Indestructible Strong Compactness but not Supercompactness [*†‡]

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Abstract

Starting from a supercompact cardinal κ, we force and construct a model in which κ is both the least strongly compact and least supercompact cardinal and κ's strong compactness, but not its supercompactness, is indestructible under arbitrary κ-directed closed forcing.

1 Introduction and Preliminaries

One of the most celebrated results in large cardinals and forcing is due to Laver [13], who showed that any supercompact cardinal κ can have its supercompactness forced to be indestructible under arbitrary κ-directed closed forcing. This raises the following

Question: Is it possible to force a supercompact cardinal κ to have its strong compactness, but not its supercompactness, indestructible under arbitrary κ-directed closed forcing?

The purpose of this paper is to answer the above question in the affirmative. Specifically, we will prove the following theorem.

Theorem 1 *Let $V \models$ "ZFC + κ is supercompact". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "κ is both supercompact and the least strongly compact cardinal". For any $\mathbb{Q} \in V^{\mathbb{P}}$ which is κ-directed closed, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact". Further, there is $\mathbb{R} \in V^{\mathbb{P}}$ which is κ-directed closed and nontrivial such that $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ is not supercompact". Moreover, for this $\mathbb{R}$, $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ has trivial Mitchell rank".*

Forcing to obtain a model in which the least strongly compact cardinal is the same as the least supercompact cardinal was of course first done by Magidor in [14].

Key to the proof of Theorem 1 (specifically the fact that κ's supercompactness is not indestructible) is the following theorem due to the second author, which we will prove as well.

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Theorem 2 *Suppose κ is a Mahlo cardinal and $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha \leq \kappa\rangle$ is an Easton support iteration of length $\kappa + 1$ satisfying the following properties.*

1. $\mathbb{P}_0 = \{\emptyset\}$.

2. *For each $\alpha < \kappa$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| < \kappa$".*

3. $\Vdash_{\mathbb{P}_\kappa}$ *"$\dot{\mathbb{Q}}_\kappa$ is $<\kappa$-strategically closed".*

4. *For some $\alpha, \delta < \kappa$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ adds a new subset of δ".*

5. *κ is Mahlo in $V^{\mathbb{P}_{\kappa+1}} = V^{\mathbb{P}}$.*

Then in $V^{\mathbb{P}}$, there are no fresh subsets of κ.

We note that Theorem 2 is an analogue of results due to Hamkins (see [9, 8, 7]). Adopting the terminology of these papers, Hamkins shows that for a suitably large cardinal κ (measurable, supercompact, etc.) and an iteration $\mathbb{P}$ *admitting a gap below* κ, after forcing with $\mathbb{P}$, there are no fresh subsets of κ. The iterations we consider need not be gap forcings, yet they retain this crucial property vital to the proof of Theorem 1.

Before beginning the proofs of our theorems, we briefly mention some preliminary information and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in the usual interval notation. If $\kappa \geq \omega$ is a regular cardinal, then $\mathrm{Add}(\kappa, 1)$ is the standard partial ordering for adding a single Cohen subset of κ.

When forcing, $q \geq p$ will mean that q *is stronger than* p. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If φ is a formula in the forcing language with respect to $\mathbb{P}$ and $p \in \mathbb{P}$, then $p \parallel \varphi$ means that p *decides* φ. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. Any term for trivial forcing will always be taken as a term for the partial ordering $\{\emptyset\}$. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

If $\mathbb{P}$ is an arbitrary partial ordering and κ is a regular cardinal, $\mathbb{P}$ is *κ-directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha \mid \alpha < \delta \rangle$ of elements of $\mathbb{P}$ (where $\langle p_\alpha \mid \alpha < \delta \rangle$ is *directed* if every two elements p_ρ and p_ν have a common upper bound of the form p_σ) there is an upper bound $p \in \mathbb{P}$. $\mathbb{P}$ is *κ-strategically closed* if in the two person game of length $\kappa + 1$ in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \le \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$<\kappa$-strategically closed* if $\mathbb{P}$ is δ-strategically closed for every $\delta < \kappa$. Note that if $\mathbb{P}$ is κ-directed closed, then $\mathbb{P}$ is $<\kappa$-strategically closed (so since $\mathrm{Add}(\kappa, 1)$ is κ-directed closed, $\mathrm{Add}(\kappa, 1)$ is $<\kappa$-strategically closed as well). We adopt Hamkins' terminology of [9, 8, 7] and say that $x \subseteq \kappa$ *is a fresh subset of κ with respect to* $\mathbb{P}$ if $\mathbb{P}$ is nontrivial forcing, $x \in V^{\mathbb{P}}$, $x \notin V$, yet $x \cap \alpha \in V$ for every $\alpha < \kappa$.

