A weakly normal ultrafilter amenable to its ultrapower.

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June 25, 2020

Abstract

We build a weakly normal ultrafilter which is amenable to its ultrapower. This answers a question of G. Goldberg [1].

1 Introduction.

In [1], G. Goldberg gives a surprising construction of a $\sigma-$complete ultrafilter amenable to its own ultrapower. The ultrafilter that he constructs is not weakly normal. Goldberg asked if it is possible to produce a weakly normal ultrafilter which is amenable to its own ultrapower.

The purpose of this note is to give an affirmative answer.

Let us state basic definitions.

Definition 1.1 A set A is called *amenable to* M iff $A \cap M \in M$.

It is a basic fact that a $\sigma-$complete ultrafilter U over a cardinal κ cannot belong to the transitive collapse M_U of its ultrapower, see 1.14 of [6].

A natural weakening of the property "$U \in M_U$" is an amenability, i.e., "$U \cap M_U \in M_U$".

It follows from [6], 1.14 that if U is a $\sigma-$complete ultrafilter over a cardinal κ and $M_U \supseteq \mathcal{P}(\kappa)$, then U cannot be amenable to M_U, since then $U \cap M_U = U$. In particular, a $\kappa-$complete ultrafilter on κ is not amenable to its own ultrapower.

However, by Goldberg [1], it is not true in general, and we may have amenability for a $\sigma-$complete ultrafilter U over a cardinal κ, if the assumption $M_U \supseteq \mathcal{P}(\kappa)$ is dropped. Goldberg used $\kappa^{++}-$supercompact cardinal κ in his construction.

*The work was partially supported by ISF grant no. 1216/18. We are grateful to Gabriel Goldberg for asking the question and to the referee of the paper for his kind suggestions and corrections.

Definition 1.2 An ultrafilter U over a cardinal κ is called *uniform* iff for every $A \in U$, $|A| = \kappa$.

Definition 1.3 An ultrafilter U over a regular cardinal κ is called *weakly normal* iff for every $A \in U$ and for every regressive function $f : A \to \kappa$ there is $\alpha < \kappa$ such that $\{\nu \in A \mid f(\nu) < \alpha\} \in U$.

We prove the following:

Theorem 1.4 *Assume GCH and suppose that κ is κ^+-supercompact cardinal of Mitchell order 2, i.e., κ is κ^+-supercompact in the ultrapower by a normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$. Then, in a cardinal preserving generic extension, there is a uniform $\kappa-$complete weakly normal ultrafilter over κ^+ which is amenable to its ultrapower.*

In the last section, starting with a stronger assumption, a forcing free construction of a uniform $\kappa-$complete weakly normal ultrafilter over κ^+ which is amenable to its ultrapower is given.

Our notation are standard. We refer to the classical books of T. Jech [3] and A. Kanamori [4] for facts on large cardinals and to the article by J. Cummings [2] for forcing with large cardinals.

Following G. Goldberg [1], we denote by $j_U : V \to M_U \simeq \mathrm{Ult}(V, U)$ the elementary embedding corresponding to an ultrafilter U.

2 The construction

Assume GCH. Suppose that W is a normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$ such that some normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$ belongs to M_W (the ultrapower by W), i.e., W has a Mitchell order at least 1 among normal ultrafilters over $\mathcal{P}_\kappa(\kappa^+)$.
Note that ${}^{\kappa^+}M_W \subseteq M_W$, and so, each normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$ in M_W is such also in V.
Consider $U = \{X \subseteq \kappa \mid \kappa \in j_W(X)\}$, i.e., the normal ultrafilter over κ to which W projects. The function $P \mapsto P \cap \kappa$ is a projection. Let $k : M_U \to M_W$ be the canonical elementary embedding, i.e., $k(j_U(f)(\kappa)) = j_W(f)(\kappa)$.
Note that $\mathrm{crit}(k) = (\kappa^{++})^{M_U}$ and $k((\kappa^{++})^{M_U}) = (\kappa^{++})^{M_W} = \kappa^{++}$.
Also, $|j_U(\kappa)| = \kappa^+$ and $|j_W(\kappa)| = \kappa^{++}$.

