Arbitrary gap: Lectures June-August

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1 One difference between gap 3 and higher gaps

Let $\mathcal{P}'(3)$ denotes the preparation forcing for the gap 3. Let G be a generic subset of $\mathcal{P}'(3)$. Consider

$$S = \{A \mid \exists \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G \quad A = A^{0\kappa^+}\}.$$

It was shown that S is a stationary subset of $[H(\kappa^{+3}]^{\leq\kappa^+}$. Let us point out in addition the following:

Proposition 1.1 *If $A, B \in S$ and $otp(A\cap\kappa^{+3}) = otp(B\cap\kappa^{+3})$, then A and B are isomorphic by an isomorphism which is an identity over $A \cap B$.*

Proof. Induction on walks complexity.
$\square$

The purpose of this note will be to show that this proposition fails already in the gap 4 case.

Theorem 1.2 *Let $\lambda < \mu$ be cardinals such that*

1. *μ is regular,*
2. *$\lambda^{++} < \mu$,*
3. *$2^\lambda = \lambda^+$,*
4. *for every $\delta, \lambda^+ < \delta < \mu$, $\delta^{\lambda^+} = \delta$.*

Suppose that S is an unbounded subset of $[H(\mu)]^\lambda$.
Then there are $A, B \in S$ with $otp(A \cap \mu) = otp(B \cap \mu)$, but the isomorphism between A and B is not the identity on $A \cap B$.

Proof. Suppose otherwise. Let S be an unbounded subset of $[H(\mu)]^\lambda$ witnessing this. Consider a sequence $\langle M_\alpha \mid \alpha < \mu \rangle$ such that for every $\alpha < \mu$

1. $\langle M_\alpha, \in, <, M_\alpha \cap S \rangle \prec \langle H(\mu), \in, <, S \rangle$,
2. $|M_\alpha| = \lambda^+$,
3. $M_\alpha \supseteq \lambda^+$,
4. ${}^\lambda M_\alpha \subseteq M_\alpha$,
5. $\beta \neq \alpha$ implies $M_\beta \neq M_\alpha$.

Form a Δ-system and shrink the sequence $\langle M_\alpha \mid \alpha < \mu \rangle$ to a sequence $\langle M_\alpha \mid \alpha \in Z \rangle$ such that for every $\alpha, \beta \in Z, \alpha < \beta$ the following hold:

1. $M_\alpha \cap \alpha = M_\beta \cap \beta$,
2. $\sup(M_\alpha \cap \mu)\beta$,
3. $\langle M_\alpha, \in, <, M_\alpha \cap S \rangle \simeq \langle M_\beta, \in, <, M_\beta \cap S \rangle$ and the isomorphism is the identity on the common part.

Fix some $\alpha \neq \beta$ in Z. Pick an ordinal $\tau \in M_\alpha$ above $\sup(M_\alpha \cap M_\beta \cap \mu)$.
Now we use unboundedness S and find $A \in S$ with $\tau, \pi_{M_\alpha, M_\beta}(\tau) \in A$.
Consider $A \cap M_\alpha$. This set belongs to M_α, since M_α is closed under λ-sequences of its elements. By elementarity it is possible to find $A_\alpha \in M_\alpha$ such that

- $A_\alpha \supseteq M_\alpha \cap A$,
- $otp(A_\alpha \cap \mu) = otp(A \cap \mu)$,
- $A_\alpha \in S$.

Set $A_\beta = \pi_{M_\alpha, M_\beta}(A_\alpha)$. Then $otp(A_\alpha \cap \mu) = otp(A_\beta \cap \mu)$ and $A_\beta \in S$, by (3) above. Note also that the isomorphism π_{A_α, A_β} is just $\pi_{M_\alpha, M_\beta}(A_\alpha) \restriction A_\alpha$. By (1) above and the choice of τ we have $A_\alpha \cap A_\beta \cap \mu \subseteq A_\alpha \cap \tau$. Hence $\tau' := \pi_{A_\alpha, A_\beta}(\tau) \neq \tau$. But $\pi_{A_\alpha, A_\beta}(\tau) = \pi_{M_\alpha, M_\beta}(\tau)$ and the last component is in A. So, $\tau' \in A \cap A_\beta$.
Now,

$$\pi_{A, A_\beta}(\tau) = \pi_{A_\alpha, A_\beta}(\pi_{A, A_\alpha}(\tau)).$$

But $\tau \in A \cap A_\alpha, A, A_\alpha \in S$, so $\pi_{A,A_\alpha}(\tau) = \tau$. Then

$$\pi_{A,A_\beta}(\tau) = \pi_{A_\alpha,A_\beta}(\tau) = \tau'.$$

Which is impossible, since $\tau' \in A \cap A_\beta, A, A_\beta \in S$ and $\tau \neq \tau'$.
$\square$

Without GCH type assumptions it looks like the theorem above consistently fails. Thus one can try to use a "baby " version of the arbitrary gap preparation forcing:

$$\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \in s \rangle,$$

with only requirement that models of the same order type are isomorphic over their intersection.

We do not know if for the gap 3 always there is S as in Proposition 1.1 (or even only unbounded set like this). Our conjecture will be -no. On the other hand in L-like models it may exist due to morass structures inside.
Note also that once we have such S, then it is quite hard to eliminate it. Cardinals should be collapsed or change their cofinality.

2 The Preparation Forcing

We assume GCH. Fix two cardinals κ and θ such that $\kappa < \theta$ and θ is regular.

We define a set which is parallel to $\mathcal{P}''$ of Gap 3, i.e. the set of central lines.

Definition 2.1 The set $\mathcal{P}'''$ consists of sequences of the form $\langle C^\tau \mid \tau \in s \rangle$ such that

1. s is a closed set of cardinals from the interval $[\kappa^+, \theta]$ satisfying the following:
   (a) $|s \cap \delta| < \delta$ for each inaccessible $\delta \in [\kappa^+, \theta]$
   (b) $\kappa^+, \theta \in s$
   (c) if $\rho^+ \in s$ and $\rho \geq \kappa^+$, then $\rho \in s$
   (d) if $\rho \in s$ is singular, then s is unbounded in ρ and $\rho^+ \in s$.

   If there is no inaccessible cardinals inside the interval $[\kappa^+, \theta]$, then s can be taken to be the set of all the cardinals of this interval.

2. For every $\tau \in s$, C^τ is a continuous closed chain of a length less than τ^+ of elementary submodels of $\langle H(\theta^+), \in, <, \subseteq, \kappa\rangle$ each of cardinality τ

   such that

   (a) for each element $X \in C^\tau$ we have $X \cap \tau^+ \in On$ and, hence $X \supseteq \tau$,
   Further we shall denote $\mathrm{otp}(X \cap \theta^+)$ by simply $\mathrm{otp}(X)$.

   (b) If $X \in C^\tau$ and there is $Y \in C^\rho, Y \supset X$, for some $\rho \in s \setminus \tau + 1$, then there is $Y \in C^{\tau^*}, Y \supset X$ such that for each $\rho \in s \setminus \tau + 1$ if $Z \in C^\rho$ and $Z \supset X$, then $Z \supseteq Y$, where $\tau^* = \min(s \setminus \tau + 1)$.

   (c) If X is a non-limit element of the chain C^τ then

      i. $C^\tau \restriction X := \{Y \mid Y \subset X, Y \in C^\tau\} \in X$,

      ii. ${}^{\mathrm{cof}(\tau)>}X \subseteq X$,

      iii. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^\rho$ with $\sup(Y) \geq \sup(X)$, then $X \subseteq Y$,

      iv. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^\rho$ with $\sup(Y) < \sup(X)$, then there are $\rho' \in (s \setminus \rho) \cap X$ and $Y' \in C^{\rho'} \cap X$ such that $Y' \supseteq Y$ and $Y \cap X = Y' \cap X$.
      Note that $\rho' = \rho$, unless there are inaccessible cardinals.

      v. If $\xi \in (s \setminus \tau + 1) \cap X$ and $C^\xi \cap X \neq \emptyset$, then
      $$\bigcup\{Y \in C^\xi \mid Y \in X\} \in X.$$
      Denote this union by $(X)_\xi$.
      Note that if for some $\tau \in s, \xi \in s \cap \tau$ and $Z \in C^\tau$ there is no $\rho \in s \setminus \tau, A \in C^\xi$ with $(A)_\rho$ defined and so that $Z \subseteq (A)_\rho$, then $Z \supseteq B$ for each $B \in C^\xi$. Since, if for some $B \in C^\xi$ we have $\sup(Z \cap \theta^+) < \sup(B \cap \theta^+)$, then, by the condition (iv) above, there are $\rho \in s \setminus \tau, Y \in C^\rho \cap B$ such that $Z \subseteq Y$ and $Z \cap B = Y \cap B$. So, $(B)_\rho$ exists and $Z \subseteq (B)_\rho$.

      vi. $\langle C^\xi \cap (X)_\xi \mid \xi \in s \setminus \tau + 1, (X)_\xi \text{ is defined } \rangle \in X$. *?It implies the previous one.*

3. If $\langle \xi_j \mid j < i\rangle$ is an increasing sequence of elements of s, $\xi = \bigcup_{j<i} \xi_j$ and $\langle X_j \mid j < i\rangle$ is an increasing (under the inclusion) sequence such that $X_j \in C^{\xi_j}$ for each $j < i$, then $X = \bigcup_{j<i} X_j$ is in C^ξ.

The next set will be needed here in order to define a Δ-system type triple.

Definition 2.2 The set $\mathcal{P}''$ consists of all sequences of triples

$$\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$$

such that for every $\tau \in s$ the following hold:

1. $|A^{1\tau}| \leq \tau$,
2. $A^{0\tau} \in A^{1\tau}$,
3. every $X \in A^{1\tau}$ is either equal to $A^{0\tau}$ or belongs to it,
4. $C^{\tau} : A^{1\tau} \to P(A^{1\tau})$,
5. $\langle C^{\tau}(A^{0\tau}) \mid \tau \in s\rangle \in \mathcal{P}'''$,
6. (Coherence)
   if $X, Y \in C^{\tau}(A^{0\tau})$ and $X \in C^{\tau}(Y)$, then $C^{\tau}(X)$ is an initial segment of $C^{\tau}(Y)$ with X being the largest element of it.
7. Let $B \in C^{\tau}(A^{0\tau})$ and $s' = \{\rho \in s \cap \tau \mid \exists X \in C^{\rho}(A^{0\rho}) \quad X \subseteq B\}$. For each $\rho \in s'$ let B_{ρ} be the largest element of $C^{\rho}(A^{0\rho})$ contained in B. Then

$$\langle C^{\rho}(B_{\rho}) \mid \rho \in s'\rangle^{\frown}\langle C^{\tau}(B)\rangle^{\frown}\langle C^{\xi}(A^{0\xi}) \mid \xi \in s \setminus \tau + 1\rangle \in \mathcal{P}'''.$$

Now we define Δ-system type triples. The definition is more involved than those in the gap 3 case. The basic reason is that instead of using a single central line consisting of ordinals there, we may have here many other central lines. Over each of them Δ-system type triple may appear (thus, for example for the gap 4: there will be Δ-system type triples for κ^{+} relatively to lines of models of cardinality κ^{++}, and those of cardinality κ^{++} relatively to lines of cardinality κ^{+3}, i.e. ordinals). We define simultaneously also switching using the induction on the rank of sets.

Definition 2.3 Suppose that $p = \langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle \in \mathcal{P}''$, $F \in C^{\tau}(A^{0\tau})$, for some $\tau \in s, \tau < \theta$ and $F_0, F_1 \in F$. We say that the triple F_0, F_1, F is of Δ-*system type* iff

1. F_0 is the immediate predecessor of F in $C^{\tau}(A^{0\tau})$
2. $F_1 \prec F$,
3. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^{\rho}(A^{0\rho})$ with $\sup(Y) \geq \sup(F_1)$, then $F_1 \subseteq Y$,

4. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^{\rho}(A^{0\rho})$ with $\sup(Y) < \sup(F_1)$, then there are $\rho' \in (s \setminus \rho) \cap F_1$ and $Y' \in C^{\rho'}(A^{0\rho'}) \cap F_1$ such that $Y' \supseteq Y$ and $Y \cap F_1 = Y' \cap F_1$.
   Here we need to consider two possibilities: $\tau^+ \in s$ or $\tau^+ \notin s$ and then $\min(s \setminus \tau + 1)$ is an inaccessible cardinal. Let shall treat both possibilities similar. Denote $\min(s \setminus \tau + 1)$ by τ^*. So τ^* is either τ^+ or τ^* is an inaccessible.

5. There is $H_i \in A^{1\tau^*} \cap F_i$ which the maximal under inclusion, where $i \in \{0, 1\}$. Moreover $H_0 \in C^{\tau^*}(A^{0\tau^*})$.
   Note that we do not require that also H_1 is in $C^{\tau^*}(A^{0\tau^*})$. The reason is that, already in the gap 4 case, H_1 may correspond to some $H_1' \in C^{\tau^*}(A^{0\tau^*})$ as a Δ-system triple, but $F_1, \pi_{H_1', H_1}(F_0)$ are not of a Δ-system type.

6. There are $G_0, G_1 \in A^{1\tau^*} \cap F$ such that

   (a) $\mathrm{cof}(G_0 \cap (\tau^*)^+) = \mathrm{cof}(G_1 \cap (\tau^*)^+) = \tau^*$,

   (b) $G_0 \in F_0$ and $G_1 \in F_1$

   (c) $F_0 \cap F_1 = F_0 \cap G_0 = F_1 \cap G_1$,

   (d) either $G_0 \in G_1$ or $G_1 \in G_0$,

   (e) there is a switch of $p \setminus \tau + 1 := \langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \setminus \tau + 1 \rangle$ which involves models only with supremums below $\max(\sup(F_0 \cap \theta^+), \sup(F_1 \cap \theta^+))$ which leaves H_0 on the central line for τ^* and moves H_1, G_0, G_1 to the central line. Moreover, all the models involved in the switch are in F.
   Here we use the induction on the ranks of sets.

   Further let us call G_0, G_1 the witnessing models for F_0, F_1, F.
   ? May be add also H_0, H_1 and the models used in the switch.

   The next condition will require more similarity:

7. (isomorphism condition)
   the structures

   $$\langle F_0, \in, <, \subseteq, \kappa, \tau, C^{\tau}(F_0), \langle A^{1\rho} \cap F_0 \mid \rho \in (s \setminus \tau) \cap F_0 \rangle, \langle C^{\rho} \restriction A^{1\rho} \cap F_0 \mid \rho \in s \setminus \tau \rangle, f_{F_0} \rangle$$

   and

   $$\langle F_1, \in, <, \subseteq, \kappa, \tau, C^{\tau}(F_1), \langle A^{1\rho} \cap F_1 \mid \rho \in (s \setminus \tau) \cap F_1 \rangle, \langle C^{\rho} \restriction A^{1\rho} \cap F_1 \mid \rho \in s \setminus \tau \rangle, f_{F_1} \rangle$$

are isomorphic over $F_0 \cap F_1$, i.e. the isomorphism $\pi_{F_0 F_1}$ between them is the identity on $F_0 \cap F_1$, where $f_{F_0} : \tau \longleftrightarrow F_0$, $f_{F_1} : \tau \longleftrightarrow F_1$ are some fixed in advance bijections.

In particular, we will have that $\mathrm{otp}(F_0) = \mathrm{otp}(F_1)$ and $F_0 \cap \tau^* = F_1 \cap \tau^*$.
Note that here we use $C^\rho \restriction A^{1\rho} \cap F_i$ $(i < 2)$. In the gap 3 case we had only $A^{1\kappa^{++}}$, but it was just an increasing sequence and so served as a replacement of $C^{\kappa^{++}}$ as well.

8. ?For each $\xi \in s$, if $X \in A^{1\xi}$ (?or $X \in C^\xi(A^{0\xi})$) and $X \supseteq F_0, F_1$, then $X \supseteq F$.

Define the switch q of p by F_0, F_1, F to be

$$\langle \langle A^{0\xi}, A^{1\xi}, D^\xi \rangle \mid \xi \in s \rangle,$$

where D^ξ, for $\xi \in s \setminus \tau + 1$ is determined by switching in $p \setminus \tau + 1$ below $\max(\sup(F_0 \cap \theta^+), \sup(F_1 \cap \theta^+))$ which turns $C^{\tau^*}(H_1)$ into an initial segment of τ^*-central line. $D^\tau(F) = C^\tau(F_1)^\frown F$ and $D^\tau(A^{0\tau}) = D^\tau(F)^\frown \langle X \in C^\tau(A^{0\tau}) \mid X \supset F \rangle$. The rest is defined in the obvious fashion by taking images under isomorphisms π_{F_0,F_1} etc.
Further let denote such q by $swt(p, F)$.
Denote by $swt(p, B_1, \ldots, B_n)$ the result of an application of the switch operation n-times: $p_{i+1} = swt(p_i, B_i)$, for each $1 \leq i \leq n$, where $p_1 = p$ and $swt(p, B_1, \ldots, B_n) = p_{n+1}$.

Note that there is no Δ-system type triples in the cardinality θ.

Now we define the preparation forcing $\mathcal{P}'$.

Definition 2.4 The set $\mathcal{P}'$ consists of elements of the form

$$\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$$

so that the following hold:

1. $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle \in \mathcal{P}''$,

   We call $C^\tau(A^{0\tau})$ *τ-central line* of $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$.
   The following conditions describe a special way in which $A^{1\tau}$ is generated from the central line, for each $\tau \in s$.

