Around accumulation points and maximal sequences of indiscernibles.

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Abstract

Answering a question of Mitchell [7] we show that a limit of accumulation points can be singular in $\mathcal{K}$. Some additional constructions are presented.

1 Introduction

W. Mitchell [7] stated the following problem (2.12 there):

Suppose that $\vec{\kappa} = \langle \kappa_i \mid i < \omega \rangle$ *is an increasing sequence of measurable cardinals in the core model such that* $o(\kappa_{i+1}) = \kappa_i$. *Is there a larger model* M *in which each* κ_i *is still measurable and such that if* $\vec{\gamma}$ *and* $\vec{\beta}$ *are any sequences such that* $\gamma_i < \kappa_i$ *and* $\beta_i < o(\kappa_i)$, *for all* $i < \omega$, *then there is a sequence* $\vec{c}$ *which is an indiscernible sequence for* $(\vec{\kappa}, \vec{\beta})$ *such that* $\gamma_i < c_i$ *for all* $i < \omega$*?*

The problem is tightly related to the problem of existence of accumulation points. Such points are one of the basic components of the Mitchell Covering Lemma (see, for example, [8]). Mitchell showed that if accumulation points exist then there is a cardinal α such that $\{o(\beta) \mid \beta < \alpha\}$ is unbounded in α. On the other hand in [5], starting with such measurable α, a model with accumulation points was constructed.
It remained open whether a measurability of α can be removed and if a limit of accumulation points can be singular in the core model. Mitchell note in [7] that an affirmative answer to the problem above would answer both of these questions affirmatively.

The main purpose of the paper is to give an affirmative answer to the Mitchell problem. This is done in Section 2. A new variation of short extenders forcing, based on names in

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order to compensate incompleteness of extenders, is used for this. It may of an interest by its own.

In Section 3, we address a question of maximality of sequences of indiscernibles. By results of Jensen -Dodd and Mitchell (see for example [8]), if $o(\kappa) < \kappa$, κ changes its cofinality, then there is such sequence. We will show that this not true anymore if $o(\kappa) = \kappa$.

2 A variation of short extenders forcing.

We would like now to present the extender based Prikry forcing using very short extenders. This will allow to give an affirmative answer to a problem of Mitchell, 2.12 [7].

Assume GCH. Let $\kappa = \bigcup_{n<\omega} \kappa_n$ with $\langle \kappa_n \mid n < \omega \rangle$ increasing.
Assume that each κ_n is measurable and there exists a coherent sequence
$\vec{U}^n = \langle U(\alpha, \beta) \mid \kappa_{n-1} < \alpha \leq \kappa_n, \beta < o^{\vec{U}^n}(\alpha) \rangle$ with $o^{\vec{U}^n}(\kappa_n) = \kappa_{n-1}$, where $\kappa_{-1} = 1$.

Force first with the forcing of [4] and turn the Mitchell order into the Rudin -Keisler order. Denote the extension by V_1. For each $n < \omega$, the $\triangleleft-$increasing sequence $\langle U(\kappa_n, \beta) \mid \beta < \kappa_{n-1} \rangle$ is turned into a $\leq_{R-K}$ $-$sequence $\langle U^*(\kappa_n, \beta) \mid \beta < \kappa_{n-1} \rangle$ in V_1.
Let $j_n : V_1 \to M_n$ be the corresponding direct limit embedding.
Then ${}^{\kappa_{n-1}>}M_n \subseteq M_n$.
Note that M_0 will be just the ultrapower by $U^*(\kappa_0, 0)$, and so it is closed under κ_0-sequences of its elements.
We view $\langle U^*(\kappa_n, \beta) \mid \beta < \kappa_{n-1} \rangle$ as extender over κ_n. Just define a $(\kappa_n, j_n(\kappa_n))-$extender E_n by setting

$$X \in E_n(\beta) \Leftrightarrow \beta \in j_n(X).$$

Our next task will be to force, using $\langle E_n \mid n < \omega \rangle$, κ^+-many $\omega-$sequences in $\prod_{n<\omega} \kappa_n$.
We will use a variation of a forcing of [6] and [1] for this purpose.
A slight complication here is that the extenders E_n are only $< \kappa_{n-1}-$closed.

We are now ready to define our first forcing notion. It will resemble the one element Prikry forcing and will be built from two pieces.
Fix $n < \omega$.

Definition 2.1 Let

$$Q_{n1} = \{f \mid f \text{ is a partial function from } \kappa^+ \text{ to } \kappa_n \text{ of cardinality at most } \kappa\}.$$

We order Q_{n1} by inclusion. Denote this order by $\leq_1$.

Thus Q_{n1} is basically the usual Cohen forcing for adding a function from κ^+ to κ_n.

Definition 2.2 Let Q_{n0} be the set of triples $p = \langle a, A, f\rangle$ so that

(1) $f \in Q_{n1}$

(2) a is an order preserving partial function from κ^+ to $j_n(\kappa_n)$ such that

(2)(i) $|a| < \kappa_{n-1}$

(2)(ii) $\mathrm{dom}(a) \cap \mathrm{dom}(f) = \emptyset$

(2)(iii) $\mathrm{rng}(a)$ has a $\leq_{E_n}$-maximal element, i.e. an element $\alpha \in \mathrm{rng}(a)$ such that $\alpha \geq_{E_n} \beta$ for every $\beta \in \mathrm{rng}(a)$

(3) $A \in U_{n\,\max(\mathrm{rng}(a))}$

(4) for every $\alpha, \beta, \gamma \in \mathrm{rng}(a)$, if $\alpha \geq_{E_n} \beta \geq_{E_n} \gamma$, then $\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$ for every $\rho \in \pi_{max(\mathrm{rng}(a)),\alpha}$ “A

(5) for every $\alpha > \beta$ in a and every $\nu \in A$

$$\pi_{\max(\mathrm{rng}(a)),\alpha}(\nu) > \pi_{\max(\mathrm{rng}(a)),\beta}(\nu) \ .$$

Further we will often denote a by $a(p)$, A by $A(p)$ and f by $f(p)$.

Definition 2.3 Let $\langle a, A, f\rangle, \langle b, B, g\rangle \in Q_{n0}$. Then

$$\langle a, A, f\rangle \geq_0 \langle b, B, g\rangle$$

($\langle a, A, f\rangle$ is stronger than $\langle b, B, g\rangle$) iff

(1) $f \supseteq g$

(2) $a \supseteq b$

(3) $\pi_{\max(a),\max(b)}$ “$A \subseteq B$

We now define a forcing notion Q_n:

Definition 2.4 $Q_n = Q_{n0} \cup Q_{n1}$.

Definition 2.5 The direct extension ordering $\leq^*$ on Q_n is defined to be $\leq_0 \cup \leq_1$.

