: For every $a\in E_s$ with $a>\delta_t$, $a\in C_t$. Such a set E_s exists: If $|Z_s|<\kappa$, simply take $E_s=\bigcap\limits_{t\in Z_s} C_t$, and $\delta_t=0$. Else, assume that $Z_s=\{t_\alpha\colon \alpha<\kappa\}$. For every $\alpha<\kappa$, choose $\delta_{t_\alpha}=\alpha$, and take $E_s=\mathop{\triangle}\limits_{\alpha<\kappa} C_{t_\alpha}$ (note that δ_t depends only on t).

Now, we shrink T^* above every $t\in Z_s$ twice. First, shrink T^* such that $\mathrm{Succ}_{T^*}(t)\subseteq B_t$. Then, shrink T^* such that for every $t'\in \mathrm{Lev}_{n_{i+1}}(T^*)$ with $t'\rhd t$, $f^*_{i+1}(t')>\delta_t$: This is possible, since otherwise, by κ-completeness, $\langle t,T^*_t\rangle$ would have had a direct extension which decides the value $\underset{\sim}{\sigma}(i+1)$, contradicting the minimality of n_{i+1}.

Let us describe a dense set in $P_{\vec{U}}$:

Claim 1.2.4. *The set $D=\{\langle s,S\rangle\in P_{\vec{U}}\colon mc(s)\notin f''_{lh(s)-1}T^*\}$ is dense in $P_{\vec{U}}$.*

Proof. Let $\langle s,S\rangle\in P_{\vec{U}}$. Assume that $\mathrm{lh}(s)=i+1$ for some $i<\omega$, such that $n_{i+1}>n_i$ and n_i,i are above k (else, extend $\langle s,S\rangle$). Take $q\in \mathrm{Succ}_S(s)\cap E_s$. Denote $s'=s^\frown\langle q\rangle$.

Now, assume, for contradiction, that for some $t' \in T^*$, $q' = f_{i+1}(t')$. By extending or shrinking the sequence t', we can assume that $t' \in \mathrm{Lev}_{n_{i+1}}(T^*)$. There exists $t \in \mathrm{Lev}_{n_{i+1}-1}T^*$, such that $t' \rhd t$. In particular, $\mathrm{mc}(t') \in B_t$. Therefore, $q' = f_{i+1}(t') \notin C_t$. On the other hand, $q' > \delta_t$ and $q' \in E_s$, so $q' \in C_t$. A contradiction. $\square$

Now, take a generic H such that $\langle \langle\rangle, T^* \rangle \in H$. Assume that $\langle q'_i : i < \omega \rangle = (\underset{\sim}{\sigma})_H$. Then, for every $m < \omega$, $q'_m \in f''_m(T^*)$. Therefore, $\langle q'_0, \dots, q'_m, S \rangle \notin D$, for every $\langle F^*_t : t \in [Q]^{<\omega} \rangle$-tree S with trunk $\langle q'_0, \dots, q'_m \rangle$. A contradiction, since D is dense in $P_{\vec{U}}$. $\square$

1.2.2 Trees With Arbitrary Measures

Motivated by theorem 1.2.1, it's reasonable to ask whether a similar result exists under more general settings. It turns out that the situation is much more involved without the normality of the ultrafilters. Our goal will be to prove the following theorem:

Theorem 1.2.5. *It's consistent, from κ^+-supercompactness of κ, that for every separative, κ-distributive notion of forcing Q with $|Q| = \kappa$, there exists a choice of pairwise distinct ultrafilters $\vec{F}^* = \langle F^*_t : t \in [Q]^{<\omega} \rangle$, such that $P_{\vec{F}^*}$ has the following property: For every pair of Prikry sequences for $P_{\vec{F}^*}$, $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$,*

$$\langle q_n : n < \omega \rangle \in V\left[\langle p_n : n < \omega \rangle\right] \iff \langle p_n : n < \omega \rangle = \langle q_n : n < \omega \rangle$$

The proof of theorem 1.2.5 will be presented in two steps: First, we assume that $\langle q_n : n < \omega \rangle \in V\left[\langle p_n : n < \omega \rangle\right]$, and show that a certain connection between the ultrafilters $\langle F^*_t : t \in [Q]^{<\omega} \rangle$ is induced (theorem 1.2.7). Then, we prove that the existence of a sequence of measures $\langle F^*_t : t \in [Q]^{<\omega} \rangle$ without this connection is consistent from κ^+-supercompactness of κ. The first step is presented in this section; The second step will be presented in the next section.

Definition 1.2.6. *Let $t \in [Q]^{<\omega}$, and $n < \omega$ such that $lh(t) < n$. Denote $n' = n - lh(t)$. Define an ultrafilter $U_n(t)$ as follows: $A \in U_n(t)$ if and only if –*

$$\{\nu_1 \in Q : \{\nu_2 \in Q : \dots \{\nu_{n'} \in Q : t^\frown \langle \nu_1, \dots, \nu_{n'} \rangle \in A\} \in F^*_{t^\frown \langle \nu_1, \dots, \nu_{n'-1} \rangle} \dots\} \in F^*_{t^\frown \langle \nu_1 \rangle}\} \in F^*_t$$

$U_n(t)$ is a non-trivial κ-complete ultrafilter on a set of cardinality κ –

$$\{t^\frown\nu\colon \nu \in [Q/mc(t)]^{n'}\}$$

Our goal, in this section, will be to prove the following theorem:

Theorem 1.2.7. *Let $\langle p_n\colon n<\omega\rangle$, $\langle q_n\colon n<\omega\rangle$ be two different Prikry sequences for $P_{\vec{F}^*}$, such that $\langle q_n\colon n<\omega\rangle \in V\left[\langle p_n\colon n<\omega\rangle\right]$. Assume:*

1. *$\pi\colon Q\to\kappa$ is a function such that, for every $t\in[Q]^{<\omega}$,*

$$[\pi\restriction Q/mc(t)]_{F^*_t} = \kappa$$

2. *The ultrafilters $\langle F^{*nor}_t\colon t\in[Q]^{<\omega}\rangle$ are pairwise distinct, where –*

$$F^{*nor}_t = \{X\subseteq\kappa\colon \pi^{-1}{}''X \in F^*_t\}$$

Then there are $\lhd$-incompatible sequences $s,t\in[Q]^{<\omega}$, $n>lh(t)$ and functions $f,g\colon \cup U_n(t)\to Q$, such that –

$$f_*U_n(t) = g_*U_n(t)\text{-}lim\langle F^*_{s^\frown\langle q\rangle}\colon q>_Q mc(s)\rangle$$

and both $f_*U_n(t)$, $g_*U_n(t)$ are non-trivial ultrafilters.

Proof. First, we note that there exists a partition $\langle A_s\colon s\in[Q]^{<\omega}\rangle$ of Q, such that $A_s\in F^*_s$: Indeed, fix a disjoint partition $\langle A^{nor}_s\colon s\in[Q]^{<\omega}\rangle$ such that for every $s\in[Q]^{<\omega}$, $A^{nor}_s\in F^{*nor}_s$, and take $A_s=\pi^{-1}A^{nor}_s$.

We start with the same arguments applied in theorem 1.2.1. Assume that $\langle p_n\colon n<\omega\rangle$, $\langle q_n\colon n<\omega\rangle$ are two Prikry sequences, such that $\langle q_n\colon n<\omega\rangle \in V\left[\langle p_n\colon n<\omega\rangle\right]$. Apply lemmas 1.1.8 and 1.1.9 to find $k<\omega$ such that for every $i\geq k$, $q_{i+1}\in A_{\langle q_0,\ldots,q_i\rangle}$, $p_{i+1}\in A_{\langle p_0,\ldots,p_i\rangle}$, and $\{p_n\colon k\leq n<\omega\},\{q_n\colon k\leq n<\omega\}$ are disjoint. Denote by G the generic set over V which corresponds to $\langle p_n\colon n<\omega\rangle$. Let $\underset{\sim}{\sigma}$ be a P-name for the sequence $\langle q_n\colon n<\omega\rangle$. Let $\langle r,T\rangle\in G$ be an element which forces the following:

1. $\underset{\sim}{\sigma}$ is a name of a P-generic Prikry sequence.
2. $\langle\underset{\sim}{\sigma}(n)\colon k\leq n<\omega\rangle$ is disjoint from $\langle p_n\colon k\leq n<\omega\rangle$ (we use the canonical name of the generic set to express $\langle p_n\colon k\leq n<\omega\rangle$).
3. For every $k\leq i<\omega$, $\underset{\sim}{\sigma}(i)\in A_{\underset{\sim}{\sigma}\restriction i}$.

For notational simplicity, let us assume that $r = \langle\rangle$.

For every $i < \omega$, define a partial function f_i, from some subset of T to Q, just as in the proof of theorem 1.2.1: Given $t \in T$,

$$f_i(t) = q \iff \exists \langle t, T_i(t) \rangle \geq^* \langle t, T \rangle \text{ s.t. } \langle t, T_i(t) \rangle \Vdash \underset{\sim}{g}(\check{i}) = \check{q} \tag{1.2}$$

Lemma 1.2.2 holds here as well. Now, let us shrink T to a $\langle F_t^* : t \in [Q]^{<\omega} \rangle$-tree, T^*, as follows:

Lemma 1.2.8. *There exists a $\langle F_t^* : t \in [Q]^{<\omega} \rangle$-tree, $T^* \subseteq T$, and strictly increasing sequences $\langle n_i : i < \omega \rangle$, $\langle m_i : i < \omega \rangle$, such that –*

1. *$Lev_{n_i}(T^*) \subseteq dom(f_{m_i})$, and for every $t \in Lev_{n_i}(T^*)$, $\langle t, T^* \rangle \Vdash \underset{\sim}{g}(\check{m}_i) = \widetilde{f_{m_i}(t)}$.*
2. *$Lev_{n_i}(T^*)$ and $dom(f_{m_i+1})$ are disjoint sets.*
3. *$k < m_0, n_0$.*

Proof. First, fix some $i < \omega$. By applying lemma 1.1.5, there exists a $\langle F_t^* : t \in [Q]^{<\omega} \rangle$-tree, $T_i \subseteq T$, with the following property: For every $n < \omega$, $\mathrm{Lev}_n(T_i)$ is entirely contained in $\mathrm{dom}(f_i)$, or disjoint to $\mathrm{dom}(f_i)$. Since all the trees T_i for $i < \omega$ have the same trunk, $T^* = \bigcap_{i<\omega} T_i$ is a $\langle F_t^* : t \in [Q]^{<\omega} \rangle$-tree.

Assume that $\langle n_j : j < i \rangle$ were defined, and let us define n_i. Take some extension $\langle t, T_t^* \rangle$ of $\langle \langle\rangle, T^* \rangle$ such that $t \in \mathrm{dom}(f_i)$ (such an extension exists, by extending the given condition to one which decides the value of $\underset{\sim}{g}(\check{i})$). Assume that $\mathrm{lh}(t) > \sup\{n_j : j < i\}$ (Else - extend it). Set $n_i = \mathrm{lh}(t)$.

For every $i < \omega$ and $t \in \mathrm{Lev}_{n_i}(T^*)$, let $m_t < \omega$ be the maximum value of m such that $f_m(t)$ exists (such maximal value exists, by part 2 of lemma 1.2.2). By applying lemma 1.1.5 again, we can assume that m_t is constant on every level of T^* (else, shrink T^*). Let m_i be the constant value on level n_i. Then the sequence $\langle m_i : i < \omega \rangle$ is weakly-increasing, and for every $i < \omega$, $m_i \geq i$. By passing to a subsequence of $\langle n_i : i < \omega \rangle$, let us assume that $\langle m_i : i < \omega \rangle$ is strictly increasing, and $n_0, m_0 \geq k$. We note that $\mathrm{Lev}_{n_i}(T^*) \subseteq \mathrm{dom}(f_{m_i})$. Moreover, by maximality of m_i, for every $t \in \mathrm{Lev}_{n_i}(T^*)$, $t \notin \mathrm{dom}(f_{m_i+1})$.

Now, for every $i < \omega$ and $t \in \mathrm{Lev}_{n_i}(T^*)$, shrink T^* above t such that $\{t' \in T^* : t' \rhd t\} \subseteq T_i(t)$, where $T_i(t)$ is as in equation 1.2. It follows that $\langle t, T_t^* \rangle \Vdash \underset{\sim}{g}(\check{m}_i) = \widetilde{f_{m_i}(t)}$. □

For every $s \in [Q]^{<\omega}$, with $\mathrm{lh}(s) = m_i + 1$ for some $i < \omega$, define –

$$Z_s = \{t \in \mathrm{Lev}_{n_i}(T^*) \colon s = \langle f_0(t), \ldots, f_{m_i}(t) \rangle\}$$

Inductively, for very $i < \omega$, let us shrink T^* above $\mathrm{Lev}_{n_i}(T^*)$. Fix such i and $t \in \mathrm{Lev}_{n_i}(T^*)$. Assume that $s = \langle f_0(t), \ldots, f_{m_i}(t) \rangle$. Then s, t are $\lhd$-incompatible (they at least differ in the coordinate k, since $n_0, m_0 > k$).

We construct a set $B''(t) \in U_{n_{i+2}}(t)$, and shrink T^* above t, such that, after shrinking, we have $\{t' \in \mathrm{Lev}_{n_{i+2}}(T^*) \colon t' \rhd t\} \subseteq B''(t)$. The first step to construct $B''(t)$ will be the following observation: if –

$$(f_{m_i+2})_* U_{n_{i+2}}(t) = (f_{m_i+1})_* U_{n_{i+2}}(t)\text{-}\lim\langle F^*_{s^\frown\langle q\rangle} \colon q >_Q \mathrm{mc}(s)\rangle$$

(we assume that f_{m_i+2} and f_{m_i+1} were extended arbitrarily on elements of $\cup U_{n_{i+2}}(t)$ which don't belong to T^*), then this proves theorem 1.2.7. Indeed, s, t are $\lhd$-incompatible, and $(f_{m_i+2})_* U_{n_{i+2}}(t), (f_{m_i+1})_* U_{n_{i+2}}(t)$ are non-trivial (otherwise, t would have had a direct extension which decides the value of $\underset{\sim}{\sigma}(m_i + 1)$. This is not possible, since $t \in \mathrm{Lev}_{n_i}(T^*)$). Thus, we can assume, for contradiction, that for every t, s as above,

$$(f_{m_i+2})_* U_{n_{i+2}}(t) \neq (f_{m_i+1})_* U_{n_{i+2}}(t)\text{-}\lim\langle F^*_{s^\frown\langle q\rangle} \colon q >_Q \mathrm{mc}(s)\rangle$$

so there exists $B(t) \in U_{n_{i+2}}(t)$ such that –

$$X(t) = \{q \in Q/\mathrm{mc}(s) \colon f''_{m_i+2} B(t) \cap A_{s^\frown\langle q\rangle} \in F^*_{s^\frown\langle q\rangle}\} \notin (f_{m_i+1})_* U_{n_{i+2}}(t)$$

Denote $B'(t) = B(t) \setminus f^{-1}_{m_i+1}{}'' X(t)$.

Now, for every $q \in Q/\mathrm{mc}(s)$ and for every $t \in Z_s$, define a set $C_q(t)$ as follows: If $q \notin X(t)$, then we know that $f''_{m_i+2} B(t) \cap A_{s^\frown\langle q\rangle} \notin F^*_{s^\frown\langle q\rangle}$. Let $C_q(t) \in F^*_{s^\frown\langle q\rangle}$ be a set disjoint from $f''_{m_i+2} B(t)$. Otherwise, take $C_q(t) = Q/q$.

Let us define a set $C_q \in F^*_{s^\frown\langle q\rangle}$, and, for every $t \in Z_s$, an ordinal δ_t (which depends only on t), such that for every $a \in C_q$ with $\pi(a) > \delta_t$, $a \in C_q(t)$. Such a set exists: If $|Z_s| < \kappa$, simply take $C_q = \bigcap_{t \in Z_s} C_q(t)$, and $\delta_t = 0$. Else, assume that $Z_s = \{t_\alpha \colon \alpha < \kappa\}$. For every $\alpha < \kappa$, choose $\delta_{t_\alpha} = \alpha$. Fix $q \in Q/\mathrm{mc}(s)$. Since $[\pi \restriction Q/q]_{F^*_{s^\frown\langle q\rangle}} = \kappa$, the required set C_q could be defined as follows:

$$C_q = \{a \in Q/q \colon \forall \alpha < \pi(a),\ a \in C_q(t_\alpha)\} \in F^*_{s^\frown\langle q\rangle}$$

Now, we finally define the set $B''(t)$ described above. Given $t \in \text{Lev}_{n_i}(T^*)$ and $s = \langle f_0(t), \ldots, f_{m_i}(t) \rangle$, we claim that there exists a set $B'(t) \in U_{n_{i+2}}(t)$, such that $B''(t) \subseteq B'(t)$, and for every $t' \in B''(t)$, $\pi(f_{m_i+2})(t') > \delta_t$: Indeed, else, by κ-completeness, there exists a direct extension of $\langle t, T^* \rangle$ which forces that $\pi(\underset{\sim}{\sigma}(m_i + 2)) = \alpha^*$ for some $\alpha^* < \kappa$; There exists a unique s' such that $\alpha^* \in A^{nor}_{s'}$. $m_i + 2 \geq k$, so the above direct extension forces in particular that $\underset{\sim}{\sigma} \restriction m_i + 2 = s'$, and therefore $t \in \text{dom}(f_{m_i+1})$. This is a contradiction, since $t \in \text{Lev}_{n_i}(T^*)$. Therefore, there exists $B''(t)$ as described above. Shrink T^* above t using $B''(t) \in U_{n_{i+2}}(t)$.

Now, let us describe a dense subset D of $P_{\vec{F}^*}$. The density of D is a contradiction, just as in the end of the proof of theorem 1.2.1.

Claim 1.2.9. *The set $D = \{\langle s, S \rangle \in P \colon mc(s) \notin f''_{lh(s)-1} T^*\}$ is dense in P.*

Proof. Let $\langle s, S \rangle \in P$. Assume that $\text{lh}(s) = m_i + 1$ for some $i < \omega$ (else, extend it). Take $q \in \text{Succ}_S(s)$, and $q' \in \text{Succ}_S(s^\frown \langle q \rangle) \cap C_q$. Denote $s' = s^\frown \langle q, q' \rangle$.

Now, assume, for contradiction, that for some $t' \in T^*$, $q' = f_{m_i+2}(t')$. By extending or shrinking the sequence t', we can assume that $t' \in \text{Lev}_{n_{i+2}}(T^*)$. There exists $t \in T^*$, $\text{lh}(t) = n_i$, such that $t' \rhd t$. In particular, $t' \in B''(t)$. Therefore, $\pi(q') > \delta_t$, so $q' \in C_q(t)$. On the other hand, $q' = f_{m_i+2}(t') \in f''_{m_i+2} B(t)$. Therefore $C_q(t)$ and $f''_{m_i+2} B(t)$ are not disjoint, so $q \in X(t)$. But $q = f_{m_i+1}(t')$, so $t' \in f^{-1}_{m_i+1}{}'' X(t)$. This is a contradiction to the definition of $B'(t)$. $\square$

This finishes the proof of theorem 1.2.7. $\square$

1.3 Extension Of The Kunen-Paris Construction

Our goal in this section will be to prove the following:

Theorem 1.3.1. *The following is consistent from κ^+-supercompactness of κ: For every separative, κ-distributive forcing notion Q with $|Q| = \kappa$, and for every $t \in [Q]^{<\omega}$, there exists a κ-complete ultrafilter F^*_t extending the filter of dense open subsets above $mc(t)$, such that there are no connections of the form:*

$$f_* U_n(t) = g_* U_n(t) - lim\langle F^*_{s^\frown\langle q \rangle} \colon q \in Q/mc(s) \rangle \tag{1.3}$$

for any pair of non-empty, $\lhd$-incompatible sequences $s, t \in [Q]^{<\omega}$, and for every $f, g \colon \cup U_n(t) \to Q$.

Assume GCH, and let κ be a κ^+-supercompact cardinal. Assume that $j \colon V \to M$ is an elementary embedding which witnesses the κ^+-supercompactness of κ, i.e., $\mathrm{crit}(j) = \kappa$, ${}^{\kappa^+}M \subseteq M$ and $j(\kappa) > \kappa^+$. Assume that this embedding is derived from a fine, normal measure on $\mathcal{P}_\kappa \kappa^+$; Thus,

Lemma 1.3.2. *The following properties hold:*

1. $V \models |j(\kappa)| = \kappa^{++}$
2. *sup* $j''\kappa^{++} = j(\kappa^{++})$
3. $j\left(\kappa^{+3}\right) = \kappa^{+3}$

This is a standard lemma; A detailed proof is presented, for example, in [1], section 4.

We would like to build a model which carries, for every $t \in [Q]^{<\omega}$, an elementary embedding j_t, which witnesses the κ^+-supercompactness of κ. Then, use the embedding j_t to extend F_t to a κ-complete ultrafilter F^*_t (the exact way in which this is done will be explained later). The main idea here is that using different elementary embeddings should prevent dependence between the ultrafilters $\langle F^*_t \colon t \in [Q]^{<\omega} \rangle$.

One possible way to construct many elementary embeddings, is to push forward a well known construction of Kunen and Paris, which maximalizes the number of normal measures on κ: Using κ^+-supercompactness of κ, we will construct a model which carries a definable sequence of elementary embeddings $\langle j_\alpha \colon \alpha < \kappa^{++} \rangle$, each one witnesses the κ^+-supercompactness of κ; This could be done such that the derived normal measures, $U_\alpha = \{X \subseteq \kappa \colon \kappa \in j_\alpha(X)\}$ are pairwise distinct, and, in a way, are "far" from each other.

Before we describe the construction, we fix a standard notation:

Notation. *For a set S of ordinals, define –*

$$\mathit{Cohen}\left(\kappa^+, S\right) = \{f \colon \kappa^+ \times S \to 2 \colon f \text{ is a partial function, } |f| \leq \kappa\}$$

Define an iteration of length $\kappa+1$, $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \kappa+1, \beta \leq \kappa\rangle$ with Easton support (direct limits are taken in regular limit stages, inverse limits elsewhere). For every inaccessible $\alpha \leq \kappa$, take $\underset{\sim}{Q}_\alpha$ to be a P_α-name for the forcing:

$$\mathrm{Cohen}(\alpha^+, \alpha^{++}) = \{f \colon \alpha^+ \times \alpha^{++} \to 2 \colon f \text{ is a partial function, } |f| < \alpha^+\}$$

For every other value of α, let $\underset{\sim}{Q}_\alpha$ name the trivial forcing. Denote for convenience $P = P_{\kappa+1}$.

Let G be P_κ-generic over V, and g be $\mathrm{Cohen}(\kappa^+, \kappa^{++})$-generic over $V[G]$. We will prove that the model $V[G,g]$ has a definable κ^{++}-sequence of elementary embeddings, as described above:

Theorem 1.3.3. *The model $V[G,g]$ carries, for every $\alpha < \kappa^{++}$, a definable elementary embedding, $j_\alpha \colon V \to M_\alpha$, such that $j_\alpha \supseteq j$ and ${}^{\kappa^+}M_\alpha \subseteq M_\alpha$, and the derived normal measures, $U_\alpha = \{X \subseteq \kappa \colon \kappa \in j_\alpha(X)\}$, are pairwise distinct.*

As it turns out, constructing the ultrafilters F^*_t from the embeddings j_α will not be enough to rule out (1.3). Thus, we will construct another sequence of elementary embeddings, $\langle j_t \colon t \in [Q]^{<\omega}\rangle$, where, for every $t \in [Q]^{<\omega}$, j_t is definable in some intermediate model $V[G, g_t] \subseteq V[G,g]$ (so, j_t will be an elementary embedding with domain $V[G,g_t]$ and not $V[G,g]$). Then, we will define the corresponding ultrafilters, F^*_t, each derived from j_t in $V[G,g_t]$. This method will reduce the amount of Cohen functions which F^*_t depends on; This will be necessary for our purposes.

In this section, we describe the constructions of the embeddings j_α and j_t, for $\alpha < \kappa^{++}$ and $t \in [Q]^{<\omega}$. This will be done in subsections 1.3.1 and 1.3.2. The embeddings $\langle j_\alpha \colon \alpha < \kappa^{++}\rangle$ will be applied to prove theorem 1.3.3; The embeddings $\langle j_t \colon t \in [Q]^{<\omega}\rangle$ will be applied to construct the ultrafilters $\langle F^*_t \colon t \in [Q]^{<\omega}\rangle$. In subsection 1.3.3 we confirm that (1.3) cannot hold.

In subsections 1.3.1, 1.3.2 we will use standard methods for extending elementary embeddings. We follow mainly Cummings' handbook article [1].