From time to time within the course of our discussion, we will refer to partial orderings $\mathbb{P}$ as being *Gitik iterations*. By this we will mean an Easton support iteration as first given by the second author in [6], to which we refer readers for a discussion of the basic properties of and terminology associated with such an iteration. For the purposes of this paper, at any stage δ at which a nontrivial forcing is done in a Gitik iteration, we assume the partial ordering $\mathbb{Q}_\delta$ with which we force is either δ-directed closed or is Prikry forcing defined with respect to a normal measure over δ (although other types of partial orderings may be used in the general case — see [6] for additional details).

We recall for the benefit of readers the definition given by Hamkins in [10, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p \rangle \le \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [10], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[1]

Finally, we mention that we are assuming familiarity with the large cardinal notions of mea-

[1]The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

surability, strongness, strong compactness, and supercompactness. Interested readers may consult [11] or [15] for further details. We do note, however, that the measurable cardinal κ is said to have *trivial Mitchell rank* if there is no elementary embedding $j : V \to M$ generated by a normal measure $\mathcal{U}$ over κ such that $M \vDash$ "κ is a measurable cardinal". We explicitly observe that if κ has trivial Mitchell rank, then κ is not supercompact (and in fact, if κ has trivial Mitchell rank, then κ is not even 2^κ supercompact).

2 The Proofs of Theorems 1 and 2

We begin with the proof of Theorem 2.

Proof: Suppose κ is a Mahlo cardinal, $\mathbb{P} = \mathbb{P}_{\kappa+1} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha \le \kappa \rangle$ is an Easton support iteration of length $\kappa + 1$ with $\mathbb{P}_0 = \{\emptyset\}$, for each $\alpha < \kappa$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| < \kappa$", $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}_\kappa$ is $<\kappa$-strategically closed", and κ is Mahlo in $V^{\mathbb{P}}$. Assume without loss of generality that forcing with $\mathbb{Q}_0$ over V adds a new subset of ω (so $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}_0$ adds a new subset of ω"). Let $G = G_{\kappa+1} = G_\kappa * G(\kappa)$ be V-generic over $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ (so G_1 is V-generic over $\mathbb{P}_1 = \mathbb{P}_0 * \dot{\mathbb{Q}}_0$), and let $\dot{x}$ be a term such that $\Vdash_{\mathbb{P}}$ "$\dot{x}$ is a fresh subset of κ".

Work for the time being in $V[G]$. Because $\Vdash_{\mathbb{P}}$ "$\dot{x}$ is a fresh subset of κ", it is the case that $\Vdash_{\mathbb{P}}$ "For every $\eta < \kappa$, $\dot{x} \cap \eta \in \check{V}$". Therefore, for every inaccessible cardinal $\eta < \kappa$, we may choose a condition $p(\eta) = \langle \dot{p}_\nu \mid \nu \in \mathrm{supp}(p(\eta)) \rangle \in G$ deciding $\dot{x} \cap \eta$. Since $\mathbb{P}$ is an Easton support iteration, $\mathrm{supp}(p(\eta)) \cap \eta$ is a bounded subset of η. Now, using the fact that κ is Mahlo in $V[G]$ and Fodor's Theorem applied to the function $f(\eta) =$ The least ξ such that $\mathrm{supp}(p(\eta)) \cap \eta \subseteq \xi$, let $S \subseteq \kappa$, $S \in V[G]$ be stationary such that for $\eta_1, \eta_2 \in S$, $\mathrm{supp}(p(\eta_1)) \cap \eta_1 = \mathrm{supp}(p(\eta_2)) \cap \eta_2$. Since $\langle p(\eta) \restriction \eta \mid \eta \in S \rangle$ forms a Δ-system, working in $V[G]$, we can find $\eta^* < \kappa$ and $p^* \in \mathbb{P}_{\eta^*}$ such that for κ many values of η, it is the case that $p(\eta) \restriction \eta = p^*$. This means we can choose a condition $q^{**} \in G$, $q^{**} \ge p^*$ such that $q^{**} \Vdash_{\mathbb{P}}$ "For κ many ordinals $\eta < \kappa$, there is $p(\eta) \in \dot{G}$ such that $p(\eta) \restriction \eta = q^{**} \restriction \eta^*$ and $p(\eta) \parallel \dot{x} \cap \eta$". For the remainder of the proof of Theorem 2, let $q^{**} \restriction \eta^* = p^{**}$.