Definition 2.6 (One step extension) Let $p = \langle a, A, f\rangle \in Q_{n0}$ and $\nu \in A$. Define $p{}^\frown\nu$ to be an element $g \in Q_{n1}$ such that

1. $\mathrm{dom}(g) = \mathrm{dom}(a) \cup \mathrm{dom}(f)$,
2. for every $\alpha \in \mathrm{dom}(f)$, $g(\alpha) = f(\alpha)$,
3. $g(\max(\mathrm{dom}(a)) = \nu$,
4. for every $\beta \in \mathrm{dom}(a)$ $\;g(\beta) = \pi_{\max(\mathrm{rng}(a)), a(\beta)}(\nu)$.

Definition 2.7 Let $p, q \in Q_n$.

Then $p \le q$ iff either

(1) $p \le^* q$ or

(2) $p = \langle a, A, f\rangle \in Q_{n0}, q \in Q_{n1}$ and the following holds for some $\nu \in A$, $q \ge_1 p{}^\frown\nu$.

Clearly, the forcing $\langle Q_n, \le\rangle$ is equivalent to $\langle Q_{n1}, \le_1\rangle$, i.e. the Cohen forcing. However, the following basic facts relate it to the Prikry type forcing notion.

The next two lemmas are standard.

Lemma 2.8 *$\langle Q_n, \le^*\rangle$ is κ_{n-1}-closed.*

Lemma 2.9 *$\langle Q_n, \le, \le^*\rangle$ satisfies the Prikry condition, i.e. for every $p \in Q_n$ and every statement σ of the forcing language there is $q \ge^* p$ deciding σ.*

Now let us put the blocks $Q_n, n < \omega$ together. A difference from the usual short extenders forcings is now that Q_n's are not complete enough (only $\kappa_{n-1}-$complete due to incompleteness of the extender E_n) and in order to overcome this names (but rather simple ones) will be used.

Let us deal first with two - Q_0, Q_1.

Definition 2.10 The set $\mathcal{P}_{\le 1}$ consists of sequences $p = \langle p_0, p_1\rangle$ so that either

1. $p_0 \in Q_{01}$ and $p_1 \in Q_{11}$, i.e. $p \in Q_{01} \times Q_{11}$.
   Or
2. $p_0 \in Q_{01}$ and $p_1 \in Q_{10}$, i.e. $p \in Q_{01} \times Q_{10}$.
   Or

3. $p_0 \in Q_{00}$ and $p_1 = \langle \underset{\sim}{a}_1, \underset{\sim}{A}_1, \underset{\sim}{f}_1 \rangle$,
where $\underset{\sim}{a}_1 = \{\langle \nu, a_1^\nu \rangle \mid \nu \in A_0\}$, $\underset{\sim}{A}_1 = \{\langle \nu, A_1^\nu \rangle \mid \nu \in A_0\}$ and $\underset{\sim}{f}_1 = \{\langle \nu, f_1^\nu \rangle \mid \nu \in A_0\}$ are such that

   (a) for every $\nu \in A_0$, $\langle a_1^\nu, A_1^\nu, f_1 \rangle \in Q_{10}$,

   (b) for every $\nu \in A_0$, $\mathrm{dom}(a_1^\nu) \supseteq \mathrm{dom}(a(p_0))$,

   (c) let *the potential domain of* a_1, $ptdom(a_1) = \bigcup_{\nu \in A_0} \mathrm{dom}(a_1^\nu)$. We require that for every $\nu, \mu \in A_0$, $f_1^\nu \upharpoonright (\kappa^+ \setminus ptdom(a_1)) = f_1^\mu \upharpoonright (\kappa^+ \setminus ptdom(a_1))$.
   Intuitively, this means that only inside the potential domain of a_1 the Cohen function f_1 is a name.

The intuitive meaning is that a choice of an element in A_0 determines the function a_1, the set of measure one A_1 for $E_{1\,\max(\mathrm{rng}(a_1))}$ and the Cohen function f_1.

Define now orders $\leq^*$ and $\leq$ on $\mathcal{P}_{\leq 1}$. Start with $\leq^*$.

Definition 2.11 Let $p = \langle p_0, p_1 \rangle$ and $q = \langle q_0, q_1 \rangle$ be in $\mathcal{P}_{\leq 1}$. Set $p \geq^* q$ iff either
If

1. $p, q \in Q_{01} \times Q_{11}$ and $p \geq_{Q_{01} \times Q_{11}} q$,
or

2. $p, q \in Q_{01} \times Q_{10}$ and $p \geq_{Q_{01} \times Q_{10}} q$,
or

3. $p_0 = \langle a(p)_0, A(p)_0, f(p)_0 \rangle, q_0 = \langle a(p)_0, A(p)_0, f(p)_0 \rangle \in Q_{00}$,
$p_1 = \langle \underset{\sim}{a}(p)_1, \underset{\sim}{A}(p)_1, \underset{\sim}{f}(p)_1 \rangle, q_1 = \langle \underset{\sim}{a}(q)_1, \underset{\sim}{A}(q)_1, \underset{\sim}{f}(q)_1 \rangle$,
and then the following hold:

   (a) $q_0 \leq_{Q_{00}} p_0$,

   (b) for every $\nu \in A(p)_0$,
   $\langle a(q)_1^\rho, A(q)_1^\rho, f(q)_1^\rho \rangle \leq_{Q_{10}} \langle a(p)_1^\nu, A(p)_1^\nu, f(p)_1^\nu \rangle$,
   where $\rho = \pi_{\max(\mathrm{rng}(a(p)_0)), \max(\mathrm{rng}(a(q)_0))}(\nu)$.

Let now $p = \langle p_0, p_1 \rangle \in \mathcal{P}_{\leq 1}$, $p_0 = \langle a(p)_0, A(p)_0, f(p)_0 \rangle$, $p_1 = \langle \underset{\sim}{a}(p)_1, \underset{\sim}{A}(p)_1, \underset{\sim}{f}(p)_1 \rangle$ and $\nu \in A(p)_0$.
We set $p^\frown \nu = \langle p_0{}^\frown \nu, \langle a(p)_1^\nu, A(p)_1^\nu, f(p)_1^\nu \rangle$.
If $\nu_1 \in A(p)_1^\nu$, then set $p^\frown \langle \nu, \nu_1 \rangle = \langle p_0{}^\frown \nu, p_1^\nu{}^\frown \nu_1 \rangle$, where $p_1^\nu = \langle a(p)_1^\nu, A(p)_1^\nu, f(p)_1^\nu \rangle$.

Definition 2.12 Let $p = \langle p_0, p_1 \rangle$ and $q = \langle q_0, q_1 \rangle$ be in $\mathcal{P}_{\leq 1}$. Set $p \geq q$ iff either

1. $p \geq^* q$,
   or

2. for some ν, $q^\frown \nu$ is defined and $p \geq^* q^\frown \nu$,
   or

3. for some $\langle \nu_0, \nu_1 \rangle$, $q^\frown \langle \nu_0, \nu_1 \rangle$ is defined and $p \geq^* q^\frown \langle \nu_0, \nu_1 \rangle$.

Define for every $n < \omega$, $\mathcal{P}_{\leq n}$ in a similar fashion.