2. Let $B \in A^{1\tau}$. Then $B \in C^\tau(A^{0\tau})$ (i.e. it is on the central line) or there there is a finite sequence $w(B)$ of models in $\bigcup_{\rho \in s \setminus \tau} A^{1\rho}$ that terminates with B. We call this sequence a walk to B and define it recursively as follows.

If $B \in C^\tau(A^{0\tau})$, then $w(B) = \langle B \rangle$. If $B \notin C^\tau(A^{0\tau})$, then pick the least element $A \in C^\tau(A^{0\tau})$ with $B \in A$. It will be the first element of the walk to B.
In general, suppose that the walk $w(B)$ reaches a point A in $A^{1\tau}$ and $B \notin C^\tau(A)$. The following possible continuations are allowed. The walk to B terminates once B is reached.

First Continuation.

There are models $A_0, A_1 \in A \cap A^{1\tau}$ such that

(a) the triple A_0, A_1, A is of a Δ-system type with respect to $\langle \langle A^{0\xi}, A^{1\xi}, C^\xi \rangle \mid \xi \in s \backslash \tau \rangle$,

(b) $A_0 \in C^\tau(A)$,

(c) $B \in A_1 \cup \{A_1\}$.

Then we add A_0, A_1 and the models witnessing the Δ system to $w(B)$.
The walk continues from A_1.

Second Continuation.

There are $\rho \in s \cap A, \rho > \tau$ and $F_0, F_1, F \in A^{1\rho} \cap A$ so that

(a) F_0, F are on the central line relatively to A, i.e. once we make the switches along the walk up to A which move A to the central line, then F_0, F move their as well; other way to state this: if Z is the largest model of $A^{1\rho} \cap A$, then $F_0, F \in C^\rho(Z)$. In particular, if A is the first model of the walk or only the first continuation was used on the way to A, then $F_0, F \in C^\rho(A^{0\rho})$.

(b) the triple F_0, F_1, F is of a Δ-system type with respect to $\langle \langle A^{0\xi}, A^{1\xi}, C^\xi \rangle \mid \xi \in s \backslash \rho \rangle$ with witnessing pair of models G_0, G_1 in A,

(c) there is no $\eta \in s \setminus \tau$ and $Z \in A^{1\eta}$ such that $F \in Z \in A$.
This condition insures a kind of minimality of A above F.

(d) $A^- \in F_0$ and $B \subseteq \pi_{F_0,F_1}[A^-]$, where A^- denotes the immediate predecessor of A in $C^\tau(A)$.

We add then F_0, F_1, F, models witnessing the Δ-system, A^- and $\pi_{F_{10},F_{11}}[A^-]$ to $w(B)$.
The walk continues from $\pi_{F_{10},F_{11}}[A^-]$.
In this case we use directly F_0, F_1 to move a model A^- from $C^\tau(A)$ to one that contains B. In other words a switch is preformed using models of cardinality above τ.

If one does not care about GCH, then there is no need in additional possibility. The further arguments work parallel to the gap 3 case. But already for the gap 4 (i.e. if $\theta = \kappa^{+3}$), we will have $2^{\kappa^{++}} = \kappa^{+4}$ in a generic extension by $\mathcal{P}'$.
Let us allow further possibilities in order to preserve GCH.

Third Continuation.

There are $\rho \in s \cap A, \rho > \tau, F_0, F_1, F \in A^{1\rho} \cap A,\ A_0, A_0', A_1 \in A \cap A^{1\tau}$ so that

(a) $F_0, F_1, F \in A_1$,

(b) F is on the central line relatively to A_1, i.e. once we make the switches along the walk up to A which move A to the central line, then F moves their as well; other way to state this: if Z is the largest model of $A^{1\rho} \cap A_1$, then $F \in C^\rho(Z)$. In particular, if A is the first model of the walk or only the first continuation was used on the way to A, then $F \in C^\rho(A^{0\rho})$.

(c) the triple F_0, F_1, F is of a Δ-system type with respect to $\langle\langle A^{0\xi}, A^{1\xi}, C^\xi\rangle \mid \xi \in s \setminus \rho\rangle$ with witnessing pair of models G_0, G_1 in A,

(d) A_0, A_1, A is of a Δ-system type,

(e) $A_0 \cap A_1 = A_1 \cap F_0$, i.e. F_0 is one of the Δ-system witnesses.

(f) $A_0' = \pi_{F_0,F_1}(A_0)$,

(g) for every $Z \in C^\tau(A_1)$ either $F_0, F_1, F \in Z$ or $Z \in F_0$ (and then in $A_0 \cap A_1$),

(h) if $M \in C^\tau(A_1)$ is the least with $F \in M$, then there is no $\eta \in s \setminus \tau$ and $Z \in A^{1\eta}$ such that $F \in Z \in M$,

(i) $B \subseteq A_0' \setminus (A_0 \cup A_1)$.

We add then $F_0, F_1, F, A_0, A_0', A_1$, models witnessing the Δ-system to $w(B)$.
The walk continues from A_0'.

Further we shall refer to models A_0, A_1 of the first continuation, A^- of the second and A_0, A_0', A_1 of the third as the immediate predecessors of A (?probably better: *true immediate predecessors*). There may be other $\in$-immediate predecessors of A which can be generated in the last case below, but the most important will be the described above.

Fourth Continuation.

There are $A_0, A_1 \in A^{1\tau} \cap A, \rho \in s \cap A, \rho > \tau,\ T_0, T_1, T \in A^{1\rho} \cap A$ such that

(a) A_0, A_1, A are of a Δ-system type,

(b) A_1 is above A_0 in the Δ-system, i.e. if $F_0 \in A_0, F_1 \in A_1$ are the witnessing models, then $F_0 \in F_1$ and so $F_1 \supseteq A_0$.

(c) T_0, T_1, T are of a Δ-system type,

(d) $T_0, T_1, T \in A_1$,

(e) T_0, T are on the central line relatively to A_1, i.e. once we make the switches along the walk up to A_1 which move A_1 to the central line, then T_0, T move their as well; other way to state this: if Z is the largest model of $A^{1\rho} \cap A_1$, then $T_0, T \in C^\rho(Z)$. In particular, if A is the first model of the walk or only the first continuation was used on the way to A, then $F \in C^\rho(A^{0\rho})$.

(f) for every $Z \in C^\tau(A_1)$ either $T_0, T_1, T \in Z$ or $Z \in T_0$,

(g) if $M \in C^\tau(A_1)$ is the least with $F \in M$, then there is no $\eta \in s \setminus \tau$ and $Z \in A^{1\eta}$ such that $F \in Z \in M$,

(h) $A_0 \in T_0$,

(i) $B \subseteq \pi_{T_0,T_1}(A_0)$.

We add all the relevant models above, i.e. $A_0, A_1, T_0, T_1, T, F_0, F_1, \pi_{T_0,T_1}(A_0)$ etc. to $w(B)$. Continue further from $\pi_{T_0,T_1}(A_0)$.

This case formally speaking includes the third one. Thus, for example, let $T_0 = F_0, T_1 = F_1$, for F's as in the third one and $B = \pi_{F_0,F_1}(A_0) = A_0'$. But note that here T's need not be the witnesses of A_0, A_1, A, also they may be of a large cardinality than those of the witnesses.

The next two conditions strengthen a bit the isomorphism condition (7) of Definition 2.3.

3. (isomorphism condition 1) Let $F_0, F_1, F \in A^{1\tau}$ be of a Δ-system type and $X \in A^{1\tau}$. Then $X \in F_0$ iff $\pi_{F_0F_1}[X] \in F_1 \cap A^{1\tau}$.

4. (isomorphism condition 2) Let $F_0, F_1, F \in A^{1\tau}$ be of a Δ-system type, $F_0, F \in C^\tau(A^{0\tau})$. If for some $\xi \in s \cap \tau$, $A^{1\xi} \cap (F_1 \setminus F_0) \neq \emptyset$, then $F \in A^{0\xi}$ and for each $X \in C^\xi(A^{0\xi})$ either $F_0, F_1, F \in X$ or $X \in F_0$.
We require the following for such ξ:

   - for every $Y \in A^{1\xi}$, $Y \in F_0$ iff $\pi_{F_0F_1}[Y] \in F_1 \cap A^{1\xi}$.

The above condition is a new strong requirement which restricts largely the number possibilities to move small models via Δ-system triples.

If one do not care about GCH, then we require the above only for Y's which are in the least $X \in C^\xi(A^{0\xi})$ with $F \in X$. We do not move the rest of Y's from F_0 to F_1. Just the lack of the third possibility in (2) prevents such moving. Here basically the place where GCH breaks. Thus F_0 and F_1 will have different sets of elements of $A^{1\xi}$ inside.

5. Let $F_0, F_1, F \in A^{1\tau}$ be of a Δ-system type, $F_0, F \in C^\tau(A^{0\tau})$. Suppose that $\xi \in s \cap \tau$, $(A^{0\xi})_\tau$ exists and $(A^{0\xi})_\tau \supseteq F_0$. Let $X \in C^\xi(A^{0\xi})$ be the least with $(X)_\tau \supseteq F_0$. Then $(X)_\tau \supseteq F$.

   The meaning of this condition is that it is impossible to have a small model in between models of a Δ-system type of larger cardinality. It will not be very restrictive for our further purposes, since we will be always able to increase first elements of $\mathcal{P}'$ by adding models of cardinality τ at the top, and only then to make a Δ-system type triple.

   The next condition is relevant once inaccessibles are present.

6. Let $F_0, F_1, F \in A^{1\tau}$ be of a Δ-system type, $F_0, F \in C^\tau(A^{0\tau})$. Suppose that $\xi \in s \cap \tau$, $X \in C^\xi(A^{0\xi})$, for some $\rho \in s \setminus \tau$, $(X)_\rho$ exists and $(X)_\rho \supseteq F_0$. Then $(X)_\rho \supseteq F$.

7. (uniqueness) Let $F_0, F_1, F_1', F \in A^{1\tau}$. If both triples F_0, F_1, F and F_0', F_1', F are of a Δ-system type, then $\{F_0, F_1\} = \{F_0', F_1'\}$.

   Note that conditions 3,4 and 7 can be stated equivalently only in the case when F is on the central line.

The following lemma follows directly from the definition.

Lemma 2.5 *Let $\langle\langle A^{0\xi}, A^{1\xi}, C^\xi\rangle \mid \xi \in s\rangle \in \mathcal{P}'$. Then $A^{1\theta}$ is a chain.*

Proof. Just note that we have no Δ-system triples in the cardinality θ. Hence each model in $A^{1\theta}$ is on the θ-central line, i.e. on $C^\theta(A^{0\theta})$.

□

Lemma 2.6 *Let $\langle\langle A^{0\xi}, A^{1\xi}, C^\xi\rangle \mid \xi \in s\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Then it is possible to move B to the κ^+-central line using finitely many switches.*

Proof. Consider the walk from $A^{0\kappa^+}$ to B. Use induction on its length and make switches to make it into the central line.

□

Let $p = \langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$ and $\eta \in s$. Set $p \setminus \eta = \langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s \setminus \eta\rangle$. Define $\mathcal{P}'_{\geq\eta}$ to be the set of all $p \setminus \eta$ for $p \in \mathcal{P}'$.

The next lemma is similar to Lemma 2.6.

Lemma 2.7 *Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'_{\geq\eta}$ and $B \in A^{1\eta}$. Then it is possible to move B to the η-central line using finitely many switches.*

Lemma 2.8 *Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$ and $B, B' \in A^{1\tau}$, for some $\tau \in s$. If $B' \subsetneq B$, then $B' \in B$.*

Proof. If both B and B' are on the central line, then we are done, by Definition 2.1. Suppose that it is not the case. Consider the walks from $A^{0\tau}$ to B and to B'. Let $A \in A^{1\tau}$ be the last common point of this walks. We need to consider three cases according to the possibilities in (2) of 2.4.
Case 1. There is $B_1 \in A^{1\tau}$ such that A^-, B_1, A is a Δ-system type triple and the walk to B' goes via A^-, the walk to B via B_1.

Note that it is impossible that the walk to B goes via A^- and those to B' via B_1, since $B' \subseteq B$.
Then $B' \subseteq A^- \cap B_1$. So we can replace B by $\pi_{B_1,A^-}[B]$ and move everything below A^-. Note that $\pi_{B_1,A^-} \restriction A^- \cap B_1 = id$, since the triple A^-, B_1, A is of a Δ-system type. Now the walks are simpler, so an induction applies. Hence $B' \in \pi_{B_1,A^-}[B]$. Moving back, we obtain $B' \in B$.

Suppose now that the case (b) of Definition 2.4(2 occurs. Then there are $\rho \in s \cap A, \rho > \tau$ and $F_0, F_1, F \in A^{1\rho} \cap A$ so that

- $F_0, F \in C^{\rho_1}(A^{0\rho})$,
- the triple F_0, F_1, F is of a Δ-system type with respect to $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s \setminus \rho_1\rangle$ with witnessing pair of models G_0, G_1 in A,
- if $Z \in C^{\tau}(A)$, then either $Z \in F_0$ or $F_0, F_1, F \in Z$, as well as the witnessing models for them.

Case 2. $A^- \in F_0$, $B' \subseteq A^-$ and $B \subseteq \pi_{F_0,F_1}[A^-]$.
Then $B' \subseteq F_0 \cap F_1$. So we can replace B by $\pi_{F_1,F_0}[B]$ and move everything below A^-. Note that by Definition 2.4 (4), $\pi_{F_1,F_0}[B] \in A^{1\tau}$. Also, $\pi_{F_1,F_1} \restriction F_0 \cap F_1 = id$, since the triple

F_0, F_1, F is of a Δ-system type. Now the walks are simpler, so an induction applies. Hence $B' \in \pi_{F_1,F_0}[B]$. Moving back, we obtain $B' \in B$.

Case 3. There is triple $Y_0, Y_1, Y \in A^{1\tau}$ of a Δ-system type with $Y_0, Y \in C^\tau(A^{0\tau})$, $A \in C^\tau(Y_0)$, $Y_1 \in F_0$, $B' \subseteq A^-$,$B \not\subseteq \pi_{Y_0,Y_1}[A]$ and $B \subseteq \pi_{F_0,F_1}(\pi_{Y_0,Y_1}[A])$.

Denote $A_1 = \pi_{Y_0,Y_1}[A]$ and $A_2 = \pi_{F_0,F_1}[A_1]$. Note that $A_2 \in A^{1\tau}$, by 2.4 and $\pi_{A_1,A_2} = \pi_{F_0,F_1} \restriction A_1$. Hence $\pi_{A_1,A_2} \restriction A_1 \cap A_2 = id$, but $\pi_{A,A_2} \restriction A \cap A_2$ need not be the identity.

Consider $E = \pi_{A_2,A_1}[B], E' = \pi_{A_2,A_1}[B']$ and $S = \pi_{A_1,A}[E], S' = \pi_{A_1,A}[E']$. Then $S, S' \in A^{1\tau} \cap A$, $S \supsetneq S'$, and so the induction applies. Hence $S' \in S$. This implies $E' \in E$, and then also $B' \in B$.

Case 4. There is triple $Y_0, Y_1, Y \in A^{1\tau}$ of a Δ-system type with $Y_0, Y \in C^\tau(A^{0\tau})$, $A \in C^\tau(Y_0)$, $Y_1 \in F_0$, $B = A$ and $B' \subseteq \pi_{F_0,F_1}(\pi_{Y_0,Y_1}[A])$.

Then, as in the previous case, denote $A_1 = \pi_{Y_0,Y_1}[A]$ and $A_2 = \pi_{F_0,F_1}[A_1]$.

The walk to B' continues via A_2. But $A_2 \in F_1 \in A$. Hence the rank of one of the sets is reduced here and we can argue by induction that $B' \in A_2$.

Consider $A \cap A_2$. Clearly, $B' \subset A \cap A_2$. Let us argue that $B' \in A$. There are $X, X_1 \in C^{\tau^*}(A^{0\tau^*})$ such that $X \in A, X_1 \in A_1$ witnessing a Δ-system type, where $\tau^* = \min(s \setminus \tau+1)$. Clearly, $\tau^* \leq \rho$. Then $X \in F_0$. Just otherwise, by Definitions 2.1, 2.4 we must have $F_0 \in X$, but then $F_0 \in A \cap F_0 = A \cap A_1$. Which is impossible, since $A_1 \in F_0$. Clearly also $X_1 \in F_0$, since $X_1 \in A_1 \in F_0$. Hence $\pi_{F_0,F_1}[X], \pi_{F_0,F_1}[X_1]$ are defined. Note that $\pi_{F_0,F_1}[X] \in A$, since $F_0, F_1 \in A$. Also $\pi_{F_0,F_1}[X_1] \in A_2$, since $\pi_{F_0,F_1}[X_1] = \pi_{A_1,A_2}[X_1]$. Let us show the following:

Claim 1 $A \cap A_2 = A \cap \pi_{F_0,F_1}[X] = A_2 \cap \pi_{F_0,F_1}[X_1]$.

Proof. Let $a \in A \cap A_2$. Then $b = \pi_{F_1,F_0}[a] \in A \cap A_1$. So, $b \in A \cap X$ and $b \in A_1 \cap X_1$. Then $a = \pi_{F_0,F_1}(b) \in A \cap \pi_{F_0,F_1}[X]$ and $a \in \pi_{F_0,F_1}[X_1]$. We use here that $\pi_{F_0,F_1} \in A$.