Work now in V. Using the properties of q^{**} given in the last sentence of the preceding paragraph, we define by induction a sequence $\langle p^t \mid t \in 2^{<\omega} \rangle$ of conditions in $\mathbb{P}$, a sequence of ordinals $\langle \eta^t \mid t \in 2^{<\omega} \rangle$, and a sequence $\langle z^t \mid t \in 2^{<\omega} \rangle$ such that the following hold.

1. If t extends s, then $\kappa > \eta^t > \eta^s$.
2. $p^t \restriction \eta^t = p^{**}$.
3. If t extends s, then $\sup(\operatorname{supp}(p^s) \cap \kappa) < \eta^t$.
4. If t extends s, then $p^t \geq p^s - \eta^t$.
5. $p^t \Vdash$ "$\dot{x} \cap \eta^t = z^t$".
6. If p^t and p^s are incompatible, then for some $\alpha \leq \max(\sup(z^t), \sup(z^s))$, it is the case that $z^t \cap \alpha \neq z^s \cap \alpha$.

We begin the induction by picking $p^{\langle\rangle}$ to be a condition such that for $\eta < \kappa$ and some $z \subseteq \eta$, $p^{\langle\rangle} \restriction \eta = p^{**}$ and $p^{\langle\rangle} \Vdash$ "$\dot{x} \cap \eta = z$". Set $\eta^{\langle\rangle} = \eta$ and $z^{\langle\rangle} = z$.

Suppose now that p^t, η^t, and z^t are all defined. We define $p^{t^\frown 0}$, $\eta^{t^\frown 0}$, $z^{t^\frown 0}$, $p^{t^\frown 1}$, $\eta^{t^\frown 1}$, and $z^{t^\frown 1}$. Note that there must be $\eta > \eta^t$, $z, y \subseteq \eta$ with $z \neq y$, and $p, q \in \mathbb{P}$ such that $p \restriction \eta = p^{**} = q \restriction \eta$, $p, q \geq p^t - \eta$, $p \Vdash$ "$\dot{x} \cap \eta = z$", and $q \Vdash$ "$\dot{x} \cap \eta = y$". If not, then it is possible to decide $\dot{x}$ completely in V, contradicting our hypothesis that $\Vdash_{\mathbb{P}}$ "$\dot{x}$ is a fresh subset of κ". We consequently choose such η, p, q, z, and y and define $p^{t^\frown 0} = p$, $\eta^{t^\frown 0} = \eta$, $z^{t^\frown 0} = z$, $p^{t^\frown 1} = q$, $\eta^{t^\frown 1} = \eta$, and $z^{t^\frown 1} = y$, going under the assumption that we have fixed at the beginning of the construction a term $\dot{\mathcal{S}}$ such that $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathcal{S}}$ is a strategy for $\dot{\mathbb{Q}}_\kappa$ which has been used to choose $\dot{p}_\kappa$ and $\dot{q}_\kappa$". This completes our induction. Note that for every $V[G_1]$-branch $f : \omega \to 2$ through $2^{<\omega}$, there is an upper bound $p^f \in \mathbb{P}/G_1$ for the sequence $\langle p^{f \restriction n} \mid n < \omega \rangle$. This is since conditions (2) and (3) ensure that if t extends s, then $p^t \restriction \eta^t = p^s \restriction \eta^s = p^{**}$ and $\operatorname{supp}(p^s) \cap \kappa < \eta^t$. Thus, the only common elements of the supports of p^t and p^s below κ are those of p^{**}. By condition (2), these agree on $\operatorname{supp}(p^{**})$. By condition (4), p^t extends p^s on κ, the only common coordinate where conditions extend. Hence, because $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}_\kappa$ is $<\kappa$-strategically closed and $\langle \dot{p}_\kappa^{f \restriction n} \mid n < \omega \rangle$ was constructed using $\dot{\mathcal{S}}$", it is possible to define an upper bound p^f in $V[G_1]$ roughly speaking by putting together

p^{**}, everything in the union of the supports of each $p^{f\restriction n}$ below κ, and an upper bound to those conditions occurring at coordinate κ. Because $(p^{f\restriction n})_0 = p^{\langle\rangle} \restriction 1 \in G_1$ for each $n < \omega$, p^f is a well-defined condition in $\mathbb{P}/G_1$.