Notation. *We fix some notations, which will be used throughout the entire section:*

1. Assume that $Q \in V[G,g]$ is a non-trivial, κ-distributive forcing notion, with $|Q| = \kappa$. Since g is a generic set for a κ^+-distributive forcing notion, we can assume that $Q \in V[G]$ (by identifying Q with an isomorphic order on κ).

2. Denote, for every $\alpha < \kappa^{++}$, the function $g_\alpha\colon \kappa^+ \to 2$, defined as follows:
$$\forall x \in \kappa^+ \quad g_\alpha(x) = g(x,\alpha)$$
This is the α-th Cohen function which g adds.

3. Let $N = \mathrm{Ult}(V,U)$, where $U = \{X \subseteq \kappa \colon \kappa \in j(X)\}$. Let $i\colon V \to N$ be the corresponding elementary embedding. Then $\mathrm{crit}(i) = \kappa$ and ${}^{\kappa}N \subseteq N$. We note that $(\kappa^+)^N = \kappa^+$ and $(\kappa^{++})^N < \kappa^{++}$.

4. Fix, in $V[G,g]$, a subset $X \subseteq \kappa^{++} \setminus (\kappa^{++})^N$ with $|X| = \kappa$, and a bijection $\phi\colon [Q]^{<\omega} \to X$. By identifying Q with κ, we can actually assume that $\phi, Q, X \in V[G]$, since g is generic for a κ^+-closed forcing notion.

5. Denote $g \setminus X = g \cap (\kappa^+ \times (\kappa^{++} \setminus X) \times 2)$. This is the set of the Cohen functions indexed by an element of $\kappa^{++} \setminus X$, i.e., not of the form $g_{\phi(t)}$ for some $t \in [Q]^{<\omega}$.

6. For every $t \in [Q]^{<\omega}$, we would like to extend $g \setminus X$ to a generic set for $\mathrm{Cohen}(\kappa^+, \kappa^{++})$, using only one Cohen function, $g_{\phi(t)}$. This could be done as follows: In $V[G]$, fix an isomorphism–
$$\sigma\colon \mathrm{Cohen}\left(\kappa^+, \{\phi(t)\}\right) \to \mathrm{Cohen}\left(\kappa^+, X\right)$$
For every $t \in [Q]^{<\omega}$, define a function $g_t\colon \kappa^+ \times \kappa^{++} \to 2$,
$$g_t = (\cup (g \setminus X)) \cup \left(\cup \sigma'' \left(g_{\phi(t)}\right)\right)$$
We identify g_t with the generic set for $\mathrm{Cohen}(\kappa^+, \kappa^{++})$, over $V[G]$ it defines. Clearly, $V[G, g_t] \subseteq V[G,g]$.

1.3.1 Extending N With a Generic Set For $i(P)$

Let us extend N with a generic set for $i(P)$. $i(P)$ is an Easton iteration of length $i(\kappa)+1$. $i(P)_\alpha = P_\alpha$ for every $\alpha \leq \kappa$ (if $\alpha < \kappa$ this holds because $P_\alpha \in V_\kappa$. If $\alpha = \kappa$, this holds because a direct limit is taken at κ).

Since $N \subseteq V$, G is P_κ-generic over N. We would like to extend $N[G]$ to a model of the form $N[G, g', H_\alpha, h_\alpha]$, for every $\alpha < \kappa^{++}$. Here:

1. g' will be $(\text{Cohen}(\kappa^+, \kappa^{++}))^N$-generic over $N[G]$.

2. H_α will be $P_{[\kappa+1, i(\kappa))}$-generic over $N[G, g']$.

3. h_α will be Cohen$(i(\kappa^+), i(\kappa^{++}))$-generic over $N[G, g', H_\alpha]$.

Remark 1.3.4. *Every construction which will be done in this subsection could be applied on $V[G, g_t]$ instead of $V[G, g]$. So, in this subsection, we also extend $N[G]$ to a model of the form $N[G, g', H_t, h_t]$, for every $t \in [Q]^{<\omega}$. Here:*

1. g' will be the same $(\text{Cohen}(\kappa^+, \kappa^{++}))^N$-generic over $N[G]$.

2. H_t will be $P_{[\kappa+1, i(\kappa))}$-generic over $N[G, g']$.

3. h_t will be Cohen$(i(\kappa^+), i(\kappa^{++}))$-generic over $N[G, g', H_t]$.

Claim 1.3.5. *Given $X \in V[G]$ such that $|X| \leq \kappa$ and $X \subseteq N[G]$, it follows that $X \in N[G]$. In particular, $V[G] \vDash {}^\kappa N[G] \subseteq N[G]$.*

Proof. First, let us show that it suffices to prove the claim for X a set of ordinals: Given $X \subseteq N[G]$, define $X' = \{\text{rank}(x) \colon x \in X\}$. Then $X' \in N[G]$, and let $\alpha > \sup(X')$. In N, fix a cardinal μ and a bijection $\phi \colon V_\alpha \to \mu$; Then define $X'' = \{\phi(x) \colon x \in X\}$. So X'' is a set of ordinals, and therefore $X'' \in N[G]$. Thus, $X = \phi^{-1} X'' \in N[G]$.

Now, let us prove the claim for a set of ordinals X. So $X \in V[G]$, and $|X| \leq \kappa$. Since P_κ is κ-c.c., there exists a set of ordinals $X' \in V$ such that $|X'| \leq \kappa$ and $X \subseteq X'$. Since ${}^\kappa N \subseteq N$, $X' \in N$. Assume that $\underset{\sim}{\sigma}$ is a P_κ-name for X. For every $\alpha \in X'$, let A_α be an antichain, maximal among the antichains contained in $\{p \in P_\kappa \colon p \Vdash \check{\alpha} \in \underset{\sim}{\sigma}\}$. Then, for every $\alpha < \kappa$, $|A_\alpha| < \kappa$, so $A_\alpha \in N$. It follows that $\vec{A} = \langle A_\alpha \colon \alpha \in X' \rangle \in N$. Now, define in $N[G]$ the set $\{\alpha \in X' \colon G \cap \vec{A}(\alpha) \neq \emptyset\}$. We claim that this set is X (and, therefore, $X \in N[G]$). Clearly, $\{\alpha \in X' \colon G \cap \vec{A}(\alpha) \neq \emptyset\} \subseteq X$. On the other hand, given $\alpha \in X$, there exists $p \in G$ such that $p \Vdash \check{\alpha} \in \underset{\sim}{\sigma}$; Now, we note that the following set is dense in P_κ–

$$D = \{q \colon q \text{ extends some } r \in A_\alpha\} \cup \{q \colon q \text{ and } p \text{ are incompatible}\}$$

So there exists $q \in G \cap D$, and since $p \in G$, q extends some element in A_α. Therefore, $G \cap A_\alpha \neq \emptyset$. $\square$

Claim 1.3.6. *Let* $g' = \left\{f \cap \left(\kappa^+ \times (\kappa^{++})^N \times 2\right) : f \in g\right\}$. *Then* $g' \subseteq N[G]$. *Moreover,* g' *is* $Cohen\,(\kappa^+, \kappa^{++})^{N[G]}$*-generic over* $N[G]$.

Proof. Let us prove that $g' \subseteq N[G]$, i.e., for every $f \in g'$, $f \in N[G]$. Denote $\mu = |\mathrm{dom}(f)| \leq \kappa$. $\mathrm{dom}(f) \in V[G]$ is a set of ordinals, so, by the last claim, $\mathrm{dom}(f) \in N[G]$. Let $h\colon \mu \to \mathrm{dom}(f)$ be a bijection in $V[G]$. So $h \in N[G]$. Define $F\colon \mu \to N[G]$ as follows: $F(\alpha) = f(h(\alpha))$. Then $F \in N[G]$. Therefore, $f \in N[G]$, since $f(\alpha) = F(h^{-1}(\alpha))$.

We prove that g' is Cohen $(\kappa^+, \kappa^{++})^{N[G]}$-generic over $N[G]$. Clearly g' is downwards closed, and any $f, f' \in g'$ are compatible. Given a dense subset $D' \in N[G]$ of Cohen $(\kappa^+, \kappa^{++})^{N[G]}$, we can define in $V[G]$ the set –

$$D = \left\{f \in \text{Cohen}\left(\kappa^+, \kappa^{++}\right) : f \cap \left(\kappa^+ \times \left(\kappa^{++}\right)^N \times 2\right) \in D'\right\}$$

It's routine to verify that $D \in V[G]$ is dense in Cohen (κ^+, κ^{++}). Take $f \in g \cap D$. Then $f \cap \left(\kappa^+ \times (\kappa^{++})^N \times 2\right) \in N[G]$, and $f \cap \left(\kappa^+ \times (\kappa^{++})^N \times 2\right) \in g' \cap D'$. $\square$

Claim 1.3.7. *Given* $X \in V[G, g]$ *such that* $|X| \leq \kappa$ *and* $X \subseteq N[G, g']$, *it follows that* $X \in N[G, g']$. *In particular,* $V[G, g] \vDash {}^\kappa N[G, g'] \subseteq N[G, g']$.

Proof. We can assume that X is a set of ordinals. Then, $X \in V[G, g]$, and $|X| \leq \kappa$. Since Cohen(κ^+, κ^{++}) is κ^+ closed, it follows that $X \in V[G]$. As a set of ordinals, $X \subseteq N[G]$. Therefore, $X \in N[G]$. $\square$

Remark 1.3.8. *Similarly,* $V[G, g'] \vDash {}^\kappa N[G, g'] \subseteq N[G, g']$, *and, for every* $t \in [Q]^{<\omega}$, $V[G, g_t] \vDash {}^\kappa N[G, g'] \subseteq N[G, g']$ *(we note that* $g' \subseteq g_t$, *since* $X \subseteq \kappa^{++} \setminus (\kappa^{++})^N$*).*

In $N[G, g']$, consider the quotient forcing $i(P)/G * g'$. Denote it by $i(P)_{(\kappa, i(\kappa))}$. Our goal is to construct, for every $\alpha < \kappa^{++}$, a $i(P)_{(\kappa, i(\kappa))}$-generic set over $N[G, g']$, H_α, which belongs to $V[G, g]$. To do so, we need the following standard lemma:

Lemma 1.3.9. *In* N, *let* μ *be the first inaccessible cardinal above* κ. *Then –*

1. $N[G, g'] \vDash$ *"*$i(P)_{(\kappa, i(\kappa))}$ *is* μ*-closed".*

2. $V[G,g'] \vDash$ *"$i(P)_{(\kappa,i(\kappa))}$ is κ^+-closed".*

Proof. Let us prove 1. Work in $N[G,g']$. Then:

1. Every limit in $i(P)_{(\kappa,i(\kappa))}$ is either direct or inverse.

2. Every set of ordinals of size less then μ in $N[G,g']$, is covered by a set of size less then μ in N: Indeed, in N, $i(P)_{\kappa+1}$ is μ-c.c., since it has cardinality $< \mu$.

3. μ is a regular uncountable cardinal in $N[G,g']$.

4. For every $\gamma \in [\kappa+1, i(\kappa))$, $\Vdash_{P_\gamma}$ "$\underset{\sim}{Q}_\gamma$ is κ closed": If $\underset{\sim}{Q}_\gamma$ names trivial forcing notion, this clearly holds; Else, $\gamma \geq \mu$, and then $\underset{\sim}{Q}_\gamma$ names a μ-closed forcing notion.

5. For every limit $\gamma \in [\kappa+1, i(\kappa)]$ with $\mathrm{cf}(\gamma) < \mu$, an inverse limit is taken in $i(P)_{i(\kappa)}$: Indeed, a direct limit is taken only at inaccessibles; Every inaccessible $\gamma \in [\kappa+1, i(\kappa))$ satisfies $\gamma \geq \mu$, so $\mathrm{cf}(\gamma) = \gamma \geq \mu$.

Now, for a regular uncountable μ such that conditions $1-5$ above hold, $\Vdash_{i(P)_{\kappa+1}}$ "$i(P)_{(\kappa,i(\kappa))}$ is μ closed" (see [1], section 7, for details). So–

$$N[G,g'] \vDash \text{"}i(P)_{(\kappa,i(\kappa))} \text{ is } \mu\text{-closed"}$$

As for 2, it is known that $V[G,g'] \vDash {}^{\kappa}N[G,g'] \subseteq N[G,g']$ and $V[G,g'] \vDash |\mu| = \kappa^+$. Therefore, $V[G,g'] \vDash$ "$i(P)_{(\kappa,i(\kappa))}$ is κ^+-closed". $\square$

Lemma 1.3.10. *In $N[G,g']$, let Z be the set of maximal antichains in $i(P)_{(\kappa,i(\kappa))}$. Then $V[G,g'] \vDash |Z| = \kappa^+$.*

Proof. In $N[G,g']$, $|i(P)_{(\kappa,i(\kappa))}| = i(\kappa)$, and $i(P)_{(\kappa,i(\kappa))}$ is $i(\kappa)$-c.c., since $i(\kappa)$ is Mahlo. Therefore, $N[G,g'] \vDash |Z| \leq i(\kappa)^{<i(\kappa)}$. In order to calculate $i(\kappa)^{<i(\kappa)}$ in $N[G,g']$, let us prove that $i(\kappa)$ remains inaccessible in $N[G,g']$: Note that $i(P)_{\kappa+1}$ is μ-c.c., where μ is the first inaccessible in N above κ. We use a nice name argument: First, $V \vDash \forall\beta < \kappa\ |P_\beta| < \kappa$. Therefore, $N \vDash |i(P)_{\kappa+1}| < i(\kappa)$; Denote this cardinality by $\lambda < i(\kappa)$. It follows that there are at most λ^μ antichains in $i(P)_{\kappa+1}$. Now, for every $\tau < i(\kappa)$, there are, in N, at most $(\lambda^\mu)^\tau < i(\kappa)$ nice names for subsets of τ. Therefore, $N[G,g'] \vDash \forall\tau < i(\kappa)\ 2^\tau <$

$i(\kappa)$. Also, $i(\kappa) > \mu$, so $i(\kappa)$ remains regular in $N[G, g']$. Thus, in $N[G, g']$, $i(\kappa)^{<i(\kappa)} = i(\kappa)$.

Up to now we proved that $N[G, g'] \vDash |Z| \leq i(\kappa)$. Now, $V \vDash |i(\kappa)| = \kappa^+$, and so $V[G, g'] \vDash |i(\kappa)| = \kappa^+$. Therefore, $V[G, g'] \vDash |Z| \leq \kappa^+$.

Now, $V[G, g'] \vDash |Z| = \kappa^+$: Otherwise, since–

$$V[G, g'] \vDash \text{"} i(P)_{(\kappa, i(\kappa))} \text{ is } \kappa^+ \text{ closed "}$$

there would exist a condition $p \in i(P)_{(\kappa, i(\kappa))}$ such that $\{q \in i(P)_{(\kappa, i(\kappa))} \colon q \leq p\}$ intersects every element of Z, and thus a generic set for $i(P)_{(\kappa, i(\kappa))}$, which belongs to $N[G, g']$; This is not possible since $i(P)_{(\kappa, i(\kappa))}$ is non-trivial. $\square$

Now, we can construct a generic set for $i(P)_{(\kappa, i(\kappa))}$ over $N[G, g']$, which belongs to $V[G, g]$. This is done in the next lemma.

Lemma 1.3.11. *There exists an injection $A \colon 2^{<\kappa^+} \to i(P)_{(\kappa, i(\kappa))}$, $A \in V[G, g']$, such that, for every $\alpha < \kappa^{++}$, the following set, defined in $V[G, g]$,*

$$H_\alpha = \{p \in i(P)_{(\kappa. i(\kappa))} \colon \exists \beta < \kappa^+ \ \ p \leq_{P_{(\kappa, i(\kappa))}} A(g_\alpha \restriction \beta)\}$$

is generic for $i(P)_{(\kappa, i(\kappa))}$ over $N[G, g']$ (actually, H_α belongs to $V[G, g' \cup g_\alpha]$).

Proof. Work In $V[G, g']$. Enumerate $Z = \{Z_\alpha \colon \alpha < \kappa^+\}$, where Z is as in the last claim. We construct a binary tree A of height κ^+, of conditions from $i(P)_{(\kappa, i(\kappa))}$. Each branch in A will be an increasing sequence of such conditions. We construct A as a function, $A \colon 2^{<\kappa^+} \to i(P)_{(\kappa, i(\kappa))}$.

Construction of A: Take the root $A(\langle\rangle)$ to be an arbitrary element of $i(P)_{(\kappa, i(\kappa))}$. Now, given $\alpha < \kappa^+$ and $f \in 2^\alpha$, assume that $A(f) = s$, and let us define $A(f ^\frown \langle 0 \rangle), A(f ^\frown \langle 1 \rangle)$. Take two incompatible elements $p, q \in i(P)_{(\kappa, i(\kappa))}$ above s. For p, there exists a unique $p' \in Z_\alpha$ such that p, p' are compatible. Let p'' extend both of them. Similarly, choose q'' which extends s and some element $q' \in Z_\alpha$. Set $A(f ^\frown \langle 0 \rangle) = p''$, $A(f ^\frown \langle 1 \rangle) = q''$. For limit levels of A, we use κ^+-closeness of $i(P)_{(\kappa, i(\kappa))}$: Given a limit $\beta < \kappa^+$ and $f \in 2^\beta$, assume that $A(f \restriction \alpha) = s_\alpha$, for every $\alpha < \kappa^+$. There exists $s \in i(P)_{(\kappa, i(\kappa))}$ such that, for every $\alpha < \beta$, s extends s_α. Set $A(f) = s$.

A **is injective:** Suppose that $h_1 \neq h_2 \in 2^{<\kappa^+}$. If for some $x \in \mathrm{dom}(h_2)$, $h_1 = h_2 \restriction x$, then $A(h_2)$ extends $A(h_1)$, so $A(h_1) \neq A(h_2)$. Therefore, let us assume that there exists $x \in \mathrm{dom}(h_1) \cap \mathrm{dom}(h_2)$ such that $h_1(x) \neq h_2(x)$. Take the first such x. Then $A(h_1 \restriction x+1), A(h_2 \restriction x+1)$ are incompatible. Thus, $A(h_1) \neq A(h_2)$ (Since $A(h_i)$ extends, or is equal to $A(h_i \restriction x+1)$).

Construction of H_α: Every maximal chain in the tree contains, for each $\beta < \kappa^+$, an extension of some element of Z_β. Given $\alpha < \kappa^{++}$, H_α is the downward closure of the branch which corresponds to g_α, and thus intersects every maximal antichain. H_α is defined in $V[G, g' \cup g_\alpha]$ from A and g_α, and clearly is a generic set for $i(P)_{(\kappa, i(\kappa))}$ over $N[G, g']$.

We note that different Cohen functions $g_\alpha, g_{\alpha'}$, induce different generic sets, H_α, $H_{\alpha'}$: This holds, since the first splitting point between two branches contains two incompatible elements. $\square$

Remark 1.3.12. *Over $V[G, g']$, g_α is reconstructible from H_α (this is trivial if $\alpha < (\kappa^{++})^N$). More formally, fix $\alpha < \kappa^{++}$. then g_α can be defined by a formula with parameters A and H_α.*

Proof. Fix $\alpha < \kappa^{++}$. Assume that $\beta < \kappa^+$, and let us compute $g_\alpha(\beta)$. Assume that $g_\alpha(\beta')$ was computed for every $\beta' < \beta$. Let $p = A(g_\alpha \restriction \beta)$. Assume that p_0, p_1 are the two incompatible successors of p in A.

Exactly one of p_0, p_1 belongs to H_α; Assume without loss of generality that $p_0 \in H_\alpha$. Since A is injective, there exists a unique $h \in^{\beta+1} 2$ such that $A(h) = p_0 = A(g_\alpha \restriction (\beta+1))$. Thus, $g_\alpha(\beta) = h(\beta)$. $\square$

Lemma 1.3.13. *For every $\alpha < \kappa^{++}$, $i \colon V \to N$ can be extended to an elementary embedding $i_\alpha \colon V[G] \to N[G, g', H_\alpha]$. Moreover, for every $x \in N[G, g', H_\alpha]$ there exists $f \colon \kappa \to V[G]$, $f \in V[G]$, such that $x = i_\alpha(f)(\kappa)$.*

Proof. We note that $i''G \subseteq G * g' * H_\alpha$ for every $\alpha < \kappa$: Indeed, for every $p \in G$, there exists $\alpha < \kappa$ such that for every $\beta \in [\alpha, \kappa)$, $p(\beta) = 0$. Therefore, for every $\beta \in [\alpha, i(\kappa))$, $i(p)(\beta) = 0$. Moreover, for every $\beta \leq \alpha$, $i(p)(\beta) = p(\beta)$. So $i(p) \in G * g' * H_\alpha$.

Thus $i \colon V \to N$ may be extended to an elementary embedding $i_\alpha \colon V[G] \to N[G, g', H_\alpha]$. Since $H_\alpha \in V[G, g' \cup g_\alpha]$, i_α is definable in $V[G, g' \cup g_\alpha]$.

Now, given $x \in N[G, g', H_\alpha]$, assume that $\underset{\sim}{\sigma}$ is a $i(P_\kappa)$-name that is interpreted though $G * g' * H_\alpha$ as x: $(\underset{\sim}{\sigma})_{G * g' * H_\alpha} = x$. Then there exists $F \colon \kappa \to V$

such that $i(F)(\kappa) = \underset{\sim}{\sigma}$. We can assume that for every $\alpha < \kappa$, $F(\alpha)$ is a P_κ-name. In $V[G]$, define $F' \colon \kappa \to V[G]$, by setting $F'(\alpha) = (F(\alpha))_G$ for every $\alpha < \kappa$. Then, by elementarity, $i(F')(\kappa) = (i(F)(\kappa))_{G*g'*H_\alpha} = x$. $\square$

Remark 1.3.14. *For the construction of $N[G, g', H_t, h_t]$, take $H_t = H_{\phi(t)}$, and $i_t = i_{\phi(t)}$.*

We turn to defining h_α, the Cohen $(i(\kappa^+), i(\kappa^{++}))$-generic set over $N[G, g', H_\alpha]$, for every $\alpha < \kappa^{++}$.

Lemma 1.3.15. *In $V[G, g]$, define, for every $\alpha < \kappa^{++}$,*

$$h_\alpha = \{q \in (\mathit{Cohen}\,(i(\kappa^+), i(\kappa^{++})))^{N[G, g', H_\alpha]} : q \subseteq \cup i''_\alpha g\}$$

Then h_α is Cohen $(i(\kappa^+), i(\kappa^{++}))$-generic over $N[G, g', H_\alpha]$. Moreover, there exists an elementary embedding definable in $V[G, g]$, which extends i_α (and therefore extends i),

$$i^*_\alpha \colon V[G, g] \to N[G, g', H_\alpha, h_\alpha]$$

and if $U_\alpha = \{X \subseteq \kappa \colon \kappa \in i^*_\alpha(X)\}$, then $N[G, g', H_\alpha, h_\alpha] = \mathit{Ult}(V[G, g], U_\alpha)$.

Proof. Clearly, the elements of h_α are pairwise compatible, and h_α is downwards closed. Therefore, it suffices to prove that h_α intersects any set $D \in N[G, g', H_\alpha]$ which is dense and open in Cohen $(i(\kappa^+), i(\kappa^{++}))$. Given such D, there exists $F \colon \kappa \to V[G]$ such that $D = i_\alpha(F)(\kappa)$. Assume without loss of generality that $F(\beta)$ is dense and open subset of Cohen (κ^+, κ^{++}) for every $\beta < \kappa$. Define, in $V[G]$,

$$D' = \bigcap_{\beta \in S} F(\beta)$$

D' is dense and open in Cohen(κ^+, κ^{++}). Take $p \in D' \cap g$. So $i_\alpha(p) \in i_\alpha(F)(\kappa)$. Therefore, $i_\alpha(p) \in i''_\alpha g \cap D$. This shows that h_α is indeed generic over $N[G, g', H_\alpha]$.

Now, we note that $i''_\alpha g \subseteq h_\alpha$, by the definition of h_α. Therefore, there exists an elementary embedding $i^*_\alpha \colon V[G, g] \to N[G, g', H_\alpha, h_\alpha]$ which extends i_α. Since $h_\alpha \in V[G, g]$, i^*_α is definable in $V[G, g]$.