Definition 2.13 The set $\mathcal{P}_{\leq n}$ consists of sequences $p = \langle p_0, p_1, ..., p_n \rangle$ so that either

1. for every $m \leq n$, $p_m \in Q_{m1}$, i.e. $p \in Q_{01} \times Q_{11} \times ... \times Q_{n1}$.
   Or

2. there is $\ell(p) < n$ such that

   (a) for every $m < \ell(p)$, $p_m \in Q_{m1}$,

   (b) $p_{\ell(p)} \in Q_{\ell(p)0}$,

   (c) for every $m, \ell(p) < m \leq n$, $p_m = \langle \underset{\sim}{a}_m, \underset{\sim}{A}_1, \underset{\sim}{f}_m \rangle$, where
   $\underset{\sim}{a}_m$ consists of pairs $\langle \langle \nu_k \mid \ell(p) \leq k < m \rangle, a_m^{\langle \nu_k | \ell(p) \leq k < m \rangle} \rangle$
   $\underset{\sim}{A}_m$ consists of pairs $\langle \langle \nu_k \mid \ell(p) \leq k < m \rangle, A_m^{\langle \nu_k | \ell(p) \leq k < m \rangle} \rangle$
   and $\underset{\sim}{f}_m$ consists of pairs $\langle \langle \nu_k \mid \ell(p) \leq k < m \rangle, f_m^{\langle \nu_k | \ell(p) \leq k < m \rangle} \rangle$, such that
   $\nu_{\ell(p)} \in A_{\ell(p)}$, $\nu_{\ell(p)+1} \in A_{\ell(p)+1}^{\nu_{\ell(p)}}$,..., $\nu_k \in A_k^{\nu_{\ell(p)},...,\nu_{k-1}}$,..., $\nu_{m-1} \in A_{m-1}^{\nu_{\ell(p)},...,\nu_{m-2}}$.
   Further let us call such sequences $\langle \nu_k \mid \ell(p) \leq k \leq m \rangle$- *suitable extension sequences for p*.
   We require that $\langle a_m^{\langle \nu_k | \ell(p) \leq k < m \rangle}, A_m^{\langle \nu_k | \ell(p) \leq k < m \rangle}, f_m^{\langle \nu_k | \ell(p) \leq k < m \rangle} \rangle \in Q_{m0}$.
   Moreover, if $\langle \nu_k \mid \ell(p) \leq k < m_1 \rangle$ and $\langle \mu_k \mid \ell(p) \leq k < m_2 \rangle$, are two suitable extension sequences, $m_1 < m_2$ and $\langle \nu_k \mid \ell(p) \leq k < m_1 \rangle$ is an initial segment of $\langle \mu_k \mid \ell(p) \leq k < m_2 \rangle$ then $\mathrm{dom}(a_{m_1}^{\langle \nu_k | \ell(p) \leq k < m_1 \rangle}) \subseteq \mathrm{dom}(a_{m_2}^{\langle \nu_k | \ell(p) \leq k < m_2 \rangle})$.

   (d) For every $m, \ell(p) < m \leq n$, set

   $$ptdom(a_m) = \bigcup \{\mathrm{dom}(a_m^{\langle \nu_k | \ell(p) \leq k < m \rangle}) \mid \langle \nu_k \mid \ell(p) \leq k < m \rangle$$

   $$\text{is a suitable extension sequence}\}.$$

   We require that for every two suitable extension sequences $\vec{\nu} = \langle \nu_k \mid \ell(p) \leq k < m \rangle, \vec{\mu} = \langle \mu_k \mid \ell(p) \leq k < m \rangle$, $f^{\vec{\nu}} \restriction (\kappa^+ \setminus ptdom(a_m)) = f^{\vec{\mu}} \restriction (\kappa^+ \setminus ptdom(a_m))$.

Definition 2.14 Let $\langle \nu_k \mid \ell(p) \leq k \leq m\rangle$ be a suitable extension sequence for p. Define $p^\frown\langle \nu_k \mid \ell(p) \leq k \leq m\rangle$ to be $q = \langle q_0, ..., q_n\rangle$ such that

1. for every $k < \ell(p)$, $q_k = p_k$,

2. $q_{\ell(p)} = p_{\ell(p)}{}^\frown\nu_{\ell(p)}$,

3. $q_k = \langle a_k^{\langle \nu_s|\ell(p)\leq s<k\rangle}, A_k^{\langle \nu_s|\ell(p)\leq s<k\rangle}, f_k^{\langle \nu_s|\ell(p)\leq s<k\rangle}\rangle^\frown\nu_k$, for every $k, \ell(p) < k \leq m$,

4. for every $k, m < k \leq n$, $q_k = \langle \underset{\sim}{a}_k(q), \underset{\sim}{A}_k(q), \underset{\sim}{f}_k(q)\rangle$, where
$\underset{\sim}{a}_k(q)$ consists of pairs $\langle\langle \mu_i \mid m < i < k\rangle, a_k^{\langle \mu_i|m<i<k\rangle}\rangle$,
$\underset{\sim}{A}_k(q)$ consists of pairs $\langle\langle \mu_i \mid m < i < k\rangle, A_k^{\langle \mu_i|m<i<k\rangle}\rangle$
and $\underset{\sim}{f}_k(q)$ consists of pairs $\langle\langle \mu_i \mid m < i < k\rangle, f_k^{\langle \mu_i|m<i<k\rangle}\rangle$, such that
$\langle \nu_k \mid \ell(p) \leq k \leq m\rangle^\frown\langle \mu_i \mid m < i < k\rangle$ is suitable extension sequence for p.

The orders $\leq^*$ and $\leq$ are defined on $\mathcal{P}_{\leq n}$ similarly to those on $\mathcal{P}_{\leq 1}$.

Definition 2.15 The set $\mathcal{P}$ consists of sequences $p = \langle p_n \mid n < \omega\rangle$ so that

(1) for every $n < \omega$, $p \restriction n + 1 \in \mathcal{P}_{\leq n}$.

(2) there is an $\ell(p) < \omega$ so that for every $n < \ell(p)$, $p_n \in Q_{n1}$,
and for every $n \geq \ell(p)$, $p_n = \langle \underset{\sim}{a}_n, \underset{\sim}{A}_n, \underset{\sim}{f}_n\rangle$,

(3) if for some suitable extension sequence $\vec{\nu} = \langle \nu_k \mid \ell(p) \leq k < m\rangle$, an ordinal α is in $\mathrm{dom}(f_m^{\vec{\nu}}) \cup \mathrm{dom}(a_m^{\vec{\nu}})$, then there is $m(\alpha), m \leq m(\alpha) < \omega$, for every $r, m(\alpha) \leq r < \omega$, we have $\alpha \in \mathrm{dom}(a_r^{\vec{\rho}})$, for every suitable extension sequence $\vec{\rho}$ of the length r.

Note that the total number of ordinals which appear in $\mathrm{dom}(f_m^{\vec{\nu}})$, for any $m < \omega$ and any suitable extension sequence $\vec{\nu}$, is at most κ. Recall that $\kappa = \bigcup_{n<\omega} \kappa_n$. So, we can spread such ordinals among the domains of a_n's.
Condition 3 will alow to show that for any $\alpha < \beta < \kappa^+$, a generic $\omega-$sequences produced by the forcing $\mathcal{P}$ for α is dominated (mod finite) by a generic $\omega-$sequences for β.