Let us show the opposite inclusions. Assume first that we have $a \in A \cap \pi_{F_0,F_1}[X]$. Let $b = \pi_{F_1,F_0}(a)$. Then $b \in A \cap X$, since $\pi_{F_0,F_1} \in A$. But $A \cap X = A_1 \cap X_1$. Hence, $b \in A_1 \cap X_1$, and so $a \in A_2 \cap \pi_{F_0,F_1}[X_1] \cap A$.

Let now $a \in A_2 \cap \pi_{F_0,F_1}[X_1]$. Then $b = \pi_{F_1,F_0}(a) \in A_1 \cap X_1$, since $\pi_{F_0,F_1} \restriction A_1 = \pi_{A_1,A_2}$. But $A_1 \cap X_1 = A \cap X$. Hence $b \in A$. This implies $a \in A$ since $\pi_{F_0,F_1} \in A$.

$\square$ of the claim.

Now we have $B' \in A_2$ and $B' \subset A \cap A_2 = A_2 \cap \pi_{F_0,F_1}[X_1]$. But $|B'| = \tau$, so $B' \in \pi_{F_0,F_1}[X_1]$. Then $B' \in A_2 \cap \pi_{F_0,F_1}[X_1] \subseteq A$ and we are done.

$\square$

Lemma 2.9 *Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$ and $B \in A^{1\tau}$, for some $\tau \in s$. Then $\langle\langle B, A^{1\tau}(B), C^{\tau} \restriction A^{1\tau}(B)\rangle^\frown\langle\langle A^{0\xi}, A^{1\xi}, D^{\xi}\rangle \mid \xi \in s \setminus \tau+1\rangle \in \mathcal{P}'_{\geq\tau}$, where $A^{1\tau}(B) = \{B' \in A^{1\tau} \cap \mathcal{P}(B) \mid$ there is a walk from B to $B'\}$ are D^{ξ}'s are the result of moving B to the τ-central line.*

Remark 2.10 Note that in view of the last case of Lemma 2.8, we cannot in general replace $A^{1\tau}(B)$ by $A^{1\tau} \cap \mathcal{P}(B)$.
Let us give a concrete example. Let $|A| = \kappa^+$, A^- exists $F_0, F_1, F \in A$ of cardinality κ^{++} of a Δ-system type with witnessing models G_0, G_1. Assume that $A^- \in F_0$ and $G_0 \in A^-$. Reflect A to F_0, i.e. find some $A_1 \in F_0$ which is isomorphic to A over $A \cap F_0$. Let A^* be a model of cardinality κ^+ with $A, A_1 \in A^*$ and set $C^{\kappa^+}(A^*) = \{A, A^*\}$. Then the triple A, A_1, A^* is of a Δ-system type. Set $A_2 = \pi_{F_0,F_1}(A_1)$ and $B = \pi_{F_0,F_1}(A^-)$. Then $B = A_2^-$. Also, $B \in A$, since $\pi_{F_0,F_1} \in A$. But $B \neq A^-$, since $G_0 \in A^-$ and $\pi_{F_0,F_1}(G_0) = G_1 \in B \cap F_1 \setminus F_0$.

Lemma 2.11 *Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$ and $X \in A^{1\tau}$, for some $\tau \in s$ and $Y \in A^{1\theta}$. Then*

1. $\sup(Y \cap \theta^+) \geq \sup(X \cap \theta^+)$ *implies* $Y \supseteq X$,

2. $\sup(Y \cap \theta^+) < \sup(X \cap \theta^+)$ *implies that there is* $Z \in A^{1\theta} \cap X$ *such that* $X \cap Y = X \cap Z$.

Remark 2.12 Note that the lemma will not be true in general if we replace the requirement $Y \in A^{1\theta}$ by $Y \in C^{\rho}(A^{0\rho})$, for some $\rho \in s \setminus \tau+1, \rho < \theta$. Thus, there may be a model $Y' \in A^{1\rho}$, $Y' \supset X$ which was switched to Y in a Δ-system type such that $\sup(Y' \cap \theta^+) < \sup(Y \cap \theta^+)$ and $X \not\subseteq Y \cap Y'$.

Proof. (1) We have a well order $<$ of $H(\theta^+)$ in the language and X is an elementary submodel. So it is possible to reconstruct X from its ordinals i.e. from $X \cap \theta^+$. Recall that $Y \cap \theta^+ \in \theta^+$. Hence, $Y \cap \theta^+ \supset X \cap \theta^+$ and we are done.
(2)Induction on the walk from $A^{0\tau}$ to X. Thus, if $X \in C^{\tau}(A^{0\tau})$, then the statement follows by Definition 2.1. The inductive step follows from Definition 2.4 treating each of the three possibilities there separately.
□

Further we will need to use more complicate inductions than on walks distances. Similar to Gap 3, we will define a notion of walks complexity. In order to do so we need first to define walks from $A^{0\tau}$ to elements of $A^{0\tau} \cap A^{1\rho}$, for $\rho \in s \setminus \tau + 1$. It corresponds to walks to ordinals in the gap 3 case. The definition repeats basically (2) of Definition 2.4.

Definition 2.13 (Complexity of walks)
Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$.

- Suppose that $\tau \in s$, $A, B \in A^{1\tau}$. We say that the walk from $A^{0\tau}$ to A is *simpler* than the walk from $A^{0\tau}$ to B iff

  1. $A \subset B$, or
  2. $A \not\subset B, B \not\subset A, A \neq B$ and if $L \in A^{1\tau}$ is the last common point of both walks, then $A \subseteq L^-$, where L^- is the immediate predecessor of L in $C^\tau(L)$. Note that necessarily, there is a triple of a Δ-system type F_0, F_1, F and $B \subseteq F_1$. In the gap 3 case we had $F_0 = L^-, F = L$ but here $F's$ can be models of bigger cardinality.

- Suppose that $\rho \in s \setminus \tau + 1$,$A \in A^{1\tau}$ and $B \in A^{1\rho} \cap A^{0\tau}$. We say that the walk from $A^{0\tau}$ to A is *simpler* than the walk from $A^{0\tau}$ to B iff

  1. A is one of the models of the walk to B,
     or
  2. if L is the last common model of the walks, then $A \in C^\tau(L)$, or $A \notin C^\tau(L)$ and $A \subseteq L^-$, where L^- is the immediate predecessor of L in $C^\tau(L)$. Note, if the second possibility occurs, then, necessarily, there is a triple of a Δ-system type F_0, F_1, F and $B \in F_1$.

- Suppose that $\mu, \rho \in s \setminus \tau + 1, A \in A^{1\mu} \cap A^{0\tau}$ and $B \in A^{1\rho} \cap A^{0\tau}$. We say that the walk from $A^{0\tau}$ to A is *simpler* than the walk from $A^{0\tau}$ to B iff $A \neq B$, there is $L \in A^{1\tau}$ which is the last common point of both walks and

  1. there are $D, E \in C^\tau(L)$ such that $A \in D \in E$ and $B \in E \setminus D$,
     or
  2. L is not the minimal model of $C^\tau(L)$ and $A \in L^-$.

The above defines a well-founded relation. We will use further the walks complexity in inductive arguments.

We need to allow a possibility to change the component C^τ in elements of $\mathcal{P}'$ and replace one central line by another. It is essential for the definition of an order on $\mathcal{P}'$ given below.

Definition 2.14 Let $r, q \in \mathcal{P}'$. Then $r \geq q$ (r is stronger than q) iff there is $p = swt(r, B_1, \dots, B_n)$ for some $B_1, \dots, B_n$ appearing in r so that the following hold, where

$$p = \langle \langle A^{0\xi}, A^{1\xi}, C^{\xi} \rangle \mid \xi \in s \rangle$$

$$q = \langle \langle B^{0\xi}, B^{1\xi}, D^{\xi} \rangle \mid \xi \in s' \rangle$$

1. $s' \subseteq s$,
2. $B^{0\xi} \in C^{\xi}(A^{0\xi}),$ for each $\xi \in s'$,
3. $q = p \restriction \langle B^{0\xi} \mid \xi \in s' \rangle$,
   where $p \restriction \langle B^{0\xi} \mid \xi \in s' \rangle = \langle \langle B^{0\xi}, A^{1\xi} \cap (B^{0\xi} \cup \{B^{0\xi}\}), C^{\xi} \restriction (B^{0\xi} \cup \{B^{0\xi}\}) \rangle \mid \xi \in s' \rangle$,
4. for each $\xi \in s'$ and $X \in C^{\xi}(A^{0\xi}) \setminus C^{\xi}(B^{0\xi}) \quad q \in X$,
5. for each $\xi \in s \setminus s'$ and $X \in C^{\xi}(A^{0\xi}) \quad q \in X$.

The meaning of the last two conditions is that new models over central lines supposed to be above all old ones.

Let $p = \langle \langle A^{0\xi}, A^{1\xi}, C^{\xi} \rangle \mid \xi \in s \rangle \in \mathcal{P}'$ and $\eta \in s$. Set $p \setminus \eta = \langle \langle A^{0\xi}, A^{1\xi}, C^{\xi} \rangle \mid \xi \in s \setminus \eta \rangle$. Define $\mathcal{P}'_{\geq \eta}$ to be the set of all $p \setminus \eta$ for $p \in \mathcal{P}'$.

Lemma 2.15 *The function $p \rightarrowtail p \setminus \eta$ projects the forcing $\mathcal{P}'$ onto the forcing $\mathcal{P}'_{\geq \eta}$.*

Remark. Note that we split at η only p's in $\mathcal{P}'$ with η inside s of p. The reason is that in the case of $\eta \notin s$ an extension of $p \setminus \eta$ may include models of cardinality η which for example belong to models of p of cardinalities below η. Such extensions will be incompatible with p.

Proof. Let $p \in \mathcal{P}'$ and $q \in \mathcal{P}_{\geq \eta}, q \geq p \setminus \eta$. We need to find $r \in \mathcal{P}', r \geq p$ such that $r \setminus \eta \geq q$. Let us take an equivalent to q condition q' in $\mathcal{P}'_{\geq \eta}$ (a switching of q) with the central lines of q' extending those of $p \setminus \eta$. Then $p^\frown q'$ the combination of p with q' will be in $\mathcal{P}'$, $p^\frown q' \geq p$ and $(p^\frown q') \setminus \eta = q'$.
$\square$

Lemma 2.16 *$\mathcal{P}'_{\geq \eta}$ is η^+-strategically closed.*

Proof. We define a winning strategy for the player playing at even stages. Thus suppose $\langle p_j \mid j < i \rangle$, $p_j = \langle \langle A_j^{0\tau}, A_j^{1\tau}, C_j^{\tau} \rangle \mid \tau \in s_j \rangle$ is a play according to this strategy up to an even stage $i < \eta^+$.

Split into two cases.

Case 1. $i = j + 1$.

Let $p = \langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s = s_j\rangle$ be a switch of p_j which restores $A^{0\tau}_{j-1}$ to τ-th central line, i.e. $A^{0\tau}_{j-1} \in C^{\tau}(A^{0\tau})$, for every $\tau \in s_{j-1}$.

Then pick an increasing continuous sequence $\langle A^{0\tau}_i \mid \tau \in s\rangle$ such that for every $\tau \in s$

(a) ${}^{\mathrm{cof}(\tau)>}A^{0\tau}_i \subseteq A^{0\tau}_i$,

(b) $\langle p_k \mid k < i\rangle, p, \langle A^{0\tau'}_i \mid \tau' < \tau\rangle \in A^{0\tau}_i$.

Set $p_i = \langle\langle A^{0\tau}_i, A^{1\tau}_i, C^{\tau}_i\rangle \mid \tau \in s\rangle$, where

$$A^{1\tau}_i = A^{1\tau} \cup \{A^{0\tau}_i\}, C^{\tau}_i = C^{\tau}(A^{0\tau}) \cup \{\langle A^{0\tau}_i, C^{\tau}(A^{0\tau}) \cup \{A^{0\tau}_i\}\rangle\}.$$

Case 2. i is a limit ordinal.

Set first

$$s = \text{ the closure of } \bigcup_{j<i} s_j.$$

For every $\tau \in \bigcup_{j<i} s_j$, define

$$A^{0\tau}_i = \bigcup_{j<i} A^{0\tau}_j, A^{1\tau}_i = \bigcup_{j<i} A^{1\tau}_j \cup \{A^{0\tau}_i\},$$

$$C^{\tau}_i = \bigcup_{j<i, j \text{ is even}} C^{\tau}_j \cup \{\langle A^{0\tau}_i, \{A^{0\tau}_i\} \cup \bigcup \{C^{\tau}_j(A^{0\tau}_j) \mid \ j \text{ is even}\}\rangle\}.$$

If $\tau \in s \setminus \bigcup_{j<i} s_j$, then set

$$A^{0\tau}_i = \bigcup_{\tau' \in (\cup_{j<i} s_j) \cap \tau} A^{0\tau'}_i,$$

$$A^{1\tau}_i = \{A^{0\tau}_i\} \text{ and } C^{\tau}(A^{0\tau}_i) = \{\langle A^{0\tau}_i, \{A^{0\tau}_i\}\rangle\}.$$

As an inductive assumption we assume that at each even stage $j < i$, p_j was defined in the same fashion. Then $p_i = \langle A^{0\tau}_i, A^{1\tau}_i, C^{\tau}_i\rangle \mid \tau \in s\rangle$ will be a condition in $\mathcal{P}'$ stronger than each p_j for $j < i$.

□

If we take $\eta = \theta$, then it is easy to show the following:

Lemma 2.17 *$\langle \mathcal{P}'_{\geq\theta}, \leq \rangle$ is θ^+-closed.*

Let $p = \langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$ and $\eta \in s$. Set $p \restriction \eta = \langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s \cap \eta\rangle$.

Let $G(\mathcal{P}'_{\geq\eta})$ be a generic subset of $\mathcal{P}'_{\geq\eta}$. Define $\mathcal{P}'_{<\eta}$ to be the set of all $p \restriction \eta$ for $p \in \mathcal{P}'$ with $p \setminus \eta \in G(\mathcal{P}'_{\geq\eta})$.

Lemma 2.18 $\mathcal{P}' \simeq \mathcal{P}'_{\geq\eta} * \underset{\sim}{\mathcal{P}'}_{<\eta}$.

Lemma 2.19 *If η is a regular cardinal, then the forcing $\mathcal{P}'_{<\eta}$ satisfies η^+-c.c. in $V^{\mathcal{P}'_{\geq\eta}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \Vdash_{\mathcal{P}'_{\geq\eta}} (\langle \underset{\sim}{p}_\alpha = \langle\langle \underset{\sim}{A}^{0\tau}_\alpha, \underset{\sim}{A}^{1\tau}_\alpha, \underset{\sim}{C}^{\tau}_\alpha\rangle \mid \tau \in \underset{\sim}{s}_\alpha\rangle \mid \alpha < \eta^+\rangle \text{ is an antichain in } \underset{\sim}{\mathcal{P}'}_{<\eta})$$

Without loss of generality we can assume that each $A^{0\tau}_\alpha$ is forced to be a successor model, otherwise just extend conditions by adding one additional models on the top. Define by induction, using Lemma 2.16, an increasing sequence $\langle q_\alpha \mid \alpha < \eta^+\rangle$ of elements of $\mathcal{P}'_{\geq\eta}$ and a sequence $\langle p_\alpha \mid \alpha < \eta^+\rangle$, $p_\alpha = \langle\langle A^{0\tau}_\alpha, A^{1\tau}_\alpha, C^{\tau}_\alpha\rangle \mid \tau \in s_\alpha\rangle$ so that for every $\alpha < \eta^+$

$$q_\alpha \Vdash_{\mathcal{P}'_{\geq\eta}} \langle\langle \underset{\sim}{A}^{0\tau}_\alpha, \underset{\sim}{A}^{1\tau}_\alpha, \underset{\sim}{C}^{\tau}_\alpha\rangle \mid \tau \in \underset{\sim}{s}_\alpha\rangle = \check{p}_\alpha \ .$$

For a limit $\alpha < \eta^+$ let $\overline{q}_\alpha$ be an upper bound of $\{q_\beta \mid \beta < \alpha\}$, as defined in Lemma 2.16 and q_α be its extension deciding $\underset{\sim}{p}_\alpha$. Also assume that $p_\alpha \in A^{0\eta}(q_\alpha)$, where $A^{0\eta}(q_\alpha)$ is the maximal model of q_α of cardinality η.

Note that the number of possibilities for s_α's is at most η, since if η is an inaccessible, then by Definition 2.1(1), $|s_\alpha| < \eta$ and if η is an accessible cardinal, then $\eta = (\eta^-)^+$ (remember that η is a regular cardinal). So $s_\alpha \subseteq \eta^- \cup \{\eta^-\}$. But $2^{\eta^-} = \eta$.
Hence, by shrinking if necessary, we may assume that each $s_\alpha = s^*$, for some $s^* \subseteq \eta$. Let $\eta^* = \max(s^*)$.