$N[G, g', H_\alpha, h_\alpha] = \mathrm{Ult}\,(V[G, g], U_\alpha)$ since for every $x \in N[G, g', H_\alpha, h_\alpha]$ there exists $f \colon \kappa \to V[G, g]$, $f \in V[G, g]$, such that $x = i^*_\alpha(f)(\kappa)$. This is done exactly as in lemma 1.3.13. $\square$

The ultrafilters $\langle U_\alpha \colon \alpha < \kappa^{++} \rangle$ are destined to be the normal ultrafilters derived from the extended embeddings j^*_α. The following proposition states that there are κ^{++} pairwise distinct ultrafilters among them:

Proposition 1.3.16. *Assume that $\alpha \neq \beta$ in the interval $\left[(\kappa^{++})^N, \kappa^{++} \right)$. Then $U_\alpha \neq U_\beta$.*

Proof. Assume the contrary. Then $i^*_{U_\alpha}(G) = i^*_{U_\beta}(G)$, so $G * g' * H_\alpha = G * g' * H_\beta$. Thus, $H_\alpha = H_\beta$ (Indeed, assume that $q \in H_\alpha$, and $\underset{\sim}{q}$ is an $i(P)_{\kappa+1}$-name for q, and $p \in G * g'$ forces that $\underset{\sim}{q}$ belongs to $i(P)_{(\kappa, i(k))}$. Then $\langle p, q \rangle \in G * g' * H_\alpha = G * g' * H_\beta$, so the interpretation of $\underset{\sim}{q}$ via $G * g'$ belongs to H_β). Consider $V[G, g' \cup g_\alpha]$. Since $H_\alpha \in V[G, g' \cup g_\alpha]$, it follows that $H_\beta \in V[G, g' \cup g_\alpha]$. Thus, by remark 1.3.12, $g_\beta \in V[G, g' \cup g_\alpha]$. This is a contradiction, since $\alpha \neq \beta \geq (\kappa^{++})^N$. $\square$

Finally, let us define h_t for every $t \in [Q]^{<\omega}$. Note that $h_{\phi(t)}$ and h_t are not defined in the same way.

Lemma 1.3.17. *Assume that $t \in [Q]^{<\omega}$. In $V[G, g_t]$, define –*

$$h_t = \{ q \in (Cohen(i(\kappa^+), i(\kappa^{++})))^{N[G, g', H_t]} : q \subseteq \cup i''_t g_t \}$$

Then h_t is $Cohen(i(\kappa^+), i(\kappa^{++}))$-generic over $N[G, g', H_t]$. Moreover, there exists an elementary embedding definable in $V[G, g_t]$, which extends i_t (and therefore extends i),

$$i^*_t \colon V[G, g_t] \to N[G, g', H_t, h_t]$$

and if $U_t = \{ X \subseteq \kappa \colon \kappa \in i^*_t(X) \}$, then $N[G, g', H_t, h_t] = Ult(V[G, g_t], U_t)$.

Proof. Just repeat the proof of lemma 1.3.15. $\square$

Remark 1.3.18. *$U_t = U_{\phi(t)}$, since the subsets of κ are the same in $V[G, g], V[G, g_t]$ and $V[G]$, and $i^*_{\phi(t)}, i^*_t$ both extend $i_{\phi(t)}$.*

1.3.2 Extending M With a Generic Set For $j(P)$

Now, let us define H^*_α, h^*_α, H^*_t, h^*_t. Clearly, for every $\alpha \le \kappa$, $j(P)_\alpha = P_\alpha$. Indeed, if $\alpha < \kappa$, then $j(P)_\alpha = j(P_\alpha) = P_\alpha$. If $\alpha = \kappa$, then it is inaccessible in M, so a direct limit is taken at κ, and thus $j(P)_\kappa = P_\kappa$.

Claim 1.3.19. *G is $j(P)_\kappa$ generic over M, and $V[G] \vDash {}^{\kappa^+}M[G] \subseteq M[G]$. Moreover, g is Cohen(κ^+, κ^{++})-generic over $M[G]$, and $V[G,g] \vDash {}^{\kappa^+}M[G,g] \subseteq M[G,g]$. Similarly, for every $t \in [Q]^{<\omega}$, g_t is Cohen(κ^+, κ^{++})-generic over $M[G]$, and $V[G,g_t] \vDash {}^{\kappa^+}M[G,g_t] \subseteq M[G,g_t]$.*

Proof. Every dense subset of $j(P)_\kappa = P_\kappa$ which belongs to M, belongs to V as well, so G is generic over M. $V[G] \vDash {}^{\kappa^+}M[G] \subseteq M[G]$ holds, since ${}^{\kappa^+}M \subseteq M$, and $j(P)_\kappa$ is κ-c.c.; Just follow the proof of claim 1.3.5. Therefore $g \subseteq M[G]$. Now, since $(\kappa^{++})^M = \kappa^{++}$, g is $(\text{Cohen}(\kappa^+, \kappa^{++}))^{M[G]}$-generic over $M[G]$. Finally, $V[G,g] \vDash {}^{\kappa^+}M[G,g] \subseteq M[G,g]$ follows similarly to claim 1.3.5, since Cohen(κ^+, κ^{++}) is $\kappa^{++} - c.c.$, and $M[G]$ is closed under κ^+-sequences. □

In $M[G,g]$, consider the quotient forcing $j(P)/G * g$. Similarly, in $M[G,g_t]$, consider the quotient forcing $j(P)/G * g_t$, for every $t \in [Q]^{<\omega}$. We repeat (briefly) the same arguments as before:

Lemma 1.3.20. *1. $V[G,g] \vDash$ "$j(P)/G * g$ is κ^{++}-closed".*

2. For every $t \in [Q]^{<\omega}$, $V[G,g_t] \vDash$ "$j(P)/G * g_t$ is κ^{++}-closed".

Proof. We prove only 1, since 2 is completely analogous. In M, let μ be the first inaccessible cardinal above κ. Work in $M[G,g]$. Then, just as in lemma 1.3.9, $\Vdash_{j(P)_{\kappa+1}}$ "$j(P)/G * g$ is μ closed". So $M[G,g] \vDash$ "$j(P)/G * g$ is μ-closed". It is known that $V[G,g] \vDash {}^{\kappa^+}M[G,g] \subseteq M[G,g]$, and $V[G,g] \vDash |\mu| = \kappa^{++}$. Therefore, $V[G,g] \vDash$ "$j(P)/G * g$ is κ^{++}-closed". □

Lemma 1.3.21. *In $M[G,g]$, let Z be the set of maximal antichains in $j(P)/G * g$. Then $V[G,g] \vDash |Z| \le \kappa^{++}$. Similarly, if Z_t is the set of maximal antichains in $j(P)/G * g_t$, then $V[G,g_t] \vDash |Z_t| \le \kappa^{++}$.*

Proof. In $M[G,g]$, $|j(P)/G * g| = j(\kappa)$, and $j(P)/G * g$ is $j(\kappa)$-c.c., since $j(\kappa)$ is Mahlo. Therefore, $M[G,g] \vDash |Z| \le j(\kappa)^{<j(\kappa)} = j(\kappa)$. Now, $V \vDash |j(\kappa)| = \kappa^{++}$, and so $V[G,g] \vDash |j(\kappa)| = \kappa^{++}$. Therefore, $V[G,g] \vDash |Z| \le \kappa^{++}$. □

Lemma 1.3.22. *There exist, for every $\alpha < \kappa^{++}$, a $j(P)/G * g$-generic set over $M[G,g]$, H^*_α, and two elementary embeddings, $j_\alpha \colon V[G] \to M[G,g,H^*_\alpha]$ and $k_\alpha \colon N[G,g',H_\alpha] \to M[G,g,H^*_\alpha]$, such that $j_\alpha = k_\alpha \circ i_\alpha$. The embeddings are definable in $V[G,g]$. Moreover, ${}^{\kappa^+}M[G,g,H^*_\alpha] \subseteq M[G,g,H^*_\alpha]$.*

$$\begin{tikzcd} & M[G,g,H^*_\alpha] \\ V[G] \arrow[ur, "j_\alpha"] \arrow[r, "i_\alpha"'] & N[G,g',H_\alpha] \arrow[u, "k_\alpha"'] \end{tikzcd}$$

Proof. Let $k \colon N \to M$ be the natural embedding, defined as follows:

$$k\left(i(f)\left(\kappa\right)\right) = j(f)\left(\kappa\right)$$

Then $\operatorname{crit}(k) = \left(\kappa^{++}\right)^N$: This follows because $k(\kappa) = \kappa$, and $k(\left(\kappa^+\right)^N) = \left(k\left(\kappa\right)^+\right)^M = \kappa^{+M} = \kappa^{+N}$.

First, let us extend $k \colon N \to M$ to an elementary embedding $k^* \colon N[G,g'] \to M[G,g]$. Let us show that $k''G * g' \subseteq G * g$. Given $\vec{q} \in G * g'$, $k(\vec{q})$ has length $\kappa+1$. k fixes elements of G; As for elements of g': Each $p \in \text{Cohen}\left(\kappa^+, \left(\kappa^{++}\right)^N\right)$ has cardinality $\leq \kappa$, so it's domain is bounded in $\kappa^+ \times \left(\kappa^{++}\right)^N$. So $k(p) = p$. Therefore, k acts as identity on $G * g'$, and thus $k \colon N \to M$ can be extended to $k^* \colon N[G,g'] \to M[G,g]$.

Now, let us construct the generic set H^*_α. H_α has cardinality $|i(\kappa)| = \left(\kappa^+\right)^V$. Let us consider $k^{*\prime\prime}H_\alpha$. For every $\vec{p} \in H_\alpha$, $\vec{p}$ is a condition in $i(P)/G * g'$, so, by elementarity, $k^*(\vec{p})$ is a condition in $j(P)/G * g$.

In $V[G,g]$, $k^{*\prime\prime}H_\alpha \in {}^{\kappa^+}M$, so $k^{*\prime\prime}H_\alpha \in M[G,g]$. By κ^{++}-closedness of $j(P)/G * g$, there exists a condition $p_\alpha \in j(P)/G * g$ which extends every element in $k^{*\prime\prime}H_\alpha$. We note that $V[G,g]$ thinks that $j(P)/G * g$ is κ^{++}-closed, and has at most κ^{++} antichains (which all lie in $M[G,g]$), so we can find a generic H^*_α for $j(P)/G * g$ over $M[G,g]$, which belongs to $V[G,g]$, such that $p_\alpha \in H^*_\alpha$.

Now, since $j''G \subseteq G * g * H^*_\alpha$ (this holds because a direct limit is taken at κ), we can extend j to an elementary embedding $j_\alpha \colon V[G] \to M[G,g,H^*_\alpha]$.

As for k^*, we prove that–

$$k^{*\prime\prime}G * g' * H_\alpha \subseteq G * g * H^*_\alpha$$

Indeed, assume that $\vec{p} \in G$, $s \in g'$ and $\vec{q} \in H_\alpha$. Then $k^*\left(\vec{p}^\frown\langle s\rangle^\frown\vec{q}\right) = \vec{p}^\frown\langle s\rangle^\frown k^*(\vec{q})$; Now, $\vec{q} \in H_\alpha$, so p_α extends $k^*(\vec{q})$. Therefore, $k^*(\vec{q}) \in H^*_\alpha$. So $\vec{p}^\frown\langle s\rangle^\frown k^*(\vec{q}) \in G * g * H^*_\alpha$, as desired. Therefore, we can extend k^* to an embedding $k_\alpha \colon N\left[G, g', H_\alpha\right] \to M\left[G, g, H^*_\alpha\right]$.

Now, since $j = k \circ i$, we have, for every P_κ-name $\underset{\sim}{\sigma}$, and every $\alpha < \kappa^{++}$,

$$k_\alpha\left(i_\alpha\left(\left(\underset{\sim}{\sigma}\right)_G\right)\right) = k_\alpha\left(\left(i\left(\underset{\sim}{\sigma}\right)\right)_{G*g'*H_\alpha}\right) = \left(\left(j\left(\underset{\sim}{\sigma}\right)\right)_{G*g*H^*_\alpha}\right) = j_\alpha\left(\left(\underset{\sim}{\sigma}\right)_G\right)$$

So $j_\alpha = k_\alpha \circ i_\alpha$.

Lastly, let us claim that ${}^{\kappa^+}M\left[G, g, H^*_\alpha\right] \subseteq M\left[G, g, H^*_\alpha\right]$. Assume that $X \in V[G, g]$ is a set of ordinals of cardinality κ^+, and $X \subseteq M\left[G, g, H^*_\alpha\right]$. In particular, $X \subseteq M[G, g]$, and thus $X \in M[G, g]$. $\square$

Remark 1.3.23. *For every $t \in [Q]^{<\omega}$, the same proof yields a $j(P)/G * g_t$-generic set over $M[G, g_t]$, H^*_t, which belongs to $V[G, g_t]$. Also, two elementary embeddings, $j_t \colon V[G] \to M[G, g_t, H^*_t]$ and $k_t \colon N[G, g', H_t] \to M[G, g_t, H^*_t]$, such that $j_t = k_t \circ i_t$. The embeddings are definable in $V[G, g_t]$. Moreover, ${}^{\kappa^+}M[G, g_t, H^*_t] \subseteq M[G, g_t, H^*_t]$.*

$$\begin{tikzcd} & M[G, g_t, H^*_t] \\ V[G] \arrow[ur, "j_\alpha"] \arrow[r, "i_\alpha"] & N[G, g', H_t] \arrow[u, "k_\alpha"'] \end{tikzcd}$$

The next step will be to find a generic set for Cohen (κ^+, κ^{++}) over $M\left[G, g, H^*_\alpha\right]$. We use a technique of Magidor. The proof of the following theorem is basically given in [1], section 13:

Theorem 1.3.24. *(Magidor) There exists $h^*_\alpha \in V[G, g]$ which is a generic set for –*

$$\textit{Cohen}\left(j(\kappa^+), j(\kappa^{++})\right)$$

over $M[G, g, H_\alpha]$, for every $\alpha < \kappa^{++}$. Moreover, $j \colon V \to M$ can be extended to an elementary embedding, definable in $V[G, g]$,

$$j^*_\alpha \colon V[G,g] \to M[G,g,H^*_\alpha,h^*_\alpha]$$

Claim 1.3.25. *For every $\alpha < \kappa^{++}$, the embedding–*

$$k_\alpha \colon N[G,g',H_\alpha] \to M[G,g,H^*_\alpha]$$

can be extended to–

$$k^*_\alpha \colon N[G,g',H_\alpha,h_\alpha] \to M[G,g,H^*_\alpha,h^*_\alpha]$$

Moreover, $k^*_\alpha \circ i^*_\alpha = j^*_\alpha$.

Proof. Let us claim that $k''_\alpha h_\alpha \subseteq h^*_\alpha$: Indeed, assume that $q \in h_\alpha$. Since h_α is the downwards closure of $i''_\alpha g$, $q \subseteq i_\alpha(p)$ for some $p \in g$. Thus, $k_\alpha(q) \subseteq k_\alpha(i_\alpha(p)) = j_\alpha(p) \in h^*_\alpha$. So $k_\alpha(q) \in h^*_\alpha$. □

Claim 1.3.26. $V[G,g] \vDash {}^{\kappa^+}M[G,g,H^*_\alpha,h^*_\alpha] \subseteq M[G,g,H^*_\alpha,h^*_\alpha]$.

Proof. As usual, it's enough to consider only sets of ordinals. Assume that X is a set of ordinals, $X \subseteq M[G,g,H^*_\alpha,h^*_\alpha]$, $|X| \leq \kappa^+$ and $X \in V[G,g]$. In particular, since X is a set of ordinals, $X \subseteq M[G,g]$ (Actually, $X \subseteq M$, but we need less than that). Therefore, $X \in M[G,g]$. □

This finishes the proof of theorem 1.3.3: For every $\alpha \in \left[(\kappa^{++})^N, \kappa^{++}\right)$, there exists a definable embedding $j^*_\alpha \colon V[G,g] \to M_\alpha = M[G,g,H^*_\alpha,h^*_\alpha]$, such that $\mathrm{crit}(j_\alpha) = \kappa$, ${}^{\kappa^+}M_\alpha \subseteq M_\alpha$, and the derived normal measures, $U_\alpha = \{X \subseteq \kappa \colon \kappa \in j_\alpha(X)\}$, are pairwise distinct.

$$\begin{tikzcd}
 & M[G,g,H^*_\alpha,h^*_\alpha] \\
V[G,g] \arrow[ur, "j^*_\alpha"] \arrow[r, "i^*_\alpha"'] & N[G,g',H_\alpha,h_\alpha] \arrow[u, "k^*_\alpha"']
\end{tikzcd}$$

Now, let us extend the embeddings j_t for $t \in [Q]^{<\omega}$. Magidor's method yields a generic set $h^*_t \in [G,g_t]$ for $\mathrm{Cohen}(j(\kappa^+), j(\kappa^{++}))$ over $M[G,g_t,H^*_t]$; Just repeat the proof of theorem 1.3.24. We can extend $j_t \colon V[G] \to M[G,g_t,H^*_t]$, $k_t \colon N[G,g',H_t] \to M[G,g_t,H^*_t]$ embeddings $j^*_t \colon V[G,g_t] \to M[G,g_t,H^*_t,h^*_t]$, $k^*_t \colon N[G,g',H_t,h_t] \to M[G,g_t,H^*_t,h^*_t]$ definable in $V[G,g_t]$, and $k^*_t \circ i^*_t = j^*_t$.

$$\begin{tikzcd} & M\left[G, g, H_t^*, h_t^*\right] \\ V\left[G, g_t\right] \arrow[ur, "j_t^*"] \arrow[r, "i_t^*"] & N\left[G, g', H_t, h_t\right] \arrow[u, "k_t^*"] \end{tikzcd}$$

1.3.3 Getting The Required Property

Work $V[G,g]$, the model built in the last section. Recall our goal: Given a separative, κ-distributive notion of forcing $Q \in V[G,g]$ with cardinality κ, we describe a method to extend each F_t to a κ-complete ultrafilter, such that the following situation is ruled out:

$$f_* U_n(t) = g_* U_n(t) - \lim\langle F^*_{s^\frown\langle q\rangle} : q \in Q/\mathrm{mc}(s)\rangle$$

for any pair of non-empty, $\lhd$-incompatible sequences $s, t \in [Q]^{<\omega}$, and for every $f, g \colon \cup U_n(t) \to Q$.

We assumed that $Q \in V[G,g]$, Q is a set of ordinals (by passing to an isomorphic forcing notion). Therefore, by κ^+-closure of Cohen(κ^+, κ^{++}),

$$\langle Q, \leq_Q\rangle \in V[G]$$

Recall also the subset $X \subseteq \kappa^{++} \setminus (\kappa^{++})^N$ and the bijection $\phi\colon [Q]^{<\omega} \to X$. We assumed $X, \phi \in V[G,g]$ as well.

□

Proposition 1.3.27. *There exists a sequence $\langle F^*_t : t \in [Q]^{<\omega}\rangle$ and a function $\pi\colon Q \to \kappa$ such that, for every $t \in [Q]^{<\omega}$,*

1. *F^*_t is a κ-complete ultrafilter which extends F_t.*
2. *$[\pi]_{F^*_t} = \kappa$.*
3. *$X \in U_{\phi(t)}$ if and only if $\{p \in Q/mc(t) \colon \pi(p) \in X\} \in F^*_t$.*
4. *$F^*_t \in V[G, g_t]$.*

Proof. Fix a partition of Q to dense subsets, $\langle D_\xi : \xi < \kappa\rangle \in V[G]$, as promised in lemma 0.1.4. Fix $t \in [Q]^{<\omega}$, and let us describe the construction of F^*_t. Work in $V[G, g_t]$. $j_t'' F_t$ belongs to $M[G, g, H_t^*, h_t^*]$, by closure under κ^+-sequences. Now, each $D \in j_t'' F_t$ is a dense open subset of $j_t(Q)$, and by $j(\kappa)$-distributivity, $\bigcap j_\alpha'' F_t$ is a dense open set. Denote $j_t(\langle D_\xi \colon \xi < \kappa\rangle) = \langle D'_\xi \colon \xi < j(\kappa)\rangle$. Then each D'_ξ is dense in $j_t(Q)$. Take $q_t \in D'_\kappa \cap \bigcap j_t'' F_t$. Now, let $F_t^* = \{X \subseteq Q/\mathrm{mc}(t) \colon q_t \in j_t(X)\}$.

Then, in $V[G, g]$, F_t^* is a κ-complete ultrafilter extending F_t. Let $\pi \colon Q \to \kappa$ be the function which maps every $p \in Q$ to the unique β such that $q \in D_\beta$.

First, we note that $F_t^* \in V[G.g_t]$. Thus, $F_t^* \in V[G, g]$. It remains an ultrafilter in $V[G, g]$, since $V[G, g], V[G, g_t]$ have the same subsets of κ.

Assume that $X \subseteq Q$, $X \in F_t$. Then $q_t \in j_t(X)$, since $j_t(X) \in J_t'' F_t$. So $F_t \subseteq F_t^*$. Now, recall that U_α is the normal ultrafilter on κ generated by j_t. Thus –

$$X \in U_\alpha \iff \kappa \in j_t(X) \iff j_t(\pi)(q_t) \in j_t(X) \iff$$

$$q_t \in j_t(\{p \in Q/\mathrm{mc}(t) \colon \pi(p) \in X\}) \iff \{p \in Q/\mathrm{mc}(t) \colon \pi(p) \in X\} \in F_t^*$$

Let us claim that $[\pi \restriction Q/\mathrm{mc}(t)]_{F_t^*} = \kappa$. We identify π with $\pi \restriction Q/\mathrm{mc}(t)$. First, assume that $f \in V[G, g]$, $f \colon Q/\mathrm{mc}(t) \to \kappa$ satisfies $[f]_{F_t^*} < [\pi]_{F_t^*}$. Then –

$$\{p \in Q/\mathrm{mc}(t) \colon f(p) < \pi(p)\} \in F_t^*$$

This holds in $V[G, g]$; But we can assume that $f \in V[G]$, so this holds in $V[G, g_t]$ as well. Thus, in $V[G, g_t]$, $q_t \in j_t(\{p \in Q/\mathrm{mc}(t) \colon f(p) < \pi(p)\})$. Therefore, $j_t(f)(q_t) < j_t(\pi)(q_t) = \kappa$, so for some $\beta < \kappa$,

$$p_t \in j_t(\{p \in Q/\mathrm{mc}(t) \colon f(p) = \beta\})$$

Thus, $[f]_{F_t^*} = \beta < \kappa$. This shows that $[\pi]_{F_t^*} \leq \kappa$. Now, if for some $\beta < \kappa$, $[\pi]_{F_t^*} = \beta$, then $q_t \in j_t(\{p \in Q/\mathrm{mc}(t) \colon \pi(p) = \beta\})$; This is a contradiction since $j_t(\pi)(q_t) = \kappa > \beta$. □

Now, let us demonstrate how independent the ultrafilters F_t^* are from each other.

Proposition 1.3.28. *Assume that $s, t \in [Q]^{<\omega}$ are $\lhd$-incompatible and $n >$ lh(t). Then $F_q^* \not\leq_{RK} U_n(t)$.*

Proof. Assume for contradiction that $F^*_q \leq_{RK} U_n(t)$. Define $I \subseteq \kappa^{++}$,

$$I = \left(\kappa^{++} \setminus X\right) \cup \{\phi(r) \colon r \in [Q]^{<\omega} \text{ and } r, t \text{ are } \rhd\text{-compatible } \}$$

Denote $g \restriction I = g \cap (\kappa^+ \times I \times 2)$. First, note that $U_n(t) \in V[G, g \restriction I]$: This holds, since, for every $r \in [Q]^{<\omega}$ which is $\lhd$-incompatible with t, $F^*_r \in V[G, g \restriction I]$ (because $F^*_r \in V[G, g_r] \subseteq V[G, g \restriction I]$).

Now, denote $\alpha = \phi(q)$. Since $U_\alpha \leq F^*_q$, $U_\alpha \leq_{RK} U_n(t)$. There exists a Rudin-Keisler projection $h \in V[G, g]$ witnessing this; By κ^+-closure, $h \in V[G, g \restriction I]$. Therefore, $U_\alpha \in V[G, g \restriction I]$. We will claim that this implies that $H_\alpha \in V[G, g \restriction I]$. This is a contradiction, since, by Remark 1.3.12, it follows that $g_\alpha \in V[G, g \restriction I]$, which cannot hold since $\alpha \notin I$.

Thus, it suffices to prove the following lemma:

Lemma 1.3.29. $H_\alpha \in V[G, g \restriction I]$.

Proof. Denote $V_0 = V[G, g \restriction I]$, $V_1 = V[G.g]$. Then V_1 is a generic extension of V_0 with a generic set $g^* = g \setminus (g \restriction I)$ for $\mathrm{Cohen}\,(\kappa^+, X \setminus I)$ over V_0. So $V_1 = V_0[g^*]$.