(4) There is $\eta < \kappa^+$ which is the maximal element of p in the following sense:

- for some suitable extension sequence $\vec{\nu} = \langle \nu_k \mid \ell(p) \leq k < m\rangle$, $\eta \in \mathrm{dom}(a_m^{\vec{\nu}})$,
- for every ordinal α, if $\alpha \in \mathrm{dom}(f_m^{\vec{\nu}}) \cup \mathrm{dom}(a_m^{\vec{\nu}})$, for some suitable extension sequence $\vec{\nu} = \langle \nu_k \mid \ell(p) \leq k < m\rangle$, then $\alpha \leq \eta$,

- for every ordinal $\alpha \leq \eta$, there is a suitable sequence $\vec{\nu} = \langle \nu_k \mid \ell(p) \leq k < m \rangle$ such that $\alpha \in \mathrm{dom}(a_m^{\vec{\nu}})$.

This condition allows to avoid situations of the following type:

For some $\alpha + 1 < \beta < \kappa^+$, $\alpha, \beta \in \mathrm{dom}(a_n)$ and $a_n(\beta) = a_n(\alpha) + 1$, for every $n < \omega$.

Here no room is left in order to add $\alpha + 1$ to domain a_n, for any n, since a_n's are order preserving.

Let us denote such η by $md(p)$.

Definition 2.16 Let $p, q \in \mathcal{P}$. We set $p \geq q$ ($p \geq^* q$) iff for every $n < \omega$, $p \upharpoonright n+1 \geq_{\mathcal{P}_{\leq n}} q \upharpoonright n+1$ ($p \upharpoonright n+1 \geq^*_{\mathcal{P}_{\leq n}} q \upharpoonright n+1$).

The forcing $\langle \mathcal{P}, \leq \rangle$ does not satisfy the κ^+-c.c., however its cardinality is κ^+, and so, there is no issue of preserving cardinals above κ^+.

Then the following lemmas are obvious:

Lemma 2.17 *$\langle \mathcal{P}_{\leq n}, \leq \rangle$ has a dense subset equivalent to the Cohen forcing for κ^+, for every $n < \omega$.*

Lemma 2.18 *$\mathcal{P} \simeq \mathcal{P}_{\leq n} * \mathcal{P}_{>n}$ for every $n < \omega$.*

Lemma 2.19 *$\langle \mathcal{P}_{>n}, \leq^* \rangle$ is κ_n-closed in $V_1^{\mathcal{P}_{\leq n}}$. Moreover if $\langle p^\alpha \mid \alpha < \delta < \kappa_n \rangle$ is a $\leq^*$ $-$increasing sequence with $\kappa_{\ell(p^0)-1} > \delta$, then there is $p \geq^* p^\alpha$ for every $\alpha < \delta$.*

We will turn now to the Prikry condition.
Let us introduce first some notation. For $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$ and m with $\ell(p) \leq m < \omega$, let $p_m = \langle \underset{\sim}{a}_m, \underset{\sim}{A}_m, \underset{\sim}{f}_m \rangle$. Denote $\underset{\sim}{a}_m$ by $\underset{\sim}{a}_m(p)$, $\underset{\sim}{A}_m$ by $\underset{\sim}{A}_m(p)$ and $\underset{\sim}{f}_m$ by $\underset{\sim}{f}_m(p)$.

Lemma 2.20 *Let $q \in \mathcal{P}$ and $\alpha < \kappa^+$. Then there is $p \geq^* q$ with $\alpha \leq md(p)$.*

Proof. If $\alpha \leq md(q)$, then just take $p = q$. Suppose that $\alpha > md(q)$. Consider $\alpha + 1 \setminus md(q)$. It is a set of cardinality at most κ. So we can present it as a non-decreasing union $\bigcup_{i<\omega} x_i$ with $|x_i| < \kappa$. For every $i < \omega$, let $n(i) < \omega$ be the least with $|x_i| < \kappa_{n(i)-1}$. Now, for every $n, n(i) \leq n < \omega$, we extend a_n by adding x_i to its domain and mapping it to κ_{n-1} in an order preserving fashion.
This will easily define a desired condition p.
$\square$

Lemma 2.21 *Let $p \in \mathcal{P}$ and D be a dense open subset of $\langle \mathcal{P}, \leq \rangle$ above p. Then there are $p^* \geq^* p$ and $n^* < \omega$ such that for every suitable extension sequence $\langle \nu_0, \ldots, \nu_{n^*} \rangle$ for p^* ,*

$$p^{*\frown} \langle \nu_0, \ldots, \nu_{n^*} \rangle \in D.$$

Proof. If there is a direct extension of p inside D, then we are done. Suppose otherwise.
Assume for simplicity that $\ell(p) = 0$.
Proceed by induction on $\nu \in A_0$. Let $\nu = \min(A_0)$.
Consider $p^\frown \nu$. If there is a direct extension of $p^\frown \nu$ which is in D, then let r be such extension.
Set $p(0) = \langle p_n(0) \mid n < \omega \rangle$ to be a direct extension of p defined as follows:

1. $p_0(0) = p_0$,
2. $p_1(0) = \langle \underset{\sim}{a}_1(p(0)), \underset{\sim}{A}_1(p(0)), \underset{\sim}{f}_1(p(0)) \rangle$, where
   (a) $\underset{\sim}{a}_1(p(0)) = \langle \mu, a_1^\mu(p(0)) \rangle$, where $a_1^\mu(p(0)) = a_1^\mu(p)$ unless $\mu = \nu$ and if $\mu = \nu$, then $a_1^\mu(p(0)) = a_1^\mu(r)$,
   (b) $\underset{\sim}{f}_1(p(0)) = \langle \mu, f_1^\mu(p(0)) \rangle$, where $f_1^\mu(p(0)) = f_1^\mu(p)$ unless $\mu = \nu$ and if $\mu = \nu$, then $f_1^\mu(p(0)) = f_1^\mu(r)$,
   (c) $\underset{\sim}{A}_1(p(0)) = \langle \mu, A_1^\mu(p(0)) \rangle$, where $A_1^\mu(p(0)) = A_1^\mu(p)$ unless $\mu = \nu$ and if $\mu = \nu$, then $A_1^\mu(p(0)) = A_1^\mu(r)$.
3. for every $m, 1 < m < \omega$, $p_m(0) = \langle \underset{\sim}{a}_m(p(0)), \underset{\sim}{A}_m(p(0)), f_m(p(0)) \rangle$, where
   (a) $\underset{\sim}{a}_m(p(0)) = \langle \vec{\mu}, a_m^{\vec{\mu}}(p(0)) \rangle$,
   where $\vec{\mu} = \langle \mu_0, ..., \mu_{m-1} \rangle$ is a suitable extension
   sequence for p and $a_m^{\vec{\mu}}(p(0)) = a_m^{\vec{\mu}}(p)$,
   unless $\mu_0 = \nu$ and if $\mu_0 = \nu$, then $\langle \mu_1, ..., \mu_{m-1} \rangle$ is a suitable extension sequence for r and $a_m^{\vec{\mu}}(p(0)) = a_m^{\langle \mu_1, ..., \mu_{m-1} \rangle}(r)$,
   (b) $\underset{\sim}{f}_m(p(0)) = \langle \vec{\mu}, f_m^{\vec{\mu}}(p(0)) \rangle$,
   where $\vec{\mu} = \langle \mu_0, ..., \mu_{m-1} \rangle$ is a suitable extension
   sequence for p and $f_m^{\vec{\mu}}(p(0)) = f_m^{\vec{\mu}}(p)$,
   unless $\mu_0 = \nu$ and if $\mu_0 = \nu$, then $\langle \mu_1, ..., \mu_{m-1} \rangle$ is a suitable extension sequence for r and $f_m^{\vec{\mu}}(p(0)) = f_m^{\langle \mu_1, ..., \mu_{m-1} \rangle}(r)$,
   (c) $\underset{\sim}{A}_m(p(0)) = \langle \vec{\mu}, A_m^{\vec{\mu}}(p(0)) \rangle$,
   where $\vec{\mu} = \langle \mu_0, ..., \mu_{m-1} \rangle$ is a suitable extension