Form a Δ-system. By shrinking if necessary assume that for some stationary $S \subseteq \eta^+$ we have the following for every $\alpha < \beta$ in S:

1. $A^{0\eta^*}_\alpha \cap A^{0\eta}(\overline{q}_\alpha) = A^{0\eta^*}_\beta \cap A^{0\eta}(\overline{q}_\beta) \in A^{0\eta}(q_0)$
2. $\langle A^{0\eta^*}_\alpha, \in, \leq, \subseteq, \kappa, C^{\eta^*}_\alpha, f_{A^{0\eta^*}_\alpha}, A^{1\eta^*}_\alpha, q_\alpha \cap A^{0\eta^*}_\alpha\rangle$ and $\langle A^{0\eta^*}_\beta, \in, \leq, \subseteq, \kappa, C^{\eta^*}_\beta, f_{A^{0\eta^*}_\beta}, A^{1\eta^*}_\beta, q_\beta \cap A^{0\eta^*}_\beta\rangle$ are isomorphic over $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta$, i.e. by isomorphism fixing every ordinal below $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta$, where

$$f_{A^{0\eta^*}_\alpha} : \eta^* \longleftrightarrow A^{0\eta^*}_\alpha$$

and

$$f_{A_\beta^{0\eta^*}} : \eta^* \longleftrightarrow A_\beta^{0\eta^*}$$

are the fixed enumerations.

Note that $|A_\alpha^{0\eta^*} \cap A_\beta^{0\eta^*}| \leq \eta^*$. So we can define a function $h_\alpha : \eta^* \to \eta$ by mapping each $i < \eta$ to the order type $A_\alpha^{0\eta^*} \cap \theta^+$ between the i-th element of $A_\alpha^{0\eta^*} \cap A_\beta^{0\eta^*} \cap \theta^+$ and its immediate successor in $A_\alpha^{0\eta^*} \cap A_\beta^{0\eta^*} \cap \theta^+$. The total number of such h_α's is at most η, hence by shrinking if necessary we will get the same function. This will insure the isomorphism which is the identity on $A_\alpha^{0\eta^*} \cap A_\beta^{0\eta^*} \cap \theta^+$ and, hence, on $A_\alpha^{0\eta^*} \cap A_\beta^{0\eta^*}$.

We claim that for $\alpha < \beta$ in S it is possible to extend q_β to a condition forcing compatibility of p_α and p_β. Proceed as follows. Pick A to be an elementary submodel of cardinality η^* with $p_\alpha, p_\beta, q_\beta$ inside.

Then the triple $A_\beta^{0\eta^*}, A_\alpha^{0\eta^*}, A$ is of a Δ-system type relatively to q_β, by (2) above. Use this to construct a condition stronger than both p_α, p_β.

Let $\langle A(\tau) \mid \tau \in s^* \cup s(q_\beta)\rangle$ (where $s(q_\beta)$ denotes the support of q_β) be an increasing and continuous sequence of elementary submodels such that for each $\tau \in s^*$ the following hold:

- $p_\alpha, p_\beta, q_\beta, A \in A(\tau)$,
- $|A(\tau)| = \tau$.

Extend q_β to q by adding to it $\langle A(\tau) \mid \tau \in s(q_\beta)\rangle$, as maximal models, i.e. $A^{0\tau}(q) = A(\tau)$. Set $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s^*\rangle$, where

$$A^{0\eta^*} = A(\eta^*), A^{1\eta^*} = A_\alpha^{1\eta^*} \cup A_\beta^{1\eta^*} \cup \{A, A^{0\eta^*}\},$$

$$C^{\eta^*} = C_\alpha^{\eta^*} \cup C_\beta^{\eta^*} \cup \langle A, C_\beta^{\eta^*}(A_\beta^{0\eta^*})^\frown A\rangle \cup \langle A^{0\eta^*}, C_\beta^{\eta^*}(A_\beta^{0\eta^*})^\frown A^\frown A^{0\eta^*}\rangle,$$

and for each $\tau \in s^* \cap \eta^*$,

$$A^{0\tau} = A(\tau), A^{1\tau} = A_\alpha^{1\tau} \cup A_\beta^{1\tau} \cup \{A^{0\tau}\},$$

$$C^\tau = C_\alpha^\tau \cup C_\beta^\tau \cup \langle A^{0\tau}, C_\beta^\tau(A_\beta^{0\tau})^\frown A^{0\tau}\rangle.$$

The triple $A_\beta^{0\eta^*}, A_\alpha^{0\eta^*}, A$ is of a Δ-system type relatively to q, by (2) above. It follows that $\langle p, q\rangle \in \mathcal{P}'$. Thus the condition (2) of Definition 2.4 holds since each of $\langle p_\alpha, q\rangle, \langle p_\beta, q\rangle$ satisfies it.

$\square$

Lemma 2.20 *Suppose that $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}'$, $\eta, \tau \in s, \tau \leq \eta$, $T_0, T_1, T \in A^{1\eta}, B \in A^{1\tau}$ are so that*

1. *$B \in T_0$,*
2. *T_0, T_1, T are of a Δ system type,*
3. *$T_0 \in C^\eta(T)$,*
4. *there is $M \in A^{1\tau}$ such that*
   (a) *$T_0, T_1, T \in M$,*
   (b) *there is no $\xi \in s \setminus \tau$ and $Z \in A^{1\xi}$ such that $T \in Z \in M$.*

Then $\pi_{T_0,T_1}(B) \in A^{1\tau}$.

Proof. Without loss of generality we can assume that T is on the η-th central line (just otherwise preform the necessary switches).
Consider the walk to B.
Claim. Models of cardinalities $\geq \eta$ are never used in this walk before entering T_0.
Proof. Suppose otherwise. Let $D \in A^{1\rho}$ be the first model used in the walk with $\rho \in s \setminus \eta$ and it is not in T. Note that the central lines for all the cardinalities $\geq \eta$ remain such up to the point when D is used. In particular both D and T remain on the central lines. But then necessarily $D \supset T$.

We deal with the case of Second Continuation of Definition 2.4.
Let $D_0, D_1 \in A^{1\rho}$ be such that $D_0 \in C^\rho(D)$ and the triple D_0, D_1, D is of a Δ-system type. By the definition of the walk there are $A \in A^{1\tau}$ with $D_0, D_1, D \in A$ and its immediate predecessor A^- in $C^\tau(A)$ such that $B \in A, B \notin A^-$ and $B \in \pi_{D_0,D_1}(A^-)$. But $T \subseteq D_0$ and $B \subseteq T$, hence $B \subseteq D_0 \cap D_1$. Then $\pi_{D_0,D_1}(B) = B$. So, $B \in A^-$. Contradiction.
The rest of the cases (Third and Forth Continuations) are similar. Thus we still will have $\pi_{D_1,D_0}(B) = B$ and then $B \in A_1$ implies $B \in A_0$ since $A_0 = \pi_{D_0,D_1}[A_1]$ and B does not move.
$\square$ of the claim.

Consider M and its immediate predecessor M^- in $C^\tau(M)$. If there are $A, A' \in A^{1\tau}$ on the walk to B such that $A', M \in C^\tau(A)$ and $A' \in M^- \cup \{M^-\}$, then both $\pi_{T_0,T_1}(A')$ and $\pi_{T_0,T_1}(B)$ will be in $A^{1\tau}$, since $\pi_{T_0,T_1}(M^-) \in A^{1\tau}$, by Definition 2.4 (Second Continuation) and so images by π_{T_0,T_1} of the models of cardinality τ of walks from M^- will be in $A^{1\tau}$.

Suppose now that the walks to B and to M split. Let $A \in A^{1\tau}$ be the last common point of the walks. Then A is a splitting point. There are A_0, A_1 its immediate predecessors with $B \in A_0 \cup \{A_0\}, M \in A_1 \cup \{A_1\}$.
Assume that we are here in the case of First Continuation of Definition 2.4. Under our assumptions it will be the only possibility once dealing with Gap 4. We claim that then $A_0 \in T_0$. Thus let $F_0, F_1 \in A^{1\tau^*}$ be the witnesses for A_0, A_1, A, i.e. $F_0 \in A_0, F_1 \in A_1$ and $A_0 \cap A_1 = A_0 \cap F_0 = A_1 \cap A_1$. Note that $M \notin A_0$ implies $T \notin A_0$. Also $M \in A_1$ implies $T \in A_1$. Hence $T \notin F_1$. Then $F_1 \in T \cup \{T\}$. Now, either $F_0 \in F_1$ and then $A_0 \subseteq F_1$ or $F_1 \in F_0$ and then $A_1 \subseteq F_0$. In the former case we are done (just it is Forth Continuation of Definition 2.4). If the later case occurs then $T \in F_0$. Pick $S \in C^\eta(A^{0\eta})$ to be the least element with B inside. Then $S \in A_0$ (just we can make such a choice inside A). Remember that $B \in T_0$. Hence $S \in T_0$ (we cannot have $S = T_0$ since then T_0 and then also T will be in A_0). The above imply $S \in A_0 \cap A_1$, by the definition of a Δ-system triple (just all of the elements of central lines of A_0 are above those of A_1 except the common part which is an initial segment). If $\eta = \tau^*$ (which is true in Gap 4 case) then $A_0 \cap S = A_1 \cap S$ (just by Δ -system triples definition). In particular $B \in A_0$ which is impossible. Contradiction.
$\square$ Gap 4.

Example Gap 5.

The following example shows that if one wants to keep GCH in the extension, then already in the gap 5 case Continuation One-Four of Definition 2.4 do not suffice.

Let $\tau = \kappa^+, \rho = \kappa^{++}, \eta = \kappa^{+3}$.
Suppose we have a long continuous chain of models $\vec{T} = \langle T_\alpha \mid \alpha < \eta^+ \rangle$ of cardinality η. Suppose that each $T_{\alpha+1}$ splits into $T_{\alpha+1,0} = T_\alpha$ and $T_{\alpha+1,1}$. Let S be an element of this chain.
Let $\vec{F} = \langle F_\gamma \mid \gamma \leq \tilde{\gamma} \rangle$ be a continuous chain of models of cardinality ρ which are spread among T_α's, S belongs to some F_α and above first such α each $F_{\beta+1}$ splits into $F_{\beta+1,0} = F_\beta$ and $F_{\beta+1,1}$ which is in S.
Pick some A_0 of cardinality τ such that $S \in A_0$ and for some member $H_0 \in A_0$ of $\vec{F}$ with $S \in H_0$ we have reflection A_1 of A_0 into H_0.
Set $H_1 = \pi_{A_0,A_1}(H_0)$. Pick some T from $\vec{T}$ in A_1 such that $H_1 \in T_0$, where T_0, T_1 are the immediate predecessors of T in $\vec{T}$. We assume also that $T_0, T_1 \in A_0$. Pick a model $M \in A_0$ of cardinality τ with $T \in M$ and no elements of $\vec{T}, \vec{F}$ in between.
Pick some β with $H_0 \in F_{\beta,0} \in A_0$ and a model $B^* \in A_0$ with $H_0 \in B^* \in F_{\beta,0}$. Set $B = \pi_{F_{\beta,0},F_{\beta,1}}(B^*)$. Then $B \subset S \subseteq T_0$ and $B \notin A_1$, since $\pi_{F_{\beta,0},F_{\beta,1}}(H_0) \notin A_1$. Neither of

Continuations One-Four of Definition 2.4 can put $\pi_{T_0,T_1}(B)$ into $A^{1\tau}$.

Lemma 2.21 *Let η, $\kappa < \eta \leq \theta$, be a regular cardinal. Then in $V^{\mathcal{P}'}$ we have $2^\eta = \eta^+$.*

Proof. Fix $N \prec H((2^\lambda)^+)$, for λ large enough, such that $\mathcal{P}' \in N$, $|N| = \eta^+$ and ${}^\eta N \subseteq N$. We find $p^N_{\geq\eta^+} \in \mathcal{P}'_{\geq\eta^+}$ which is N-generic for $\mathcal{P}'_{\geq\eta^+}$, using η^{++}-strategic closure of $\mathcal{P}'_{\geq\eta^+}$. Let $G(\mathcal{P}'_{\geq\eta^+})$ be a generic subset of $\mathcal{P}'_{\geq\eta^+}$ with $p_{\geq\eta^+} \in G(\mathcal{P}'_{\geq\eta^+})$. Then, $N[p_{\geq\eta^+}] \prec V_\lambda[G(\mathcal{P}'_{\geq\eta^+})]$. By Lemma 2.19, $\mathcal{P}'_{<\eta^+}$ satisfies η^{++}-c.c in $V[G(\mathcal{P}'_{\geq\eta^+})]$. In particular, $\mathcal{P}_{=\eta}$ satisfies η^{++}-c.c. Let $G(\mathcal{P}'_{=\eta})$ be a generic subset of $\mathcal{P}_{=\eta}$ over $V[G(\mathcal{P}'_{\geq\eta^+})]$. Denote $N[p_{\geq\eta^+}]$ by N_1. Then $N_1[N_1 \cap G(\mathcal{P}'_{=\eta})] \prec V[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})]$, since each antichain for $\mathcal{P}'_{=\eta}$ has cardinality at most η^+. Hence, if it belongs to N_1 then it is also contained in N_1. Denote $N_1[N_1 \cap G(\mathcal{P}'_{=\eta})]$ by N_2.

Consider $\mathcal{P}'_{<\eta} \cap N_2$. Clearly this is a forcing of cardinality η^+. By Lemma 2.19, $\mathcal{P}'_{<\eta}$ satisfies η^+-c.c., so $\mathcal{P}'_{<\eta} \cap N_2$ is a nice suborder of $\mathcal{P}'_{<\eta}$. Thus, let $G \subseteq \mathcal{P}'_{<\eta}$ be generic over $V[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})]$ and $H = G \cap N_2$. Then H is $\mathcal{P}'_{<\eta} \cap N_2$ generic over $V[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})]$, since, if $A \subseteq \mathcal{P}'_{<\eta} \cap N_2$ is a maximal antichain, then A is a maximal antichain also in $\mathcal{P}'_{<\eta}$. This follows due to the fact that N_2 is an elementary submodel closed under η-sequences of its elements. Namely, $|A| \leq \eta$, so $A \in N_2$. Then

$$N_2 \models A \text{ is a maximal antichain in } \mathcal{P}'_{<\eta}.$$

Now, by elementarity, A is a maximal antichain in $\mathcal{P}'_{<\eta}$. So there is $p \in G \cap A$. Finally, $A \subseteq N_2$ implies that $p \in N_2$ and hence $p \in H$.

We claim that each subset of η in $V[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})][G]$ is already in $N_2[G]$. It is enough since $|N_2[G]| = |N| = \eta^+$.

Work in V. The construction below can be preformed above any condition of $\mathcal{P}'$ stronger than $p^N_{\geq\eta^+} \in \mathcal{P}'_{\geq\eta^+}$ (which is needed in order to preserve the elementarity of N in generic extensions). So, by density arguments, we will obtain the desired conclusion.

Let $\underset{\sim}{a}$ be a name of a function from η to 2. Define by induction (using the strategic closure of the forcings and η^+-c.c. of $\mathcal{P}'_{<\eta}$) sequences of ordinals

$$\langle \delta_\beta \mid \beta < \eta \rangle, \langle \gamma(\alpha, \beta) \mid \beta < \eta, \alpha < \delta_\beta \rangle$$

and sequences of conditions

$$\langle p_\beta(\alpha) \mid \alpha < \delta_\beta \rangle (\beta < \eta), \langle p(\beta) \mid \beta < \eta \rangle$$

such that

(1) for each $\beta < \eta$, $\delta_\beta < \eta^+$,

(2) for each $\beta < \eta$, $\langle p_\beta(\alpha)_{\geq \eta} \mid \alpha < \delta_\beta \rangle$ is increasing sequence of elements of $\mathcal{P}'_{\geq \eta}$ and $p(\beta)$ is its upper bound obtained as in the Strategic Closure Lemma 2.16,

(3) $p_0(0)_{\geq \eta^+} \geq p^N_{\geq \eta^+}$,

(4) the sequence $\langle p(\beta) \mid \beta < \eta \rangle$ is increasing,

(5) for each $\beta < \eta$ and $\alpha < \delta_\beta$, $p_\beta(\alpha) \Vdash \underset{\sim}{a}(\beta) = \gamma(\alpha, \beta)$,

(6) if for some $p \in \mathcal{P}'$ we have $p \setminus \eta \geq_{\mathcal{P}'_{\geq \eta}} p(\beta)_{\geq \eta}$, then there is $\alpha < \delta$ such that the conditions $p, p_\beta(\alpha)$ are compatible. (I.e. $\{p_\beta(\alpha)_{<\eta} \mid \alpha < \delta_\beta\}$ is a pre-dense set as forced by $p(\beta)_{\geq \eta}$).

Set $p(\eta)$ to be the upper bound of $\langle p(\beta) \mid \beta < \eta \rangle$ as in the Strategic Closure Lemma 2.16. Let L denotes the top model of cardinality η of $p(\eta)$, i.e. $A^{0\eta}(p(\eta))$. By the construction in 2.16, we have $\delta_\beta, p(\beta) \in L$ and $\gamma(\alpha, \beta), p_\beta(\alpha) \in L$, for each $\beta < \eta$ and $\alpha < \delta_\beta$. Alternatively, we can just extent the model L to one which includes this sequences. Extend L further if necessary to include $p(\eta)$ as an element.

Turn for a moment to a generic extension. Let $G(\mathcal{P}'_{\geq \eta^+})$ be a generic subset of $\mathcal{P}'_{\geq \eta^+}$ with $p(\eta) \setminus \eta^+ \in G(\mathcal{P}'_{\geq \eta^+})$. Pick $K \in N$ realizing the same type as those of L in $H(2^\lambda)[G(\mathcal{P}'_{\geq \eta^+})]$ over $N \cap L$. Note that $N \cap L$ is a subset of N of cardinality η and, hence, it is in N.