Now, $U_\alpha \in V_0$ is a normal, κ-complete ultrafilter on κ; Thus, there are a definable model $N_0 \simeq \mathrm{Ult}\,(V_0, U_\alpha)$ and an elementary embedding $i_{U_\alpha} \colon V_0 \to N_0$. By the same methods of the previous subsections, the downwards closure of $i_{U_\alpha}{}''g^*$ in $i_{U_\alpha}(\mathrm{Cohen}\,(\kappa^+, X \setminus I))$ is generic over N_0; Denote $N_1 = N_0[i_{U_\alpha}{}''g^*]$, and extend i_{U_α} to an elementary embedding $i^*_{U_\alpha} \colon V_1 \to N_1$, such that $i^*_{U_\alpha} \supseteq i_{U_\alpha}$. Then, again, by the same methods of the previous subsections, $i^*_{U_\alpha}$ is the ultrapower embedding of the normal, κ-complete ultrafilter $\{X \subseteq \kappa \colon \kappa \in i^*_{U_\alpha}(X)\}$; This ultrafilter is exactly U_α, since for every $X \in V_1$, $X \subseteq \kappa$, it holds that $X \in V_0$ (by κ^+-closure). Thus, $i^*_{U_\alpha} = i^*_\alpha$.

Now, $G * g' * H_\alpha = i^*_\alpha(G)$. Thus, $G * g' * H_\alpha = i_{U_\alpha}(G)$, so $G * g' * H_\alpha \in V_0$. Thus, $H_\alpha \in V_0$. □

□

Remark 1.3.30. *It was crucial, in the last proposition, that $F^*_t \in V[G, g \restriction I]$. This might not hold if F^*_t depends on more Cohen function of g. This is the reason why we developed the embeddings j_t and used them to extend F_t.*

Now, we can generalize proposition 1.3.28 and give a stronger evidence for the independence between the ultrafilters in $\langle F^*_t : t \in [Q]^{<\omega}\rangle$. The following theorem, together with theorem 1.2.7, finishes the proof of theorem 1.2.5.

Theorem 1.3.31. *Assume that s, t are $\lhd$-incompatible. Then there are no $n > lh(t)$ and functions f, g such that –*

$$f_* U_n(t) = g_* U_n(t)\text{-}lim\langle F^*_{s^\frown\langle q\rangle} : q \geq_Q mc(s)\rangle$$

Proof. First, let us deal with the case that $g_* U_n(t)$ is trivial. This case is less significant, since theorem 1.2.7 promises that $g_* U_n(t)$ is non-trivial; But the majority of work for this case was already done: If $g_* U_n(t)$ is trivial, then for some $q \in Q/\text{mc}(s)$, $f_* U_n(t) = F^*_{s^\frown\langle q\rangle}$. So $F^*_{s^\frown\langle q\rangle} \leq_{RK} U_n(t)$, and this is impossible by proposition 1.3.28.

We move forward to the general case. Recall that $U_r = U_{\phi(r)}$ for every $r \in [Q]^{<\omega}$. It would be simpler to work with the normal ultrafilters $U_{s^\frown\langle q\rangle}$ instead $F^*_{s^\frown\langle q\rangle}$.

Lemma 1.3.32. *By modifying f, we can assume, without loss of generality, that $f_* U_n(t) = g_* U_n(t)$-$lim\langle U_{s^\frown\langle q\rangle} : q \geq_Q mc(s)\rangle$.*

Proof. Assume that $f_* U_n(t) = g_* U_n(t)\text{-lim}\langle F^*_{s^\frown\langle q\rangle} : q \geq_Q \text{mc}(s)\rangle$. Then –

$$X \in f_* U_n(t) \iff \{q \in Q/\text{mc}(s) : X \in F^*_{s^\frown\langle q\rangle}\} \in g_* U_n(t)$$

Therefore,

$$X \in (\pi \circ f)_* U_n(t) \iff \{q \in Q/\text{mc}(s) : X \in U_{\phi(s^\frown\langle q\rangle)}\} \in g_* U_n(t)$$

So –

$$(\pi \circ f)_* U_n(t) = g_* U_n(t)\text{-lim}\langle U_{s^\frown\langle q\rangle} : q \geq_Q \text{mc}(s)\rangle$$

$\square$

So assume that $f_* U_n(t) = g_* U_n(t)\text{-lim}\langle U_{s^\frown\langle q\rangle} : q \geq_Q \text{mc}(s)\rangle$, and $g_* U_n(t)$ is non-trivial. Denote –

$$I = (\kappa^{++} \setminus X) \cup \{\phi(r) : r \in [Q]^{<\omega} \text{ and } r, t \text{ are } \rhd\text{-compatible }\}$$

$$J = \{\phi(s^\frown\langle q\rangle) : q \geq_Q \text{mc}(s)\}$$

Then I, J are disjoint, $J \subseteq X \setminus I$ and $|J| = \kappa$. Denote $V_0 = V[G, g \restriction I]$, $V_1 = V[G, g]$, where $g \restriction I = g \cap (\kappa^+ \times I \times 2)$. Then $V_1 = V_0[g^*]$, where $g^* = g \cap (\kappa^+ \times (X \setminus I) \times 2)$ is generic for Cohen $(\kappa^+, X \setminus I)$.

Note that for every $t \in \phi^{-1} I$, $F^*_t \in V_0$, so $U_n(t) \in V_0$.

Denote $U = f_* U_n(t)$, $W = g_* U_n(t)$. Then f, g can be identified with functions $\in \kappa^\kappa$, so $f, g \in V[G]$. Thus, $U, W \in V_0$. Moreover, $W <_{RK} U$ by the discreteness of $F^*_{s^\frown \langle q \rangle}$. The Rudin-Keisler projection $h \colon \cup U \to \cup W$ belongs to $V[G]$, and thus to V_0.

Now, let $N_W = \mathrm{Ult}(V_0, W)$, $N_U = \mathrm{Ult}(V_0, U)$. Let $i_W \colon V_0 \to N_W$, $i_U \colon V_0 \to N_U$ be the corresponding elementary embeddings. Define $k \colon N_W \to N_U$ as follows:

$$k\left(i_W(f)([Id]_W)\right) = i_U(f)([h]_U)$$

this is an elementary embedding, defined in V_0.

$$\begin{tikzcd} & N_U \\ V_0 \arrow[ur, "i_U"] \arrow[r, "i_W"] & N_W \arrow[u, "k"'] \end{tikzcd}$$

The downwards closure of $i''_W g^*$ is generic for $i_W(\text{Cohen}(\kappa^+, \kappa^{++} \setminus I))$ over N_W (by the same methods of previous subsections). Denote $N^2_W = N_W[i'' g^*]$ (we identified $i''_W g^*$ with it's downwards closure). Let $i^2_W \colon V[G, g] \to N^2_W$ be an elementary embedding which extends i_W. Every element $x \in N^2_W$ is of the form $i^2_W(F)([Id]_W)$ for some $F \colon \kappa \to V[G, g]$, $F \in V[G, g]$. Thus, $N^2_W = \mathrm{Ult}(V[G, g], W)$.

Similarly, define $i^2_U \colon V[G, g] \to N^2_U$, where $N^2_U = N_U[i''_U g*] = \mathrm{Ult}(V[G, g], U)$.

$$\begin{tikzcd} & N^2_U \\ V_1 \arrow[ur, "i^2_U"] \arrow[r, "i^2_W"] & N^2_W \arrow[u, "k^2"'] \end{tikzcd}$$

Note that for $p < i_W(q)$, where $q \in g^*$, $k(p) < i_U(q)$, and $i_U(q) \in i''_U g^*$.

Therefore, $k''i''_U g^* \subseteq i''_W g^*$. So we can extend k to $k^2\colon N^2_W \to N^2_U$. It can be easily checked that, in V_1, $i^2_W \circ k^2 = i^2_U$.

We note that $\left([h]_U\right)^{V_1} = \left([h]_U\right)^{V_0}$ (here we identify the equivalence class and the transitive collapse): Both are ordinals in V_1 (recall that we identified Q with κ), and are isomorphic, since for every f such that $[f]_U < [h]_U$ in V_1, there exists $f^* \in V_0$ such that, in V_1, $[f^*]_U = [f]_U$. Thus, we identify $[h]^{V_0}_U = [h]^{V_1}_U = [h]_U$. Similarly, $\left([Id]_W\right)^{V_0} = \left([Id]_W\right)^{V_1}$. Thus, in V_1, $k^2([Id]_W) = [h]_U$.

The following properties uniquely define k^2:

1. $k^2\colon N^2_W \to N^2_U$ is elementary.

2. $i^2_W \circ k^2 = i^2_U$.

3. $k^2([Id]_W) = [h]_U$.

There exists another embedding which satisfies properties $1-3$ above, which is the ultrapower embedding of $\mathrm{Ult}(N^2_W, F)$, where –

$$F = i^2_W\left(\langle U_{s^\frown\langle q\rangle} \colon q \geq_Q \mathrm{mc}(s)\rangle\right)\left([Id]_W\right)$$

(Recall that W is an ultrafilter on $Q/\mathrm{mc}(s)$, so $[Id]_W \in i_W(Q)$, and the last line makes sense). So k^2 is the ultrapower embedding of F.

Lemma 1.3.33. *Denote $\bar{g} = i^2_W\left(\langle g_\alpha \colon \alpha \in J\rangle\right)\left(i^2_W(\phi)\left([Id]_W\right)\right)$. Then $\bar{g} \in V_0$.*

Proof. Work in V_1. Let $A, Z \in V[G, g']$ respectively be the binary tree and the set of antichains from lemma 1.3.11. By Remark 1.3.12, for every $\alpha < \kappa^{++}$, g_α is reconstructible from A and H_α. Note that $H_\alpha = i_\alpha(G) \restriction (\kappa, i(\kappa))$, where i_α is the ultrapower embedding of U_α.

Thus, for every $q \in Q/\mathrm{mc}(s)$, $g_{\phi(s^\frown\langle q\rangle)}$ is reconstructible from A, Z and $i_{s^\frown\langle q\rangle}(G)$. This is true in $V[G, g]$. Recall that W is an ultrafilter on $Q/\mathrm{mc}(s)$. By elementarity, in N^2_W, the function –

$$\bar{g} = i^2_W\left(\langle g_\alpha \colon \alpha \in J\rangle\right)\left(i^2_W(\phi)\left([Id]_W\right)\right)$$

can be reconstructed from $i^2_W(A)$ and $i^2_F(i^2_W(G)) = i^2_U(G)$.

But $A, Z, G \in V_0$. Thus $\bar{g}$ can be reconstructed from $i_W(A)$, $i_W(Z)$ and $i_U(G)$, which all belong to V_0. $\square$

Now, let us finish the proof by deriving a contradiction. Recall, from the beginning of the proof, the generic set g^* for Cohen $(\kappa^+, X \setminus I)$ over V_0. Recall that $J \subseteq X \setminus I$. Define, in V_0, a dense set in Cohen $(\kappa^+, X \setminus I)$:

$$D = \{p \in \text{Cohen}\left(\kappa^+, X \setminus I\right) : \exists j \in \{0,1\}\ \exists \xi < \kappa^+\ \bar{g}\left(i_W(\xi)\right) = j \text{ and } \forall \beta \in J\ p(\xi, \beta) \neq j\}$$

Let us prove that, indeed, D is dense in Cohen $(\kappa^+, X \setminus I)$: Take a condition $p \colon \kappa^+ \times (X \setminus I) \to 2$ with $|p| \leq \kappa$. There exists $\xi < \kappa^+$ such that, for every $\beta \in J$, $(\xi, \beta) \notin \text{dom}(p)$. Denote $j = \bar{g}\left(i_W(\xi)\right)$. Define:

$$p' = p \cup \{(\xi, \beta, 1-j) : \alpha \in J\}$$

then $\bar{p} \in D$, $\bar{p} \supseteq p$.

Thus, D is dense in Cohen $(\kappa^+, X \setminus I)$. Then $g^* \cap D \neq \emptyset$. Take some element r in the intersection, and let $j \in \{0,1\}$ and $\xi < \kappa^+$ be the parameters promised by $r \in D$. Then for every $\beta \in J$, $g^*(\xi, \beta) = 1 - j$. Thus, for every $\beta \in J$, $g_\beta(\xi) = 1 - j$.

On the other hand, $\bar{g}\left(i_W(\xi)\right) = j$, so $\{q \in Q/\text{mc}(s) \colon g_{\phi(s^\frown\langle q\rangle)}(\xi) = j\} \in W$. Take q in this set, and denote $\beta = \phi\left(s^\frown\langle q\rangle\right)$. Then $\beta \in J$, a contradiction. $\square$

1.4 Concluding Remarks

Given pair of different generic Prikry sequences for $P_{\vec{F}^*}$,

$$\langle q_n \colon n < \omega\rangle, \langle p_n \colon n < \omega\rangle$$

we proved that–

$$\langle q_n \colon n < \omega\rangle \in V\left[\langle p_n \colon n < \omega\rangle\right]$$

implies some connection between the ultrafilters $\langle F^*_t \colon t \in [Q]^{<\omega}\rangle$; We are not sure that this is the optimal connection.

Question 1.4.1. *Does there exist any connection between the ultrafilters $\langle F^*_t \colon t \in [Q]^{<\omega}\rangle$, which promises that, for pair of disjoint Prikry sequences as above,*

$$\langle q_n \colon n < \omega\rangle \in V\left[\langle p_n \colon n < \omega\rangle\right]$$

As for the quotient forcing, by combining theorem 1.2.5 and proposition 1.1.13, it follows that the quotient forcing $P_{\vec{F}^*}/H$, described in the last section, is not homogeneous:

Corollary 1.4.2. *It's consistent from κ^+-supercompactness of κ that for every separative, κ-distributive forcing notion Q with $|Q| = \kappa$, there exists a choice of measures $\vec{F}^*$, such that for every $H \subseteq Q$ generic over V, the the quotient forcing $P_{\vec{F}^*}/H$ is not homogeneous.*

Question 1.4.3. *Is it consistent, from some large cardinal assumption, that for every separative, κ-distributive forcing notion Q with $|Q| = \kappa$, there exists a choice of measures $\vec{F}^*$ such that for every $H \subseteq Q$ generic over V, the quotient forcing $P_{\vec{F}^*}/H$ is homogeneous?*

Chapter 2

Prikry Forcing With One Ultrafilter

2.1 Definitions and Basic Properties

Let κ be a κ-compact cardinal. Consider a separative, κ-distributive forcing notion $\langle Q, <_Q \rangle$, with $|Q| = \kappa$. Let us assume that $h\colon Q \to \kappa$ is some function which satisfy –

$$\forall \alpha < \kappa \ , \ |\{q \in Q\colon h(q) = \alpha\}| < \kappa$$

Remark 2.1.1. *For example, assuming that $Q \subseteq V_\kappa$, we may always take $h(q) = rank(q)$. Alternatively, identify Q with κ and take h to be the identity map.*

Let F be the κ-complete filter generated by the dense-open subsets of Q –

$$F = \{E \subseteq Q \ : \ D \subseteq E \text{ for some dense open subset } D \text{ of } Q\}$$

By κ-compactness of κ, there is a κ-complete ultrafilter F^* extending F. Let $j_{F^*}\colon V \to Ult(V, F^*)$ be the elementary embedding of V in it's ultrapower. Assume that $\pi\colon Q \to \kappa$ satisfies $[\pi]_{F^*} = \kappa$ (where $[f]_{F^*}$ is the equivalence class of the function $f\colon Q \to V$, under the natural equivalence relation derived from F^*). Let $U \leq_{RK} F^*$ be the non-trivial, normal, κ-complete ultrafilter on κ, derived from the Rudin-Keisler projection π, i.e. –

$$\forall X \subseteq \kappa \ , \ X \in U \iff \pi^{-1}(X) \in F^*$$

In this section, we will develop a Prikry-type forcing P_{F^*}, which depends on the choice of F^*, the function h and the Rudin-Keisler projection π.

Definition 2.1.2. *Let $\langle P_{F^*}, \leq, \leq^* \rangle$, consist of elements of the form $\langle p_1, \ldots, p_n, A \rangle$, where –*

1. $p_i \in Q$

2. $A \in F^*$

3. For every $1 < i \leq n$, $\pi(p_i) > h(p_{i-1})$

We say that $\langle p_1, \ldots, p_n, A\rangle \geq \langle q_1, \ldots, q_m, B\rangle$*, namely* $\langle p_1, \ldots, p_n, A\rangle$ *extends* $\langle q_1, \ldots, q_m, B\rangle$*, if and only if –*

1. $n \geq m$

2. $\forall 1 \leq i \leq m \quad q_i = p_i$

3. $\forall m < i \leq n \quad p_i \in B$

4. $A \subseteq B$

If $n = m$*, we say that* $\langle p_1, \ldots, p_n, A\rangle$ *is a Direct Extension of* $\langle q_1, \ldots, q_m, B\rangle$*, and denote it by* $\langle p_1, \ldots, p_n, A\rangle \geq^* \langle q_1, \ldots, q_m, B\rangle$*.*

If $Q = \langle \kappa, \in\rangle$, h is the identity, and F^* is some normal ultrafilter on κ, then P_{F^*} is the standard Prikry forcing.

Remark 2.1.3. *1.* $\{p \in Q \colon \pi(p) \leq h(p)\} \in F^*$.

2. For very $q \in Q$*,* $\{p \in Q \colon \pi(p) > h(q)\} \in F^*$*. In particular,* P_{F^*} *is separative.*

Proof. 1. Otherwise, we would have had $[h]_{F^*} < \kappa$, so, for some $\alpha < \kappa$, $\{q \in Q \colon h(q) = \alpha\} \in F^*$, and in particular, $|\{q \in Q \colon h(q) = \alpha\}| = \kappa$.

2. given an element $q \in Q$, $\{\alpha < \kappa \colon \alpha > h(q)\} \in U$, and thus $\pi^{-1}\{\alpha < \kappa \colon \alpha > h(q)\} = \{p \in Q \colon \pi(p) > h(q)\} \in F^*$.

□

We would like to prove some Prikry-type properties of P_{F^*}. Given a generic $G \subseteq P_{F^*}$, we may define a corresponding ω-sequence $\langle p_i \colon i < \omega\rangle \in V[G]$ of elements of Q, derived from –

$$\bigcup\{\vec{p} \colon \exists A \in F^*, \langle \vec{p}, A\rangle \in G\}$$

By a simple density argument, the sequence $\langle h(p_i)\colon i<\omega\rangle \in V[G]$ is cofinal in κ, so in $V[G]$, κ changes it's cofinality to ω. Moreover, P_{F^*} preserves cardinals: For cardinals above κ this easily follows from $\kappa^+ - c.c.$. For κ and below, this will follow, by a standard argument, from the Prikry condition (Claim 2.1.8 below). Towards the proof of the Prikry condition, let us show that F^* admits some kind of a diagonal intersection:

Lemma 2.1.4. *Let $A \in F^*$, and assume that for every $p \in A$, $A_p \in F^*$. Let –*

$$\underset{p\in A}{\Delta^*} A_p = \{x \in A \,:\, \forall p \in A\ h(p) < \pi(x) \to x \in A_p\}$$

Then $\underset{p\in A}{\Delta^*} A_p \in F^*$.

Proof. For every $\gamma < \kappa$, let –

$$B_\gamma = \begin{cases} \bigcap\limits_{h(p)=\gamma} A_p & \exists p \in Q\ h(p)=\gamma \\ Q & \text{else} \end{cases}$$

By the κ-completeness of F^*, $B_\gamma \in F^*$. Now, we may easily verify that –

$$\underset{p\in A}{\Delta^*} A_p \supseteq \{x \in Q \,:\, \forall \gamma < \kappa\ \gamma < \pi(x) \to x \in B_\gamma\} \cap A$$

So it suffices to prove that $\{x \in Q \,:\, \forall \gamma < \kappa\ \gamma < \pi(x) \to x \in B_\gamma\} \in F^*$. We note that by Los's theorem, it suffices to prove the following property in the ultrapower $Ult(V, F^*)$:

$$\forall \gamma < \kappa\ \ [Id]_{F^*} \in j(B_\gamma)$$

Where $j\colon V \to Ult(V, F^*)$ is the corresponding elementary embedding. Indeed, this property trivially holds since $B_\gamma \in F^*$. □

Notation. *For $A \subseteq Q$ and $n \in \omega$, denote by $[\![A]\!]^n$ the set of all finite sequences of the form $\langle q_1, \dots, q_n\rangle \in A^n$, where –*

1. $\forall i,\ q_i \in Q$
2. $\forall 1 < i \le n\ ,\ \pi(q_i) > h(q_{i-1})$

Set $[\![A]\!]^0 = \{\langle\rangle\}$ (the empty sequence). Denote $[\![A]\!]^{<\omega} = \bigcup\limits_{n<\omega} [\![A]\!]^n$.

Remark 2.1.5. *We can generalize our form of diagonal intersection for sets indexed by finite sequences of elements of Q. Assume that for every $\vec{a} \in [\![Q]\!]^{<\omega}$ there exists a set $A_{\vec{a}} \in F^*$. Let –*

$$\mathop{\Delta^*}_{\vec{a}\in[\![Q]\!]^{<\omega}} A_{\vec{a}} = \{x \in A_{\langle\rangle} \ : \ \forall \vec{a} = \langle a_1, \ldots, a_n\rangle \in [\![Q]\!]^{<\omega} \quad h(a_n) < \pi(x) \to x \in A_{\vec{a}}\}$$

Then $\mathop{\Delta^*}_{\vec{a}\in[\![Q]\!]^{<\omega}} A_{\vec{a}} \in F^*$.

Proof. For every $p \in A$, denote $S_p = \{\vec{a} = \langle a_1, \ldots, a_n\rangle \in S \ : \ a_n = p\}$, and let –

$$H_p = \bigcap_{\vec{a}\in S_p} A_{\vec{a}}$$

Note that $|S_p| < \kappa$, since $|\{a \in Q \colon h(a) \leq \pi(p)\}| < \kappa$, so there are $< \kappa$ options for a_{n-1}; For each one of them, there are $< \kappa$ options for a_{n-2}, and so on. Therefore, by κ-completeness, $H_p \in F^*$. Now, set –

$$H = \mathop{\Delta^*}_{p\in Q} H_p = \{x \in A \ : \ \forall p \in A \ h(p) < \pi(x) \to x \in H_p\}$$

Then $H \in F^*$, and $H \subseteq \mathop{\Delta^*}_{\vec{a}\in[\![Q]\!]^{<\omega}} A_{\vec{a}}$. □

Recall that, given a measure F on κ and $1 < n < \omega$,

$$F^n = \{A \subseteq \kappa^n \colon \ \{\alpha_1 < \kappa \colon \{\alpha_2 < \kappa \colon \ \ldots \{\alpha_n < \kappa \colon \langle \alpha_1, \ldots, \alpha_n\rangle \in A\} \in F \ldots\} \in F\} \in F\}$$

This is a κ-complete ultrafilter on κ^n. F^n, where F is an ultrafilter on Q, is defined similarly. The following property will be useful:

Lemma 2.1.6. *For every* $n < \omega$ *and* $Z \in F^{*n}$, *there exists* $A \in F^*$ *such that* $[\![A]\!]^n \subseteq Z$.

Proof. $Z \in F^{*n}$ means that –

$$A_{\langle\rangle} = \{x_1 \in Q \colon \{x_2 \in Q \colon \ \ldots \{x_n \in Q \colon \langle x_1, \ldots, x_n\rangle \in Z\} \in F^* \ldots\} \in F^*\} \in F^*$$

We define sets $A_{\vec{x}}$ recursively: Assume that $A_{\langle x_1,\ldots,x_k\rangle}$ was defined for $k < n-1$. For every $x_{k+1} \in A_{\langle x_1,\ldots,x_k\rangle}$, set –

$$A_{\langle x_1,\ldots x_k, x_{k+1}\rangle} = \{x_{k+2} \colon \{x_{k+3} \ldots \{x_n \colon \langle x_1, \ldots, x_n\rangle \in Z\} \in F^* \ldots\} \in F^*\} \in F^*$$

For every $\vec{a} \in [\![Q]\!]^{<\omega}$ such that $A_{\vec{a}}$ has not been defined, take $A_{\vec{a}} = Q \in F^*$. Now just take $A = \mathop{\Delta^*}_{\vec{a}\in[\![Q]\!]^{<\omega}} A_{\vec{a}}$. □

We will use the following generalization of Rowbottom's theorem:

Lemma 2.1.7. *(Rowbottom's theorem for F^*)*
Assume $f \colon [\![Q]\!]^{<\omega} \to \alpha$ *is a partition of* $[\![Q]\!]^{<\omega}$, *for some* $\alpha < \kappa$. *Then there exists* $H \in F^*$ *such that for every* $n \in \mathbb{N}$, f *is constant on* $[\![H]\!]^n$.

Proof. It suffices to prove that for every $n \in \mathbb{N}$, for every partition f, there exists $H_n \in F^*$ such that f is constant on $[\![H_n]\!]^n$ (and then set $H = \bigcap_{n\in\mathbb{N}} H_n$).