Working now in $V[G_1]$ (or $V[H_1]$ where $H_1 \subseteq \mathbb{P}_1$ is V-generic over $\mathbb{P}_1$ with $p^{\langle\rangle} \restriction 1 \in H_1$), define $\eta^f = \bigcup_{n<\omega} \eta^{f\restriction n}$ and $z^f = \bigcup_{n<\omega} z^{f\restriction n}$. Since $V[G_1] \models$ "κ is a Mahlo cardinal", $\eta^f < \kappa$. Consequently, because $p^f \Vdash_{\mathbb{P}/G_1}$ "$\dot{x} \cap \eta^f = z^f$" and $\Vdash_{\mathbb{P}}$ "For every $\eta < \kappa$, $\dot{x} \cap \eta \in \check{V}$", $z^f \in V$. In addition, by condition (6), if $g : \omega \to 2$, $g \in V[G_1]$, $g \neq f$ is another branch through $2^{<\omega}$ such that $g \neq f$, then $z^g \neq z^f$. Moreover, for every $s \in 2^{<\omega}$, z^s is an initial segment of z^f iff s is an initial segment of f.

Let H_1 be V-generic over $\mathbb{P}_1$ such that $p^{\langle\rangle} \restriction 1 \in H_1$, and let $r : \omega \to 2$ be a new real added. Let H be a V-generic subset of $\mathbb{P}$ extending H_1 such that $p^r \in H$. Consider z^r. By our observations in the preceding paragraph, $z^r \in V$, and r can be reconstructed from z^r as the set $\bigcup\{s \in 2^{<\omega} \mid z^s$ is an initial segment of $z^r\}$. It thus immediately follows that $r \in V$. This contradiction to the fact that $(\dot{x})_H$ is a fresh subset of κ with respect to $\mathbb{P}$ completes the proof of Theorem 2.

$\square$

Suppose that for some $\alpha, \delta < \kappa$ with $\delta \neq \omega$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ adds a new subset of the cardinal δ". Assume without loss of generality that $\alpha = 0$ and δ is the least cardinal to which a new subset is added. We explicitly observe that since $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}_\kappa$ is $<\kappa$-strategically closed", the same inductive construction as given above remains valid, with every occurrence of ω replaced by an occurrence of δ.

Having completed the proof of Theorem 2, we turn now to the proof of Theorem 1.

Proof: Let $V \models$ "ZFC + κ is supercompact". Without loss of generality, we assume that $V \models$ GCH as well. The partial ordering $\mathbb{P}$ to be used in the proof of Theorem 1 is now defined as follows. For any ordinal δ, let δ' be the least V-strong cardinal above δ. $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa\rangle$ is the Gitik iteration of length κ which begins by forcing with $\mathrm{Add}(\omega, 1)$, i.e., $\mathbb{P}_0 = \{\emptyset\}$ and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}_0 = \dot{\mathrm{A}}\mathrm{dd}(\omega, 1)$". $\mathbb{P}$ then (possibly) does nontrivial forcing only at those ordinals δ which are, in V, Mahlo limits of strong cardinals. At such a stage δ, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{L}}_\delta * \dot{\mathbb{R}}_\delta$, where $\dot{\mathbb{L}}_\delta$ is a term for the lottery sum of all δ-directed closed partial orderings having rank below δ'. If either $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} = V^{\mathbb{P}_\delta}$, i.e., the lottery selects trivial forcing at stage δ, or $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is not measurable", then $\dot{\mathbb{R}}_\delta$ is a term for trivial forcing. If $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is measurable" and $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \neq V^{\mathbb{P}_\delta}$, i.e., the lottery selects nontrivial forcing at stage δ, then $\dot{\mathbb{R}}_\delta$ is a term for Prikry forcing defined with respect to some normal measure over δ.

The intuition behind the above definition of $\mathbb{P}$ is as follows. The fact that nothing is done at stage δ when the lottery selects trivial forcing, i.e., that no Prikry sequence is added, ensures that $V^{\mathbb{P}} \models$ "κ is supercompact". Since a Prikry sequence is added when a nontrivial forcing at stage δ preserves the measurability of δ, there will be a partial ordering $\mathbb{R} \in V^{\mathbb{P}}$ such that $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ is not supercompact". The lottery sum at stage δ, in conjunction with the Prikry forcing, will allow us to show that in $V^{\mathbb{P}}$, κ's strong compactness is preserved by nontrivial forcing. Because unboundedly many in κ Prikry sequences will have been added by $\mathbb{P}$, $V^{\mathbb{P}} \models$ "No cardinal below κ is strongly compact", i.e., $V^{\mathbb{P}} \models$ "κ is the least strongly compact cardinal".