Let

$$\langle q(\beta) \mid \beta < \eta \rangle, \langle q_\beta(\alpha) \mid \alpha < \delta_\beta \rangle (\beta < \eta)$$

be the sequences corresponding to

$$\langle p_\beta(\alpha) \mid \alpha < \delta_\beta \rangle (\beta < \eta), \langle p(\beta) \mid \beta < \eta \rangle$$

and let $q(\eta)$ corresponds to $p(\eta)$. Note that $q(\beta) \setminus \eta^+, q_\beta(\alpha) \setminus \eta^+$ are in $G(\mathcal{P}'_{\geq \eta^+})$, since $p(\beta) \setminus \eta^+, p_\beta(\alpha) \setminus \eta^+$ are in $G(\mathcal{P}'_{\geq \eta^+})$. Then,

$$q(\beta) \setminus \eta^+, q_\beta(\alpha) \setminus \eta^+ \leq_{\mathcal{P}'_{\leq \eta^+}} p^N_{\geq \eta^+},$$

by the choice of $p^N_{\geq \eta^+}$ and since $p^N_{\geq \eta^+} \leq_{\mathcal{P}'_{\leq \eta^+}} p(\eta) \setminus \eta^+ \in G(\mathcal{P}'_{\geq \eta^+})$.

Combine now K, L into one condition making them a splitting point. Let M be a model of cardinality η such that $K, L \in M$. Then the triple L, K, M will be of a Δ-system type relatively to $p(\eta)^\frown L^\frown M$ (which is defined in the obvious fashion with $L \in C^\eta(M)$). Now, we add $q(\eta)^\frown K$ to $p(\eta)^\frown L^\frown M$ and turn this into condition in $\mathcal{P}'$, exactly the same way as it was done at the end of the proof of Lemma 2.19. Denote such condition by r.

Define a name $\underset{\sim}{b}$ of a subset of η to be

$$\{\langle q_\beta(\alpha), \gamma(\alpha,\beta)\rangle \mid \alpha < \delta_\beta, \beta < \eta\}.$$

Clearly, $\underset{\sim}{b}$ is in N.

Claim 2.21.1 $r \Vdash \underset{\sim}{a} = \underset{\sim}{b}$.

Proof. Let G be a generic subset of $\mathcal{P}'$ with $r \in G$. Then also $p(\eta)_{\geq\eta}, q(\eta)_{\geq\eta} \in G$. Now, for each $\beta < \eta$ there is $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$ (just otherwise there will be a condition t in G forcing that for some β there is no $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$. Extend it to t' deciding the value $\underset{\sim}{a}(\beta)$. By (6) there is α such that $t', p_\beta(\alpha)$ are compatible). Let $r' \in G$ be a common extension of r and $p_\beta(\alpha)$. Recall that L, K, M is a triple of a Δ-system type in r and the isomorphism π_{LK} moves $p_\beta(\alpha)$ to $q_\beta(\alpha)$. Hence $q_\beta(\alpha) \leq r'$. But then $q_\beta(\alpha) \in G$.
$\square$ of the claim.
$\square$

Remark 2.22 It is not hard to modify the proof of 2.21 and show that in $V[G(\mathcal{P}_{\geq\eta})]$ the forcing $\mathcal{P}_{<\eta}$ is equivalent to the forcing $N_2 \cap \mathcal{P}_{<\eta}$ of cardinality η^+. Thus, instead of a name $\underset{\sim}{a}$ of a subset of η take a $\mathcal{P}'_{\geq\eta}$-name of a maximal antichain of $\mathcal{P}'_{<\eta}$. By η^+-c.c. of $\mathcal{P}'_{<\eta}$, the antichain has cardinality $\leq \eta$. Using the strategic closure of $\mathcal{P}'_{\geq\eta}$ we produce a condition deciding all the elements of the antichain. Let L be its top model of cardinality η. Find K as in the proof of 2.21 and copy the antichain to N_2. Finally, any $N_2 \cap \mathcal{P}_{<\eta}$-generic will intersect this image, which in turn will imply that on the L-side the same happens.

Let us show that $2^\eta = \eta^+$ for singular η's as well. Note that it is possible to deduce this appealing to Core Models arguments (provided that there is no inner model with too large cardinals).

Lemma 2.23 *(a) Let η be a singular cardinal in $[\kappa^+, \theta]$. Then in $V^{\mathcal{P}'}$ we have $2^\eta = \eta^+$.*

(b) $V^{\mathcal{P}'}$ satisfies GCH.

Proof. It is enough to proof (a) since then (b) will follow by the previous lemma 2.21.
Fix a singular cardinal $\eta \in [\kappa^+, \theta]$. Let N, $p_{\geq\eta^+}, N_1, N_2, \underset{\sim}{a}$ be as in the proof of 2.21.
Pick an increasing sequence $\langle \eta_i \mid i < \mathrm{cof}(\eta)\rangle$ of regular cardinals cofinal in η. Let $\langle L_i \mid i < \mathrm{cof}(\eta)\rangle$ be an increasing sequence of elementary submodels of $H((2^\lambda)^+)$ such that

1. $|L_i| = \eta_i$,
2. $L_i \supseteq \eta_i$,
3. ${}^{\eta_i>}L_i \subseteq L_i$,
4. $\langle L_j \mid j < i \rangle \in L_i$,
5. $N, p_{\geq \eta^+}, \underset{\sim}{a} \in L_0$.

Set $L = \bigcup_{i<\mathrm{cof}(\eta)} L_i$.
Now we construct a sequence $\langle p(i) \mid i < \mathrm{cof}(\eta) \rangle$ of elements of $\mathcal{P}'$ such that

1. $p(0) \geq p_{\geq \eta^+}$,
2. $p(i)_{\geq \eta_i}$ is $(L_i, \mathcal{P}')$-generic over $p(i)_{<\eta_i}$, i.e. for any maximal antichain $A \subseteq \mathcal{P}'$ with $A \in L_i$, if some q is in A and is compatible with $p(i)$, then there is $r \geq q, p(i)$ such that for some $r' \leq r$ we have $r' \in A \cap L_i$.
3. $p(j) \restriction \eta_i = p(i)_{<\eta_i}$, for every $j > i$,
4. $p(i) \in L_{i+1}$.

The construction is by recursion and uses that at each $i < \mathrm{cof}(\eta)$ strategic closure of $\mathcal{P}'_{\geq \eta_i}$ together with η_i^+-c.c. of $\mathcal{P}'_{<\eta_i}$.
Now let p be the result of putting $\langle p_i \mid i < \mathrm{cof}(\eta) \rangle$ together as in the strategic closure lemma 2.16 with L the top model of cardinality η. Note that if $G \subseteq \mathcal{P}'$ with $p \in G$, then $L[G \cap L] \prec H((2^\lambda)^+)[G]$. Thus, if $A \in L$ is a maximal antichain, then $A \in L_i$ for some $i < \mathrm{cof}(\eta)$ and by (2) above some $r' \in G$ is in $A \cap L_i$.
In particular, $\underset{\sim}{a}$ can be computed correctly inside L. We continue further as in 2.19 define K etc.,with p replacing $p(\eta)$ of 2.19.
$\square$

3 The Intersection Property- Gap 4

We turn now to the intersection properties. They are somewhat more complicated here than those in the gap 3 case.

Let us give a general definition, but further we shall concentrate at Gap 4. The property as defined fails already at Gap 5. In further sections we present an argument that avoids it. Nerveless intersection properties seem to us to be interesting on their own.

Definition 3.1 Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$, $\tau \leq \rho$ and $A \in A^{1\tau}, B \in A^{1\rho}$. We say that *$A$ satisfies the intersection property with respect to B* or shortly *$ip(A, B)$* iff either

1. $A \subseteq B$, or
2. $B \in A$, or
3. $A \nsubseteq B, B \notin A$, and then there are pairwise different ordinals $\eta_1, ..., \eta_n \in s \setminus \rho$ and sets $A_1 \in A^{1\eta_1} \cap A, ..., A_n \in A^{1\eta_n} \cap A, A' \in (A \cup \{A\}) \cap A^{1\tau}$ such that

$$A \cap B = A' \cap A_1 \cap ... \cap A_n.$$

If $\rho = \tau$, then let $ipb(A, B)$ denotes that both $ip(A, B)$ and $ip(B, A)$ hold.

Lemma 3.2 *Let $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle \in \mathcal{P}'$, $\tau \in s$ and $A \in A^{1\tau}$ be a successor model. Then for every $\xi \in s \setminus \tau + 1$, if $A \cap A^{1\xi} \neq \emptyset$, then there is $(A)_\xi \in A \cap A^{1\xi}$ such that*

1. *for every $B \in A \cap A^{1\xi}$, $B \in (A)_\xi \cup \{(A)_\xi\}$,*
2. *if $B \in A^{1\xi}$ and $(A)_\xi \in C^{\xi}(B)$, then $B \supseteq A$.*

Remark 3.3 We cannot in general allow ξ's below τ. Thus, say there are A_0, A_1 such that the triple A_0, A_1, A is of a Δ-system type. Suppose that A_0, A are on the τ-central line, there a maximal model $B \in C^{\xi}(A^{0\xi}) \cap A_0$ and A_0, A_1, A belong to the immediate successor B^+ of B in $C^{\xi}(A^{0\xi})$. Then $\pi_{A_0,A_1}(B) \in A$, but there is no $X \in A \cap A^{1\xi}$ which includes both B and $\pi_{A_0,A_1}(B)$, since $B^+ \notin A$.

Proof. Induction on complexity of the walk from $A^{0\tau}$ to A.
Suppose first that A is on the τ-central line. By Definition 2.1,

$$\bigcup\{Y \in C^{\xi}(A^{0\xi}) \mid Y \in A\} \in A.$$

Set $(A)_\xi$ to be this union. Let $B \in A \cap A^{1\xi}$. We prove by induction on the walk to B from $A^{0\xi}$ that $B \in (A)_\xi \cup \{(A)_\xi\}$. If B is on the ξ-central line or the walk goes via $(A)_\xi$, then it follows from the choice of $(A)_\xi$ or it is obvious. Suppose otherwise. Let then X be the least model from the ξ-central line with $X \supseteq B$. Then $B \in A$ implies necessarily that $X = ((A)_\xi)^+$ the immediate successor of $(A)_\xi$ in $C^{\xi}(A^{0\xi})$. Also X must be a splitting point. But then there is no models of small cardinalities between $(A)_\xi$ and X. *(in gap 4 case, in general the may be bigger than ξ model with splitting and the statement of the lemma is a* *bit weaker)*

Suppose now that the walk to A goes to some Y which is an immediate predecessor of A and $A \not\in C^\tau(Y)$. Then either there is $Y^- \in C^\tau(Y)$ such that Y^-, A, Y is a triple of a Δ-system type or there are there are $Y^- \in C^\tau(Y), Y_1 \in Y \cap A^{1\tau}$ which are immediate predecessor of Y and satisfy the last possibility of Definition 2.4.

Assume first that Y^-, A, Y is a triple of a Δ-system type. Then the induction applies to Y^-. By Definition 2.3 (7), then $(A)_\xi$ will be as desired.

Suppose now that there are $Y^- \in C^\tau(Y), Y_1 \in Y \cap A^{1\tau}$ which are immediate predecessor of Y and satisfy the last possibility of Definition 2.4. Then the only case to consider is when the triple Y^-, Y_1, Y is of a Δ-system type and A is obtained from Y^- or from Y_1 by moving it by isomorphism of models of bigger cardinality. Then the induction applies to both Y^- and Y_1. So the isomorphic image A will satisfy the statement as well.

□

Lemma 3.4 *Let* $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}'$, $\tau \in s$ *and* $A \in A^{1\tau}$. *Suppose that* $B \in A^{1\theta}$ *and* $\sup(B) < \sup(A)$. *Then*

1. $(A)_\theta$ *exists,*

2. $B \subseteq (A)_\theta$,

3. *if* X *is the least model in* $C^\theta(A^{0\theta}) \cap A$ *which includes* B, *then* $A \cap X = A \cap B$.

Proof. Move A to τ-central line. Note that no switch can change θ-central line, since $A^{1\theta}$ itself is such a line. Once A is on the τ-central line, then Definition 2.1 applies.

□

Remark 3.5 It is possible to have a situation when $B \in A^{1\xi}$ and $\sup(B) < \sup(A)$, for some $\xi \in s, \tau < \xi < \theta$, but $(A)_\xi$ does not exist. Thus suppose that we are in gap 4 case, $\tau = \kappa^+, \xi = \kappa^{++}, \theta = \kappa^{+3}$. Let $A^{0\tau}, A^{0,\xi}, A^{0\theta}$ be the only models of a condition. Assume that $A^{0\theta} \in A^{0\tau} \in A^{0\xi}$. Now inside $A^{0\theta}$ find X which realizes the same type as $A^{0\xi}$ over $A^{0\xi} \cap A^{0\theta}$. Let Y be a model of cardinality ξ such that $A^{0\xi}, X, A^{0\theta}, A^{0\tau} \in Y$ and Z be a model of cardinality τ such that $A^{0\xi}, Y, A^{0\theta}, A^{0\tau}, Y \in Z$. Let $S = \pi_{A^{0\xi},X}(A^{0\theta})$ and $T = \pi_{A^{0\xi},X}(A^{0\tau})$. Consider now the following condition $p = \langle\langle A^{0\mu}(p), A^{1\mu}(p), C^\mu(p)\rangle \mid \mu \in \{\tau, \xi, \theta\}\rangle$, where $A^{0\theta}(p) = A^{0\theta}, A^{1\theta} = \{A^{0\theta}, S\}, A^{0\xi}(p) = Y, A^{1\xi} = \{Y, A^{0\xi}, X\}, C^\xi(p)(Y) = \langle A^{0\xi}, Y\rangle, A^{0\tau}(p) = Z, A^{1\tau} = \{A^{0\tau}, T, Z\}, C^\tau(Z) = \langle Z, A^{0\tau}\rangle$. Then $\sup(A^{0\tau}) > \sup(X)$, but $(A^{0\tau})_\xi$ does not exists due to minimality of $(A^{0\tau})_\xi$ in p.

Lemma 3.6 *Let $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle \in \mathcal{P}'$, $\tau \in s$ and $A \in A^{1\tau}$. Suppose that $A^* \in C^{\tau}(A^{0\tau})$ is the model of the same order type as those of A. Then $(A)_\xi$ exists iff $(A^*)_\xi$ exists.*

Proof. *(if no inaccessibles are present)* Just isomorphisms needed to move from A^* to A will preserve such existence due to Definitions 2.3 and 2.4.
□

Let us prove the intersection property for the gap 4. Thus, for models in $A^{1\kappa^{++}} \cup A^{1\kappa^{+3}}$ it is exactly as in the gap 3 case. Now, if $A \in A^{1\kappa^{+}}$ and $B \in A^{1,\kappa^{+3}}$, then then this follows by Lemma 3.4.

Lemma 3.7 *Suppose that $A \in A^{1\kappa^{+}}$ and $B \in A^{1\kappa^{++}} \cap A^{0\kappa^{+}}$. Then either $A \subset B$ or there are $B' \in A \cap A^{1\kappa^{++}}$ and $C' \in A \cap A^{1\kappa^{+3}}$ such that $A \cap B = A \cap B'$ or $A \cap B = A \cap B' \cap C'$.*

Proof. Suppose that $A \not\subset B$. We prove the lemma by induction on walks complexity. Suppose that $X \in C^{\kappa^{+}}(A^{0\kappa^{+}})$ is the last common point of the walks from $A^{0\kappa^{+}}$ to A and to B. We split the argument into few cases. Let us start with the most complicated one.
Case 1. X has three immediate predecessors.
Let X_0', X_0, X_1 be this predecessors of X. Let $F_0, F_1, F \in X_1 \cap A^{1\kappa^{++}}$ be a witnessing triple of a Δ-system type.
Case 1.1. $A \subseteq X_0'$ and $B \in X_1$.
Compare B with F_1. There are $B' \in A^{1\kappa^{++}} \cap (F_1 \cup \{F_1\}, G' \in A^{1\kappa^{+3}} \cap F_1$ such that

$$B \cap F_1 = B' \cap G'.$$

Then

$$A \cap B = A \cap F_1 \cap B = A \cap B' \cap G'.$$

Now the induction applies.
Case 1.2. $A \subseteq X_1$ and $B \in X_0'$.
Case 1.2.1. $F_0 \in A$ (or $F_1 \in A$).
Then also $F \in A$ since there is no models of small cardinality between F and its immediate predecessors. $F \in A$ implies $F_0, F_1 \in A$ and so $\pi_{F_0,F_1} \in A$. Set $B_0 = \pi_{F_1,F_0}[B]$. Now

$$\alpha \in A \cap B \text{ iff } \pi_{F_1,F_0}(\alpha) \in A \cap B_0.$$

Consider A, B_0. The triple X_0, X_1, X is of a Δ-system type and $X_0 \cap X_1 = X_1 \cap F_0$. So,

$$A \cap B_0 = B_0 \cap X_0 \cap A \cap X_1 = B_0 \cap \pi_{X_1,X_0}(A) \cap \pi_{X_1,X_0}(F_0).$$