We prove this by induction on n. The case $n = 1$ follows from κ-completeness. For $n+1$, given a partition $f\colon [\![Q]\!]^{<\omega} \to \alpha$, define, for every sequence $\vec{a} = \langle a_1, \ldots, a_n\rangle \in [\![Q]\!]^n$, a function $f_{\vec{a}}\colon Q \to \alpha$, as follows –

$$f_{\vec{a}}(q) = \begin{cases} f(\vec{a}, q) & \text{if } \pi(q) > h(a_n) \\ 0 & \text{otherwise} \end{cases}$$

By the κ-completeness of F^*, for every $\vec{a} \in [\![Q]\!]^n$, there exist an ordinal $\gamma_{\vec{a}} \in \kappa$ and a set $H_{\vec{a}} \in F^*$ such that $f_{\vec{a}}$ gets the constant value $\gamma_{\vec{a}}$ on $H_{\vec{a}}$. Now, apply the induction hypothesis on the function $\vec{a} \mapsto \gamma_{\vec{a}}$: There is $\gamma < \kappa$ and a large set $Z \in F^*$ such that for all $\vec{a} \in [\![Z]\!]^n$, $\gamma_{\vec{a}} = \gamma$. Let –

$$H = Z \cap \underset{\vec{a}\in[\![A]\!]^n}{\Delta^*} H_{\vec{a}}$$

We claim that f gets the constant value γ on $[\![H]\!]^{n+1}$.

Indeed, Let $\vec{a} = \langle a_1, \ldots, a_n, a_{n+1}\rangle \in [\![H]\!]^{n+1}$. We note that, by the definition of the diagonal intersection, $a_{n+1} \in H_{\langle a_1,\ldots,a_n\rangle}$. Therefore:

$$f(\vec{a}) = f_{\langle a_1,\ldots,a_n\rangle}(a_{n+1}) = \gamma_{\langle a_1,\ldots,a_n\rangle} = \gamma$$

(the last equation follows from the fact that $\langle a_1, \ldots, a_n\rangle \in [\![Z]\!]^n$). $\square$

The next lemma follows in a standard fashion:

Lemma 2.1.8. *(The Prikry Condition) Let σ be a statement in the forcing language of P_{F^*}. Let $\langle p_1, \ldots, p_n, B\rangle \in P_{F^*}$. Then there exists $A \in F^*$, $A \subseteq B$ such that $\langle p_1, \ldots, p_n, A\rangle \parallel \sigma$ (i.e. $\langle p_1, \ldots, p_n, A\rangle \Vdash \sigma$ or $\langle p_1, \ldots, p_n, A\rangle \Vdash \neg\sigma$).*

Lemma 2.1.9. *Assume $\langle p_n\colon n < \omega\rangle$ is a Prikry sequence for P_{F^*}, generated from some generic $G \subseteq P_{F^*}$. Let $E \in F^*$. Then there exists $n_0 \in \omega$ such that for every $n > n_0$, $p_n \in E$.*

Proof. Let $D = \{\langle a_0, \ldots, a_n, A\rangle \in P_{F^*} \,:\, A \subseteq E\}$. D is clearly dense in P_{F^*}. Therefore, there exists some $n_0 < \omega$ and some $A \in F^*$, $A \subseteq E$, such that –

$$\langle p_0, \ldots, p_{n_0}, A\rangle \in G$$

Therefore, for every $n > n_0$, $p_n \in E$. $\square$

Remark 2.1.10. *Assume that $\langle p_n\colon n < \omega\rangle$ is a generic Prikry sequence for P_{F^*}, with a corresponding generic set G over V. Then $V[G] = V[\langle p_n\colon n < \omega\rangle]$.*

Proof. It suffices to prove that $G \in V[\langle p_n \colon n < \omega \rangle]$. Let us argue that –

$$G = \{\langle p_0, \ldots, p_n, A \rangle \colon n < \omega, A \in F^* \text{ and for every } m > n, p_m \in A\}$$

The inclusion $\subseteq$ is clear; Now, given $\langle p_0, \ldots, p_n, A \rangle$ such that for every $m > n$, $p_m \in A$, it follows that $\langle p_0, \ldots, p_n, A \rangle$ is compatible with every element of G, and thus, belongs to G. $\square$

We will use the following observation as well:

Remark 2.1.11. *Suppose that $\langle p_n \colon n < \omega \rangle$ is a Prikry sequence for P_{F^*}. Then, for every $m < \omega$, $\langle p_n \colon n > m \rangle$ is a Prikry sequence for P_{F^*} as well.*

Proof. Denote $t = \langle p_0 \ldots, p_m \rangle$. Let G be the generic set corresponding to $\langle p_n \colon n < \omega \rangle$. Define –

$$G' = \{\langle p_{m+1}, \ldots, p_n, A \rangle \colon m < n < \omega \text{ and for every } k > n, p_k \in A\}$$

We claim that G' is P_{F^*}-generic over V. It suffices to prove that G intersects every dense open set. Given $D' \subseteq P_{F^*}$ dense and open, denote–

$$D = \{t^\frown \langle q_0, \ldots, q_n, A \rangle \colon \langle q_0, \ldots, q_n, A \rangle \in D' \text{ and } \pi(q_0) > h(p_m)\}$$

Then D is dense above $\langle t, Q \rangle \in G$. Thus, G contains an element of the from $t^\frown \langle q_0, \ldots, q_n, A \rangle$ where $\langle q_0, \ldots, q_n, A \rangle \in D'$. In particular, $\langle q_0, \ldots, q_n \rangle = \langle p_{m+1}, \ldots, p_{n+m+1} \rangle$. Also, for every $k > n + m + 1$, $p_k \in A$. Therefore, $\langle q_0, \ldots, q_n, A \rangle \in G' \cap D'$. $\square$

Remark 2.1.12. *Assume that F^* is Rudin-Keisler equivalent to a normal ultrafilter on κ. Then every generic extension of V, obtained by forcing with P_{F^*}, is a generic extension of V obtained by forcing with the standard Prikry forcing.*

Proof. Indeed, the function $h \colon Q \to \kappa$ defines a κ-complete, non-principal ultrafilter –

$$W = h_*(F^*) = \{X \subseteq \kappa \ : \ h^{-1}X \in F^*\}$$

Therefore $W \leq_{RK} F^*$, and by minimality of F^* in the Rudin-Keisler order, $W \equiv_{RK} F^*$, and $h \colon Q \to \kappa$ is injective on a large set $A \in F^*$. We can assume that W is normal (if not, take an injection $f \colon \kappa \to \kappa$ such that f_*W is a normal ultrafilter, and replace W with f_*W and h with $f \circ h$ for the rest of the proof).

Let $\langle p_n : n < \omega \rangle$ be a generic Prikry sequence for P_{F^*}. By lemma 2.1.9, we can assume without loss of generality that $p_n \in A$ for every $n < \omega$. Thus, $V[\langle p_n : n < \omega \rangle] = V[\langle h(p_n) : n < \omega \rangle]$. $\langle h(p_n) : n < \omega \rangle$ is an increasing sequence (this is true from some index, and we may cut the initial segment). For every $C \in W$, there exists $n_0 < \omega$ such that, for every $n \geq n_0$, $h(p_n) \in C$ (this follows by lemma 2.1.9, again). Therefore, by the Mathias criterion (see [2], 1.12), $\langle h(p_n) : n < \omega \rangle$ is a Prikry sequence for P_W, the standard Prikry forcing with the normal ultrafilter W. $\square$

2.2 Prikry Sequences Inside Generic Extensions

Fix a measure F on κ. A function $f \colon \kappa^n \to \kappa$ is called a projection of F^n onto F, if it's a Rudin-Keisler projection, i.e.,

$$X \in F \iff f^{-1}X \in F^n$$

Given $1 \leq i \leq n$, let $\rho_i \colon \kappa^n \to \kappa$ be the projection on the i-th coordinate: $\rho_i(x_1, \dots x_n) = x_i$. Clearly, every such a projection is a Rudin-Keisler projection of F^n onto F, since,

$$A \in F \iff \{x_1 \in \kappa \colon \{x_2 \in \kappa \colon \dots \{x_n \in \kappa \colon x_i \in A\} \in F \dots\} \in F$$

Definition 2.2.1. *A projection $f \colon \kappa^n \to \kappa$ of F^n onto F is called non-trivial, if for every $1 \leq i \leq n$, $\{\vec{x} \in \kappa^n \colon f(\vec{x}) \neq \rho_i(\vec{x})\} \in F^n$.*

Every projection $f \colon \kappa \to \kappa$ of F onto itself is trivial, i.e., $\{x \in \kappa \colon f(x) = x\} \in F$. Therefore, the last definition makes sense for $n > 1$.

Let Q, F^*, P_{F^*} be as in the last section.

Theorem 2.2.2. *Assume $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$ are two Prikry sequences for P_{F^*}, with a finite intersection, such that –*

$$\langle q_n : n < \omega \rangle \in V[\langle p_n : n < \omega \rangle]$$

Then there exists $n > 1$ and a non-trivial projection of ${F^*}^n$ onto F^*.

Proof. By cutting a large enough initial segment, we may assume that the sequences $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$ are disjoint. In V, assume $\underset{\sim}{\sigma}$ is a P_{F^*}-name for $\langle q_n : n < \omega \rangle \in V[\langle p_n : n < \omega \rangle]$. We will use the following lemma:

Lemma 2.2.3. *There are $m, n \in \omega$, $\vec{r} \in [\![Q]\!]^{<\omega}$ and $A \in F^*$, which satisfy the following property: For every $\vec{\nu} = \langle \nu_1, \dots \nu_n \rangle \in [\![A]\!]^n$, there exists $p_{\vec{\nu}} \in Q$ and $B_{\vec{\nu}} \in F^*$, such that –*

1. *$\langle \vec{r}, \vec{\nu}, B_{\vec{\nu}} \rangle \in P_{F^*}$, and $\langle \vec{r}, \vec{\nu}, B_{\vec{\nu}} \rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \check{p}_{\vec{\nu}}$*

2. *For every $B \in F^*$, $B \subseteq A$, $\{\ p_{\vec{\nu}}\ :\ \vec{\nu} \in [\![B]\!]^n\ \} \in F^*$*

Proof. Assume otherwise. First, take $\langle \vec{r}, X \rangle \in P_{F^*}$, which forces that $\underset{\sim}{\sigma}$ is a generic Prikry sequence for P_{F^*}, disjoint from $\langle p_n\ : n < \omega \rangle$ (which could be expressed as the sequence generated from the canonical name for the generic set). In this proof, we work in P_{F^*} above the condition $\langle \vec{r}, X \rangle$. For notational simplicity, let us assume that $\langle \vec{r}, X \rangle = \langle \langle \rangle, Q \rangle$ is the weakest condition. We will build an increasing sequence $\langle n_i \colon i < \omega \rangle$, and, for every $i \leq \omega$, large sets $B_i \in F^*$, $E_i \in F^*$, which satisfy the following property: For every $\vec{\nu} \in [\![B_i]\!]^{n_i}$, there exists $p_i(\vec{\nu}) \in Q$ and a set $B_i(\vec{\nu}) \in F^*$, such that:

1. $\langle \vec{\nu}, B_i(\vec{\nu}) \rangle \Vdash \underset{\sim}{\sigma}(\check{i}) = \widetilde{p_i(\vec{\nu})}$

2. $\{\ p_i(\vec{\nu})\ :\ \vec{\nu} \in [\![B_i]\!]^{n_i}\ \} \cap E_i = \emptyset$

We build those elements in the following way: On stage i, define a function $f_i \colon [\![Q]\!]^{<\omega} \to 2$ as follows: For every $\vec{\nu} \in [\![Q]\!]^{<\omega}$,

$$f_i(\vec{\nu}) = \begin{cases} 1 & \exists p_i(\vec{\nu}) \in Q, B_i(\vec{\nu}) \in F^*, \text{ s.t. } \langle \vec{\nu}, B_i(\vec{\nu}) \rangle \Vdash \underset{\sim}{\sigma}(\check{i}) = \widetilde{p_i(\vec{\nu})} \\ 0 & \text{otherwise} \end{cases}$$

Let $H_i \in F^*$ be homogeneous for f_i. We use the following claim:

Claim. *For every $n_0 < \omega$, there exists some $n \geq n_0$, such that $f_i \restriction_{[\![H_i]\!]^n} = 1$.*

Proof. Let $\vec{\nu} \in [\![H_i]\!]^{n_0}$. Take $p_i(\vec{\nu}) \in Q$ such that for some $\vec{\nu}'$, $H' \subseteq H_i$,

$$\langle \vec{\nu}, H_i \rangle \leq \langle \vec{\nu}', H' \rangle \Vdash \underset{\sim}{\sigma}(\check{i}) = \widetilde{p_i(\vec{\nu})}$$

Let $n = \mathrm{lh}(\vec{\nu}')$ be the length of $\vec{\nu}'$. Then $f_i(\vec{\nu}') = 1$. By the homogeneity of H_i, we get $f_i \restriction_{[\![H_i]\!]^n} = 1$. $\square$

Applying the claim, for every i, there exists some $n_i > \sup\{n_j \colon j < i\}$ such that, for every $\vec{\nu} \in [\![H_i]\!]^{n_i}$, there exists $p_i(\vec{\nu}) \in Q$ and a large set $B_i(\vec{\nu}) \in F^*$ which satisfy $\langle \vec{\nu}, B_i(\vec{\nu}) \rangle \Vdash \underset{\sim}{\sigma}(\check{i}) = \widetilde{p_i(\vec{\nu})}$. By our assumption, there are some $B_i \subseteq H_i$, $B_i \in F^*$ such that –

$$\{\ p_i(\vec{\nu})\ :\ \vec{\nu} \in [\![B_i]\!]^{n_i}\ \} \notin F^*$$

So, we may assume that this set is disjoint from some $E_i \in F^*$. This concludes stage i in the construction. Now, take –

$$B = \bigcap_{i<\omega} B_i\ ,\ E = \bigcap_{i<\omega} E_i$$

Let us argue that –

$$\langle\langle\rangle, B\rangle \Vdash \forall n < \check{\omega}\ \exists i \geq n\ ,\ \underset{\sim}{g}(i) \notin \check{E} \quad (*)$$

$(*)$ finishes the proof of the lemma, since it contradicts claim 2.1.9. To prove $(*)$, it suffices to show that the following sets are dense above $\langle\langle\rangle, B\rangle$:

$$D_n = \{\ p \in P_{F^*}\ :\ p \Vdash \exists i \geq \check{n}\ ,\ \underset{\sim}{g}(i) \notin \check{E}\}$$

Indeed, fix $n < \omega$. Assume $\langle\vec{\nu}, B'\rangle \geq \langle\langle\rangle, B\rangle$. By extending $\vec{\nu}$ if necessary, there exists some $i \geq n$ such that $\vec{\nu} \in [\![B_i]\!]^{n_i}$. Therefore:

$$\langle\vec{\nu}, B' \cap B_i(\vec{\nu})\rangle \Vdash \underset{\sim}{g}(\check{i}) = \widetilde{p_i(\vec{\nu})}$$

(as an extension of $\langle\vec{\nu}, B_i(\vec{\nu})\rangle$). But $p_i(\vec{\nu}) \notin E$ by our construction. So –

$$\langle\vec{\nu}, B'\rangle \leq^* \langle\vec{\nu}, B' \cap B_i(\vec{\nu})\rangle \Vdash \underset{\sim}{g}(\check{i}) \notin \check{E}$$

$\square$

Now, fix $n, m, A, \vec{r}$ as in the lemma, and denote by $f\colon\ [\![A]\!]^n \to Q$ the function $\vec{\nu} \mapsto p_{\vec{\nu}}$. We identify f with one of it's arbitrary extensions to the domain Q^n. We note that condition 2 of lemma 2.2.3 implies that $n > 0$. Let us argue that f is a projection of ${F^*}^n$ onto F^*:

Claim 2.2.4. $Y \in F^* \iff \{\vec{\nu}\colon f(\vec{\nu}) \in Y\} \in {F^*}^n$.

Proof. First, let us assume that for some $Y \in F^*$, $\{\vec{\nu} \in [\![Q]\!]^n : f(\vec{\nu}) \notin Y\} \in {F^*}^n$. Applying remark 2.1.6, let $X \in F^*$ be chosen such that for every $\vec{\nu} \in [\![X]\!]^n$, $f(\vec{\nu}) \notin Y$. By intersecting, assume $X \subseteq A$. By condition 2 of lemma 2.2.3, it follows that $Z = \{f(\vec{\nu}) : \vec{\nu} \in [\![X]\!]^n\} \in F^*$, therefore $Z \cap Y \neq \emptyset$, a contradiction.

For the other direction, assume that $\{\vec{\nu}\colon f(\vec{\nu}) \in Y\} \in {F^*}^n$. Take $Z \subseteq A$ such that $f''[\![Z]\!]^n \subseteq Y$. By condition 2 of lemma 2.2.3, $\{f(\vec{\nu})\colon \vec{\nu} \in [\![Z]\!]^n\} \in F^*$. Therefore $Y \in F^*$. $\square$

The non-triviality of f follows from the following claim:

Claim 2.2.5. *For every $i \leq n$, $\{\vec{\nu} \in Q^n\colon\ f(\vec{\nu}) = \nu_i\ \} \notin {F^*}^n$.*

Proof. Assume otherwise. So $\{\vec{\nu} \in Q^n\colon\ f(\vec{\nu}) = \nu_i\ \} \in {F^*}^n$. Therefore, by Remark 2.1.6, there exists a set $C \in F^*$ such that, for every $\vec{\nu} \in [\![C]\!]^n$, $f(\vec{\nu}) = \nu_i$. Assume that –

$$C \subseteq \left(\mathop{\Delta^*}_{\vec{\nu} \in [\![A]\!]^{<\omega}} B_{\vec{\nu}} \right) \cap A$$

(else, intersect). Here, if $B_{\vec{\nu}}$ has not been defined, take $B_{\vec{\nu}} = Q$. Let us claim that –

$$D = \{\langle \vec{r}, \vec{\nu}, S\rangle \in P_{F^*} : \mathrm{lh}(\vec{\nu}) \geq n \text{ and } \langle \vec{r}, \vec{\nu}, S\rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \check{\nu}_i\}$$

is dense above $\langle \vec{r}, C\rangle$. Once we prove this, we are done: Just take $G \subseteq P_{F^*}$ such that $\langle \vec{r}, C\rangle \in G$. Choose $\langle \vec{r}, \vec{\nu}, S\rangle \in D \cap G$, where $\vec{\nu} = \langle \nu_1, \ldots, \nu_k\rangle$, for some $k < \omega$, $k \geq n$. So $\langle \vec{r}, \vec{\nu}, S\rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \check{\nu}_i$, contradicting the disjointness of $\underset{\sim}{\sigma}$ and the Prikry sequence of G (recall that this disjointness was forced by $\langle \vec{r}, X\rangle$, where $C \subseteq X$. We assumed that $X = Q$; without this assumption, in the definition of C, we should intersect with X).

Therefore, it suffices to prove the density of D above $\langle \vec{r}, C\rangle$. Let $\langle \vec{r}, \vec{\nu}, S\rangle \in P_{F^*}$ extend $\langle \vec{r}, C\rangle$, and assume that $\mathrm{lh}(\vec{\nu}) \geq n$ (else, extend it). Now, since –

$$\langle \vec{r}, \vec{\nu}, S \cap B_{\langle \nu_1, \ldots, \nu_n\rangle}\rangle \geq \langle \vec{r}, \nu_1, \ldots, \nu_n, B_{\langle \nu_1, \ldots, \nu_n\rangle}\rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \check{p}_{\langle \nu_1, \ldots, \nu_n\rangle}$$

we get $\langle \vec{r}, \vec{\nu}, S\rangle \leq^* \langle \vec{r}, \vec{\nu}, S \cap B_{\langle \nu_1, \ldots, \nu_n\rangle}\rangle \Vdash \underset{\sim}{\sigma}(\check{m}) = \check{p}_{\langle \nu_1, \ldots, \nu_n\rangle} = \check{\nu}_i$. □

This shows that $f\colon Q^n \to Q$ is, indeed, a non-trivial projection. Clearly $n \neq 1$ (else, f was trivial). □

Remark 2.2.6. *Assume that F^* is Rudin-Keisler equivalent to a normal ultrafilter on κ. Then the assumptions of theorem 2.2.2 cannot hold. More precisely, if $\vec{p} = \langle p_n : n < \omega\rangle$, $\vec{q} = \langle q_n : n < \omega\rangle$ are two Prikry sequences for P_{F^*}, such that $\langle q_n : n < \omega\rangle \in V[\langle p_n : n < \omega\rangle]$, then $\vec{p}, \vec{q}$ have infinitely many common elements. This follows from theorem 2.2.2 and from the following proposition:*

Proposition 2.2.7. *Let U be a normal ultrafilter on κ, and $1 \leq n < \omega$. Then any projection $f\colon \kappa^n \to \kappa$ of U^n onto U is trivial.*

Proof. Assume the contrary. Let $n < \omega$ be the first such that U^n is projected on U via a non-trivial projection $f\colon \kappa^n \to \kappa$. We prove that $\{\vec{x} \in \kappa^n : f(\vec{x}) = \rho_n(\vec{x})\} \in U^n$. This follows from the following two claims:

Claim. $\{\vec{x}\colon f(\vec{x}) < \rho_n(\vec{x})\} \notin U^n$

Proof. Otherwise, for some set $A \in U$, and for every $\vec{x} = \langle x_1, \ldots, x_n \rangle \in [A]^n$, $f(\vec{x}) < x_n$ (this follows from lemma 2.1.6; here $[A]^n$ is the set of increasing n-sequences of elements of A).

Fix $\langle x_1 \ldots, x_{n-1} \rangle \in [A]^{n-1}$. Then $\{x\colon f(x_1, \ldots, x_{n-1}, x) < x\} \in U$. By normality, for some $\alpha(x_1, \ldots, x_{n-1}) < \kappa$, and $A_{\langle x_1, \ldots, x_{n-1} \rangle} \in U$, for every $x \in A_{\langle x_1, \ldots, x_{n-1} \rangle}$,

$$f(x_1, \ldots, x_{n-1}, x) = \alpha(x_1, \ldots, x_{n-1})$$

Thus, the function $\alpha\colon [A]^{n-1} \to \kappa$ is a projection of U^{n-1} onto U: Indeed, given $B \in U$, there exists $C \in U$ such that $[C]^n \subseteq f^{-1}B$. We can assume that $C \subseteq A \cap \underset{\vec{x} \in [A]^{n-1}}{\triangle} A_{\vec{x}}$ (else, intersect). So for every $\vec{x} = \langle x_1, \ldots, x_n \rangle \in [C]^n$, $x_n \in A_{\langle x_1, \ldots, x_{n-1} \rangle}$, so $f(x_1, \ldots, x_n) = \alpha(x_1, \ldots, x_{n-1})$. Thus, $\alpha^{-1}B \supseteq [C]^{n-1}$. So $\alpha^{-1}B \in U^{n-1}$.

The projection α is non-trivial (else, f was trivial), contradicting the minimality of n. $\square$

Claim. $\{\vec{x}\colon f(\vec{x}) > \rho_n(\vec{x})\} \notin U^n$.

Proof. Assume otherwise. Fix $A \in U$ such that, for every $\vec{x} = \langle x_1, \ldots, x_n \rangle \in [A]^n$, $f(\vec{x}) > x_n$. Since f is a projection, and $[A]^n \in U^n$, $f'' [A]^n \in U$.

Define a function g from some subset of κ to κ as follows: For every $y < \kappa$, if there exists $\vec{x} = \langle x_1, \ldots, x_n \rangle \in [A]^n$ such that $f(\vec{x}) = y$, let –

$$g(y) = \min\{x \in A\colon \exists \vec{t} \in [A \cap x]^{n-1} \quad f(\vec{t}, x) = y\}$$

Note that $\mathrm{dom}(g) \supseteq f'' [A]^n$, so $\mathrm{dom}(g) \in U$. Also, for every $y \in f'' [A]^n$, there exists $\vec{x} = \langle x_1, \ldots, x_n \rangle \in [A]^n$ such that $f(\vec{x}) = y$. Therefore, $x_n < y$. Thus, $g(y) \leq x_n < y$. So, on a set in U, g is regressive. By the normality of U, there exists $\alpha < \kappa$ such that, for some $Y \in U$, $g''Y = \{\alpha\}$. In particular, for every $y \in Y$, there exists $\vec{x} \in [A \cap (\alpha + 1)]^n$ such that $f(\vec{x}) = y$. In particular, $f'' [A \cap (\alpha + 1)]^n \supseteq Y$. But $|[A \cap (\alpha + 1)]^n| < \kappa$, so $|Y| < \kappa$, a contradiction. $\square$

$\square$

Let us recall our general context: $\langle Q, <_Q \rangle$ is a κ-distributive forcing notion, with $|Q| = \kappa$. We consider the forcing P_{F^*}, where F^* extends the filter of dense open subsets of Q. Assume that $\langle p_n : n < \omega \rangle$ is a Prikry sequence for P_{F^*}.