The following lemmas show that $\mathbb{P}$ is as desired.

Lemma 2.1 *$V^{\mathbb{P}} \models$ "κ is supercompact".*

Proof: We follow the proof of [1, Lemma 2.1]. Let $\lambda \geq \kappa^+ = 2^\kappa$ be any regular cardinal. Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of κ such that $M \models$ "κ is not λ supercompact". By [3, Lemma 2.1], in M, κ is a Mahlo limit of strong cardinals. This means by the definition of $\mathbb{P}$ that it is possible to opt for trivial forcing in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Further, $M \models$ "No cardinal $\delta \in (\kappa, \lambda]$ is strong". This is since otherwise, in M, κ is supercompact up to a strong cardinal, so by the proof of [3, Lemma 2.4], κ is supercompact in M. Consequently, in M, above the appropriate condition, $j(\mathbb{P})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}}$, where the first nontrivial stage in $\dot{\mathbb{Q}}$ takes place well after λ.

We may now apply the argument of [6, Lemma 1.5]. Specifically, let G be V-generic over $\mathbb{P}$. Since GCH in V implies that $V \models$ "$2^\lambda = \lambda^+$", we may let $\langle \dot{x}_\alpha \mid \alpha < \lambda^+ \rangle$ be an enumeration in V of all of the canonical $\mathbb{P}$-names of subsets of $P_\kappa(\lambda)$. By [6, Lemmas 1.4 and 1.2] and the fact that $M^\lambda \subseteq M$, we may define an increasing sequence $\langle p_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $j(\mathbb{P})/G$ such that if $\alpha < \beta < \lambda^+$, p_β is an Easton extension of p_α, every initial segment of the sequence is in M, and for every $\alpha < \lambda^+$, $p_{\alpha+1} \parallel$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x}_\alpha)$". The remainder of the argument of [6, Lemma 1.5] remains valid and shows that a supercompact ultrafilter $\mathcal{U}$ over $(P_\kappa(\lambda))^{V[G]}$ may be defined in $V[G]$ by $x \in \mathcal{U}$ iff $x \subseteq (P_\kappa(\lambda))^{V[G]}$ and for some $\alpha < \lambda^+$ and some $\mathbb{P}$-name $\dot{x}$ of x, in $M[G]$, $p_\alpha \Vdash_{j(\mathbb{P})/G}$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x})$". (The fact that $j''G = G$ tells us $\mathcal{U}$ is well-defined.) Thus, $\Vdash_{\mathbb{P}}$ "κ is λ supercompact". Since λ was arbitrary, this completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is a partial ordering which is κ-directed closed. Then $V^{\mathbb{P}*\dot{\mathbb{Q}}} \vDash$ "κ is strongly compact".*

Proof: We follow the proof of [2, Lemma 2.2]. Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is κ-directed closed. Let $\lambda > \max(\kappa, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ be an arbitrary regular cardinal large enough so that $(2^{[\lambda]^{<\kappa}})^V = \rho = (2^{[\lambda]^{<\kappa}})^{V^{\mathbb{P}*\dot{\mathbb{Q}}}}$ and ρ is regular in both V and $V^{\mathbb{P}*\dot{\mathbb{Q}}}$, and let $\sigma = \rho^+ = 2^\rho$. Take $j : V \to M$ as an elementary embedding witnessing the σ supercompactness of κ such that $M \vDash$ "κ is not σ supercompact". By the choice of σ, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Further, as in Lemma 2.1, since $M \vDash$ "No cardinal $\delta \in (\kappa, \sigma]$ is strong", the next nontrivial forcing in the definition of $j(\mathbb{P})$ takes place well above σ. Thus, in M, above the appropriate condition, $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$ "$\dot{\mathbb{S}}_\kappa$ is a term for either Prikry forcing or trivial forcing".