Denote $\pi_{X_1,X_0}(A)$ by A_0 and $\pi_{X_1,X_0}(F_0)$ by F_0^0. Then $A_0 \in (X_0 \cup \{X_0\}) \cap A^{1\kappa^+}$ and $F_0^0 \in X_0 \cap A^{1\kappa^{++}}$. We can apply the induction to A_0, B_0, since the common part of the walks to them is longer than those to A, B. So there are $B_0' \in A_0 \cap A^{1\kappa^{++}}$ and $C_0' \in A_0 \cap A^{1\kappa^{+3}}$ such that $A_0 \cap B_0 = A_0 \cap B_0'$ or $A_0 \cap B_0 = A_0 \cap B_0' \cap C_0'$. Suppose that $A_0 \cap B_0 = A_0 \cap B_0' \cap C_0'$. Set $B' = \pi_{X_0,X_1}(B_0')$ and $C' = \pi_{X_0,X_1}(C_0')$. Then

$$A \cap B_0 = A \cap F_0 \cap B_0 = A_0 \cap B_0 \cap F_0 =$$

$$A_0 \cap B_0' \cap C_0' \cap F_0 = A \cap B' \cap C' \cap F_0.$$

Now $B', C', F_0 \in A$. It remains only to replace $B' \cap C' \cap F_0$ by intersection of the form $B'' \cap C''$ for some $B'' \in A \cap A^{1\kappa^{++}}$ and $C'' \in A^{1\kappa^{+3}}$, and it is easy. So

$$A \cap B_0 = A \cap B'' \cap C''.$$

Then

$$A \cap B = \pi_{F_0,F_1}[A \cap B'' \cap C''].$$

If $C'' \supset F_0$, then

$$\pi_{F_0,F_1}[A \cap B'' \cap C''] = \pi_{F_0,F_1}[A \cap B''],$$

and we can drop it. If $C'' \not\supset F_0$, then pick some $C''' \in A \cap F_0 \cap A^{1\kappa^{+3}}$ such that $F_0 \cap C''' = F_0 \cap C''$. Let $D = \pi_{F_0,F-1}(C''')$. Then

$$\pi_{F_0,F_1}[A \cap B'' \cap C''] = \pi_{F_0,F_1}[A \cap B'' \cap \cap F_0 \cap C''] = \pi_{F_0,F_1}[A \cap B'' \cap C'''] = \pi_{F_0,F_1}[A \cap B''] \cap D.$$

Hence it remains to deal with $\pi_{F_0,F_1}[A \cap B'']$. Compare F_0 with B''. There are $B''' \in (F_0 \cup \{F_0\}) \cap A \cap A^{1\kappa^{++}}$ and $H \in F_0 \cap A \cap A^{1\kappa^{+3}}$ such that

$$B'' \cap F_0 = B''' \cap H.$$

Note that we use here (the only place) that $B'' \in A^{1\kappa^{++}}$ and so it is possible to find such B''' and H. This breaks down once $B'' \in A^{1\kappa^+}$ and makes intersections of this type more complicated.
Let $E = \pi_{F_0,F_1}(B''')$ and $S = \pi_{F_0,F-1}(E)$. Then

$$\pi_{F_0,F_1}[A \cap B''] = \pi_{F_0,F_1}[A \cap B'' \cap \cap F_0] = \pi_{F_0,F_1}[A \cap B''' \cap H] = \pi_{F_0,F_1}[A] \cap E \cap S = A \cap E \cap S.$$

So

$$A \cap B = A \cap E \cap S.$$

Case 1.2.2. $F_0 \notin A$ (or $F_1 \notin A$).
Then, also $F \notin A$ and so $F_1 \notin A$. Consider $H = (X)_{\kappa^{++}}$ and $H_1 = (X_1)_{\kappa^{++}}$. Then $H_1 \supseteq F$ and $F \in C^{\kappa^{++}}(H_1)$. Let $T \in C^{\kappa^{++}}(H_1)$ be the least model which includes A.
Case 1.2.2.1. T is a splitting point.
So, let T_0, T_1 be the immediate predecessors of T with $T_0 \in C^{\kappa^{++}}(T)$ such that the triple T_0, T_1, T is of a Δ-system type.
Subcase 1.2.2.1.1. $F \subsetneq T$.
Then $F \subseteq T_0$. Let $G_0 \in T_0 \cap A^{1\kappa^{+3}}$ be so that $T_0 \cap T_1 = T_0 \cap G_0$. Clearly

$$A \cap B = A \cap F \cap B = A \cap T_0 \cap B.$$

Set $A_0 = \pi_{T_1,T_0}(A)$. Then $A_0 \in A^{1\kappa^+}$, since the walk to A from $A^{0\kappa^+}$ proceeds via X, X_1 continues through $C^{\kappa^+}(X_1)$ and cannot move out of $C^{\kappa^{++}}(H_1)$ before getting to T.
Now $A \cap T_0 = A_0 \cap G_0$. Hence

$$A \cap B = A \cap F \cap B = A \cap T_0 \cap B = B \cap A_0 \cap G_0.$$

The induction applies to A_0, B. Hence there are $B'_0 \in A_0 \cap A^{1\kappa^{++}}, C'_0 \in A_0 \cap A^{1\kappa^{+3}}$ such that

$$A_0 \cap B = A_0 \cap B'_0 \cap C'_0.$$

Set $B' = \pi_{T_0,T_1}(B'_0)$ and $C' = \pi_{T_0,T_1}(C'_0)$. Then

$$A \cap B = B \cap A_0 \cap G_0 = A_0 \cap B'_0 \cap C'_0 \cap G_0$$

$$= A \cap B' \cap C' \cap G_1,$$

where $G_1 = \pi_{T_0,T_1}(G_0)$. Replace finally $C' \cap G_1$ by their maximum.
Subcase 1.2.2.1.3. $F \supsetneq T$.
Then $T \subseteq F_0$ or $T \subseteq F_1$. The arguments of the previous case apply.
Case 1.2.2.1. T is not a splitting point.
Let T^- be the unique immediate predecessor of T. Then any further splitting on the way to A, if there is such at all, involves only models of Δ-system type of cardinality κ^+. Hence relevant models of cardinality κ^{++} form here a chain. This implies $T^- \in A$, and hence, $T^- = (A)_{\kappa^{++}}$. Then $C^{\kappa^{++}}(T^-) \in A$ as well. We assume that $F \in T^-$, just otherwise the arguments of the previous cases work.
Let $R \in A \cap C^{\kappa^{++}}(T^-)$ be the least model which includes F. Consider

$$R_* = \bigcup\{S \in C^{\kappa^{++}}(R) \mid S \neq R, S \in A\}.$$

Then

$$A \cap F = A \cap F_0 = A \cap R_*.$$

Hence

$$A \cap B = A \cap F \cap B = A \cap B \cap R_*.$$

But $R_* \subseteq F_0$, hence

$$B \cap R_* = B_0 \cap F_0 \cap F_1 \cap R_*,$$

where $B_0 = \pi_{F_1,F_0}[B]$. So

$$A \cap B = A \cap B \cap R_* = A \cap B_0 \cap F_0 \cap F_1 \cap R_*.$$

The induction applies to A, B_0 and the rest is easy here.
$\square$

Lemma 3.8 *Let* $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s = \{\kappa^+, \kappa^{++}, \kappa^{+3}\}\rangle \in \mathcal{P}', A, B \in A^{1\kappa^+}$. *Then* $ipb(A, B)$.

Proof. Consider the walks from $A^{0\kappa^+}$ to A and to B. Let $X \in A^{1\kappa^+}$ be the least common point of this walks. X must be a splitting point. We preform switching in order to move X to the κ^+-central line. So, let us assume that $X \in C^{\kappa^+}(A^{0\kappa^+})$ and it is the least common model of the walks.
Let us concentrate on the new case. Thus there are $X_0, X_1, X_2 \in X \cap A^{1\kappa^+}$ which are the immediate predecessors of X, $F_0, F_1, F \in C^{\kappa^{++}}$ such that

1. $F \in C^{\kappa^{++}}(A^{0\kappa^{++}})$,
2. $F_0 \in C^{\kappa^{++}}(F)$,
3. F_0, F_1, F is a triple of a Δ-system type,
4. $X_0 \in C^{\kappa^+}(X)$,
5. $X_0 \in F_0$,
6. X_0, X_1, X is a triple of a Δ-system type,
7. $X_0' = \pi_{F_0,F_1}(X_0)$,
8. $F_0, F_1, F \in X_1$,

9. $A \subseteq X'_0$,

10. $B \subseteq X_1$.

Let $Y^0_0, Y^0_1, Y^0 \in X_0$ be the images of F_0, F_1, F under π_{X_1,X_0} and $Y_0, Y_1, Y \in X'_0$ be the images of F_0, F_1, F under π_{F_0,F_1}.

Using more switching if necessary we may assume that the central line was chosen so that the models in $C^{\prime\kappa^+}(X_1)$ either have F (and hence also F_0, F_1) inside or are the members of F_0. !Also assume the least model of $C^{\prime\kappa^+}(X_1)$ with F inside has at most one immediate predecessor. It is possible by Definition 2.4.

We split the argument into few cases.

Case 1. $A \in C^{\prime\kappa^+}(X'_0)$.
Set $A^0 = \pi_{F_1,F_0}(A)$ and $A^1 = \pi_{X_0,X_1}(A_0)$. Then $A^0 \in C^{\prime\kappa^+}(X_0)$, and so $A^1 \in C^{\prime\kappa^+}(X_1)$. Now either $A^1 \in F_0$ or $F_0, F_1, F \in A^1$ by the assumption above.

Subcase 1.1. $A^1 \in F_0$.
Then

$$A^1 \in X_1 \cap F_0 = X_1 \cap X_0.$$

This implies that $A^0 = A^1$, and then $A^0 \in X_1$. We have $\pi_{F_0,F_1} \in X_1$. Hence $A = \pi_{F_0,F_1}(A^0)$ is in X_1. Note that $A^1 \in C^{\prime\kappa^+}(X_1)$. So we obtain a walk from X_1 to A by taking the image under π_{F_0,F_1} of the walk from X_1 to A^1 after it enters F_0.

Subcase 1.2. $A^1 \notin F_0$.
Then $F_0, F_1, F \in A^1$. Let $X_1 \cap A^0 = X_1 \cap A^1 \cap (X_1 \cap X_0) = A^1 \cap H$, for some $H \in X_1 \cap C^{\prime\kappa^{++}}(F_0)$. As in the previous case we have

$$\alpha \in A \cap X_1 \text{ iff } \pi_{F_1,F_0}(\alpha) \in A^0 \cap X_1 \text{ iff } \pi_{F_1,F_0}(\alpha) \in A^1 \cap H.$$

Now we cannot apply π_{F_0,F_1} to A^1, since it is not in the domain. Instead, $\pi_{F_0,F_1} \in A^1$. So

$$\pi_{F_1,F_0}(\alpha) \in A^1 \cap H \text{ iff } \alpha \in A^1 \cap \pi_{F_0,F_1}[H].$$

Putting together we obtain that

$$\alpha \in A \cap X_1 \text{ iff } \alpha \in A^1 \cap \pi_{F_0,F_1}[H].$$

Hence

$$A \cap X_1 = A^1 \cap \pi_{F_0,F_1}[H].$$

Then

$$A \cap B = A \cap X_1 \cap B = A^1 \cap \pi_{F_0,F_1}[H] \cap B.$$

Now the induction applies to the right side and we obtain $ip(B, A)$.
Let us show $ip(A, B)$. Apply the induction to A^1, B and find $A' \in (A^1 \cup \{A^1\}) \cap A^{1\kappa^+}$, $H' \in A^1 \cap A^{1\kappa^{++}}$, $G' \in A_1 \cap A^{1\kappa^{+3}}$ such that

$$A^1 \cap B = A' \cap H' \cap G'.$$

Then

$$A \cap B = A \cap X_1 \cap B = A \cap A' \cap H' \cap G' \cap \pi_{F_0,F_1}[H].$$

Hence we basically need to check $ip(A, A')$. But note that $A' \in (A^1 \cup \{A^1\}) \cap A^{1\kappa^+}$ and if $B \notin A^1$, then we are here in a simpler situation and the induction can be applied to deduce $ip(A, A')$. Suppose that $B \in A^1$. If $F \in B$ or $B \in F_0$, then we proceed as above. In general we consider the walk from A^1 to B and proceed by induction on the walk complexity. Thus, if $B \in C^{\kappa^+}(A^1)$, then either $F \in B$ or $B \in F_0$. Assume that $B \notin C^{\kappa^+}(A^1)$. Consider the least model K of this walk with $F \in K$. Note that $F \in K$ implies that $F \in (K)_{\kappa^{++}}$, since F is on κ^{++}-central line and one cannot change this moving between models.
Again we need to consider few cases.
Subcase 1.2.1. There is $K_1 \in A^{1\kappa^+}$ such that the triple K^-, K_1, K is of a Δ-system type and $B \subseteq K_1$.
Let $H = (K)_{\kappa^{++}}, H_0 = (K^-)_{\kappa^{++}}$ and $H_1 = \pi_{K^-,K_1}(H) = (K_1)_{\kappa^{++}}$. Then $H, H_0, H_1 \in C^{\kappa^{++}}(A^{\kappa^{++}})$. In addition, due to a Δ-system type of the triple, $H \supseteq K^-, K_1$. So, we may assume that $H \supsetneq F$. Just, if $H = F$, then $K^- \in F_0$ (no small models between F_0 and F). But then also $K_1 \in F_0$, since $K_1 \in K \cap F$. This implies that every element of K_1 is in F_0 and we are done.
If $H_1 \in F$, then $K_1 \in F$ too, and then $K_1 \in F_0$ and we are done.
So, let us assume that $F \in H_1$ and $F \notin K_1$. Let T be the least element of $K_1 \cap C^{\kappa^{++}}(H_1)$ which contains F. Consider

$$T_* = \bigcup\{S \in C^{\kappa^{++}}(T) \mid S \neq T, S \in K_1\}.$$

Then

$$K_1 \cap F = K_1 \cap T = K_1 \cap T_*,$$

but $T_* \subseteq F_0$ and $A \cap F_0 = A \cap A^0 = A^0 \cap F_0 \cap F_1$. Then

$$A \cap B = A \cap B \cap F = A \cap B \cap T_* = A \cap F_0 \cap B \cap T_* = A^0 \cap F_0 \cap F_1 \cap B \cap T_*.$$

Now the induction applies to A^0, B.
Subcase 1.2.2. $B \subseteq K^-$.

Then we consider $H_0 = (K^-)_{\kappa^{++}}$. If $H_0 \in F$, then $H_0 \in F_0$. This implies $K^- \in F_0$ and we are done.
Subcase 1.2.3. There are $E_0, E_1, E \in K \cap A^{1\kappa^{++}}$ of a Δ-system type with $E_0 \in C^{\kappa^{++}}(E), E \in C^{\kappa^{++}}(H)$ such that $B \not\subset K^-$, but $B \subseteq \pi_{E_0,E_1}(K^-)$.
We may assume that $F \subseteq E_0$. Just otherwise $E_0 \in F$ and then $F \supseteq E$. Which means that either $E = F$ and then $E_0 = F_0, E_1 = F_1$ or $E \in F$ and then $E_0, E_1 \in F_0$. The second possibility is impossible since $B \notin F_0$. If the first one occurs, then $\pi_{F_1,F_0}(B) \in F_0 \cap K$. But $A_1 \cap F_0 = A_0 \cap A_1$. So $\pi_{F_1,F_0}(B) \in A_0$, and then $B \in A^-$.
Set $B' = \pi_{E_1,E_0}(B)$. Then

$$B \cap F = B \cap F \cap E_0 = B \cap F \cap E_0 \cap E_1 = B' \cap F \cap E_0 \cap E_1.$$

So we are able to replace B with a simpler model B'.
Subcase 1.2.4. There are $E_0, E_1, E \in K \cap A^{1\kappa^{++}}$ of a Δ-system type with $E_0 \in C^{\kappa^{++}}(E), E \in C^{\kappa^{++}}(H)$ such that $B \not\subseteq K^-$ and $B \not\subseteq \pi_{E_0,E_1}(K^-)$.
Denote $\pi_{E_0,E_1}(K^-)$ by K_1. There must be $K_2 \in K \cap A^{1\kappa^{++}}$ such that the triple K_1, K_2, K is of a Δ-system type after switching E_0 by E_1 and $B \subseteq K_2$. Also $E_0, E_1, E \in K_2$. Consider $H_2 = (K_2)_{\kappa^{++}}$. Then $E \in H_2$. As in the previous case, we have $F \subseteq E_0$. So $F \in H_2$. But $F \notin K_2$. Proceed as in the first case. Let T be the least element of $K_2 \cap C^{\kappa^{++}}(H_2)$ which contains F. Consider

$$T_* = \bigcup \{S \in C^{\kappa^{++}}(T) \mid S \neq T, S \in K_2\}.$$

Then

$$K_2 \cap F = K_2 \cap T = K_2 \cap T_*.$$

So

$$B \cap F = B \cap T_*.$$

Now $T_* \subseteq F_0$ and $A \cap F_0 = A \cap A^0$. Hence

$$A \cap B = A \cap B \cap F = A \cap B \cap T_* = A \cap F_0 \cap B \cap T_* = A^0 \cap F_0 \cap F_1 \cap B \cap T_*.$$

Now the induction applies to A^0, B.
Case 2. $A \notin C^{\kappa^+}(X_0')$.
Let $K \in C^{\kappa^+}(X_0')$ be the least model with $A \in K$.
Subcase 2.1. There are $K_0, K_1 \in K \cap A^{1\kappa^+}, K_0 \in C^{\kappa^+}(K)$ such that the triple K_0, K_1, K is of a Δ-system type.