Our next observation is that two disjoint Prikry sequences in $V[\langle p_n : n < \omega \rangle]$, disjoint from $\langle p_n : n < \omega \rangle$, induce two different non-trivial projections. Let us define the exact way in which two projections differ:

Definition 2.2.8. *Suppose $1 \leq n < \omega$. Two projections $f : \kappa^n \to \kappa$, $g : \kappa^n \to \kappa$ of ${F^*}^n$ onto F^* are called equivalent, if $\{\vec{x} \in Q^n : f(\vec{x}) = g(\vec{x})\} \in {F^*}^n$ (i.e., f, g represent the same element in the iterated ultrapower construction of ${F^*}^n$).*

Proposition 2.2.9. *Assume that $\langle a_n : n < \omega \rangle, \langle b_n : n < \omega \rangle$ are two disjoint Prikry sequences in $V[\langle p_n : n < \omega \rangle]$, which are disjoint from $\langle p_n : n < \omega \rangle$. Then, for some $n < \omega$, there are two non-equivalent, non-trivial projections of ${F^*}^n$ onto F^*.*

Proof. Denote by $G \subseteq P_{F^*}$ the generic set corresponding to $\langle p_n : n < \omega \rangle$. Assume that $\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2$ are P_{F^*}-names for $\langle a_n : n < \omega \rangle$, $\langle b_n : n < \omega \rangle$. Choose $\langle \vec{r}, X \rangle \in G$ which forces that $\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2$ are P_{F^*}-names for disjoint Prikry sequences, both disjoint from the sequence $\langle p_n : n < \omega \rangle$.

Apply lemma 2.2.3 and get parameters $n_1, m_1, A_1 \subseteq X$ and $n_2, m_2, A_2 \subseteq X$, such that, for every $i \in \{1, 2\}$, the following property holds:

For every $\vec{\nu} = \langle \nu_1, \ldots \nu_{n_i} \rangle \in [\![A_i]\!]^{n_i}$, there exists $p^i_{\vec{\nu}} \in Q$ and $B^i_{\vec{\nu}} \in F^*$, such that –

1. $\langle \vec{r}, \vec{\nu}, B^i_{\vec{\nu}} \rangle \in P_{F^*}$, and $\langle \vec{r}, \vec{\nu}, B^i_{\vec{\nu}} \rangle \Vdash \underset{\sim}{\sigma}_i(\check{m}_i) = \check{p}^i_{\vec{\nu}}$
2. For every $B \in F^*$, $B \subseteq A_i$, $\{ p^i_{\vec{\nu}} : \vec{\nu} \in [\![B]\!]^{n_i} \} \in F^*$

Assume that $n_1 \leq n_2$. Denote, for $i \in \{1, 2\}$, $f_i(\vec{\nu}) = p^i_{\vec{\nu}}$. Extend f_1, f_2 arbitrarily to domains Q^{n_1}, Q^{n_2}, respectively. Let $\rho_{n_2,n_1} : Q^{n_2} \to Q^{n_1}$ be the function $\rho_{n_2,n_1}(\nu_1 \ldots, \nu_{n_1}, \ldots, \nu_{n_2}) = (\nu_1 \ldots, \nu_{n_1})$ (if $n_1 = n_2$, ρ_{n_2,n_1} is the identity). During the following proof, we denote $\rho = \rho_{n_2,n_1}$ for notational simplicity. Let us claim that f_2, $f_1 \circ \rho$ are two non-equivalent, non-trivial projections of ${F^*}^{n_2}$ onto F^*.

The non-triviality is similar to claim 2.2.5. Let us prove that f_2, $f_1 \circ \rho$ are non-equivalent. Assume otherwise. Apply lemma 2.1.6 to find some $C \in F^*$, such that for every $\vec{\nu} \in [\![C]\!]^{n_2}$, $f_1(\nu_1, \ldots, \nu_{n_1}) = f_2(\nu_1, \ldots, \nu_{n_2})$. By intersecting, assume that–

$$C \subseteq \left(\underset{\vec{\nu} \in [\![A_1]\!]^{<\omega}}{\Delta^*} B^1_{\vec{\nu}} \right) \cap \left(\underset{\vec{\nu} \in [\![A_2]\!]^{<\omega}}{\Delta^*} B^2_{\vec{\nu}} \right) \cap A_1 \cap A_2 \cap X$$

Now, let us claim that the following set is dense above $\langle \vec{r}, C \rangle$:

$$D = \{ \langle \vec{r}, \vec{\nu}, S \rangle \in P_{F^*} : \mathrm{lh}(\vec{\nu}) > n_2 \text{ and } \langle \vec{r}, \vec{\nu}, S \rangle \Vdash \underset{\sim}{\sigma}_1(\check{m}_1) = \underset{\sim}{\sigma}_2(\check{m}_2) \}$$

This will finish the proof: Just take a generic $H \subseteq P_{F^*}$ such that $\langle \vec{r}, C \rangle \in H$. In particular, $\langle \vec{r}, X \rangle \in H$, and it forces that $\underset{\sim}{\sigma}_1$ and $\underset{\sim}{\sigma}_2$ are disjoint. This contradicts the density of D. Therefore, it suffices to prove that D is dense. Indeed, take some $\langle \vec{r}, \vec{\nu}, S \rangle$ above $\langle \vec{r}, C \rangle$. Assume that $\mathrm{lh}(\vec{\nu}) > n_2$ (else, extend). Then –

$$\langle \vec{r}, \vec{\nu}, S \cap B^1_{\langle \nu_1, \ldots, \nu_{n_1} \rangle} \rangle \geq \langle \vec{r}, \nu_1, \ldots, \nu_{n_1}, B^1_{\langle \nu_1, \ldots, \nu_{n_1} \rangle} \rangle \Vdash \underset{\sim}{\sigma}_1(\check{m}_1) = f_1(\overparen{\nu_1, \ldots,} \nu_{n_1})$$

and –

$$\langle \vec{r}, \vec{\nu}, S \cap B^2_{\langle \nu_1, \ldots, \nu_{n_2} \rangle} \rangle \geq \langle \vec{r}, \nu_1, \ldots, \nu_{n_2}, B^2_{\langle \nu_1, \ldots, \nu_{n_2} \rangle} \rangle \Vdash \underset{\sim}{\sigma}_2(\check{m}_2) = f_2(\overparen{\nu_1, \ldots,} \nu_{n_2})$$

Therefore, $\langle \vec{r}, \vec{\nu}, S \cap B^1_{\langle \nu_1, \ldots, \nu_{n_1} \rangle} \cap B^2_{\langle \nu_1, \ldots, \nu_{n_2} \rangle} \rangle \Vdash \underset{\sim}{\sigma}_1(\check{m}_1) = \underset{\sim}{\sigma}_2(\check{m}_2)$. □

It's straightforward to generalize proposition 2.2.9 to finitely many pairwise disjoint Prikry sequences in $V[\langle p_n : n < \omega \rangle]$; A generalization to infinitely many pairwise disjoint Prikry sequences could be done in the following way:

Proposition 2.2.10. *Assume that $\langle p_n : n < \omega \rangle$ is a Prikry sequence for P_{F^*}. Assume that $\langle \langle p^n_\xi : n < \omega \rangle : \xi < \kappa \rangle$ is a set of pairwise disjoint Prikry sequences for P_{F^*} in $V[\langle p_n : n < \omega \rangle]$, which are all disjoint from $\langle p_n : n < \omega \rangle$. Then for some $n < \omega$, there are κ-many non-equivalent, non-trivial projections of ${F^*}^n$ onto F^*.*

Proof. Denote by G the Prikry-generic set corresponding to $\langle p_n : n < \omega \rangle$. Let $\underset{\sim}{\sigma}$ be a P_{F^*}-name for $\langle \langle p^n_\xi : n < \omega \rangle : \xi < \kappa \rangle$. Assume that $\langle \vec{r}, X \rangle \in G$ forces that the elements of $\underset{\sim}{\sigma}$ are pairwise disjoint Prikry sequences for P_{F^*}, such that each one is disjoint from the Prikry sequence corresponding to the canonical name of the generic set. We slightly abuse the notation and denote $\underset{\sim}{\sigma}(\check{\xi})$ by $\underset{\sim}{\sigma}_\xi$.

Apply lemma 2.2.3, and get parameters $n_\xi, m_\xi, A_\xi \subseteq X$ such that, for every $\xi < \kappa$, the following property holds: For every $\vec{\nu} = \langle \nu_1, \ldots \nu_{n_\xi} \rangle \in [\![A_\xi]\!]^{n_\xi}$, there exist $p^\xi_{\vec{\nu}} \in Q$ and $B^\xi_{\vec{\nu}} \in F^*$, such that –

1. $\langle \vec{r}, \vec{\nu}, B^{\xi}_{\vec{\nu}} \rangle \in P_{F^*}$, and $\langle \vec{r}, \vec{\nu}, B^{\xi}_{\vec{\nu}} \rangle \Vdash \underset{\sim}{\sigma}_{\xi}(\check{m}_{\xi}) = \check{p}^{\xi}_{\vec{\nu}}$

2. For every $B \in F^*$, $B \subseteq A_{\xi}$, $\{\ p^{\xi}_{\vec{\nu}} \ : \ \vec{\nu} \in [\![B]\!]^{n_{\xi}} \ \} \in F^*$

Let $I \subseteq \kappa$ be a set cardinality κ, such that for some $n < \omega$, and for every $\xi \in I$, $n_{\xi} = n$. For simplicity, let us assume that $I = \kappa$ for the rest of the proof.

Assume that for every $\xi < \kappa$, $f_{\xi} \colon Q^n \to Q$ is a function such that, for every $\vec{\nu} \in [\![A_{\xi}]\!]^n$, $f(\vec{\nu}) = p^{\xi}_{\vec{\nu}}$. As in claim 2.2.5, each f_{ξ} is a non-trivial projection of ${F^*}^n$ onto F^*.

We now prove that the projections f_{ξ} are pairwise non-equivalent. Let $\xi_1 \neq \xi_2$. It suffices to prove that $\{\vec{\nu} \in Q^n \colon f_{\xi_1}(\vec{\nu}) = f_{\xi_2}(\vec{\nu})\} \notin {F^*}^n$. Assume the opposite, and get $C \in F^*$ such that for every $\vec{\nu} \in [\![C]\!]^n$, $f_{\xi_1}(\vec{\nu}) = f_{\xi_2}(\vec{\nu})$. By intersecting, assume that –

$$C \subseteq \left(\underset{\vec{\nu} \in [\![A_{\xi_1}]\!]^{<\omega}}{\Delta^*} B^{\xi_1}_{\vec{\nu}} \right) \cap \left(\underset{\vec{\nu} \in [\![A_{\xi_2}]\!]^{<\omega}}{\Delta^*} B^{\xi_2}_{\vec{\nu}} \right) \cap A_{\xi_1} \cap A_{\xi_2} \cap X$$

Then, as before, the following set is dense above $\langle \vec{r}, C \rangle$:

$$D = \{ \langle \vec{r}, \vec{\nu}, S \rangle \in P_{F^*} \colon lh(\vec{\nu}) > n \text{ and } \langle \vec{r}, \vec{\nu}, S \rangle \Vdash \underset{\sim}{\sigma}_{\xi_1}(\check{m}_{\xi_1}) = \underset{\sim}{\sigma}_{\xi_2}(\check{m}_{\xi_2}) \}$$

And this is a contradiction, since $\langle \vec{r}, C \rangle$ extends $\langle \vec{r}, X \rangle$, which forces that the sequences $\underset{\sim}{\sigma}_{\xi}$ are disjoint. $\square$

Remark 2.2.11. *By [3], it's consistent from large cardinals that for $Q = \langle \kappa, \in \rangle$, there exists a κ-complete ultrafilter F^*, such that the forcing P_{F^*} has a generic extension $V[\langle p_n \colon n < \omega \rangle]$, which carries a sequence–*

$$\langle \langle p^n_{\xi} \colon n < \omega \rangle \colon \xi < \kappa \rangle$$

of pairwise disjoint Prikry sequences for P_{F^*}, which are also disjoint from $\langle p_n \colon n < \omega \rangle$.

2.3 The Quotient Forcing

Assume that G is P_{F^*}-generic over V, with a corresponding Prikry sequence $\langle p_n \colon n < \omega \rangle$. Assume that $H \in V[\langle p_n \colon n < \omega \rangle]$ is RO(Q)-generic over V. Let us consider the quotient forcing P_{F^*}/H (more details about the existence and definition of the quotient forcing are included in the preliminaries).

Definition 2.3.1. *We say that two elements $\langle a_1, \ldots, a_n, A\rangle$, $\langle b_1, \ldots, b_m, B\rangle$ of P_{F^*}/H* ***can be balanced*** *if they have extensions (in P_{F^*}/H), $\langle a_1, \ldots, a_{n'}, A\rangle$ and $\langle b_1 \ldots, b_{m'}, B\rangle$, such that $h(a_{n'}) = h(b_{m'})$.*

Definition 2.3.2. *We say that a forcing notion $\langle P, <_P\rangle$ is cone homogeneous, if for every $a, b \in P$ there are extensions $a' >_P a$, $b' >_P b$ such that P/a' and P/b' are isomorphic.*

Lemma 2.3.3. *Assume that the quotient forcing P_{F^*}/H is cone homogeneous. Suppose that $\langle a_1, \ldots, a_n, A\rangle, \langle b_1, \ldots, b_m, B\rangle \in P_{F^*}/H$ can't be balanced. Then $\langle a_1, \ldots, a_n\rangle$, $\langle b_1, \ldots, b_m\rangle$ could be extended to Prikry sequences $\langle a_n \colon n < \omega\rangle$, $\langle b_n \colon n < \omega\rangle$ for P_{F^*}, which have a finite intersection, such that –*

$$V[\langle a_n \colon n < \omega\rangle] = V[\langle b_n \colon n < \omega\rangle]$$

In particular, for some $n < \omega$, there exists a non-trivial projection of ${F^*}^n$ onto F^*.

Proof. Let $p = \langle a_1, \ldots, a_{n'}, A'\rangle$, $q = \langle b_1, \ldots, b_{m'}, B'\rangle$ be some extensions of the given sequences, in P_{F^*}/H, such that, there exists an isomorphism $\sigma \in V[H]$, $\sigma \colon (P_{F^*}/H)/p \to (P_{F^*}/H)/q$. Extend both p, q to generic Prikry sequences for P_{F^*}/H, $\langle a_n \colon n < \omega\rangle, \langle b_n \colon n < \omega\rangle$, such that the image of one under σ gives the other. Then $V[\langle a_n \colon n < \omega\rangle] = V[\langle b_n \colon n < \omega\rangle]$, since $\sigma \in V[H]$. But the Prikry sequences $\langle a_n \colon n < \omega\rangle$, $\langle b_n \colon n < \omega\rangle$ have a finite intersection (because the initial sequences cannot be balanced). Therefore, by theorem 2.2.2, there exists a non-trivial projection of ${F^*}^n$ onto F^*, for some $n < \omega$. $\square$

Lemma 2.3.4. *Assume that every pair of elements of P_{F^*}/H can be balanced, and that P_{F^*}/H satisfies the following property:*

($*$) *For every $\langle a_1, \ldots, a_n, X\rangle$,$\langle b_1, \ldots, b_m, X\rangle \in P_{F^*}/H$ with $h(a_n) = h(b_m)$, and for every $x_1, \ldots x_k \in X$ and $C \subseteq X$, $\langle a_1, \ldots, a_n, x_1, \ldots, x_k, C\rangle \in P_{F^*}/H$ if and only if $\langle b_1, \ldots, b_m, x_1, \ldots, x_k, C\rangle \in P_{F^*}/H$.*

Then P_{F^*}/H is cone-homogeneous.

Proof. Assume that $\langle a_1, \ldots, a_n, A\rangle$, $\langle b_1, \ldots, b_m, B\rangle$ are two elements in P_{F^*}/H. $\langle a_1, \ldots, a_n, A\rangle$ can be extended to a Prikry sequence of the quotient forcing,

$\langle a_i : i < \omega\rangle$. By lemma 2.1.9, there exists $n_0 > n$ such that for every $i \geq n_0$, $a_i \in B$. Therefore, $\langle a_1, \ldots, a_{n_0}, A \cap B\rangle$ belongs to the quotient forcing (indeed, if G is the generic set for P_{F^*}/H, which corresponds to $\langle a_i : i < \omega\rangle$, then G is generic over P_{F^*} as well; Thus $\langle a_1, \ldots, a_{n_0}, A \cap B\rangle \in G$. In particular, $\langle a_1, \ldots, a_{n_0}, A \cap B\rangle$ belongs to P_{F^*}/H). Similarly, $\langle b_1, \ldots, b_m, B\rangle$ can be extended to $\langle b_1, \ldots, b_{m_0}, A \cap B\rangle$ that belongs to the quotient forcing. We can balance $\langle a_1, \ldots, a_{n_0}, A \cap B\rangle$ and $\langle b_1, \ldots, b_{m_0}, A \cap B\rangle$, and find extensions $\langle a_1, \ldots, a_{n'}, A \cap B\rangle$ and $\langle b_1 \ldots, b_{m'}, A \cap B\rangle$, such that $h(a_{n'}) = h(b_{m'})$. Now we simply apply $(*)$ to get the required isomorphism:

$$\langle a_1, \ldots, a_{n'}, x_1, \ldots, x_k, C\rangle \mapsto \langle b_1, \ldots, b_{m'}, x_1, \ldots, x_k, C\rangle$$

$\square$

The condition $(*)$ of lemma 2.3.4 will hold in the natural examples which will be considered.

2.4 Forcing A Club Disjoint From Inaccessibles

Let us consider an example. In this section, consider–

$$Q = \{X \subseteq \kappa \colon X \text{ is closed, bounded in } \kappa, \\ \text{and doesn't contain any inaccessible cardinal}\}$$

Ordered by $X_1 <_Q X_2 \iff X_2 \cap (\max X_1 + 1) = X_1$. This forcing is designed to turn κ into a non-Mahlo cardinal, preserving inaccessibles below it.

Notation. *We use the following notation throughout this section: For every set Z of ordinals,*

$$\overline{Z} = Z \cup \{\alpha \leq \sup(Z) : \sup(Z \cap \alpha) = \alpha\}$$

Lemma 2.4.1. *$\langle Q, <\rangle$ is κ-distributive.*

Proof. Assume $\xi < \kappa$, and let $f \colon \xi \to ON$ belong to $M[H]$, where $H \subseteq Q$ is Q-generic over V. It suffices to prove that $f \in V$. Assume without loss of generality that the weakest condition in Q forces that $\underset{\sim}{f}$ is a Q-name for a sequence on ordinals of length $\check{\xi}$. Let $q \in Q$, and let $p_0 \in P$ be such that $\max(p_0) > \xi$, $p_0 \geq q$ and $p_0 \Vdash \underset{\sim}{f}(\check{0}) = \check{\tau}_0$ for some ordinal τ_0. Proceed by induction. Assume

p_β, τ_β are chosen for every $\beta < \alpha$, where $\alpha \leq \xi$. If $\alpha = \alpha^* + 1$ is a successor, pick some ordinal τ_α and $p_\alpha \geq p_{\alpha^*}$, such that $p_\alpha \Vdash \underset{\sim}{f}(\check{\alpha}) = \check{\tau}_\alpha$. If α is limit, let–

$$p^*_\alpha = \overline{\bigcup_{\xi<\alpha} p_\xi} = \left(\bigcup_{\xi<\alpha} p_\xi\right) \cup \sup\left(\bigcup_{\xi<\alpha} p_\xi\right)$$

(we claim that p^*_α is a legitimate element of Q: It suffices to prove that $\max(p^*_\alpha)$ is not an inaccessible. We note that $\max(p^*_\alpha) > \xi$, since $\max(p_0) > \xi$. If $\max(p^*_\alpha)$ was an inaccessible, it was above ξ, with cofinality $\leq cf(\alpha) \leq \alpha \leq \xi$, contradicting regularity). Now, pick some $p_\alpha \geq p^*_\alpha$ such that $\underset{\sim}{f}(\check{\alpha})$ is decided to be some ordinal τ_α. This finishes the construction.

We can repeat this construction above any $q \in Q$, so the elements of Q which force that $f \in V$ form a dense subset, and therefore intersect the generic set H. It follows that $f \in V$. $\square$

Note that $|Q| = \kappa$. Let F^* be a κ-complete ultrafilter which extends F, the filter of dense open subsets of Q. As before, let $\pi \colon Q \to \kappa$ be the such that $[\pi]_{F^*} = \kappa$. Define the mapping $h \colon Q \to \kappa$ by $x \mapsto \sup(x) = \max(x)$. Let $G \subseteq P_{F^*}$ be generic over V, and assume that $\langle p_n \colon n < \omega\rangle$ is the corresponding Prikry sequence.

Proposition 2.4.2. *In $V[G]$, define $H^* = \{\overline{C^* \cap \alpha} \colon \alpha < \kappa\}$, where $C^* = \bigcup_{n<\omega} p^*_n$, and p^*_n is defined recursively, as follows:*

$$p^*_n = \begin{cases} p_0 & n = 0 \\ p_n \setminus \max(p^*_{n-1}) & n > 0 \end{cases} \tag{2.1}$$

Then H^* is Q-generic over V. In particular, there exists a P_{F^*}-name $\underset{\sim}{H}^*$, such that the weakest condition in P_{F^*} forces that $\underset{\sim}{H}^*$ is Q-generic over V. Moreover, $\left(\underset{\sim}{H}^*\right)_G = H^*$.

Proof. We prove first that H^* is Q-generic over V. The only non trivial property is that H^* intersects every dense open subset of Q. Let $D \subseteq Q$ be a dense open subset. Let –

$$E = \{\langle q_1, \ldots, q_n, A\rangle \in P_{F^*} \ : \ \textstyle\bigcup_{i=1}^n q^*_i \in D\}$$

where q^*_i are defined as in equation 2.1. We claim that $E \subseteq P_{F^*}$ is dense. This promises that $H^* \cap D \neq \emptyset$: Simply take some $\langle p_1, \ldots, p_m, A\rangle \in G \cap E$. So, for

$\alpha = \max(p_m)$,

$$\overline{C^* \cap \alpha} = \bigcup_{i=1}^{m} p_i^* \in H^* \cap D$$

as desired.

As for the density of E: Assume that $\langle q_1, \ldots, q_n, A \rangle \in P_{F^*}$, and let $\delta = \max(q_n) + 1$. Define a subset of q:

$$D_\delta = \{p \in D \,:\, \forall Z \subseteq \delta \ (p \setminus \delta) \cup \overline{Z} \in D\}$$

Then –

$$D_\delta = \bigcap_{Z \subseteq \delta} D_\delta(Z)$$

Where $D_\delta(Z) = \{p \in D\colon\ (p \setminus \delta) \cup \overline{Z} \in D\}$.

Now, given $Z \subseteq \delta$, $D_\delta(Z)$ is dense and open. It's simple to prove that $D_\delta(Z)$ is open. For density, take $p \in Q$, let $p' \in D$ be some extension of $(p \setminus \delta) \cup \overline{Z}$. Now, let $p'' \in D$ be some extension of $(p' \setminus \delta) \cup p$. Then $p'' >_Q p$, and $p'' \in D_\delta(Z)$.

If $\delta < \kappa$, then $2^{|\delta|} < \kappa$, since κ is inaccessible; Thus, by κ-distributivity of Q, D_δ is dense and open. Therefore $D_\delta \in F^*$. Choose some $q \in D_\delta \cap A$, such that $\pi(q) > \delta$. So –

$$\langle q_1, \ldots, q_n, A \rangle \leq \langle q_1, \ldots, q_n, q, A \rangle$$

and –

$$\left(\bigcup_{i=1}^{n} q_i^* \right) \cup (q \setminus \delta) \in D$$

so $\langle q_1, \ldots, q_n, q, A \rangle \in E$. This shows that E is dense in P_{F^*}, and proves that H^* is, indeed, Q-generic over V.