The remainder of the proof of Lemma 2.2 is as in the proof of [4, Lemma 2]. As in the proof of Lemma 2.1, we outline the argument, and refer readers to the proof of [4, Lemma 2] for any missing details. By the last two sentences of the preceding paragraph, as in [4, Lemma 2], there is a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P})$ such that if $G * H$ is either V-generic or M-generic over $\mathbb{P} * \dot{\mathbb{Q}}$, $\Vdash_{j(\mathbb{P})}$ "τ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". In other words, τ is a term for a "master condition" for $\dot{\mathbb{Q}}$. Thus, if $\langle \dot{A}_\alpha \mid \alpha < \rho < \sigma \rangle$ enumerates in V the canonical $\mathbb{P} * \dot{\mathbb{Q}}$ names of subsets of $(P_\kappa(\lambda))^{V[G*H]}$, we can define in M a sequence of $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa$ names of elements of $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, $\langle \dot{p}_\alpha \mid \alpha \le \rho \rangle$, such that $\dot{p}_0$ is a term for $\langle 0, \tau \rangle$ (where 0 represents the trivial condition with respect to $\mathbb{R}$), $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{S}}_\kappa}$ "$\dot{p}_{\alpha+1}$ is a term for an Easton extension of $\dot{p}_\alpha$ deciding '$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{A}_\alpha)$' ", and for $\eta \le \rho$ a limit ordinal, $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa}$ "$\dot{p}_\eta$ is a term for an Easton extension of each member of the sequence $\langle \dot{p}_\beta \mid \beta < \eta \rangle$". If we then in $V[G * H]$ define a set $\mathcal{U} \subseteq 2^{[\lambda]^{<\kappa}}$ by $X \in \mathcal{U}$ iff $X \subseteq P_\kappa(\lambda)$ and for some $\langle r, q \rangle \in G * H$ and some $q' \in \mathbb{S}_\kappa$ either the trivial condition (if $\mathbb{S}_\kappa$ is trivial forcing) or of the form $\langle \emptyset, B \rangle$ (if $\mathbb{S}_\kappa$ is Prikry forcing), in M, $\langle r, \dot{q}, \dot{q}', \dot{p}_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in \dot{X}$" for some name $\dot{X}$ of X, then as in [4, Lemma 2], $\mathcal{U}$ is a κ-additive, fine ultrafilter over $(P_\kappa(\lambda))^{V[G*H]}$, i.e., $V[G * H] \models$ "κ is λ strongly compact". Since λ was arbitrary, this completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 $V^{\mathbb{P}} \models$ "*No cardinal $\delta < \kappa$ is strongly compact*".

Proof: Let $\lambda = \beth_\omega(\kappa)$. Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of κ. Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is $\mathrm{Add}(\kappa, 1)$. By Lemma 2.2, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is measurable" (since $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact"). Because λ has been chosen large enough, it therefore follows that $M^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is measurable". In addition, as in Lemma 2.2, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Therefore, by the definition of $\mathbb{P}$, above the appropriate condition, $(j(\mathbb{P} * \dot{\mathbb{Q}}))_{\kappa+1} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa = \mathbb{P}_{\kappa+1}$ is forcing equivalent in M to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa$, where $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{S}}_\kappa$ is Prikry forcing defined over κ". This means that in M, $\Vdash_{\mathbb{P}_\kappa}$ "By forcing above a condition $\dot{p}^*_\kappa$ ensuring that $\mathrm{Add}(\kappa, 1)$ is chosen in the stage κ lottery held in the definition of $j(\mathbb{P})$, $\dot{\mathbb{Q}}_\kappa$ is forcing equivalent to a partial ordering adding a Prikry sequence to κ". Consequently, by reflection, for unboundedly many $\delta < \kappa$, $\Vdash_{\mathbb{P}_\delta}$ "By forcing above a condition $\dot{p}^*_\delta$ ensuring that $\mathrm{Add}(\delta, 1)$ is chosen in the stage δ lottery held in the definition of $\mathbb{P}$, $\dot{\mathbb{Q}}_\delta$ is forcing equivalent to a partial ordering adding a Prikry sequence to δ".

It now follows that $\Vdash_{\mathbb{P}}$ "Unboundedly many $\delta < \kappa$ contain Prikry sequences". To see this, let $\gamma < \kappa$ be fixed but arbitrary. Choose $p = \langle \dot{p}_\alpha \mid \alpha < \kappa \rangle \in \mathbb{P}$. Since $\mathbb{P}$ is an Easton support iteration, let $\rho > \gamma$ be such that for every $\alpha \ge \rho$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{p}_\alpha$ is a term for the trivial condition". We may now find $\delta > \rho > \gamma$ such that $\Vdash_{\mathbb{P}_\delta}$ "By forcing above a condition $\dot{p}^*_\delta$ ensuring that $\mathrm{Add}(\delta, 1)$ is chosen in the stage δ lottery held in the definition of $\mathbb{P}$, $\dot{\mathbb{Q}}_\delta$ is forcing equivalent to a partial ordering adding a Prikry sequence to δ". This means that we may find $q \geq p$ such that $q \Vdash$ "δ contains a Prikry sequence". Thus, $\Vdash_{\mathbb{P}}$ "Unboundedly many $\delta < \kappa$ contain Prikry sequences". Hence, by [5, Theorem 11.1], $V^{\mathbb{P}} \models$ "Unboundedly many $\delta < \kappa$ (i.e., the successors of those cardinals having Prikry sequences) contain non-reflecting stationary sets of ordinals of cofinality ω". By [15, Theorem 4.8] and the succeeding remarks, it thus follows that $V^{\mathbb{P}} \models$ "No cardinal $\delta < \kappa$ is strongly compact". This completes the proof of Lemma 2.3.