The elementarity of k implies that

$$M_U \models \kappa \text{ is a } \kappa^+ - \text{supercompact cardinal.}$$

Let $W_0^U \in M_U$ be such that

$$M_U \models W_0^U \text{ is a normal ultrafilter over } \mathcal{P}_\kappa(\kappa^+).$$

Set $W_0 = k(W_0^U)$. Then

$$M_W \models W_0 \text{ is a normal ultrafilter over } \mathcal{P}_\kappa(\kappa^+).$$

Hence, W_0 is a normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$ in V, as well.
Let us fix a set $K_0^U \in W_0^U$ such that the function $P \mapsto \sup(P)$ is one-to-one on it.
It exists by a classical result of R. Solovay [5].
Let $K = k(K_0^U)$. Further, dealing with extensions of W_0, we will restrict to this K.

Force a Cohen function to every inaccessible non-measurable cardinal $\nu < \kappa$ with the usual Easton support.
Formally, we define the Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle,$$

where $\underset{\sim}{Q}_\beta$ is trivial, unless β is an inaccessible non-measurable cardinal, and in this case let $\underset{\sim}{Q}_\beta$ be the Cohen forcing over β in V^{P_β}, i.e.,

$$Q_\beta = \{ f \in V^{P_\beta} \mid f : \beta \to 2, |f| < \beta \}.$$

Let G_κ be a generic subset of P_κ.
Denote $V[G_\kappa]$ by V^*.
Let us extend $U, W, W_0^U, W_0, j_U, j_W, k, j_{W_0^U}$ and j_{W_0}.

Start with $j_U : V \to M_U$. Construct in $V[G_\kappa]$ a master condition sequence $\{p_\nu \mid \nu < \kappa^+\} \subseteq M_U[G_\kappa]$ for the forcing $j_U(P_\kappa)/G_\kappa$[1].
Let $G_{j_U(\kappa)}$ be an M_U-generic subset of $j_U(P_\kappa)$ that it generates.
Then j_U extends to

$$j_U^* : V[G_\kappa] \to M_U[G_{j_U(\kappa)}]$$

and U extends to

$$U^* = \{ X \subseteq \kappa \mid \kappa \in j_U^*(X) \}.$$

[1]It is an increasing sequence of elements of $j_U(P_\kappa)/G_\kappa$ which meets every dense open subset of $j_U(P_\kappa)/G_\kappa$ belonging to $M_U[G_\kappa]$.

It follows that $M_{U^*} = M_U[G_{j_U(\kappa)}]$ and $j_{U^*} = j_U^*$. We refer to Cummings [2] for more details.
Now we extend j_W and k.
Proceed as follows.
Consider $k''\{p_\nu \mid \nu < \kappa^+\}$ in $M_W[G_\kappa]$. It is a set in $M_W[G_\kappa]$, due to the closure of $M_W[G_\kappa]$ under κ^+-sequences of its elements. Also this set consists of κ^+-many compatible conditions in $P_{j_W(\kappa)}/G_\kappa$.
The forcing $P_{j_W(\kappa)}/G_\kappa$ is at least $\kappa^{++}-$closed (in $M_W[G_\kappa]$).
Hence there is a condition q which is stronger than every $k(p_\nu), \nu < \kappa^+$. Now, we construct in $V[G_\kappa]$ a master condition sequence $\{q_\nu \mid \nu < \kappa^{++}\} \subseteq M_W[G_\kappa]$ for the forcing $j_W(P_\kappa)/G_\kappa$ with $q_0 \geq q$.
Then j_W extends to

$$j_W^* : V[G_\kappa] \to M_U[G_{j_W(\kappa)}]$$

and W extends to

$$W^* = \{X \subseteq P_\kappa(\kappa^+) \mid j_W''\kappa^+ \in j_W^*(X)\}.$$

It follows that $M_{W^*} = M_U[G_{j_W(\kappa)}]$ and $j_{W^*} = j_W^*$.
Also, k extends to

$$k^* : M_U[G_{j_U(\kappa)}] \to M_W[G_{j_W(\kappa)}],$$

since $k''G_{j_U(\kappa)} \subseteq G_{j_W(\kappa)}$.
Again, we refer to Cummings [2] for more details.

