On changing cofinality of partially ordered sets

Moti Gitik *

July 29, 2009

Abstract

It is shown that under GCH every poset preserves its cofinality in any cofinality preserving extension. On the other hand, starting with ω measurable cardinals, a model with a partial ordered set which can change its cofinality in a cofinality preserving extension is constructed.

1 Introduction

Let $\mathcal{P} = \langle P, \preceq \rangle$ be a partially ordered set and $A \subseteq P$. We denote by $\mathrm{cof}(\mathcal{P})$ *the cofinality* of $\mathcal{P}$, i.e. $\min\{|S| \mid S \subseteq P, \forall a \in P \exists b \in S \quad a \preceq b\}$ and by $\mathrm{cof}_{\mathcal{P}}(A)$ *the outer cofinality* of A in $\mathcal{P}$, i.e. $\min\{|S| \mid S \subseteq P, \forall a \in A \exists b \in S \quad a \preceq b\}$.

Let W be an extension of V, i.e. $W \supseteq V$ and they have the same ordinals. It is called *a cofinality preserving extension* if every regular cardinal of V remains such in W. Clearly that if W is a cofinality preserving extension of V, then it preserves cofinality of ordinals and linear ordered sets.

A natural question that was raised by S. Watson, A. Dow and appears as Problem 5.8 on A. Miller list [3] asks:

whether a cofinality preserving extension always preserves cofinalities of partially ordered sets.

We address here this question. It is shown (Theorem 2.7) that an affirmative answer follows from GCH or even from the assumption $\forall \kappa \quad 2^\kappa < \kappa^{+\omega}$. In the last section, starting with ω measurable cardinals, we force a partially ordered set which changes its cofinality in a cofinality preserving extension.

*We like to thank to Assaf Rinot for pointing our attention to the problem of changing cofinality of poset and his remarks and corrections. We are grateful to the referee of the paper for his long list of corrections and requests for clarification of the exposition. The author was partially supported by ISF Grant 234/08.

Definition 1.1 Let $\mathcal{P} = \langle P, \preceq \rangle$ be a partially ordered set (further poset).

1. $\mathcal{P}$ is called *a cofinality changeable poset* if there is a cofinality preserving extension W such that
$$(\mathrm{cof}(\mathcal{P}))^V \neq (\mathrm{cof}(\mathcal{P}))^W.$$

2. $\mathcal{P}$ is called *an outer cofinality changeable poset* if there is a cofinality preserving extension W such that for some $A \subseteq P$
$$(\mathrm{cof}_{\mathcal{P}}(A))^V \neq (\mathrm{cof}_{\mathcal{P}}(A))^W).$$

3. $\mathcal{P}$ is called *an unboundedly outer cofinality changeable poset* if there is a cofinality preserving extension W such that for every $\lambda < |P|$ there is $A \subseteq P$ of outer cofinality above λ such that
$$(\mathrm{cof}_{\mathcal{P}}(A))^V \neq (\mathrm{cof}_{\mathcal{P}}(A))^W).$$

Taking the negations we define *a cofinality preserving poset*, *an outer cofinality preserving poset* and *an unboundedly outer cofinality preserving poset*.

Clearly (1) $\rightarrow$ (3) $\rightarrow$ (2).

2 The strength and GCH type assumptions.

Let us start with the following simple observation:

Proposition 2.1 *Suppose that there is an outer cofinality changeable poset. Then there is an inner model with a measurable cardinal.*

Proof. Let $\mathcal{P} = \langle P, \preceq \rangle$ be an outer cofinality changeable poset. Suppose $|P| = \kappa$. Without loss of generality we can assume that $P = \kappa$. Pick a cofinality preserving extension W and $A \subseteq \kappa$, $A \in V$ such that $(\mathrm{cof}_{\mathcal{P}}(A))^V \neq (\mathrm{cof}_{\mathcal{P}}(A))^W)$. Then there is $S \in W, S \subseteq \kappa$ such that

1. for every $\tau \in A$ there is $\nu \in S$ with $\tau \preceq \nu$

2. $(\mathrm{cof}_{\mathcal{P}}(A))^V > |S|$.

Note that if $X \supseteq S$, then for every $\tau \in A$ there is $\nu \in X$ with $\tau \preceq \nu$. This means that any $X \in V$ which covers S has cardinality above those of S. Remember that W is cofinality preserving extension, hence $\mathcal{K}^W$- the core model computed in W must be the same as $\mathcal{K}^V$-the one computed in V. Just any disagreement between $\mathcal{K}^W$ and $\mathcal{K}^V$ will imply in turn that this two models have completely different structure of cardinals, they will disagree about regularity of cardinals, about successors of singular cardinals etc. This in turn will imply that V and W disagree about their cardinals, since by [1], [4] $\mathcal{K}^W$ computes correctly successors of singular in W cardinals and the same is true about $\mathcal{K}^V$ and V. So we must to have $\mathcal{K}^V = \mathcal{K}^W$.