Clearly, there exists a P_{F^*}-name $\underset{\sim}{H}^*$ which is forced, by some condition in P_{F^*}, to be Q-generic over V; But we would like to choose $\underset{\sim}{H}^*$ such that it's genericity is forced by the weakest condition of P_{F^*}. This could be done using the maximal principle (see [6]): Let $\phi(x)$ be a formula which defines x from the canonical name of the generic set, in the same way H^* was defined from $\langle p_n \colon n < \omega \rangle$. The weakest condition of P_{F^*} forces $\exists x \phi(x)$, so by the maximal principle, there exists a P_{F^*}-name $\underset{\sim}{H}^*$, for which $\phi(\underset{\sim}{H}^*)$ is forced by every condition. $\square$

By distributivity of Q, every inaccessible cardinal below κ in V, remains inaccessible in $V[H^*]$. It's not hard to see that $C^* = \cup H^*$ is a club disjoint from the set of inaccessibles below κ. So κ is not Mahlo in $V[H]$.

Proposition 2.4.2 implies the existence of the quotient forcing, which we will denote P_{F^*}/C^*. Note that P_{F^*}/C^* is a non-trivial forcing notion, since, in $V[H^*]$, κ is still regular.

Remark 2.4.3. *Let us define the quotient forcing in a formal way. Denote by $RO(Q)$ the completion of Q to a complete boolean algebra, and let $i\colon Q \to RO(Q)$ be the corresponding dense embedding (to simplify notations, we write $RO(Q)$ instead $RO(Q) \setminus \{0_{RO(Q)}\}$). Then –*

$$\{q \in RO(Q)\colon \text{ for some } p \in H^*,\ i(p) \text{ extends } q\}$$

is $RO(Q)$-generic over V, and belongs to $V[H^*]$. Thus, there exists a projection $\pi\colon P_{F^*} \to RO(Q)$, and we can define in $V[H^*]$ the quotient forcing:

$$P_{F^*}/C^* = \{q \in P_{F^*}\colon \text{ for some } p \in H^*,\ i(p) \text{ extends } \pi(q)\}\} \tag{2.2}$$

The Definition of the quotient forcing in formula 2.2 is rather abstract, and it's hard to give a more explicit characterisation of P_{F^*}/C^*. Nevertheless, we can state some useful properties:

Lemma 2.4.4. *Assume that $\langle a_0, \ldots, a_n, A\rangle \in P_{F^*}/C^*$. Define, for every $i \leq n$, an element $a_i^* \in Q$, as follows:*

$$a_i^* = \begin{cases} a_0 & i = 0 \\ a_i \setminus \max(a_{i-1}^*) & i > 0 \end{cases}$$

Then –

$$\bigcup_{i=1}^{n} a_i^* = C^* \cap (\max(a_n) + 1)$$

Moreover, for every $\alpha < \kappa$, there exists an extension $\langle a_0, \ldots, a_{n'}, A'\rangle \in P_{F^*}/C^*$ of $\langle a_0, \ldots, a_n, A\rangle$, such that –

$$\left(\bigcup_{i=1}^{n'} a_i^*\right) \cap \alpha = C^* \cap \alpha$$

Proof. We prove the "moreover" part, which implies that–

$$\bigcup_{i=1}^{n} a_i^* = C^* \cap (\max(a_n) + 1)$$

by taking $\alpha = \max(a_n) + 1$.

Assume that $\alpha < \kappa$. Let $G' \subseteq P_{F^*}/C^*$ be a generic set for the quotient forcing, such that $\langle a_0, \ldots, a_n, A\rangle \in G'$. Assume that $\langle a_i : i < \omega\rangle$ is the corresponding Prikry sequence. By claim 0.2.5,

$$C^* = \bigcup_{i<\omega} a_i^*$$

(a_i^* for $i > n$ are defined in the same way). Let $\langle a_0, \ldots, a_{n'}, A'\rangle \in G'$ be some element with $\max a_{n'} \geq \alpha$. Then –

$$\left(\bigcup_{i=1}^{n'} a_i^*\right) \cap \alpha = C^* \cap \alpha$$

□

Our goal in this section is to show that in P_{F^*}/C^* there are many pairs of elements which cannot be balanced. This will be proved in proposition 2.4.7, and will be applied in theorem 2.4.8.

We use standard notations: Consider the ultrapower $\mathrm{Ult}(V, F^*)$. For a function $f : Q \to \kappa$, we denote by $[f]_{F^*}$ the standard equivalence class of f in the ultrapower construction. Recall that $\pi : Q \to \kappa$ is a function such that $[\pi]_{F^*} = \kappa$. Let $Id : Q \to Q$ be the identity function. For ordinals α, β, denote $[\alpha, \beta] = \{\gamma \leq \beta : \gamma \geq \alpha\}$, $(\alpha, \beta) = \{\gamma < \beta : \gamma > \alpha\}$.

Proposition 2.4.5. *There exists a function $\pi^* : Q \to \kappa$, an ordinal $\alpha^* < \kappa$ and a set $E \in F^*$ such that:*

1. *For every $x \in E$, $\max(x) > \pi^*(x) \geq \pi(x)$*
2. *For every $x \in E$, $x \cap \pi^*(x) = x \cap \pi(x)$*
3. *For every $x \in E$, $\pi^*(x) < \min(x \setminus \alpha^*)$*
4. *For every $p, q \in E$, if $\max(p) = \max(q)$, and –*

$$p \cap [\pi(p), \max(p)] = q \cap [\pi(p), \max(p)]$$

then $\pi^*(p) = \pi^*(q)$.

Proof. We begin by constructing a sequence of functions, $h_i: Q \to \kappa$, $\pi_i: \kappa \to \kappa$ for every $i \leq n$, where $n < \omega$ will be decided in the construction. We make sure during the construction that $[h_i]_{F^*} > \kappa$ and for every $\alpha < \kappa$, $\pi_i(\alpha) < \alpha$. Also, we define $\pi_i^* = \pi_i \circ h_i$.

Take $h_0(x) = \max(x)$. Note that $[h_0]_{F^*} > \kappa$ (equality cannot hold since the image of h doesn't contain any inaccessible cardinals). Define $W_0 = (h_0)_* F^*$, and let $\pi_0: \kappa \to \kappa$ be a function such that $[\pi_0]_{W_0} = \kappa$. Then $[\pi_0]_{W_0} < [Id]_{W_0}$ (since otherwise, W_0 was a normal ultrafilter, concentrating on the set of inaccessibles, and thus, for some $x \in Q$, $h_0(x) = \max(x)$ was inaccessible). Therefore, $\{\alpha < \kappa: \pi_0(\alpha) < \alpha\} \in W_0$, and by changing π_0 on a set outside W_0, we can assume that for every $\alpha < \kappa$, $\pi_0(\alpha) < \alpha$.

Assume that h_i was constructed, such that $[h_i]_{F^*} > \kappa$. Let us define h_{i+1}. Set $W_i = (h_i)_* F^*$. Then W_i is a non-trivial ultrafilter. Let $\pi_i: \kappa \to \kappa$ be a function, such that $[\pi_i]_{W_i} = \kappa$. Denote $\pi_i^* = \pi_i \circ h_i$. Note that –

$$\{\alpha \in \kappa: \pi_i(\alpha) \text{ is inaccessible}\} \in W_i$$

and thus –

$$\{x \in Q: \pi_i^*(x) \text{ is inaccessible}\} \in F^*$$

so $[\pi_i^*]_{F^*}$ is inaccessible, and therefore $[Id]_{F^*} \cap [\pi_i^*]_{F^*}$ is bounded in $[\pi_i^*]_{F^*}$ (since $[Id]_{F^*}$ is closed and disjoint from inaccessibles). If –

$$\max\left([Id]_{F^*} \cap [\pi_i^*]_{F^*}\right) < \kappa$$

finish the construction, and fix some $\alpha^* < \kappa$ such that –

$$[Id]_{F^*} \cap [\pi_i^*]_{F^*} \subseteq \alpha^*$$

Else, define, for every $x \in Q$, $h_{i+1}(x) = \max\left(x \cap \pi_i^*(x)\right)$, and note that $[h_{i+1}] > \kappa$ (equality cannot hold, since κ is inaccessible).

We claim that this construction must stop after finitely many steps. It's enough to argue that if the construction doesn't stop, then for every $i < \omega$, $\left[\pi_{i+1}^*\right]_{F^*} < [\pi_i^*]_{F^*}$ (so $[\pi_i^*]_{F^*}$ is a strictly decreasing sequence of ordinals in the ultrapower, and thus necessarily finite). Indeed, for every x in some set in F^*,

$$\pi_{i+1}^*(x) = \pi_{i+1}\left(h_{i+1}(x)\right) < h_{i+1}(x) = \max\left(x \cap \pi_i^*(x)\right) < \pi_i^*(x)$$

Assume that $n < \omega$ is the maximal such that π^*_n is defined. Denote $\pi^* = \pi^*_n$. Note that $[\pi^*_n]_{F^*} \geq \kappa$, since π^*_n projects F^* into a non-trivial ultrafilter. Thus,

$$[Id]_{F^*} \cap [\pi^*]_{F^*} = [Id]_{F^*} \cap \kappa$$

Take a set $E \in F^*$ such that for every $x \in E$,

1. $\pi^*(x)$ is inaccessible.

2. $x \cap \pi^*(x) \subseteq \alpha^*$

3. $x \cap \pi^*_i(x) = x \cap \pi(x) \iff i = n$

4. $\max(x) \geq \pi^*_1(x) > \ldots > \pi^*_n(x) \geq \pi(x)$

Assume that $p, q \in E$ and $\max(p) = \max(q)$. Suppose that $i < n$, $\pi^*_i(p) = \pi^*_i(q)$, and let us prove that $\pi^*_{i+1}(p) = \pi^*_{i+1}(q)$. It suffices to prove that $h_i(p) = h_i(q)$. This is clear for $i = 0$. For $i > 0$, note that –

$$\max\left(p \cap \pi^*_i(p)\right) = \max\left(q \cap \pi^*_i(q)\right) \tag{2.3}$$

indeed, since $i < n$, $p \cap \pi^*_i(p) \neq p \cap \pi(p)$, so $p \cap [\pi(p), \pi^*_i(p)] \neq \emptyset$. But –

$$p \cap [\pi(p), \max(p)] = q \cap [\pi(p), \max(p)]$$

and $\pi^*_i(q) = \pi^*_i(p) \geq \pi(p)$, so –

$$p \cap [\pi(p), \pi^*_i(p)] = q \cap [\pi(p), \pi^*_i(q)] \neq \emptyset$$

and 2.3 follows.

Now, let us prove that for every $x \in E$, $\pi^*(x) < \min\left(x \setminus \alpha^*\right)$. It's clear that the equality $\pi^*(x) = \min\left(x \setminus \alpha^*\right)$ cannot hold, since $\pi^*(x)$ is inaccessible. Thus, it suffices to prove that $\pi^*(x) \leq \min\left(x \setminus \alpha^*\right)$. This is clear as well, since otherwise,

$$\min\left(x \setminus \alpha^*\right) \in x \cap \pi^*(x) \subseteq \alpha^*$$

Lastly, for every $x \in E$, $\max(x) \neq \pi^*(x)$ (because $\pi^*(x)$ is inaccessible), and thus $\max(x) > \pi^*(x)$. $\square$

Lemma 2.4.6. *The following set is dense in P_{F^*}/C^*:*

$$D = \{\langle \vec{q}, X\rangle \colon \{\max(a) \colon \langle \vec{q}, a, X\rangle \in P_{F^*}/C^*\} \textit{ is unbounded in } \kappa\}$$

Proof. Suppose otherwise. Let $\langle \vec{q}, X\rangle \in P_{F^*}/C^*$ be an element which has no extension in D. Define, for every $n < \omega$,

$$S_n = \{\max(s)\colon \text{for some } \vec{p} \in [\![Q]\!]^n, \langle \vec{q}^\frown \vec{p}^\frown \langle s\rangle, X\rangle \in P_{F^*}/C^*\}$$

Let us argue, by induction on n, that $|S_n| < \kappa$. For $n = 0$ this is clear. Assume that $|S_n| < \kappa$. Let $\alpha < \kappa$ be some upper bound of S_n (we work in $V[H^*]$, where κ is still regular, so S_n is bounded in κ). Let–

$$A = \{\vec{p} \in [\![Q]\!]^n\colon\ \max(\mathrm{mc}(\vec{p})) \leq \alpha \text{ and } \langle \vec{q}^\frown \vec{p}, X\rangle \in P_{F^*}/C^*\}$$

Note that $|A| < \kappa$, since there are less then κ sequences $\vec{p} \in [\![Q]\!]^n$ with $\max(\mathrm{mc}(\vec{p})) \leq \alpha$. Also, for every $\vec{p} \in A$, $\langle \vec{q}^\frown \vec{p}, X\rangle$ extends $\langle \vec{q}, X\rangle$, and thus doesn't belong to D. Therefore, there exists an upper bound $\tau(\vec{p}) < \kappa$ for the set–

$$\{\max(a)\colon \langle \vec{q}^\frown \vec{p}^\frown \langle a\rangle \in P_{F^*}/C^*\rangle\}$$

Let $\tau < \kappa$ be an upper bound for the set $\{\tau(\vec{p})\colon \vec{p} \in A\}$. Thus–

$$|S_{n+1}| < \kappa$$

(since every element in S_{n+1} has a maximum less then τ), as required.

Denote $S = \bigcup_{n<\omega} S_n$. Then S is bounded in κ, assume that by some $\beta < \kappa$. Extend $\langle \vec{q}, X\rangle$ to a generic set G' for P_{F^*}/C^*. Then G' is generic for P_{F^*} as well, but is disjoint from the dense set $\{\langle \vec{p}, A\rangle \in P_{F^*}\colon\ \max(\mathrm{mc}(\vec{p})) > \beta\}$. □

Proposition 2.4.7. *In P_{F^*}/C^*, every element has at least two extensions which cannot be balanced.*

Proof. Let $\langle q_0, \ldots, q_l, X\rangle \in P_{F^*}/C^*$ be an arbitrary element. Fix a set $E \in F^*$ and an ordinal $\alpha^* < \kappa$ as in proposition 2.4.5, i.e., such that –

1. For every $x \in E$, $\max(x) > \pi^*(x) \geq \pi(x)$
2. For every $x \in E$, $x \cap \pi^*(x) = x \cap \pi(x)$
3. For every $x \in E$, $\pi^*(x) < \min(x \setminus \alpha^*)$
4. For every $p, q \in E$, if $\max(p) = \max(q)$, and –

$$p \cap [\pi(p), \max(p)] = q \cap [\pi(p), \max(p)]$$

then $\pi^*(p) = \pi^*(q)$.

Extend $\langle q_0, \ldots, q_l, X\rangle \in P_{F^*}/C^*$ to a Prikry generic sequence for P_{F^*}/C^*, $\langle q_n : n < \omega\rangle$. By claim 2.1.9, there exists $k < \omega$ such that for every $k' \geq k$, $q_{k'} \in E$. Therefore, $\langle q_0, \ldots, q_k, E'\rangle$ belongs to P_{F^*}/C^* and extends $\langle q_0, \ldots, q_l, X\rangle$, for some $E' \subseteq E$, $E' \in F^*$. Assume that $\max(q_k) > \alpha^*$ (else, extend). Let D be the dense subset from lemma 2.4.6. Assume that $\langle q_0, \ldots, q_k, E'\rangle \in D$ (else, extend). Denote $\vec{q} = \langle q_0, \ldots, q_k\rangle$. Let–

$$s = \bigcup_{i=0}^{k} q_i^*$$

where q_i^* is defined as in equation 2.1.

Assume that $\langle \vec{q}, a_1, \ldots, a_n, A\rangle$ and $\langle \vec{q}, b_1, \ldots, b_m, B\rangle$ both extend $\langle \vec{q}, E'\rangle$ in P_{F^*}/C^*, such that $\max(a_1) \neq \max(b_1)$ (such a_1, b_1 exists since $\langle \vec{q}, E'\rangle \in D$). We prove that $\max(a_i) \neq \max(b_j)$ for every i, j. Assume the contrary, and let $n \in \mathbb{N}$ be the least index such that for some $m \in \mathbb{N}$, $\max(a_n) = \max(b_m)$. Take the least such m. It follows that –

$$s \cup \left(\bigcup_{i=1}^{n} a_i^* \setminus \max(s)\right) = s \cup \left(\bigcup_{i=1}^{m} b_i^* \setminus \max(s)\right) \tag{2.4}$$

(by lemma 2.4.4). Consider the following cases:

1. $m = 1, n > 1:$ By equation (2.4),

$$a_n \cap (\max(a_{n-1}), \max(a_n)] = b_1 \cap (\max(a_{n-1}), \max(a_n)]$$

   Now, since $\pi(a_n) \in (\max(a_{n-1}), \max(a_n))$, it follows that –

$$a_n \cap [\pi(a_n), \max(a_n)] = b_1 \cap [\pi(a_n), \max(a_n)]$$

   and thus $\pi^*(a_n) = \pi^*(b_1)$. But this is a contradiction because –

$$\pi^*(b_1) < \min(b_1 \setminus \alpha^*) \leq \max(a_{n-1}) < \pi^*(a_n)$$

   ($\min(b_1 \setminus \alpha^*) \leq \max(a_{n-1})$ follows since $\max(a_{n-1}) \in b_1 \setminus \alpha^*$, by 2.4).

2. $n = 1, m > 1:$ Simply use a symmetric argument to get a contradiction.

3. $m > 1, n > 1:$ By minimality of m, n,

$$\max(a_{n-1}) \neq \max(b_{m-1})$$

Assume without loss of generality that $\max(a_{n-1}) > \max(b_{m-1})$. By equation (2.4),

$$a_n \cap (\max(a_{n-1}), \max(a_n)] = b_m \cap (\max(a_{n-1}), \max(a_n)]$$

and since $\pi(a_n) \in (\max(a_{n-1}), \max(a_n))$, it follows that –

$$a_n \cap [\pi(a_n), \max(a_n)] = b_m \cap [\pi(a_n), \max(a_n)]$$

and thus $\pi^*(a_n) = \pi^*(b_m)$. Therefore,

$$\max(a_{n-1}) = \max(C^* \cap \pi^*(a_n)) = \max(C^* \cap \pi^*(b_m)) = \max(b_{m-1})$$

a contradiction.

□

Theorem 2.4.8. *Suppose that P_{F^*}/C^* is homogeneous. Then P_{F^*} has a generic extension which contains a set $\langle\langle\xi_n^\alpha\colon n < \omega\rangle\colon \alpha < \kappa\rangle$ of pairwise disjoint Prikry sequences for P_{F^*}.*

Proof. Begin as in the last proposition: Let $\langle\xi_0, \ldots, \xi_l, X\rangle \in P_{F^*}/C^*$ be an arbitrary element in the quotient forcing. Take $E \in F^*$ and $\alpha^* < \kappa$ as in proposition 2.4.5. Find an extension $\langle\vec{\xi}, E'\rangle = \langle\xi_0, \ldots, \xi_m, E'\rangle \in P_{F^*}/C^*$ of $\langle\xi_0, \ldots, \xi_l, X\rangle$ such that $E' \subseteq E$, and such that the following holds: There exists a set A,

$$A \subseteq \{a\colon \langle\vec{\xi}^\frown\langle a\rangle, E'\rangle \in P_{F^*}/C^*\}$$

for which $\{\max(a)\colon a \in A\}$ is unbounded in κ (this is possible due to lemma 2.4.6). Then $|A| = \kappa$, since κ is still regular in $V[H^*]$, and by shrinking A, we can assume that $a \neq a' \in A \to \max(a) \neq \max(a')$. Enumerate $A = \langle a_\alpha\colon \alpha < \kappa\rangle$.

For every $\alpha < \kappa$, denote $p_\alpha = \langle\vec{\xi}^\frown\langle a_\alpha\rangle, E'\rangle \in P_{F*}/C^*$. As in the last proposition, note that for $\alpha \neq \alpha'$, p_α, $p_{\alpha'}$ cannot be balanced. Moreover, if we extend such $p_\alpha, p_{\alpha'}$ to generic Prikry sequences for the quotient forcing, those sequences will be disjoint (aside from the constant initial segment $\vec{\xi}$ that they share).

Define–

$$D_\alpha = \{q \in P_{F^*}/C^*\colon \text{ for some extension } p' \text{ of } p_\alpha,\ (P_{F^*}/C^*)/q \simeq (P_{F^*}/C^*)/p'\}$$

(where $\simeq$ denotes isomorphism between forcing notions). Then for every $\alpha < \kappa$, D_α is dense in P_{F^*}/C^*, by homogeneity of P_{F^*}/C^*. Enumerate–

$$\vec{D} = \langle D_\alpha \colon \alpha < \kappa \rangle$$

For every $\alpha < \kappa$ and $q \in D_\alpha$, fix an isomorphism $\sigma_\alpha(q) \in V[H^*]$ between $(P_{F^*}/C^*)/q$ and $(P_{F^*}/C^*)/p'$, for some p' above p_α.

Extend p_0 to a generic Prikry sequence for P_{F^*}/C^*, with a corresponding generic set G_0. G_0 is a generic set for P_{F^*} over V as well. Work in $V[G_0]$. Note that the enumerations $\langle p_\alpha \colon \alpha < \kappa \rangle$, $\langle D_\alpha \colon \alpha < \kappa \rangle$ and $\langle \sigma_\alpha(q) \colon \alpha < \kappa,\ q \in D_\alpha \rangle$ belong to $V[G_0]$.

For every $0 < \alpha < \kappa$, $G_0 \cap D_\alpha \neq \emptyset$, because G_0 is generic for P_{F^*}/C^* over $V[C^*]$, and $D_\alpha \in V[C^*]$ is a dense subset. Let $g_\alpha \in P_{F^*}/C^*$ be an element in the intersection. Then the downwards closure, in P_{F^*}/C^*, of the set–

$$\{(\sigma_\alpha(g_\alpha))(p) \colon p \in G_0 \text{ and } p \text{ extends } g_\alpha\}$$

is a generic set for P_{F^*}/C^* over $V[C^*]$ which contains p_α; Denote it by G_α. Note that–

$$\langle G_\alpha \colon 0 < \alpha < \kappa \rangle \in V[G_0]$$

Each G_α induces a generic Prikry sequence $\langle \xi^\alpha_n \colon n < \omega \rangle$ for the quotient forcing. Those are generic Prikry sequences for P_{F^*} over V as well; We can assume that the sequences $\langle \langle \xi^\alpha_n \colon n < \omega \rangle \colon \alpha < \kappa \rangle$ are pairwise disjoint, by removing, from each one, the constant initial segment of length m that they all share (after removing the initial segments, each sequence will remain a Prikry sequence for P_{F^*}, not for P_{F^*}/C^*). □

Corollary 2.4.9. *Suppose that F^* is an ultrafilter which extends the filter the dense open subsets of Q, and such that the quotient forcing P_{F^*}/C^* is homogeneous. Then for some $n < \omega$, there are κ-many non-equivalent, non-trivial projections of ${F^*}^n$ onto F^*.*

Proof. This is immediate from theorem 2.4.8 and proposition 2.2.10. □

2.5 Cohen's Forcing

In this section, let us consider $Q = \{X \subseteq \kappa \colon \sup(X) < \kappa\}$, ordered by $X_1 <_Q X_2 \iff X_2 \cap (\max X_1 + 1) = X_1$. Clearly, Q is κ-closed. This forcing could be densely embedded in the standard Cohen's forcing,

$$\text{Cohen}(\kappa) = \{f\colon A \to 2 \ : \ A \subseteq \kappa \text{ and } |A| < \kappa\}.$$

(which is ordered by inclusion), so it generates the same generic extensions. In our context, it's simpler to use $\langle Q, <_Q\rangle$; Therefore, in this section, we refer to it as Cohen's forcing, instead of Cohen(κ).

As before, let F be the filter generated by the dense open subsets of Q. Assume κ is κ-compact, and let F^* be a κ-complete ultrafilter extending F. Let $\pi\colon Q \to \kappa$ represent κ in the ultrapower. Let $h\colon Q \to \kappa$ be the function $h(x) = \sup(x)$.