Now, inside $M_{U^*} = M_U[G_{j_U(\kappa)}]$, we extend W_0^U to a normal ultrafilter W_0^{U*} in the usual fashion, i.e., as above.
Let $\langle r_\alpha \mid \alpha < (\kappa^{++})^{M_U}\rangle$ be a master condition sequence used for this.
Let us move to M_{W^*} using k^*. Consider $\langle k(r_\alpha) \mid \alpha < (\kappa^{++})^{M_U}\rangle$.
The forcing which is used in M_{W^*} in order to construct a master condition sequence is $\kappa^{++}-$closed and $(\kappa^{++})^{M_U} < \kappa^{++}$.
So there is a single condition $r \geq k(r_\alpha)$, for every $\alpha < (\kappa^{++})^{M_U}$.
Now, let us use extensions of r and define, in M_{W^*}, $\kappa-$many different
extensions $\langle W_{0\alpha} \mid \alpha < \kappa\rangle$ of W_0.
Then, for every $\alpha < \kappa$, $W_{0\alpha} \cap \mathcal{P}(\mathcal{P}_\kappa(\kappa^+))^{M_{U^*}} = W_0^{U*}$.
So W_0^{U*} is in $V^* = V[G_\kappa]$ a $\kappa-$complete filter over $\mathcal{P}_\kappa(\kappa^+)$ such that
(*) for every $\alpha < \kappa$, $W_0^{U*} \subseteq W_{0\alpha}$.

In M_{U^*}, let $\langle W_{0\alpha}^* \mid \alpha < j_U(\kappa)\rangle = j_{U^*}(\langle W_{0\alpha} \mid \alpha < \kappa\rangle)$.
Consider

$$j_{W_{0\kappa}^*} : M_{U^*} \to M_{W_{0\kappa}^*} \text{ and } i = j_{W_{0\kappa}^*} \circ j_{U^*} : V^* \to M_{W_{0\kappa}^*}.$$

The embedding i is actually the ultrapower embedding by $U^* - \lim \langle W_{0\alpha} \mid \alpha < \kappa \rangle$. Note that the family $\langle W_{0\alpha} \mid \alpha < \kappa \rangle$ consists of normal ultrafilters, and so, it is a discrete family. In particular, every element of this ultrapower is definable from $j^*_{W_{0\kappa}}[j_U(\kappa^+)]$ and points in the range of i.

Note that if $X \in W_0^{U*}$, then by (*), for every $\alpha < \kappa$, $X \in W_0^{U*} \subseteq W_{0\alpha}$. Hence $j_{U^*}(X) \in W^*_{0\kappa}$. So,

$$j^*_{W_{0\kappa}}[j_U(\kappa^+)] \in i(X).$$

Set, for every $\alpha < \kappa$,

$$W'_{0\alpha} = \{\{\sup(P) \mid P \in A\} \mid A \in W_{0\alpha}\} \text{ and } W' = \{\{\sup(P) \mid P \in A\} \mid A \in W_0^{U*}\}.$$

Then again:
(**) for every $\alpha < \kappa$, $W' \subseteq W'_{0\alpha}$.

Also, if $X \in W'$, then by (**), for every $\alpha < \kappa$, $X \in W' \subseteq W'_{0\alpha}$. Hence $j_U(X) \in W^{*\prime}_{0\kappa}$. So,

$$\sup(j^*_{W_{0\kappa}}[j_U(\kappa^+)]) \in i(X).$$

Define an ultrafilter $\mathcal{V}$ over κ^+ by setting

$$X \in \mathcal{V} \text{ iff } \sup(i[\kappa^+]) \in i(X).$$

Clearly, such defined $\mathcal{V}$ is a weakly normal.
Note that $j_U[\kappa^+]$ is unbounded in $j_U(\kappa^+)$.
Hence,

$$\sup(i[\kappa^+]) = \sup(j_{W^*_{0\kappa}}[j_U(\kappa^+)]).$$

So,

$$X \in \mathcal{V} \text{ iff } \sup(i[\kappa^+]) = \sup(j_{W^*_{0\kappa}}[j_U(\kappa^+)]) \in i(X).$$

Then $\mathcal{V} \supseteq W'$ and $\mathcal{V} \cap M_{W^*_{0\kappa}} = \mathcal{V} \cap M_{U^*} = W'$.