Then $A = K_1$ or $A \in K_1$.
Remember that

$$X_1 \cap X_0' = X_1 \cap F_1 = X_0' \cap Y_0,$$

since

$$a \in X_0' \cap X_1 \iff \pi_{F_1,F_0}(a) \in X_0 \cap X_1 \iff \pi_{F_1,F_0}(a) \in X_1 \cap F_0 \iff$$

$$\pi_{F_1,F_0}(a) \in X_0 \cap \pi_{X_1,X_0}(F_0) \iff a \in X_0' \cap \pi_{F_0,F_1}(\pi_{X_1,X_0}(F_0)) \text{ and } Y_0 = \pi_{F_0,F_1}(\pi_{X_1,X_0}(F_0)).$$

Subcase 2.1.1. $A \in Y_0$.
Then $A \in F_1$. If in addition $K \in Y_0$, then also $K \in F_1$. Let $K^0 = \pi_{F_1,F_0}(K)$ and $K^1 = \pi_{X_0,X_1}(K^0)$. It follows that $K^0 \in Y_0^0$ and $K^1 \in F_0$. So, $K^0 = K^1$. Also $K^1 \in C^{\kappa^+}(X_1)$, as $K \in C^{\kappa^+}(X_0')$. Then we obtain the walk from X_1 to A by taking the image under π_{F_0,F_1} of the walk from X_1 to A^1 after it enters F_0.
Suppose now that $K \notin Y_0$. Then $Y_0 \in K$, since by Definition 2.4 each element of $C^{\kappa^+}(X_0')$ either in Y_0 or Y_0 (and Y) belongs to it. Now K is a splitting point, so K_0 cannot be inside Y_0. Then $Y_0 \in K_0$ and hence also $Y \in K_0$, since there is no models of small cardinalities between Y and Y_0. Consider $(K_0)_{\kappa^{++}}$. We have $Y \subseteq (K_0)_{\kappa^{++}}$ and so $Y_0 \in (K_0)_{\kappa^{++}}$. Remember that $A \in K_1 \cup \{K_1\}$ and $A \in Y_0$. Let $T_1 \in K_1 \cap A^{1\kappa^{++}}$ be the Δ-system witness, i.e. $K_0 \cap K_1 = K_1 \cap T_1$. If $Y_0 \subseteq T_1$, then $A \in T_1$. Hence $A \in K_1 \cap T_1$ and so $A \in K_0$. Which is impossible by the choice of K. So we must have $T_1 \in Y_0$. Then, by the definition of a Δ-system type triple, $(K_1)_{\kappa^{++}} \in Y_0$ and then $K_1 \in Y_0$.
Set $K^0 = \pi_{F_1,F_0}(K), K^1 = \pi_{X_0,X_1}(K^0), K_0^0 = \pi_{F_1,F_0}(K_0), K_0^1 = \pi_{X_0,X_1}(K_0^0), K_1^0 = \pi_{F_1,F_0}(K_1)$, $K_1^1 = \pi_{X_0,X_1}(K_1^0)$. Then $F_0, F_1, F \in K_0^1$, as $Y_0, Y_1, Y \in K_0$. Also $K_1^0 \subseteq F_0$, as $K_1 \subseteq Y_0$. Hence $K_1^0 = K_1^1$. Then we obtain the walk from X_1 to A by going down to K^1 then to K_1^0 and taking the image under π_{F_0,F_1} of the walk from K_1^0 to A^1.
Subcase 2.1.2. $A \notin Y_0$.
Consider $(X_0')_{\kappa^{++}}$. We have $Y \in C^{\kappa^{++}}((X_0')_{\kappa^{++}})$. Also $(K)_{\kappa^{++}} \in C^{\kappa^{++}}((X_0')_{\kappa^{++}})$.
Subcase 2.1.2.1. $Y_0 \in A$ (or equivalently $Y_1 \in A$).
Then also $Y_1, Y \in A$. Hence $F_0, F_1, F \in A^1 = \pi_{F_0,F_1}(\pi_{X_0',X_0}(A))$. Now, as was shown in Case 1.2,

$$A \cap B = A^1 \cap B \cap F_1.$$

Subcase 2.1.2.2. $Y_0 \notin A$ (or equivalently $Y_1 \notin A$).

Then also $Y_1, Y \notin A$.
If $Y \in K_1$, then $Y \in K_0$ as well, since K as a model with two immediate predecessors cannot be the least model in $C^{\kappa^+}(X_0')$ with Y inside. So $Y_0, Y_1, Y \in K_0 \cap K_1$. Then

$$K_0 \cap Y = K_1 \cap Y,$$

by the definition of a Δ-system type triple (just compare $(K_0)_{\kappa^{++}}$ and $(K_1)_{\kappa^{++}}$). Consider $\underset{\sim}{A} = \pi_{K_1,K_0}[A]$. Then

$$A \cap Y_0 = \underset{\sim}{A} \cap Y_0.$$

But

$$A \cap Y_0 = A \cap X_1,$$

since

$$A \cap X_1 = A \cap F_1 \cap X_1 = A \cap Y_0 \cap X_0' = A \cap Y_0.$$

Hence

$$A \cap B = A \cap X_1 \cap B = A \cap Y_0 \cap B = \underset{\sim}{A} \cap B \cap Y_0.$$

The induction applies now to $\underset{\sim}{A}, B$.

Suppose now that $Y \notin K_1$. We have $Y \in K_0$, since K as a model with two immediate predecessors cannot be the least model in $C^{\kappa^+}(X_0')$ with Y inside. Also $(K_1)_{\kappa^{++}} \not\subseteq Y$, since otherwise K_1 will be a subset of Y_0, as K_0, K_1, K are of a Δ-system type and Y_0, Y are on the κ^{++}-central line. Hence $Y \in C^{\kappa^{++}}((K_1)_{\kappa^{++}})$ and so $(K_0)_{\kappa^{++}} \in C^{\kappa^{++}}((K_1)_{\kappa^{++}})$. Then

$$K_1 \cap Y = K_1 \cap Y_0 = K_1 \cap K_0 \cap Y_0.$$

So

$$A \cap Y_0 = A \cap K_1 \cap Y_0 = A \cap K_1 \cap K_0 \cap Y_0 = \underset{\sim}{A} \cap Y_0 \cap G_0,$$

where $G_0 \in K_0 \cap C^{\kappa^{++}}((K_0)_{\kappa^{++}})$ is so that $K_0 \cap K_1 = K_0 \cap G_0$. Hence

$$A \cap B = A \cap Y_0 \cap B = \underset{\sim}{A} \cap B \cap Y_0 \cap G_0.$$

Now the induction applies.
Subcase 2.2. There are K_0, K_0', K_1 which are the immediate predecessors of K.
Let $G_0, G_1, G \in A^{1\kappa^{++}} \cap K_1$, $G \in C^{\kappa^{++}}((K)_{\kappa^{++}}), G_1 \in C^{\kappa^{++}}(G)$ be the corresponding witnessing triple of a Δ-system type.
Split into two subcases.
Subcase 2.2.1. $A \subseteq K_1$ and $K_0' \in C^{\kappa^+}(K)$.

Then $Y_0, Y_1, Y \in K_0'$, since K splits and so it cannot be the least model on $C^{\kappa^+}(X_0')$ with Y's inside. Then $Y \subseteq G_1$. Hence

$$K_1 \cap Y_0 = K_1 \cap G_1 \cap Y_0 = K_0' \cap G_0' \cap Y_0,$$

where $G_0' = \pi_{G_0,G_1}(\pi_{K_1,K_0}(G_1))$. So,

$$A \cap Y_0 = A \cap K_1 \cap Y_0 = A \cap K_0' \cap G_0' \cap Y_0.$$

Apply the induction to A and K_0'. So, there are $A' \in K_0' \cup \{K_0'\}, T \in K_0' \cap A^{1\kappa^{++}}, S \in K_0' \cap A^{1\kappa^{+3}}$ such that

$$A \cap Y_0 = A' \cap T \cap S.$$

Then

$$A \cap B = A \cap Y_0 \cap B = A' \cap B \cap T \cap S$$

and the induction applies to A' and B.

Subcase 2.2.2. $A \subseteq K_0'$ and $K_1 \in C^{\kappa^+}(K)$.

Then $Y_0, Y_1, Y \in K_1$, since K splits and so it cannot be the least model on $C^{\kappa^+}(X_0')$ with Y's inside.

Subcase 2.2.2.1 $Y_0 \subseteq G_1$.

Then

$$A \cap Y_0 = A \cap K_0' \cap G_1 \cap Y_0 = A \cap K_1 \cap Y_0.$$

Apply the induction to A, K_1. So, there are $A' \in K_1 \cup \{K_1\}, T \in K_1 \cap A^{1\kappa^{++}}, S \in K_1 \cap A^{1\kappa^{+3}}$ such that

$$A \cap Y_0 = A' \cap T \cap S.$$

Then

$$A \cap B = A \cap Y_0 \cap B = A' \cap B \cap T \cap S$$

and the induction applies to A' and B.

Subcase 2.2.2.2 $Y_0 \not\subseteq G_1$.

Then $G_1 \subsetneq Y_0$. So $K_0' \in Y_0$ and then $A \in Y_0$. Move everything to X_1 and copy the walks as it was done in the previous cases.

$\square$.

4 Suitable structures and assignment functions

We address first the new splitting possibility, which is crucial for GCH and does not appear in the gap 2, 3 cases.

Definition 4.1 Let $\nu < \xi < \mu$ be cardinals, A, X, Y_0, Y_1, Y be models, $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$. We call triples F_0, F_1, F and A_0', A_0, A_1 *splitting triples* over A, X, Y_0, Y_1, Y inside C_ν, C_ξ iff

1. $|A_0| = \nu$,
2. $|Y_0| = \xi$,
3. $|X| = \mu$,
4. $A_0, A_0', A_1 \in C_\nu$,
5. $Y_0, Y_1, Y, F_0, F_1, F \in C_\xi$,
6. $F_0, F_1 \in F$,
7. F_0, F_1 are isomorphic over $F_0 \cap F_1$,
8. $F_0, F_1, F \in A_1$,
9. $X \in F_1$,
10. $F_0 \cap F_1 = F_1 \cap X$,
11. $A_0 \in F_0$,
12. $A_1 \cap A_0 = A_1 \cap F_0$,
13. A_1, A_0 are isomorphic over $A_1 \cap A_0$,
14. $A_0' = \pi_{F_0,F_1}(A_0)$,
15. $A \subseteq A_0'$,
16. $Y_0 = \pi_{F_0,F_1}(\pi_{A_1,A_0}(F_0)), Y_1 = \pi_{F_0,F_1}(\pi_{A_1,A_0}(F_1)), Y = \pi_{F_0,F_1}(\pi_{A_1,A_0}(F))$.
    Note that $A_0 \cap A_1 = A_0 \cap \pi_{A_1,A_0}(F_0)$, since $\alpha \in A_0 \cap A_1$ iff $\alpha \in A_1 \cap F_0$ iff $\pi_{A_1,A_0}(\alpha) \in A_0 \cap \pi_{A_1,A_0}(F_0)$, but for $\alpha \in A_0 \cap A_1, \pi_{A_1,A_0}(\alpha) = \alpha$.
    Then $A_0' \cap A_1 = A_1 \cap F_1 = A_0' \cap Y_0$, since $\pi_{F_1,F_0} \in A_1$. Hence Y is a model which corresponds to F_0 in A_0'.

Normally, we will have $|A_0| < |F|$ and $|X| = |F|^*$.

Lemma 4.2 *Suppose that all the models of Definition 4.1 are members of a condition in $\mathcal{P}'$. Then $Y_0 \in A$ implies $Y_1, Y, X \in A$.*

Proof. Set $A_1^* = \pi_{A_0,A_1}(\pi_{F_1,F_0}(A))$. If $Y_0 \in A$, then $\pi_{F_1,F_0}(Y_0) \in \pi_{F_1,F_0}(A)$, and hence $\pi_{A_0,A_1}(\pi_{F_1,F_0}(Y_0)) = F_0 \in A_1^*$. Then $F \in A_1^*$, since there are no models of small cardinality between F_0 and F. Hence, $F_1 \in A_1^*$. So, their pre-images Y and Y_1 are in A.
Now, there is $G_0 \in F_0 \cap A_1^*$ such that $F_0 \cap F_1 = F_0 \cap G_0$. Then $G_0 \in A_1 \cap F_0 = A_0 \cap A_1$. Moreover, $G_0 \in A_1^* \cap F_0 = A_0^* \cap A_1^*$, where $A_0^* = \pi_{F_1,F_0}(A)$.
Set $G_1 = \pi_{F_0,F_1}(G_0)$. Then $G_1 \in A \cap A_1^*$ and $F_0 \cap F_1 = F_1 \cap G_1$, i.e. $G_1 = X$ and $X \in A$.
□

Lemma 4.3 *(Existence of splitting triples). Let $\mu > \xi > \nu$ be regular cardinals in $[\kappa^+, \theta]$. Then for every closed unbounded sets $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ there is a closed unbounded $C_\mu \subseteq \mathcal{P}_{\mu^+}(H(\theta^+))$ such that for every model $X \in C_\mu$, with ${}^\xi X \subseteq X$, there are $Y_0, Y_1, Y \in C_\xi, {}^\nu Y_0 \subseteq Y_0, {}^\nu Y_1 \subseteq Y_1, {}^\nu Y \subseteq Y$ so that for every model A with $|A| \leq \nu$ there are splitting triples over A, X, Y_0, Y_1, Y inside C_ν, C_ξ.*

Proof. Suppose otherwise. Then there are clubs $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ such that for every club $C_\mu \subseteq \mathcal{P}_{\mu^+}(H(\theta^+))$ there is a model $X \in C_\mu$ so that for every models $Y_0, Y_1, Y \in C_\xi$ there is a model $A(X, Y_0, Y_1, Y)$ without splitting triples over $A(X, Y_0, Y_1, Y), X, Y_0, Y_1, Y$ inside C_ν, C_ξ.
Let $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ be such clubs. Define a function

$$I : \mathcal{P}_{\mu^+}(H(\theta^+)) \times C_\xi \times C_\xi \times C_\xi \to \mathcal{P}_{\nu^+}(H(\theta^+))$$

by setting $I(X, Y_0, Y_1, Y)$ to be the least model $A \in \mathcal{P}_{\nu^+}(H(\theta^+))$ without splitting triples over $A(X, Y_0, Y_1, Y), X, Y$ inside C_ν, C_ξ, if there is one and 0 otherwise.
Fix functions $h_\nu : [H(\theta^+)]^{<\omega} \to \mathcal{P}_{\nu^+}(H(\theta^+)), h_\xi : [H(\theta^+)]^{<\omega} \to \mathcal{P}_{\xi^+}(H(\theta^+))$ such that

$$C_\nu \supseteq \{t \in \mathcal{P}_{\nu^+}(H(\theta^+)) \mid h_\nu(e) \subseteq t \text{ whenever } e \in [t]^{<\omega}\},$$

$$C_\xi \supseteq \{t \in \mathcal{P}_{\xi^+}(H(\theta^+)) \mid h_\xi(e) \subseteq t \text{ whenever } e \in [t]^{<\omega}\}.$$

Turn to submodels of $\langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ for λ much bigger than θ. Consider

$$C = \{Z \in \mathcal{P}_{\mu^+}(H(\lambda^{+5})) \mid Z \prec \langle H(\lambda), \in, <, \theta^+, h_\nu, h_\xi, I\rangle\}.$$

Then

$$C \restriction H(\theta^+) = \{Z \cap H(\theta^+) \mid Z \in C\}$$

contains a club in $\mathcal{P}_{\mu^+}(H(\theta^+))$. Let C_μ be such a club. Pick $X \in C_\mu, {}^\xi \subseteq X$, to be a counterexample.

Find $X^* \in C$ with $X^* \cap H(\theta^+) = X$. Note that X^* may be not closed under ξ-sequences of its elements (even $\sup(X^* \cap \theta^{++})$ can have cofinality ω).

Let $F_1^* \prec \langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ be a model of cardinality ξ, closed under ν-sequences of its elements and with X^* inside. Then $F_1 = F_1^* \cap H(\theta^+)$ is closed under h_ξ and hence $F_1 \in C_\xi$. Let F_0^* be obtained from F_1^* via a reflection to X^*. Here $F_1^* \cap X^*$ need not be an element of X^* due the possible lack of closure, but $F_1 = F_1^* \cap H(\theta^+)$ is in $X = X^* \cap H(\theta^+)$, since ${}^\xi X \subseteq X$. We pick $F_0^* \prec \langle H(\lambda^{+4}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ to be a model realizing the same type as F_1^* over $F_1 \cap X$. So F_1^*, F_0^* are isomorphic by the isomorphism which is the identity over $F_1 \cap X$, but probably not the identity over $F_1^* \cap F_0^*$.

Let $F^* \prec \langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ be a model with F_0^*, F_1^* inside and closed under ν-sequences of its elements. Pick now $A_1^* \prec \langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ to be a model of cardinality ν with $F_0^*, F_1^*, F^*, X^* \in A_1^*$. Reflect A_1^* to F_0^*. Let $A_0^* \subseteq F_0^* \cap H(\lambda^{+3})$ be a result. Then $A_0^* \prec \langle H(\lambda^{+3}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$, the isomorphism $\pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}$ is the identity on $A_1^* \cap H(\theta^+) \cap A_0^*$ and $A_1^* \cap H(\theta^+) \cap F_0^* = A_1^* \cap A_0^* \cap H(\theta^+)$.