$\square$

Lemma 2.4 *For* $\mathbb{R} = (\mathrm{Add}(\kappa, 1))^{V^{\mathbb{P}}}$, $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "*κ is not supercompact". In fact, in* $V^{\mathbb{P} * \dot{\mathbb{R}}}$, *$\kappa$ has trivial Mitchell rank.*

Proof: Let $G * H$ be V-generic over $\mathbb{P} * \dot{\mathbb{R}}$. If $V[G * H] \models$ "κ does not have trivial Mitchell rank", then let $j : V[G * H] \to M[j(G * H)]$ be an elementary embedding generated by a normal measure $\mathcal{U} \in V[G*H]$ over κ such that $M[j(G*H)] \models$ "κ is measurable". Note that since $M = \bigcup_{\alpha \in \mathrm{Ord}} j(V_\alpha)$, $j \restriction V : V \to M$ is elementary. Therefore, because $j \restriction \kappa = \mathrm{id}$, we may infer that $(V_\kappa)^V = (V_\kappa)^M$. However, by Theorem 2, we may further infer that $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$. To see this, let $x \subseteq \kappa$, $x \in M$. Since $M \subseteq M[j(G * H)] \subseteq V[G * H]$, $x \in V[G * H]$. In addition, because $(V_\kappa)^V = (V_\kappa)^M$, we know that $x \cap \alpha \in V$ for every $\alpha < \kappa$. This means that if $x \notin V$, then x is a fresh subset of κ with respect to $\mathbb{P} * \dot{\mathbb{R}}$. Since by Lemma 2.2, κ is strongly compact and hence both measurable and Mahlo in $V[G * H]$, this contradicts Theorem 2. Thus, $x \in V$, so $(\wp(\kappa))^M \subseteq (\wp(\kappa))^V$. From this, it of course immediately follows that $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$.

Let $I = j(G * H)$. Note that if $V \models$ "$\delta < \kappa$ is a strong cardinal", then $M \models$ "$j(\delta) = \delta$ is a strong cardinal". Also, $M \models$ "κ is a Mahlo limit of strong cardinals", since $M[j(G * H)] \models$ "κ is a Mahlo cardinal", and forcing can't create a new Mahlo cardinal. Hence, by the results of the preceding paragraph, it follows as well that $j(\mathbb{P}) \restriction \kappa = \mathbb{P}_\kappa = \mathbb{P}$ and $I_\kappa = G$. Further, as $V[G * H] \models$ "$M[I]^\kappa \subseteq M[I]$", $H \in M[I]$. We know in addition that in M, $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa}$ "The forcing beyond stage κ adds no new subsets of 2^κ" and κ is a stage at which nontrivial forcing in $j(\mathbb{P})$ can take place. Consequently, $H \in M[I_{\kappa+1}] = M[G][I(\kappa)]$, and $M[I_{\kappa+1}] \models$ "κ is measurable".

Note that since $\mathbb{P}$ is defined by taking Easton supports, $\mathbb{P}$ is κ-c.c. in both V and M. Because $\mathbb{P}$ is a Gitik iteration of suitably directed closed partial orderings together with Prikry forcing and $(V_\kappa)^V = (V_\kappa)^M$, $V_\kappa[G]$ is the same when calculated in either $V[G]$ or $M[G]$. It must therefore be the case that $(\mathrm{Add}(\kappa, 1))^{V[G]} = (\mathrm{Add}(\kappa, 1))^{M[G]}$. In addition, since $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$, the fact $\mathbb{P}$ is κ-c.c. in M yields that $(V_{\kappa+1})^{M[G]} \subseteq (V_{\kappa+1})^{V[G]}$. This means that H is $M[G]$-generic over $(\mathrm{Add}(\kappa, 1))^{M[G]}$, since H is $V[G]$-generic over $(\mathrm{Add}(\kappa, 1))^{V[G]} = (\mathrm{Add}(\kappa, 1))^{M[G]}$, and a dense open subset of $(\mathrm{Add}(\kappa, 1))^{M[G]}$ in $M[G]$ is a member of $(V_{\kappa+1})^{M[G]}$. Hence, H must be added by the stage κ forcing done in $M[G] = M[I_\kappa]$, i.e., the stage κ lottery held in $M[I_\kappa]$ must opt for some nontrivial forcing. By the definition of $\mathbb{P}$ and $j(\mathbb{P})$, we must then have that $M[I_{\kappa+1}] \models$ "κ contains a Prikry sequence". This contradiction to the fact that $M[I_{\kappa+1}] \models$ "κ is measurable" completes the proof of Lemma 2.4.