sequence for p and $A^{\vec{\mu}}_m(p(0)) = A^{\vec{\mu}}_m(p)$,

unless $\mu_0 = \nu$ and if $\mu_0 = \nu$, then $\langle \mu_1, ..., \mu_{m-1} \rangle$ is a suitable extension sequence for r and $A^{\vec{\mu}}_m(p(0)) = A^{\langle \mu_1,...,\mu_{m-1} \rangle}_m(r)$.

If there is no direct extension of $p^\frown \nu$ in D then set $p(0) = p$.

Turn next the second element $\nu(1)$ of A_0. If there is no direct extension of $p(0)^\frown \nu(1)$ in D, then set $p(1) = p(0)$. Otherwise, pick such a direct extension $r \geq^* p(0)$ in D and define $p(1)$ as above replacing p by $p(0)$ and ν by $\nu(1)$.

Continue by induction. We have enough completeness at coordinates 2 and above, at coordinate 1 the lack of completeness (recall that there we have only κ_0-completeness) is compensated by taking names.

Denote by $p^*(0)$ the resulting direct extension of p, i.e. the one obtained after passing through all ν's in A_0.

Then, for every $\nu \in A_0$, if there is a direct extension of $p^*(0)^\frown \nu$ in D, then already $p^*(0)^\frown \nu \in D$.

Shrink now A_0 to $A^*_0 \in E_{0 \max(\mathrm{rng}(a)_0}$ such that for every $\nu, \nu' \in A^*_0$,

$$p^*(0)^\frown \nu \in D \Leftrightarrow p^*(0)^\frown \nu' \in D.$$

Let $p^{**}(0)$ be obtained from $p^*(0)$ by replacing A_0 with A^*_0.

If for some (every) $\nu \in A^*_0$, $p^*(0)^\frown \nu \in D$, then $p^*(0)$ is as required and we are done.

Suppose otherwise. Proceed to the next level and repeat the process.

This way $p^{**}(n)$'s will be constructed, for $n < \omega$.

The process should stop at some stage $n^* < \omega$ and the resulting $p^{**}(n^*)$ will be as desired.

$\square$

By standard arguments it follows now:

Proposition 2.22 *The forcing $\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets to κ and preserves all the cardinals.*

In particular, each $\kappa_n (n < \omega)$ remains measurable.

Let us show that this forcing adds κ^+ ω-sequences to κ. Thus, let $G \subseteq \mathcal{P}$ be generic. For every $n < \omega$ define a function $F_n : \kappa^+ \to \kappa_n$ as follows:

$F_n(\alpha) = \nu$ if for some $p = \langle p_m \mid m < \omega \rangle \in G$ with $\ell(p) > n$, $p_n(\alpha) = \nu$.

Now for every $\alpha < \kappa^+$ set $t_\alpha = \langle F_n(\alpha) \mid n < \omega \rangle$. Let us show that the set $\{t_\alpha \mid \alpha < \kappa^+\}$ has cardinality κ^+.

Lemma 2.23 *For every $\beta < \alpha < \kappa^+$, $t_\beta(n) < t_\alpha(n)$, for all but finitely many $n < \omega$.*

Proof. Work in V_1. Let $q \in \mathcal{P}$. By Lemma 2.20 there is a direct extension p of q such that $\alpha, \beta \leq md(p)$. Then, by Definition 2.15(3,4), there is $n^*, \ell(q) = \ell(p) \leq n^* < \omega$ such that for every $n, n^* \leq n < \omega$, $\alpha, \beta \in \mathrm{dom}(\underset{\sim}{a}_n(p))$, where by $\gamma \in \mathrm{dom}(\underset{\sim}{a}_n(p))$ we mean the following:
if $n = \ell(p)$, then just $\gamma \in \mathrm{dom}(a_n)(p)$;
if $n > \ell(p)$, then for every suitable extension sequence $\vec{\nu} = \langle \nu_{\ell(p)}, ..., \nu_{n-1}\rangle$, $\gamma \in \mathrm{dom}(a_n^{\vec{\nu}}(p))$.
Now, the order preservation implies

$$p \Vdash \forall n \geq n^* (\underset{\sim}{t}_\beta(n) < \underset{\sim}{t}_\alpha(n)).$$

$\square$

Let us argue that $\langle t_\alpha \mid \alpha < \kappa^+\rangle$ is actually a scale in $\prod_{n<\omega} \kappa_n$ mod finite.
We will show a bit stronger statement which will insure that unboundedly many α's, t_α is a sequence of indiscernibles.

Definition 2.24 A sequence $\langle c_n \mid n < \omega\rangle \in \prod_{n<\omega} \kappa_n$ is called *a principle sequence* iff

1. for some $\alpha < \kappa^+$, for all but finitely many $n < \omega$, $c_n = t_\alpha(n)$,
2. there is $p \in G$ such that
   (a) for every $n, \ell(p) \leq n < \omega$, $\alpha \in \mathrm{dom}(a_n(p))$,
   (b) $a_{\ell(p)}(p)(\alpha)$ is a generator of $E_{\ell(p)}$,[1]
   (c) for every $k, \ell(p) < k < \omega$, for every suitable extension sequence $\langle \nu_{\ell(p)}, ..., \nu_{k-1}\rangle$ for p, $a_k(p)^{\langle \nu_{\ell(p)},...,\nu_{k-1}\rangle}(\alpha)$ is a generator of E_k.

Lemma 2.25 *Let $t \in \prod_{n<\omega} \kappa_n$ in $V_1[G]$. Then there is $\alpha < \kappa^+$ such that for all but finitely many $n < \omega$, $t_\alpha(n) > t(n)$.*
Moreover, such t_α can be picked to be a principle sequence.