Also, $j_{W^*_{0\kappa}}[j_U(\kappa^+)]$ is $[id]_{W^*_{0\kappa}}$ and by Solovay, in M_{U^*}, the ultrafilter

$$W^{*\prime}_{0\kappa} = \{X \subseteq j_U(\kappa^+) \mid \sup(j_{W^*_{0\kappa}}[j_U(\kappa^+)]) \in j_{W^*_{0\kappa}}(X)\}$$

is isomorphic to $W^*_{0\kappa}$, since the function $P \mapsto \sup(P)$ is one to one on a big set.
In particular they share the same ultrapower.

Then we have that $\mathcal{V}$ is $U^* - \lim \langle W'_{0\alpha} \mid \alpha < \kappa \rangle$. So, $M_{\mathcal{V}} = M_{W^*_{0\kappa}}$ and $j_{\mathcal{V}} = i$.
Recall that we have $\mathcal{V} \cap M_{W^*_{0\kappa}} = \mathcal{V} \cap M_{U^*} = W'$ and $W' \in M_{U^*}$ implies that $W' \in M_{W^*_{0\kappa}}$, since the models M_{U^*} and $M_{W^*_{0\kappa}}$ agree about subsets of $j_U(\kappa)$, and in particular, of $(\kappa^{++})^{M_{U^*}}$.
Hence, $W' \in M_{\mathcal{V}}$ and $\mathcal{V} \cap M_{\mathcal{V}} = W'$.
In addition, $\mathcal{V}$ is a uniform $\kappa-$complete ultrafilter over κ^+, by its definition.
So, $\mathcal{V}$ is as desired, i.e., it is a weakly normal $\kappa-$complete ultrafilter over κ^+ which is amenable to its ultrapower.

3 A construction without forcing.

The construction of a weakly normal ultrafilter amenable to its ultrapower of the previous section was based on the property (*).
Namely we needed U^*, $W_0^{U^*}$ and a sequence $\langle W_{0\alpha} \mid \alpha < \kappa \rangle$ such that

1. U^* is a normal ultrafilter over κ,
2. $W_0^{U^*} \in M_{U^*}$ and $M_{U^*} \models W_0^{U^*}$ is a normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$,
3. $\langle W_{0\alpha} \mid \alpha < \kappa \rangle$ is a sequence of pairwise different normal ultrafilters over $\mathcal{P}_\kappa(\kappa^+)$,
4. for every $\alpha < \kappa$, $W_0^{U^*} \subseteq W_{0\alpha}$.

Let us argue that it is possible to insure all these conditions starting with a bit stronger assumption, but without the use of forcing.

Let us assume GCH[2].

We proceed as in the previous section, but up one cardinal.
Suppose that W is a normal ultrafilter over $\mathcal{P}_\kappa(\kappa^{++})$ such that some normal ultrafilter over $\mathcal{P}_\kappa(\kappa^{++})$ belongs to M_W, i.e., W has a Mitchell order at least 1 among normal ultrafilters over $\mathcal{P}_\kappa(\kappa^{++})$.[3]
Note that ${}^{\kappa^{++}}M_W \subseteq M_W$, and so, each normal ultrafilter over $\mathcal{P}_\kappa(\kappa^{++})$ in M_W is such also in V.
Consider $U = \{X \subseteq \kappa \mid \kappa \in j_W(X)\}$, i.e., the normal ultrafilter over κ to which W projects. The function $P \mapsto P \cap \kappa$ is a projection. Let $k : M_U \to M_W$ be the canonical elementary embedding, i.e., $k(j_U(f)(\kappa)) = j_W(f)(\kappa)$.

[2] Using obvious adaptations it is possible to remove GCH assumptions.