Set $A_0'^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(A_0^*)$. Then, $A_0'^* \prec \langle H(\lambda^{+3}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$, since $A_0^* \prec F_0^* \cap H(\lambda^{+3})$ and $F_0^* \simeq F_1^* \cap H(\lambda^{+4})$. This implies in particular that $A_0' = A_0'^* \cap H(\theta^+)$ is in C_ν and $A_0'^*$ is closed under I.

Set $F_0^{0*} = \pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}(F_0^* \cap H(\lambda^{+3})), F_1^{0*} = \pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}(F_1^* \cap H(\lambda^{+3}))$
and $F^{0*} = \pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}(F^* \cap H(\lambda^{+3}))$.

Move this models to $A_0'^*$. Thus let $Y_0^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(F_0^{0*}), Y_1^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(F_1^{0*})$ and $Y^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(F^{0*})$. Then $Y_0^*, Y_1^*, Y^* \in A_0'^*$.

Define $F_0 = F_0^* \cap H(\theta^+), F_1 = F_1^* \cap H(\theta^+), F = F^* \cap H(\theta^+), Y_0 = Y_0^* \cap H(\theta^+), Y_1 = Y_1^* \cap H(\theta^+), Y = Y^* \cap H(\theta^+), A_0 = A_0^* \cap H(\theta^+)$ etc. Then $X, Y_0, Y_1, Y \in A_0'$, since $X \in A_1 \cap F_1 = A_1 \cap A_0'$ (the last equality holds because $A_1 \cap F_0 = A_1 \cap A_0$ and $\pi_{F_0, F_1} \in A_1$). The models A_0', A_0, A are in C_ν, since they are closed under h_ν. Similar $F_0, F_1, F, Y_0, Y_1, Y \in C_\xi$. Finally, $A_0'^*$ is closed under I and $X, Y_0, Y_1, Y \in A_0'^*$, hence $I(X, Y_0, Y_1, Y) \in A_0'^*$. By the choice of X, Y_0, Y_1, Y, $I(X, Y_0, Y_1, Y)$ must be a model without splitting triples over $I(X, Y_0, Y_1, Y), X, Y_0, Y_1, X$ inside C_ν, C_ξ. But $F_0, F_1, F \in C_\xi$ and $A_0', A_0, A_1 \in C_\nu$ are splitting triples over $I(X, Y_0, Y_1, Y), X, Y_0, Y_1, Y$. Contradiction.

$\square$

Lemma 4.4 *Suppose that X, Y_0, Y_1, Y satisfy the conclusion of Lemma 4.3 and they are in M for a model $M \in C_\nu$. Then there are splitting triples $A'_0, A_0, A_1, F_0, F_1, F$ over M, X, Y_0, Y_1, Y with $A'_0 = M$.*

Proof. Let $A'_0, A_0, A_1, F_0, F_1, F$ be any splitting triples over M, X, Y_0, Y_1, Y. Consider $M_0 = \pi_{F_1,F_0}(M)$ and $M_1 = \pi_{A_0,A_1}(M_0)$. Then, $F_0, F_1, F \in M_1$, since $F_0 = \pi_{A_0,A_1}(\pi_{F_1,F_0}(Y_0)), F_1 = \pi_{A_0,A_1}(\pi_{F_1,F_0}(Y_1)), F = \pi_{A_0,A_1}(\pi_{F_1,F_0}(Y))$.
So, we can replace A'_0 by M, A_0 by M_0 and A_1 by M_1. Hence M, M_0, M_1, F_0, F_1, F will be splitting triples over M, X, Y_0, Y_1, Y.
□

For every cardinal $\mu \in [\kappa^+, \theta]$ we define a closed unbounded subset C_μ of $\mathcal{P}_{\mu^+}(H(\theta^+))$ by induction as follows: $C_{\kappa^+} = \mathcal{P}_{\kappa^{++}}(H(\theta^+))$,
$C_{\kappa^{++}} = \mathcal{P}_{\kappa^{+3}}(H(\theta^+))$,
if μ is a limit cardinal, then
$C_\mu = \mathcal{P}_{\mu^+}(H(\theta^+))$,
if μ is a successor cardinal, then let C_μ be the intersection of the clubs given by Lemma 4.3 for each $\nu < \xi < \mu$.

Definition 4.5 A model M of a regular cardinality ν is called a *reliable model* iff

1. $M \cap H(\theta^+) \in C_\nu$,

2. for every regular cardinals $\xi, \mu \in M, \nu < \xi < \mu$, for every clubs $E \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), D \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ in M and there is a club $C \subseteq \mathcal{P}_{\mu^+}(H(\theta^+)), C \subseteq C_\mu, C \in M$ such that for every $X \in C \cap M$ there are $Y_0, Y_1, Y \in D \cap M$ which satisfy the conclusion of Lemma 4.3 with E and D.

Definition 4.6 A structure $\mathfrak{X} = \langle X, E, C \in, \subseteq \rangle$, where $E \subseteq [X]^2$ and $C \subseteq [X]^3$ is called *suitable structure* iff there is $p(\mathfrak{X}) = \langle \langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), C^\tau(\mathfrak{X}) \rangle \mid \tau \in s(\mathfrak{X}) \rangle \in \mathcal{P}'$ such that

1. $X = A^{0\kappa^+}(\mathfrak{X})$,

2. $s(\mathfrak{X}) \in X$,

3. $s(\mathfrak{X}) \subseteq X$,

4. $\langle a, b \rangle \in E$ iff $a \in s(\mathfrak{X})$ and $b \in A^{1a}(\mathfrak{X})$,

5. $\langle a, b, d \rangle \in C$ iff $a \in s(\mathfrak{X}), b \in A^{1a}(\mathfrak{X})$ and $d \in C^a(\mathfrak{X})(b)$.

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.

Definition 4.7 A suitable structure $\mathfrak{X} = \langle X, E, C \in, \subseteq \rangle$ is called *suitable generic structure* iff there is $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in G(\mathcal{P}')$ such that

1. $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s \setminus \{\kappa^+\}\rangle \in A^{0\kappa^+}$.
   In particular $s \in A^{0\kappa^+}$. Note that s may have cardinality above κ^+ (which is not a case in a suitable structure) and so s not necessary is contained in $A^{0\kappa^+}$.

2. $\mathfrak{X}$ is a substructure (not necessarily elementary) of the suitable structure generated by $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$, i.e. $\langle A^{0\kappa^+}, \{\langle \tau, B\rangle \mid \tau \in s, B \in A^{1\tau}\}, \{\langle \tau, B, D\rangle \mid \tau \in s, B \in A^{1\tau}, D \in C^\tau(B)\}$,

3. $X \in C^{\kappa^+}(A^{0\kappa^+})$,

4. $p(\mathfrak{X})$ and $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ agree about the walks to members of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$. In other words we require that all the elements of walks in $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ to elements of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$ are in X.

5. If $A \in A^{1\tau}(\mathfrak{X})$, for some $\tau \in s(\mathfrak{X})$, then either A it is of one of the first three types of Definition 2.4(2) inside $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ or the models witnessing that it is of the forth type appear in $\mathfrak{X}$ as well.

Note that, as a condition in $\mathcal{P}'$, $p(\mathfrak{X})$ need not be weaker than $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$, and hence it need not be in $G(\mathcal{P}')$.
Note also, that any stronger condition $\langle\langle B^{0\tau}, B^{1\tau}, D^\tau\rangle \mid \tau \in r\rangle \in G(\mathcal{P}')$ such that

- $\langle\langle B^{0\tau}, B^{1\tau}, D^\tau\rangle \mid \tau \in r \setminus \{\kappa^+\}\rangle \in B^{0\kappa^+}$,
  and

- $C^\tau(A^{0\tau})$ is an initial segment of $D^\tau(B^{0\tau})$, for each $\tau \in s$

will witness that $\mathfrak{X}$ is a suitable generic structure.

Fix $n < \omega$. We define an analog $\mathcal{P}'_n$ of $\mathcal{P}'$ on the level n just replacing κ by κ_n^{+n} and θ by some λ_n big enough (λ_n a Mahlo will be more than enough; we can use for the gap 4 case $\lambda_n = \kappa_n^{+n+4}$, etc). An assignment function a_n will be an isomorphism between a suitable generic structure of cardinality less than κ_n over κ and a suitable structure over κ_n^{+n}.

Define Q_{n0}.

Definition 4.8 Let Q_{n0} be the set of the triples $\langle a, A, f\rangle$ so that:

1. f is partial function from θ^+ to κ_n of cardinality at most κ

2. a is an isomorphism between a suitable generic structure $\mathfrak{X}$ of cardinality less than κ_n and a suitable structure $\mathfrak{X}'$ in $\mathcal{P}'_n$ so that

   (a) every model in $\mathfrak{X}'$ is a reliable model,

   (b) X' is above every model which appears in $A^{1\tau}(\mathfrak{X}')$ for some $\tau \in s(\mathfrak{X}') \setminus \{\kappa^+\}$ and also those in $A^{1\kappa^+}(\mathfrak{X}') \setminus \{X'\}$ in the order $\leq_{E_n}$ of the extender E_n, (or actually, after codding X' by an ordinal),

   (c) if $t \in \bigcup\{A^{1\tau}(\mathfrak{X}') \mid \tau \in s(\mathfrak{X}')\}$, then for some $k, 2 < k < \omega$,
   ?$t \prec H(\chi^{+k})$, with χ big enough fixed in advance. (Alternatively, may be to work with subsets of λ_n only and further require it is a restriction of such model to λ_n.)
   We deal with elementary submodels of $H(\chi^{+k})$, instead of those of $H(\lambda_n)$.
   Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of a model $t \in X \cup Y$ will be an elementary submodel of $H(\chi^{+k})$.
   The way to compare such models $t_1 \prec H(\chi^{+k_1}), t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
   move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{n,X'}$,

4. for every ordinals α, β, γ which code models in $\bigcup\{A^{1\tau}(\mathfrak{X}') \mid \tau \in s(\mathfrak{X}')\}$ we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \quad \text{implies}$$
$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho))$$

   for every $\rho \in \pi``_{X',\alpha}(A)$.

Define a partial order on Q_{n0} as follows.

Definition 4.9 Let $\langle a, A, f\rangle$ and $\langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $a \supseteq b$,

2. $f \supseteq g$,

3. $\pi_{\max(\mathrm{rng}(a)),\max(\mathrm{rng}(b))}$"$A \subseteq B$,

4. $\mathrm{dom}(f)\cap Y^b = \mathrm{dom}(g)\cap Y^b$, where Y^b is the second component (i.e. the set of ordinals) of the suitable structure on which b is defined.
   Note that here we do not require disjointness of the domain of g and of Y^b, but as it will follow from the further definition of non-direct extension, the value given by g will be those that eventually counts.

Definition 4.10 Q_{n1} consists of all partial functions $f : \kappa^{+3} \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

Definition 4.11 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq_n^* = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f\rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown\nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{rng}(a)),a(\alpha)}(\nu) \mid \alpha \in A^{1\theta}(\mathrm{dom}(a)) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $A^{1\theta}(\mathrm{dom}(a))$ are used to produce non direct extensions, the rest of models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq_n^* q$ or $p \in Q_{n1}, q = \langle b, B, g\rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown\nu$.

Definition 4.12 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega\rangle$ so that

(1) for every $n < \omega$, $\quad p_n \in Q_n$,

(2) there is $\ell(p) < \omega$ such that

  (i) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,

  (ii) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n\rangle \in Q_{n0}$,

  (iii) there is $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in G(\mathcal{P}')$ which witnesses that $\mathrm{dom}(a_n(p))$ is a suitable generic structure (i.e. $\mathrm{dom}(a_n(p))$ and $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ satisfy 4.7), simultaneously for every $n, l(p) \leq n < \omega$.

(3) For every $n \geq m \geq \ell(p)$, $\quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,

(4) ? for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).

(5) ? For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.

Next lemma which allows to extend elements of $\mathcal{P}$ is crucial.

Lemma 4.13 *Let $p \in \mathcal{P}$ and $\langle\langle B^{0\tau}, B^{1\tau}, D^\tau\rangle \mid \tau \in r\rangle \in G(\mathcal{P}')$. Then*

1. *for every $t \in \bigcup\{B^{1\tau} \mid \tau \in r\}$ there is $q \geq^* p$ such that $t \in \mathrm{dom}(a_n(q))$ for all but finitely many n's;*

2. *for every $A \in B^{1\kappa^+}$ there is $q \geq^* p$ such that $A \in \mathrm{dom}(a_n(q))$ for all but finitely many n's. Moreover, if $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \geq \langle\langle B^{0\tau}, B^{1\tau}, D^\tau\rangle \mid \tau \in r\rangle$ witnesses a generic suitability of p and $A \in C^{\kappa^+}(A^{0\kappa^+})$, then the addition of A does not require adding of ordinals and the only models that probably will be added together with A are its images under Δ-system type isomorphisms for triples in p.*

Proof. The proof follows the proof of this lemma in a gap 3 case. Let us concentrate on the new possibility of splitting. Namely given triples $A_0', A_0, A_1 \in A$ and F_0, F_1, F as in the last case of Definition 2.4 with A_0', A and F_1, F on the central lines (other possibilities are as in a gap 3 case), we would like to add A_0, A_1, F_0. Denote by $\hat{A}$ the largest model of $C^{|A|}(A_0') \setminus \{A_0'\}$ which is in p, if such a model exists. Suppose that it exists. If it does not exist then the argument is similar and simpler. Consider $X \in F_1 \cap A^{1|F_1|^*}$ such that $F_0 \cap F_1 = F_1 \cap X$ and $Y_0, Y_1, Y \in A^{1|F_1|^*}$ as in Definition 4.1. Then $X, Y_0, Y_1, Y \in A_0'$. Using the induction we can assume that X already appears in p. Now apply Lemma 4.3 to $X^* = a_n(X)$ and appropriate C (C will depend on $a_n(\hat{A})$ and its place relatively to Y_0, Y_1, Y) and find models Y_0^*, Y_1^*, Y^* satisfying the conclusion of this lemma and which can be added to $\mathrm{rng}(a_n)$ as images of Y_0, Y_1, Y. Assume that already $a_n(Y_0) = Y_0^*, a_n(Y_1) = Y_1^*$ and $a_n(Y) = Y^*$. Pick now inside $A^* = a_n(A)$ splitting triples F_0^*, F_1^*, F^* and $A_0'^*, A_0^*, A_1^*$ over $a_n(A_0'), X^*, Y_0^*, Y_1^*, Y^*$. By Lemma 4.4, we can assume that $A'^* = a_n(A_0')$. Add this models to $\mathrm{rng}(a_n)$ as images of the corresponding models over κ. Finally extend a_n further by adding the images under isomorphisms corresponding to Δ-system types.

We need the following property:

if $A \in A^{0\kappa^+} \cap \mathrm{dom}(a_n)$, for some $n \geq \ell(p)$ big enough, and $B \in \max(\mathrm{dom}(a_n))$ is a model which is reachable by a walk from A, then

(1) it is possible to extend a_n to b_n by adding B, probably in addition also models which belong to A and then taking isomorphic images.

(2) Let $A \in \mathrm{dom}(a_n)$, B a model added to $\mathrm{dom}(a_n)$ and $\tilde{B}$ is an isomorphic image of B which belongs to A, then $b_n(\tilde{B}) \in a_n(A)$ as well all the models of the walk from A to $\tilde{B}$, where b_n denotes the extension of a_n obtained by adding B and taking isomorphic images.

This means basically that for adding such B we should take care only of models which are in A. The images of the rest of models with B inside will have the image of B inside automatically.

(1) was explained above. Let us deal with (2).

Assume that B is a model of cardinality κ^+ and B is on the central line. Note that any model involved is a member of one of cardinality κ^+.

Our first tusk will be to replace A by a model on the central line. Consider the walk to A. Let M be the last model on the central line which includes A, $M_1 \in M$ the next model of the walk of the same cardinality with $A \in M_1 \cup \{M_1\}$ and $M_0 \in C^{|M|}(M)$ isomorphic to it model. By the definition of the walk (Definition 2.4, One to Four Continuations), the models M_0, M_1 are the immediate predecessors of M. Replace A by $A_1 = \pi_{M_1,M_0}[A]$. Note that $\tilde{B}_1 := \pi_{M_1,M_0}[\tilde{B}]$ is an isomorphic image of B. If A_1 and $\tilde{B}_1$ satisfy (2), then also A and $\tilde{B}$ do.

Replace A by A_1 and consider the walk to A_1. After finitely many steps we will reach the desired situation.

Assume now that both A and B are on the central line. Then $B \in A$, since both are on the central line and $otp(B) = otp(\tilde{B}) < otp(A)$.

Consider now the walk to $\tilde{B}$. Let M be the last model on the central line which includes $\tilde{B}$,M_0, M_1 its immediate predecessors with $\tilde{B} \in M_1 \cup \{M_1\}$ and $M_0 \in C^{|M|}(M)$.

If $A \in M_0 \cup \{M_0\}$, then we move everything to M_1 putting M_1 on the central line and apply an appropriate inductive assumption (the number of steps required to move from B to $\tilde{B}$ is now reduced, since B is replaced by $\pi_{M_0,M_1}(B)$ which is needed to move to the same $\tilde{B}$).

If $M_0 \in A$, then $M \subseteq A$. So $M_1 \in A$. We make a switch below A (actually below M) to move M_1 to the central line. Then $\pi_{M_0,M_1}(B)$ will be on the new central line as well as A (and M). As above the induction applies here to A and $\pi_{M_0,M_1}(B)$.

□