Proof. Work in V_1. Let $p \Vdash \underset{\sim}{t} \in \prod_{n<\omega} \kappa_n$. Suppose for simplicity that $\ell(p) = 0$. Let $\underset{\sim}{\eta}_n$ be a name of the ordinal $t(n) < \kappa_n$.
Start with η_0.

[1] Recall that an ordinal η is called a generator of an extender F over λ, if for every $n < \omega$ and $h : [\lambda]^n \to \lambda$, $\eta \neq j_F(h)(a)$, for any $a \in [\eta]^n$, where j_F denotes the corresponding elementary embedding.

Set $D_0 = \{q \in \mathcal{P} \mid q \geq p \wedge q || \underset{\sim}{\eta}_0\}$.
Clearly, D_0 is a dense open subset of $\mathcal{P}$.
Apply Lemma 2.21 to p and D_0.
Let $n_0 < \omega$ be the least possible which satisfies the conclusion of Lemma 2.21.
Let $p^*(0)$ be a witnessing direct extension of p.

Claim 1 $n_0 = 0$.

Proof. First note that due to degrees of completeness $n_0 \leq 1$.
Suppose that $n_0 = 1$.
Let $\nu \in A_0(p^*(0))$. Consider $A_1^\nu((p^*(0))$.
It is a set in κ_1-complete ultrafilter $E_{1\max(\mathrm{rng}(a_1^\nu(p^*(0))))}$.
For every $\rho \in A_1^\nu((p^*(0))$, $p^*(0)^\frown\langle\nu,\rho\rangle \in D_0$, hence there is $\eta(\nu,\rho) < \kappa_0$ such that

$$p^*(0)^\frown\langle\nu,\rho\rangle \Vdash \underset{\sim}{\eta}_0 = \eta(\nu,\rho).$$

Now, using κ_1-completeness of $E_{1\max(\mathrm{rng}(a_1^\nu(p^*(0))))}$, we can shrink $A_1^\nu((p^*(0))$ to a set $A_1^\nu \in E_{1\max(\mathrm{rng}(a_1^\nu(p^*(0))))}$ such that for every $\rho \in A_1^\nu$

$$p^*(0)^\frown\langle\nu,\rho\rangle \Vdash \underset{\sim}{\eta}_0 = \eta(\nu),$$

where $\eta(\nu) < \kappa_0$ does not depend on ρ.
Preform the above for every $\nu \in A_0(p^*(0))$. Then replace in $p^*(0)$ the sets $A_1^\nu((p^*(0))$'s by A_1^ν.
This will reduce n_0, which is impossible by its minimality.
Contradiction.
$\square$ of the claim.

Proceed by induction on $k < \omega$ and define D_k,n_k, $p^*(k) \geq^* p^*(k-1)$ as above.
As in the claim, we will have $n_k = k$.
Let finally, p^* be a common direct extension of all $p^*(k)$'s.
Then for every $k < \omega$, for every suitable extension sequence $\langle\nu_0, ..., \nu_k\rangle$ for p^*,

$$p^{*\frown}\langle\nu_0, ..., \nu_k\rangle || \langle\underset{\sim}{\eta}_0, ..., \underset{\sim}{\eta}_k\rangle.$$

Denote the decided values by $\langle\eta_0(\nu_0, ..., \nu_k), ..., \eta_k(\nu_0, ..., \nu_k)\rangle$.

Pick now $\alpha < \kappa^+$ above everything that appears in p^*. Let us define a direct extension p^{**} of p^* by adding α to domains of $a_n(p^*)$'s and insuring that p^{**} will force $\underset{\sim}{\eta}_n < \underset{\sim}{t}_\alpha(n)$, for every $n < \omega$.

Start with $n=0$. Consider $\max(\mathrm{rng}(a_0(p^*)))$. It is an ordinal $< j_0(\kappa_0)$.
Actually, we here in a special case - E_0 is just equivalent to the normal ultrafilter $U^*(\kappa_0, 0)$, i.e. it has a single generator κ_0. Consider the function $\nu \mapsto \eta_0(\nu)$ on $A_0(p^*)$. It represents an ordinal $\gamma_0 < j_0(\kappa_0)$. Pick $\alpha_0 < j_0(\kappa_0)$ to be above both γ_0 and $\max(\mathrm{rng}(a_0(p^*)))$.
Extend $a_0(p^*)$ by adding α to its domain and setting its value to be α_0.
Turn to the next step - $n = 1$.
Fix $\nu \in A_0(p^*)$. Consider a function $\rho \mapsto \eta(\nu, \rho)$ defined on $A_1^\nu(p^*)$.
We have $A_1^\nu(p^*) \in E_{1\,\max(\mathrm{rng}(a_1^\nu(p^*)))}$. Recall that E_1 is a $(\kappa_1, j_1(\kappa_1))-$extender generated by taking a direct limit of a Rudin-Keisler increasing sequence $\langle U^*(\kappa_1, \gamma) \mid \gamma < \kappa_0\rangle$ of ultrafilters over κ_1.
Note that for every $\gamma < \kappa_0$, $j_{U(\kappa_1,\gamma)}(\kappa_1) = j_{U^*(\kappa_1,\gamma)}(\kappa_1)$, by arguments of [4].
So, $j_1(\kappa_1) = \bigcup_{\gamma<\kappa_0} j_{U(\kappa_1,\gamma)}(\kappa_1)$ and every $j_{U(\kappa_1,\gamma)}(\kappa_1)$ is a generator of E_1.
Pick $\gamma < \kappa_0$ such that $\max(\mathrm{rng}(a_1^\nu(p^*))) < j_{U(\kappa_1,\gamma)}(\kappa_1)$. Set $\alpha_1^\nu = j_{U(\kappa_1,\gamma)}(\kappa_1)$. Extend a_1^ν by adding α to its domain and setting its value to be α_1^ν.
Do the above for every $\nu \in A_0(p^*)$ (separately).
Continue up to $n = 2$ etc. We will construct a desired direct extension of p^*.
$\square$

Let us show that our construction provides an affirmative answer to the problem of W. Mitchell [7] stated the introduction.

Lemma 2.26 *Suppose that $\vec{c} = \langle c_n \mid n < \omega\rangle \in \prod_{n<\omega} \kappa_n$ is a principle sequence. Then it is a sequence of indiscernibles for $\vec{\kappa} = \langle \kappa_n \mid n < \omega\rangle$*[2].

Proof. Let N be a covering model (we refer to Mitchell [8] for definitions) with $\vec{c} \in N$. Then $N \cap \mathcal{K}_\kappa = h^N(\rho; \mathbb{C}^N)$, for some $\rho < \kappa$, sequence of indiscernibles $\mathbb{C}^N$ and a Skolem function $h^N \in \mathcal{K}$.
Suppose that infinitely many of c_n's are not indiscernibles for N.
Assume for simplicity that none of is an indiscernible.
Then for every $n < \omega$ there is a finite sequence of ordinals $\vec{d_n}$ below c_n such that $h^N(\vec{d_n}) = c_n$.
Define in $\mathcal{K}$ a function $\tilde{h} \in \prod_{n<\omega} \kappa_n$ as follows:
Let $\nu < \kappa$, pick the least $n < \omega$ such that $\nu < \kappa_n$.
Set

$$\tilde{h}(\nu) = \bigcup\{h^N(\vec{d}) \mid \vec{d} \in [\nu+1]^{<\omega}, h^N(\vec{d}) \text{ is defined and } h^N(\vec{d}) \text{ is an ordinal } < \kappa_n\}.$$

[2]Note that $o(\kappa_n) = \kappa_{n-1}$ implies that the measure over κ_n to which c_n corresponds is $U(\kappa_n, o(c_n))$.