Consider the forcing P_{F^*}. Suppose that $\langle p_n \colon n < \omega\rangle$ is a Prikry sequence for P_{F^*}, with a corresponding generic set G over V. Set–

$$H^* = \{C^* \cap \alpha \colon \alpha < \kappa\}$$

where $C^* = \bigcup_{n<\omega} p_n^*$, and p_n^* are defined recursively, as follows:

$$p_n^* = \begin{cases} p_0 & n = 0 \\ p_n \setminus \left(\sup(p_{n-1}^*) + 1\right) & n > 0 \end{cases} \tag{2.5}$$

Proposition 2.5.1. *$H^* \in V[G]$ is Q-generic over V. In particular, there exists a P-name $\underset{\sim}{H}^*$, such that the weakest condition in P_{F^*} forces that $\underset{\sim}{H}^*$ is Q-generic over V. Moreover, $\left(\underset{\sim}{H}^*\right)_G = H^*$.*

Proof. We repeat the proof of proposition 2.4.2 with minor changes. Given a dense open subset $D \subseteq Q$, let–

$$E = \{\langle q_1, \dots, q_n, A\rangle \in P_{F^*} \ : \ \textstyle\bigcup_{i=1}^n q_i^* \in D\}$$

Then it suffices to prove that E is dense in P_{F^*}. Indeed, given $\langle q_0, \dots, q_n, A\rangle \in P_{F^*}$, let $\delta = \sup(q_n) + 1$. Define a subset of q:

$$D_\delta = \{p \in D \ : \ \forall Z \subseteq \delta \ (p \setminus \delta) \cup Z \in D\}$$

then D_δ is dense and open; Take $q \in A \cap D_\delta$ with $\pi(q) > \sup(q_n)$. Then–

$$(q \setminus \delta) \cup \left(\bigcup_{i=1}^n q_i^*\right) \in D$$

as desired. $\square$

From the last proposition, it follows that the quotient forcing P_{F^*}/C^* could be defined the same way as in the last section. In particular, the following property holds:

Lemma 2.5.2. *Assume that* $\langle a_0, \ldots, a_n, A\rangle \in P_{F^*}/C^*$. *Define, for every* $i \leq n$, *an element* $a_i^* \in Q$, *as follows:*

$$a_i^* = \begin{cases} a_0 & i = 0 \\ a_i \setminus \left(\sup(a_{i-1}^*) + 1\right) & i > 0 \end{cases}$$

then –

$$\bigcup_{i=1}^{n} a_i^* = C^* \cap (\sup(a_n) + 1)$$

Moreover, for every $\alpha < \kappa$, *there exists an extension* $\langle a_0, \ldots, a_{n'}, A'\rangle \in P_{F^*}/C^*$ *of* $\langle a_0, \ldots, a_n, A\rangle$, *such that –*

$$\left(\bigcup_{i=1}^{n'} a_i^*\right) \cap \alpha = C^* \cap \alpha$$

Proof. Follow the same proof as in lemma 2.4.4 in order to prove the "moreover" part. The first part follows by taking $\alpha = \sup(a_n) + 1$. □

Let us argue that P_{F^*}/C^* satisfies the property $(*)$ of lemma 2.3.4.

Lemma 2.5.3. *For every* $\langle a_0, \ldots, a_n, X\rangle$, $\langle b_0, \ldots, b_m, X\rangle \in P_{F^*}/C^*$ *with* $\sup(a_n) = \sup(b_m)$, *and for every* $x_0, \ldots, x_k \in X$ *and* $A \subseteq X$,

$$\langle a_0, \ldots, a_n, x_0, \ldots, x_k, A\rangle \in P_{F^*}/C^* \iff \langle b_0, \ldots, b_m, x_0, \ldots, x_k, A\rangle \in P_{F^*}/C^*$$

Proof. Let $\sigma \colon P_{F^*}/\langle a_0, \ldots, a_n, X\rangle \to P_{F^*}/\langle b_0, \ldots, b_m, X\rangle$ be the isomorphism–

$$\sigma\left(\langle a_0, \ldots, a_n, x_0, \ldots, x_k, A\rangle\right) = \langle b_0, \ldots, b_m, x_0, \ldots, x_k, A\rangle$$

(note that without the assumption that $\sup(a_n) = \sup(b_m)$, σ is not an isomorphism, since $\sigma\left(\langle a_0, \ldots, a_n, x_0, \ldots, x_k, A\rangle\right)$ is not necessarily an element of P_{F^*}).

Let $p = \langle a_0, \ldots, a_n, x_0, \ldots, x_k, A\rangle$, $q = \sigma(p) = \langle b_0, \ldots, b_m, x_0, \ldots, x_k, A\rangle$ be extensions of $\langle a_0, \ldots, a_n, X\rangle$, $\langle b_0, \ldots, b_m, X\rangle$, respectively. Let us prove that $p \in P_{F^*}/C^* \iff q \in P_{F^*}/C^*$. Its enough to show that $\pi(p) = \pi(q)$, where π is the standard projection $\pi \colon P_{F^*} \to \mathrm{RO}(Q)$. It suffices to argue that, for every $a \in Q$,

$$q \Vdash \check{a} \in \underset{\sim}{H}^* \iff p \Vdash \check{a} \in \underset{\sim}{H}^*$$

By symmetry, it's enough to prove one direction only. Assume that $a \in Q$, $q \Vdash \check{a} \in \underset{\sim}{H}^*$. Let $G' \subseteq P_{F^*}$ be generic over V such that $p \in G'$. Our goal is to prove that $a \in \left(\underset{\sim}{H}^*\right)_{G'}$. Define –

$$G'' = \{r \in P_{F^*} : \exists p' \in G' \cap (P_{F^*}/p)\,, r \leq \sigma(p')\}$$

Then G'' is P_{F^*}-generic over V: Indeed, given $D \subseteq P_{F^*}$ dense,

$$\sigma^{-1}\left(D \cap (P_{F^*}/q)\right)$$

is dense above p. Now, $p \in G'$, so for some $s \in G'$, $s > p$ and $\sigma(s) \in D \cap G''$. The other properties needed to be checked for genericity of G'' are straightforward.

Since $q \in G''$, it follows that $a \in \left(\underset{\sim}{H}^*\right)_{G''}$. So its enough to argue that $\left(\underset{\sim}{H}^*\right)_{G''} = \left(\underset{\sim}{H}^*\right)_{G'}$. Assume that –

$$\langle a_0, \ldots, a_n \rangle^\frown \langle x_i \colon i < \omega \rangle$$

is the Prikry sequence corresponding to G'. Then –

$$\langle b_0, \ldots, b_m \rangle^\frown \langle x_i \colon i < \omega \rangle$$

is the Prikry sequence corresponding to G''. Let –

$$s = \bigcup_{i=0}^{n} a_i^* = \bigcup_{i=0}^{m} b_i^*$$

(the equality follows from lemma 2.4.4). Denote –

$$C^{**} = s \cup \left(\left(\bigcup_{i<\omega} x_i^* \right) \setminus (\sup(s) + 1) \right)$$

Then –

$$\left(\underset{\sim}{H}^*\right)_{G'} = \{C^{**} \cap \beta \colon \beta < \kappa\} = \left(\underset{\sim}{H}^*\right)_{G''}$$

□

Recall that property $(*)$ above could be used to prove homogeneity of P_{F^*}/C^*, under the assumption that every pair of elements in P_{F^*}/C^* could be balanced. This might depend on F^*. Currently, we don't know if under some choice of F^*, every pair of elements could indeed be balanced. We actually could modify F^* such that there are many elements in P_{F^*}/C^* which cannot be balanced, and we will do so in this section; In any case, Modifying F^* will require κ to satisfy more then κ-compactness, and we will assume 2^κ-supercompactness of κ.

Assume that κ is 2^κ-supercompact. Therefore, there exists a definable embedding $j\colon V \to M$ such that $\mathrm{crit}(j) = \kappa$, $2^\kappa < j(\kappa)$ and ${}^{2^\kappa}M \subseteq M$. Work in M. For every dense open $E \subseteq Q$, $j(E)$ is dense open in $j(Q)$, which is $j(\kappa)$-distributive. Therefore,

$$D^* = \bigcap_{E\subseteq Q \text{ dense open}} j(E) \tag{2.6}$$

is a dense open subset of $j(Q)$ (we note that it's an intersection of a 2^κ-sequence of elements of M, which belongs to M).

Definition 2.5.4. *For every $p \in D^*$, define an ultrafilter F_p on Q as follows:*

$$\forall X \subseteq j(Q) \quad X \in F_p \iff p \in j(X)$$

F_p is a κ-complete ultrafilter on Q which extends F, the κ-complete filter generated by the dense-open subsets of Q,

$$F = \{E \subseteq Q \,:\, X \subseteq E \text{ for some dense open subset } X \text{ of } Q\}$$

Given $p \in D^*$, let $M_p \simeq Ult(V, F_p)$ be the transitive collapse of the ultrapower, and $j_p\colon V \to M_p$ be the corresponding elementary embedding. Define an elementary embedding $k_p\colon M_p \to M$,

$$k_p\left(j_p(f)([Id]_{F_p})\right) = j(f)(p)$$

for every $f\colon Q \to V$. Then $k_p \circ j_p = j$, and $k_p([Id]_{F_p}) = p$.

Remark 2.5.5. *For every $p \in D^*$, $p \cap \kappa = [Id]_{F_p} \cap \kappa$.*

Proof. Clearly $p \cap \kappa \subseteq k_p(p) \cap \kappa$, since for every $\alpha < \kappa$, $k_p(\alpha) = \alpha$. Now, given $\alpha \in k_p(p) \cap \kappa$, note that $k_p(\alpha) = \alpha \in k_p(p)$, so, by elementarity, $\alpha \in p \cap \kappa$. $\square$

Before describing a general method to choose F^*, such that many elements cannot be balanced in the quotient forcing, we state the following lemma:

Lemma 2.5.6. *The following set is dense in P_{F^*}/C^*:*

$$D = \{\langle \vec{q}, X\rangle \colon \{\sup(a) \colon \langle \vec{q}, a, X\rangle \in P_{F^*}/C^*\} \text{ is unbounded in } \kappa\}$$

Proof. Repeat the proof of lemma 2.4.6. $\square$

Theorem 2.5.7. *Assume that κ is 2^κ-supercompact. There exists a κ-complete ultrafilter F^* which extends the filter of dense open sets of Q, such that $[Id]_{F^*} \cap \kappa$ is bounded in κ, and every condition in P_{F^*}/C^* has two extensions which cannot be balanced.*

Moreover, if P_{F^*}/C^* is homogeneous, then P_{F^*} has a generic extension which contains a set $\langle\langle \xi_n^\alpha : n < \omega\rangle : \alpha < \kappa\rangle$ of pairwise disjoint Prikry sequences for P_{F^*}.

Proof. We prove that there exists a set $E \in F^*$, an ordinal $\alpha^* < \kappa$ and a function $\pi : Q \to \kappa$, such that–

1. $[\pi]_{F^*} = \kappa$
2. For every $x \in E$, x has a maximum $\max(x)$.
3. For every $x \in E$, $\max(x) \geq \pi(x)$
4. For every $x \in E$, $\pi(x) < \min(x \setminus \alpha^*)$
5. For every $p, q \in E$, if $\max(p) = \max(q)$ then $\pi(p) = \pi(q)$.

Let $I \subseteq \kappa$ be a bounded subset. Let $\alpha^* < \kappa$ be such that $\sup(I) < \alpha^*$. Define–

$$D^{**} = \{p \in D^* : p \cap \kappa = I\}$$

(where D^* is the dense open subset of $j(Q)$, defined in equation (2.6)). It's clear that D^{**} is open, since D^* is open. D^{**} is not dense, but it is dense and open above $I' = I \cup \{\kappa + 1\}$.

In V, let $\langle A_\alpha : \alpha < \kappa\rangle$ be a partition of κ to pairwise disjoint unbounded subsets. Denote $j\left(\langle A_\alpha : \alpha < \kappa\rangle\right) = \langle A'_\alpha : \alpha < j(\kappa)\rangle$. There exists an extension p of I' such that p has a maximum, $\max(p) \in A'_\kappa$ and $p \in D^{**}$. Let $F^* = F_p$. Let $\pi : Q \to \kappa$ be defined as follows:

$$\pi(x) = \alpha \iff \sup(x) \in A_\alpha$$

(it's well defined, since $\langle A_\alpha : \alpha < \kappa\rangle$ is a partition of κ). Clearly, for every $p, q \in Q$,

$$\sup(p) = \sup(q) \to \pi(p) = \pi(q)$$

Note that $p \in j(\{x \in Q\colon \max(x) \text{ exists}\})$. So–

$$E_1 = \{x \in Q\colon \max(x) \text{ exists}\} \in F^*$$

Therefore, for every $p, q \in E_1$, $\max(p) = \max(q) \to \pi(p) = \pi(q)$.

Next, we note that the property $j(\pi)(p) = \kappa$ implies $[\pi]_{F^*} = \kappa$. Also, $j(\pi)(p) < \min(p \setminus \alpha^*)$ implies that–

$$p \in j(\{x \in Q\colon \pi(x) < \min(x \setminus \alpha^*)\})$$

so–

$$E_2 = \{x \in Q\colon \pi(x) < \min(x \setminus \alpha^*)\} \in F^*$$

Finally, for every $\beta < \kappa$,

$$\{x \in Q\colon \sup(x) > \beta\} \in F^*$$

so–

$$E_3 = \{x \in Q\colon \sup(x) \geq \pi(x)\} \in F^*$$

Take $E = E_1 \cap E_2 \cap E_3$ to get the required properties, $1 - 5$ above. Note that these properties, together with lemma 2.5.6, are enough to argue that, under the homogeneity assumption about P_{F^*}/C^*, P_{F^*} has a generic extension which contains a set $\langle\langle \xi_n^\alpha\colon n < \omega\rangle\colon \alpha < \kappa\rangle$ of pairwise disjoint Prikry sequences: Simply repeat the proof of theorem 2.4.8. $\square$

The last theorem deals with the case where, in $\mathrm{Ult}(V, F^*)$, $[Id]_{F^*} \cap \kappa$ is bounded in κ. We give a similar result in the other case, where $[Id]_{F^*} \cap \kappa$ is unbounded in κ.

Theorem 2.5.8. *Assume that κ is 2^κ-supercompact. There exists a κ-complete ultrafilter F^* which extends the filter of dense open sets of Q, such that $[Id]_{F^*} \cap \kappa$ is unbounded in κ, and every condition in P_{F^*}/C^* has two extensions which cannot be balanced.*

Moreover, if P_{F^*}/C^* is homogeneous, then P_{F^*} has a generic extension which contains a set $\langle\langle \xi_n^\alpha\colon n < \omega\rangle\colon \alpha < \kappa\rangle$ of pairwise disjoint Prikry sequences for P_{F^*}.

Proof. Let us choose F^*, a set $E \in F^*$ and a function $\pi\colon Q \to \kappa$, such that–

1. $[\pi]_{F^*} = \kappa$.

2. For every $x \in E$, $\sup(x) > \pi(x)$

3. For every $x \in E$, $x \cap \pi(x) = [Id]_{F^*} \cap \pi(x)$.

4. For every $x \in E$ and $\alpha < \sup(x)$, $(x \triangle [Id]_{F^*}) \cap (\alpha, \sup(x)) \neq \emptyset$ (in particular, $\sup(x)$ is limit).

5. For every $p, q \in E$, if $\sup(p) = \sup(q)$ then $\pi(p) = \pi(q)$.

(where D^* is the dense open subset of $j(Q)$, defined in equation (2.6)). Assume $I \subseteq \kappa$ is unbounded in κ, and let –

$$D^{**} = \{p \in D^* \,:\, p \cap \kappa = I \ \wedge\ \forall \alpha < \sup(p)\ \exists \beta > \alpha, \beta < \sup(p),\ \beta \in j(I) \setminus p\}$$

Since $j(I)$ is unbounded in $j(\kappa)$, D^* is dense above I.

In V, let $\langle A_\alpha \colon \alpha < \kappa \rangle$ be a partition of κ to pairwise disjoint unbounded subsets, where A_0 is the set of inaccessibles below κ. Denote $j\left(\langle A_\alpha \colon \alpha < \kappa\rangle\right) = \langle A'_\alpha \colon \alpha < j(\kappa)\rangle$. There exists an extension p of I such that $\sup(p) \in A'_\kappa$, (in particular, $\sup(p)$ is not an inaccessible cardinal) and $p \in D^{**}$. Take $F^* = F_p$. Let $I_1 = [Id]_{F^*}$. Then by remark 2.5.5, $I = p \cap \kappa = I_1 \cap \kappa$. Thus –

$$\begin{aligned} p \in j\left(\{q \in Q \,:\, \forall \alpha < \sup(q)\ \exists \beta > \alpha, \beta < \sup(q),\ \beta \in I \setminus q\}\right) = \\ j\left(\{q \in Q \,:\, \forall \alpha < \sup(q)\ \exists \beta > \alpha, \beta < \sup(q),\ \beta \in I_1 \setminus q\}\right) \end{aligned}$$

therefore, by the definition of $F^* = F_p$,

$$E_1 = \{q \in Q \,:\, \forall \alpha < \sup(q)\ \exists \beta > \alpha, \beta < \sup(q),\ \beta \in I_1 \triangle q\} \in F^*$$

Let $\pi \colon Q \to \kappa$ be defined as follows: For every $q \in Q$,

$$\pi(q) = \alpha \iff \sup(q) \in A_\alpha$$

then $[\pi]_{F^*} = \kappa$. Thus, for every $p, q \in Q$, if $\sup(p) = \sup(q)$ then $\pi(p) = \pi(q)$. Moreover, note that–

$$E_2 = \{x \in Q \colon x \cap \pi(x) = I_1 \cap \pi(x)\} \in F^*$$

Indeed, if j_{F^*} is the ultrapower embedding of F^*, then in $\mathrm{Ult}(V, F^*)$,

$$I_1 \cap \kappa = j_{F^*}(I_1) \cap \kappa$$

since $j_{F^*} \restriction \kappa$ is the identity. Finally, clearly $\{x \in Q \colon \sup(x) \geq \pi(x)\} \in F^*$; We claim that $E_3 = \{x \in Q \colon \sup(x) > \pi(x)\} \in F^*$. Else,

$$p \in j(\{x \colon \sup(x) \text{ is an inaccessible}\})$$

a contradiction.

Take $E = E_1 \cap E_2 \cap E_3$. Let $\langle q_0, \ldots, q_l, X\rangle$ be an arbitrary element in P_{F^*}/C^*; Extend it to $\langle q_0, \ldots, q_k, E'\rangle$ where $E' \subseteq X \cap E$, and such that the set–

$$A = \{\sup(a) \colon a \in E' \text{ and } \langle q_0, \ldots, q_k, a, E'\rangle \in P_{F^*}/C^*\}$$

is unbounded. Denote–

$$s = \bigcup_{i=0}^{k} q_i^*$$

(where q_i^* are defined as in (2.5)). Take $a_1, a_2 \in A$ with $\sup(a_1) \neq \sup(a_2)$, and let us prove that–

$$\langle \vec{q}, a_1, E'\rangle, \langle \vec{q}, b_1, E'\rangle$$

cannot be balanced. This suffices to finish the proof, exactly as in theorem 2.4.8. Suppose for contrary that–

$$\langle \vec{q}, a_1, \ldots, a_n, E\rangle, \langle \vec{q}, b_1, \ldots, b_m, E\rangle \in P_{F^*}/H^*$$

and $\sup(a_n) = \sup(b_m)$. Let $n \in \mathbb{N}$ be the least index such that for some $m \in \mathbb{N}$, $\sup(a_n) = \sup(a_m)$. Take the least such m. Then, by lemma 2.5.2 –

$$s \cup \left(\bigcup_{i=1}^{n} a_i^* \setminus (\sup(s)+1) \right) = s \cup \left(\bigcup_{i=1}^{m} b_i^* \setminus (\sup(s)+1) \right) \tag{2.7}$$

Let us derive a contradiction. We consider the following cases:

1. $m = 1, n > 1$: Since $\sup(a_n) = \sup(b_1)$, it follows that $\pi(a_n) = \pi(b_1)$. In particular, $\pi(b_1) > \sup(a_{n-1})$. Now, $b_1 \cap \pi(b_1) = I_1 \cap \pi(b_1)$. Take an arbitrary $\alpha \in (\pi(a_{n-1}), \sup(a_{n-1}))$. We remark that such α exists since $\sup(a_{n-1})$ is limit. By equation (2.7), and since $\pi(b_1) > \sup(a_{n-1})$,

   $$(a_{n-1} \triangle I_1) \cap (\alpha, \sup(a_{n-1})) = \emptyset$$

   a contradiction.

2. $n = 1, m > 1$: Apply the symmetric argument to get a contradiction.

3. $m > 1, n > 1$: Since $\sup(a_n) = \sup(b_m)$, it follows that $\pi(a_n) = \pi(b_m)$. By minimality of m, n, it follows that $\sup(a_{n-1}) \neq \sup(b_{m-1})$. Assume without loss of generality that $\sup(a_{n-1}) < \sup(b_{m-1})$. Take an arbitrary $\alpha \in (\sup(a_{n-1}), \sup(b_{m-1}))$. By equation (2.7),

$$a_n \cap (\alpha, \sup(b_{m-1})) = b_{m-1} \cap (\alpha, \sup(b_{m-1})) \tag{2.8}$$

But $\sup(b_{m-1}) < \pi(a_n)$, so –

$$a_n \cap (\alpha, \sup(b_{m-1})) = I_1 \cap (\alpha, \sup(b_{m-1})) \tag{2.9}$$

Combining (2.8) and (2.9),

$$(b_{m-1} \triangle I_1) \cap (\alpha, \sup(b_{m-1})) = \emptyset$$

a contradiction.

□

Corollary 2.5.9. *For the ultrafilters F^* from theorems 2.5.7, 2.5.8, suppose that P_{F^*}/C^* is homogeneous. Then there are κ-many, non-equivalent, non-trivial projections of ${F^*}^n$ onto F^*, for some $n < \omega$.*

Proof. Combine the theorems with proposition 2.2.10. □

We currently don't know if the following interesting scenario is possible: P_{F^*}/C^* is homogeneous, where F^* is one of the ultrafilters from theorems 2.5.7, 2.5.8. This promises a generic extension for P_{F^*}, which contains a set $\langle\langle \xi_n^\alpha : n < \omega\rangle : \alpha < \kappa\rangle$ of pairwise disjoint Prikry sequences for P_{F^*}. Under this scenario, P_{F^*}/C^* contains many pairs of elements which cannot be balanced, so property $(*)$ of lemma 2.3.4 can't be applied to prove homogeneity.

2.6 Concluding Remarks

Suppose that $V[p_n : n < \omega] = V[q_n : n < \omega]$, where $\langle p_n : n < \omega\rangle$, $\langle q_n : n < \omega\rangle$ are pairwise disjoint Prikry sequence for P_{F^*} over V. Then, as we proved, F^* is a non-normal ultrafilter, and, for some $n < \omega$, ${F^*}^n$ can be projected onto F^*, in a non-trivial way.

Question 2.6.1. *(Under suitable large cardinal axioms) Given a κ-complete filter F on κ, could F be extended to a κ-complete ultrafilter F^* on κ, such that ${F^*}^n$ cannot be projected onto F^* in a non-trivial way, for every $n < \omega$?*

A positive answer promises that $V[p_n : n < \omega] \neq V[q_n : n < \omega]$, whenever $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$ are pairwise disjoint Prikry sequences for P_{F^*}. Another question is natural from our analysis:

Question 2.6.2. *Even if we allow non-trivial projections of ${F^*}^n$ onto F^*, can we choose F^* such that there are less then κ such projections, up to equivalence?*

Doing this for the forcing notion which adds a club disjoint from inaccessibles, promises that the quotient forcing is non-homogeneous.

We remark that by [3], it's consistent, from large cardinals, that for some non-normal, κ-complete ultrafilter F on κ, there are κ many non-trivial projections of F^2 onto F.

Property $(*)$ from lemma 2.3.4 holds in the natural examples we considered: The proof we gave in the last section, holds both in the context of Cohen's forcing, and the forcing which adds a club disjoint from inaccessibles. This is because the generic sets for both forcing notions were created, more or less, in the same way. However, for the forcing which adds a club disjoint from inaccessibles, lemma 2.3.4 could not be applied to prove homogeneity of the quotient forcing, because there are many elements which cannot be balanced. As for Cohen's forcing:

Question 2.6.3. *Suppose that $Q = Cohen(\kappa)$ is Cohen's forcing. Does there exist a choice of a measure F^* which extends the filter of dense open subsets of Q, and a generic set $H \subseteq Q$ over V, such that every pair of elements in the quotient forcing P_{F^*}/H can be balanced?*

A positive answer to this question, results in an homogeneous quotient forcing.[1]

[1]It looks like the negative answer is consistent. We plan to address this issue in a further paper.

References

[1] James Cummings. Iterated forcing and elementary embeddings. In *Handbook of set theory*, pages 775–883. Springer, 2010.

[2] Moti Gitik. Prikry-type forcings. In *Handbook of set theory*, pages 1351–1447. Springer, 2010.

[3] Moti Gitik. Strange ultrafilters. *Archive for Mathematical Logic*, To Appear.

[4] Thomas Jech. Set theory. the third millennium edition. *Springer Monographs in Mathematics. Springer-Verlag*, 2003.

[5] Peter Koepke, Karen Rasch, and Philipp Schlicht. A minimal prikry-type forcing for singularizing a measurable cardinal. *The Journal of Symbolic Logic*, 78(1):85100, 2013.

[6] Kenneth Kunen. *Set theory an introduction to independence proofs*, volume 102. Elsevier, 2014.