$\square$

We note that in general, if $j : V \to M$ is an elementary embedding having critical point κ, then $(V_{\kappa+1})^V \subseteq (V_{\kappa+1})^M$. To see this, we begin by observing that as in the proof of Lemma 2.4, since $j \restriction \kappa = \mathrm{id}$, $(V_\kappa)^V = (V_\kappa)^M$. Without fear of ambiguity, we therefore write V_κ. However, since for every $x \subseteq \kappa$, $x \in V$, $j(x) \in M$, x is definable in M as $j(x) \cap V_\kappa$. In particular, this means that in Lemma 2.4, it is actually true that $(V_{\kappa+1})^V = (V_{\kappa+1})^M$.

It could be the case, though, that $j : V \to M$ is an elementary embedding with critical point κ, yet $(V_{\kappa+1})^V$ is a proper subset of $(V_{\kappa+1})^M$. We briefly outline two examples of this phenomenon, which are as follows:

1. Let $\kappa < \lambda$ be such that κ is a measurable cardinal and λ is a Woodin cardinal. Suppose $\mathbb{P}$ is the stationary tower forcing having critical point κ^+ which changes the cofinality of κ^+ to ω (see [12] for a discussion of this partial ordering). Since $V^{\mathbb{P}} \models$ "κ is a measurable cardinal", let $j : V^{\mathbb{P}} \to M^{j(\mathbb{P})}$ be an elementary embedding witnessing κ's measurability. Consider $j \restriction V : V \to M$. It will then be the case that $(V_{\kappa+1})^M$ contains a subset of $\kappa \times \kappa$ coding a well-ordering of $(\kappa^+)^V$ of order type κ.

2. Let κ be a measurable cardinal. Take $L[\mu]$ as our ground model. Force over $L[\mu]$ with the

reverse Easton iteration of length κ which adds a Cohen subset of κ to each inaccessible cardinal $\delta < \kappa$. Let V be the resulting generic extension. Suppose C is now V-generic over $(\mathrm{Add}(\kappa, 1))^V$. Standard arguments show that $V[C] \models$ "κ is a measurable cardinal". We can therefore once again let $j : V[C] \to M[j(C)]$ be an elementary embedding witnessing κ's measurability and consider $j \restriction V : V \to M$. It will then be the case that $C \in (V_{\kappa+1})^M$.

On the other hand, if there is no inner model with a strong cardinal, V is the core model, and $j \restriction V : V \to M$ is the restriction of $j : V^{\mathbb{P}} \to M^{j(\mathbb{P})}$ for $\mathbb{P} \in V$ (i.e., $\mathbb{P}$ is set forcing over V), then $j \restriction V$ must be an iterated ultrapower of V starting with an extender over κ. This implies that $(V_{\kappa+1})^V = (V_{\kappa+1})^M$. We note that M here need not be a class in V or even be contained in V, unless κ is not a limit of measurable cardinals.

Theorem 1 now follows from Lemmas 2.1 – 2.4. By Lemma 2.1, κ is supercompact in $V^{\mathbb{P}}$, and by Lemma 2.2, in $V^{\mathbb{P}}$, κ's strong compactness is indestructible under arbitrary κ-directed closed forcing. By Lemma 2.3, $V^{\mathbb{P}} \models$ "κ is the least strongly compact cardinal". By Lemma 2.4, there is a nontrivial κ-directed closed forcing $\mathbb{R} \in V^{\mathbb{P}}$ such that $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ has trivial Mitchell rank and hence is not supercompact". This completes the proof of Theorem 1.

$\square$

In conclusion to this paper, we ask if it is possible to get a model witnessing the conclusions of Theorem 1 in which κ is not the least strongly compact cardinal. Since Prikry forcing above a strongly compact cardinal destroys strong compactness, an answer to this question would require a different sort of iteration from the one used in the proof of Theorem 1.

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