Define $\tilde{h}_n = \tilde{h} \restriction \kappa_n$. Clearly, $\tilde{h}_n : \kappa_n \to \kappa_n$.
Then, $c_n \leq \tilde{h}(\max(\vec{d_n})) = \tilde{h}_n(\max(\vec{d_n})) < \kappa_n$ and $\max(\vec{d_n}) < c_n$, for every $n < \omega$. It is impossible to have this type of situation, since by the definition of a principal sequence, c_n corresponds to a generator of extender E_n for all but finitely many $n < \omega$.
$\square$

Now, Lemma 2.25 and Lemma 2.26 imply the following:

Lemma 2.27 *For every $\vec{\gamma} = \langle \gamma_n \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_n$ there is $\alpha < \kappa^+$ such that*

1. *t_α is a principal sequence,*
2. *$t_\alpha(n) > \gamma_n$, for all but finitely many $n < \omega$.*

The next lemma completes the proof:

Lemma 2.28 *Let $\vec{\beta} = \langle \beta_n \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_{n-1}$. Then there are arbitrary large $\alpha < \kappa^+$ such that*

1. *t_α is a principal sequence,*
2. *$o^{\mathcal{K}}(t_\alpha(n)) = \beta_n$, for all but finitely many $n < \omega$.*

Proof. We repeat the proof of Lemma 2.25 with $t = \vec{\beta}$. Find p^* which suitable extensions of the length $n-1$ decide $\underset{\sim}{\beta}_n$ for all n's. Next, $\alpha < \kappa^+$ above everything in p^* is picked and added to domains of $a_n(p^*)$'s. The final stage was to specify images of α. Generators of E_n's were picked for this purpose.
Note that for given $n, 0 < n < \omega$, there are arbitrary large generators ζ of E_n with $o^{\mathcal{K}^{M_n}}(\zeta) = \beta_n$. So we are free to choose one of them to be the image of α.
This process will produce the desired t_α.
$\square$

3 No maximal sequence of indiscernibles for a measurable.

Let $o(\kappa) = \delta \leq \kappa$ as witnessed by $\vec{U} = \langle U(\kappa, \beta) \mid \beta < \delta \rangle$.
Suppose that the Prikry or Magidor or Radin forcing with $\vec{U}$ is used. Let $C_{\vec{U}}$ be a generic club.
Then in $V[C_{\vec{U}}]$, we have

(A) for every $A \in \bigcap_{\beta<\delta} U(\kappa, \beta)$ there is $\eta < \kappa$ such that $C_{\vec{U}} \setminus \eta \subseteq A$.

(B) (Maximality) If $C' \subseteq \kappa$ is so that
for every $A \in \bigcap_{\beta<\delta} U(\kappa, \beta)$ there is $\eta < \kappa$ such that $C \setminus \eta \subseteq A$,
then a final segment of C' is subset of $C_{\vec{U}}$.

By results of Jensen -Dodd and Mitchell, if $o(\kappa) < \kappa$, κ changes its cofinality, then there is a club $C \subseteq \kappa$ which satisfies both items above.

We would like to show here that this not true anymore if $o(\kappa) = \kappa$.
Namely, we will construct a generic extension $V[G]$ which satisfies the following:

1. $\mathrm{cof}(\kappa) = \omega$,

2. there is no $C \subseteq \kappa$ which satisfies Items (A) and (B) above.

The final extension $V[G]$ will be built in three steps. The first one will be a tree Prikry forcing extracted from the Radin forcing with $\vec{U}$ which will change the cofinality of κ to ω. Let $\langle \kappa^n \mid n < \omega \rangle$ be such Prikry sequence. The next forcing will be an Easton support iteration of Prikry, Magidor forcings for changing cofinality of each measurable $\nu \in \kappa \setminus \{\kappa^n \mid n < \omega\}$ according to $o(\nu)$. Then the final one will a variant of short extenders forcing similar to those of [1]. Also the one of the previous section can be used, but this will make the matters more complicated due to use of names.

Let us describe the first one - $\mathrm{P}_{\vec{U}}$ a tree Prikry forcing with $\vec{U}$.

Definition 3.1 $\mathrm{P}_{\vec{U}}$ consists of pairs $\langle t, T \rangle$ such that

1. t is a finite increasing sequence of ordinals $< \kappa$,

2. T is a tree of finite increasing sequences of ordinals $< \kappa$ with trunk t,

3. if t is the empty sequence, then $Suc_T(t) \in U(\kappa, 0)$,

4. if $s \in T$ and s extends t, then $Suc_T(s) \in U(\kappa, \max(s))$.

The orders $\leq$ and $\leq^*$ are defined on $\mathrm{P}_{\vec{U}}$ in the standard fashion.

Let $\langle \kappa_n \mid n < \omega \rangle$ be a generic Prikry sequence.
By removing an initial segment if necessary, we can assume that for every $n < \omega$, $o(\kappa_{n+1}) > (\kappa_n)^+$ is an inaccessible cardinal $< \kappa_{n+1}$.

The second stage is to force further with Easton support iteration of Prikry, Magidor forcings for changing cofinality of each measurable $\nu \in \kappa \setminus \{\kappa_n \mid n < \omega\}$ according to $o(\nu)$. We refer to [3],[4] for details.
Note that for every $n < \omega$, the increasing in the Mitchell order sequence $\langle U(\kappa_n, \beta) \mid \beta < o(\kappa_n) \rangle$ turned a Rudin-Keisler increasing sequence $\langle U^*(\kappa_n, \beta) \mid \beta < o(\kappa_n) \rangle$. We view further this sequence as an extender E_n over κ^n.

The final stage will be to force with a short extenders forcing $\mathcal{P}$ with $\langle E_n \mid n < \omega \rangle$.
We just use the forcing of the previous section, but without names anymore.
Note that here we assumed that $o(\kappa_{n+1}) > (\kappa_n)^+$ is an inaccessible cardinal $< \kappa_{n+1}$. This will guarantee enough completeness at the final stage, and so, will eliminate the need of using names.

Let us argue that there is no maximal set of indiscernibles. An analysis of subsets of κ will be needed for this.

Denote $V[\langle \kappa_n \mid n < \omega \rangle]$ by V_0, the second extension by V_1 and the final third by V_2.

Suppose that in V_2 there is a club $C \subseteq \kappa$ which satisfies Items (A) and (B) above.
Note that for every $n < \omega$, $C_n = C \cap \kappa_n \in V_1$, since the forcing with $\mathcal{P}$ does not add new bounded subsets to κ.
We have $2^{\kappa_n} = \kappa_n^+$, so C_n can be coded (in V_1) by an ordinal $\xi_n < \kappa_n^+$.