[3] The referee found a way to weaken this assumption a bit. Namely his argument uses that the Mitchell order on normal fine $\kappa-$complete ultrafilters on $\mathcal{P}_\kappa(\kappa^+)$ has rank κ^{+++}.

Note that $\mathrm{crit}(k) = (\kappa^{++})^{M_U}$ and $k((\kappa^{++})^{M_U}) = (\kappa^{++})^{M_W} = \kappa^{++}$.
Also, $|j_U(\kappa)| = \kappa^+$ and $|j_W(\kappa)| = \kappa^{+3}$.
The elementarity of k implies that

$$M_U \models \kappa \text{ is a } \kappa^{++} - \text{supercompact cardinal.}$$

Let $W_0^U \in M_U$ be such that

$$M_U \models W_0^U \text{ is a normal ultrafilter over } \mathcal{P}_\kappa(\kappa^{++}).$$

Set $W_0 = k(W_0^U)$. Then

$$M_W \models W_0 \text{ is a normal ultrafilter over } \mathcal{P}_\kappa(\kappa^{++}).$$

Hence, W_0 is a normal ultrafilter over $\mathcal{P}_\kappa(\kappa^{++})$ in V, as well.
Consider also in M_U the projection of W_0^U to $\mathcal{P}_\kappa(\kappa^+)$. Denote it by R_0^U.
Let $R_0 = k(R_0^U)$.
Then the following hold:

1. R_0 is normal ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$ in V,
2. R_0 is a projection of W_0 to $\mathcal{P}_\kappa(\kappa^+)$,
3. $R_0 \cap M_U = R_0^U$.

The first two items follow by the elementarity of k and closure properties of M_W.
The third item follows, since $\mathrm{crit}(k) = (\kappa^{++})^{M_U} > \kappa^+$, and so, k does not move subsets of κ^+.

So, we have:
a normal ultrafilter W_0 *over* $\mathcal{P}_\kappa(\kappa^{++})$ *and a normal ultrafilter* U *over* κ *such that the projection* R_0 *of* W_0 *to* $\mathcal{P}_\kappa(\kappa^+)$ *is amenable to* M_U.

Let us argue that this enough for (*).

Lemma 3.1 *There are at least* $\kappa^{+3}-$*many different normal ultrafilters over* $\mathcal{P}_\kappa(\kappa^+)$ *which extend* R_0^U.

Proof. Suppose otherwise.
Let Z be the set of all such extensions of R_0^U.

Then $|Z| \leq \kappa^{++}$.
We have $Z \in M_{W_0}$. Hence,

$$M_{W_0} \models \exists Y (Y \text{ is a maximal family of extensions of } R_0^U \text{ and } |Y| \leq \kappa^{++}).$$

Let $k_0 : M_{R_0} \to M_{W_0}$ be the natural elementary embedding. Note that its critical point is $(\kappa^{+3})^{M_{R_0}}$. By elementarity of k_0, the same statement is true in M_{R_0}. Let $Y \in M_{R_0}$ be a witness. But now, $|Y| \leq \kappa^{++}$ implies that $k_0(Y) = Y$, and so, Y is maximal also in M_{W_0}. However, R_0 itself extends R_0^U and R_0 cannot be in Y since $R_0 \notin M_{R_0}$.
Contradiction.
□

References

[1] G. Goldberg, The internal relation, arXiv: 1810.04290v2 [math. LO]

[2] J. Cummings, Iterated forcing and elementary embeddings, Handbook of Set Theory, M. Foreman, A. Kanamori, eds., Springer, 2010, pp.775-884.

[3] T. Jech, Set Theory.

[4] A. Kanamori, The Higher Infinite.

[5] R. Solovay, Strongly compact cardinals and GCH, in Henkin, ed., Proceedings of the Tarski Symposium, Proceedings of Symposia in Pure Mathematics vol. 25, AMS 1974, pp. 365-372.

[6] R. Solovay, W. Reinhadt and A. Kanamori, Strong axioms of infinity and elementary embeddings, Annals of Mathematical Logic 13 (1978), pp. 73-116.