Work in V_1. Let $\underset{\sim}{C}$ be a name of such C, $\langle \underset{\sim}{C}_n \mid n < \omega \rangle$ and $\langle \underset{\sim}{\xi}_n \mid n < \omega \rangle$ names for $\langle C_n \mid n < \omega \rangle$ and for $\langle \xi_n \mid n < \omega \rangle$.
Let $p \in \mathcal{P}$ be a condition forcing this.
Assume for simplicity that $\ell(p) = 0$.

Proceed as in the previous section and find a direct extension p^* of p such that for every $m < \omega$, $\langle \nu_0, ..., \nu_m \rangle \in \prod_{i \leq m} A_i(p^*)$,

$$p^{*\frown} \langle \nu_0, ..., \nu_m \rangle || \langle \underset{\sim}{\xi}_0, ..., \underset{\sim}{\xi}_m \rangle,$$

and so decides $\langle \underset{\sim}{C}_i \mid i \leq m \rangle$ as well.
Note that there is no need here in suitable extension sequences due to degree of completeness

of extenders E_n's.
Split now into two cases.

Case 1. *For infinitely many $n<\omega$ there are $A_m \subseteq A_m(p^*), A_m \in E_{m\,\max(\mathrm{rng}(a_m(p^*)))}$, $m \leq n$ such that for every $\langle \nu_0,...,\nu_n\rangle \in \prod_{m\leq n} A_m$, the value of $\underset{\sim}{C}_n$ decided by $p^{*\frown}\langle \nu_0,...,\nu_n\rangle$ is bounded in κ_n.*

Then, as in Lemma 2.27, in V_2 there will be $\alpha<\kappa^+$ with t_α being a sequence of indiscernibles such that $t_\alpha(n)\not\in C$ for infinitely many $n<\omega$.
Contradiction to the maximality of C, i.e. to Item (B).

Case 2. *For all but finitely many $n<\omega$, for every $\langle \nu_0,...,\nu_n\rangle \in \prod_{m\leq n} A_m(p^*)$, the value of $\underset{\sim}{C}_n$ decided by $p^{*\frown}\langle \nu_0,...,\nu_n\rangle$ is unbounded in κ_n.*

Let show that then Item (A) breaks down.

Suppose for simplicity that the above holds for every $n<\omega$.

Fix $n<\omega$.
Denote the value decided by $p^{*\frown}\langle \nu_0,...,\nu_n\rangle$ value of $\underset{\sim}{C}_n$ by $C_n(\nu_0,...,\nu_n)$.
By the assumption, it is an unbounded subset of κ_n.
Define a function $h_n:\prod_{m\leq n}\kappa_m\to\kappa_n$ by setting $h_n(\nu_0,...,\nu_n)=\min(C_n(\nu_0,...,\nu_n)\setminus \nu_n+1)$, whenever the right side is defined and 0 otherwise.

Such $\langle h_n \mid n<\omega\rangle$ naturally defines, in V_2, an unbounded sequence $\langle \eta_n \mid n<\omega\rangle \in \prod_{n<\omega}\kappa_n$ such that $\eta_n\in C_n\subseteq C$, for every $n<\omega$.
Let us argue that it cannot be a sequence of indiscernibles in some (every) covering over V model. This easily implies a failure of Item (A).

Fix $n<\omega$. Work in V_0. Recall that the forcing below κ_n used over V_0 in order to get V_1 satisfies κ_n-c.c., nothing is done at κ_n and no new subsets are added to κ_n by iteration above it.
Apply κ_n-c.c.. So, there is a function $g_n:\prod_{m\leq n}\kappa_m\to\kappa_n$ in V_0 which dominates h_n everywhere.

Now consider $\langle g_n \mid n<\omega\rangle\in V_0$.
Recall that V_0 is an extension of V obtained by adding a generic Prikry sequence $\langle\kappa_n \mid n<\omega\rangle$ for the forcing $\mathrm{P}_{\vec{U}}$. No new bounded subsets are added to κ by such extension. In particular, each g_n is in V.

Work in V. Let $\langle \underset{\sim}{g}_n \mid n<\omega\rangle$ be the sequence of names of $\langle g_n \mid n<\omega\rangle$. Suppose that a condition $\langle\langle\rangle, T\rangle \in\mathrm{P}_{\vec{U}}$ forces this.
Use normality of ultrafilters involved and define $\langle\langle\rangle,T^*\rangle\geq^*\langle\langle\rangle,T\rangle$ and $\langle R_t \mid t\in T^*\rangle$ such that for every $t\in T^*$,

1. $R_t : [\kappa]^{<\omega} \to \kappa$,

2. for every $\nu \in Suc_{T^*}(t)$, $\langle t, T^*_t \rangle \Vdash R_t \restriction \nu = \underset{\sim}{g}_{|t|}$.

In particular,
for every $\nu \in Suc_{T^*}(\langle\rangle)$, $\langle \langle\rangle, T^* \rangle \Vdash R_{\langle\rangle} \restriction \nu = \underset{\sim}{g}_0$
and
for every $\nu \in Suc_{T^*}(\langle\rangle)$, $\langle \langle \kappa_0 \rangle, T^* \rangle \Vdash R_{\langle \kappa_0 \rangle} \restriction \nu = \underset{\sim}{g}_1$.

Now, for every $t \in T^*$ let

$$A_t = \{\nu < \kappa \mid R_t{}''\nu \subseteq \nu\}.$$

Clearly it is a club, and so, is inside every $U(\kappa, \beta), \beta < \kappa$.

Set

$$A = \Delta_{t \in T^*} A_t = \{\delta < \kappa \mid \forall t \in T^*(\max(t) < \delta \to \delta \in A_t)\}.$$

Then $A \in U(\kappa, \beta), \beta < \kappa$, due to normality.

Now, let us argue that in V_2, $C \setminus A$ will be unbounded in κ, as witnessed by $\langle \eta_n \mid n < \omega \rangle$. This will contradict Item (A).
Namely, assume that $\eta_n \in A$, for some $n < \omega$. Then

$$\kappa_{n-1} < \nu_n < \eta_n < g_n(\nu_0, ..., \nu_n) < \kappa_n.$$

We have $R_{\langle \kappa_0, ..., \kappa_{n-1} \rangle} \restriction \kappa_n = g_n$.
Hence, $R_{\langle \kappa_0, ..., \kappa_{n-1} \rangle}(\nu_0, ..., \nu_n) = g_n(\nu_0, ..., \nu_n) > \eta_n$, and so, $\eta_n \notin A_{\langle \kappa_0, ..., \kappa_{n-1} \rangle}$.
But $\eta_n > \kappa_{n-1} = \max(\langle \kappa_0, ..., \kappa_{n-1} \rangle)$ and $\eta_n \in A$, hence $\eta_n \in A_{\langle \kappa_0, ..., \kappa_{n-1} \rangle}$. Which is impossible. Contradiction.

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