Now, by the Dodd-Jensen Covering Lemma [1] there is an inner model a measurable cardinal. $\square$

Remark 2.2 Note that the argument of 2.1 implies that if $V = \mathcal{K}$ or at least every measurable cardinal of $\mathcal{K}$ is regular in V (where $\mathcal{K}$ is the core model), then there are at least ω measurable cardinals in $\mathcal{K}$. We refer to [1], [4] for the relevant stuff on Core Models and Covering Lemmas.

Proposition 2.3 *Suppose that κ is the least possible cardinality of a changeable cofinality poset then*

1. *κ is singular in V*
2. *κ is a measurable or a limit of measurable cardinals in an inner model*
3. *the cofinality of a witnessing poset is κ.*

Proof. Let $P = \langle \kappa, \preceq \rangle$ be such poset of the smallest possible cardinality. Suppose that $\mathrm{cof}(P) = \lambda$ in V and $\mathrm{cof}(P) = \eta < \lambda$ in a cofinality preserving extension W of V. Pick in W a cofinal subset $S \subseteq \kappa$ of P of the size η.

Let $A \in V, A \subseteq \kappa$ be of the smallest size including S. Then, in V, $|A| \geq \lambda$. Moreover, the inner cofinality of A (i.e. $\mathrm{cof}(\langle A, A^2 \cap \preceq \rangle)$) is at least λ, since clearly A is cofinal in P.

Suppose for a moment that κ has a cofinality above η in V and hence also in W. Then, for some $\alpha < \kappa$ we will have $S \subseteq \alpha$. Hence $|A| < \kappa$. Consider $\langle A, A^2 \cap \preceq \rangle$. This is a poset of cardinality below κ. Its cofinality is at least λ in V and $\eta < \lambda$ in W. So we have a contradiction to the minimality of κ.

Hence $\mathrm{cof}(\kappa) \leq \eta$ and every set A in V which covers S must have cardinality at least κ.

This implies the conclusions 1 and 2. For 2 note that if κ is not a measurable in the core model and measurable cardinals of it are bounded in κ by some $\delta < \kappa$, then S can be covered by a subset of $A \in V$ of cardinality δ.

Let us prove 3 now. Suppose that $\mathrm{cof}(P) < \kappa$ in V. Let X be a subset of P witnessing this. Consider a poset $P' = \langle X, \preceq \cap X^2 \rangle$. We claim that P' changes its cofinality in W, as well. Thus let $S \in W$ be cofinal in P. For each $\nu \in S$ pick $\nu' \in X$ with $\nu \preceq \nu'$ (it is possible since X is cofinal in P). Let S' be the set consisting of all this ν''s. Then, $S' \subseteq X$, S' cofinal in X and $|S'| \leq |S|$.
Which contradicts the minimality of κ.
□

Proposition 2.4 *Assume that $\aleph_\omega$ is a strong limit cardinal. Then every poset $\mathcal{P} = \langle P, \preceq \rangle$ and for every $A \subseteq P$ with $|A| \leq \aleph_\omega$ the outer cofinality of A cannot be changed in any cofinality preserving extension.*
In particular, $\aleph_\omega$ is a strong limit cardinal implies that every poset of cardinality at most $\aleph_\omega$ is outer cofinality preserving and hence cofinality preserving.

Proof. Suppose otherwise. Let $\mathcal{P} = \langle P, \preceq \rangle$ be a poset $A \subseteq P$ of cardinality at most $\aleph_\omega$, W be a cofinality preserving extension of V such that

$$(\mathrm{cof}_{\mathcal{P}}(A))^W < (\mathrm{cof}_{\mathcal{P}}(A))^V.$$

Clearly, $(\mathrm{cof}_{\mathcal{P}}(A))^V \leq |A| \leq \aleph_\omega$, and hence $(\mathrm{cof}_{\mathcal{P}}(A))^W < \aleph_\omega$. Let $(\mathrm{cof}_{\mathcal{P}}(A))^W = \aleph_{k^*}$, for some $k^* < \omega$. Pick $S \subseteq P$ of cardinality $\aleph_{k^*}$ cofinal for A. Pick in V an enumeration (possibly with repetitions) $\{a_\alpha \mid \alpha < \aleph_\omega\}$ of A. For each $n < \omega$ we set

$$A_n = \{a_\alpha \mid \alpha < \aleph_n\}.$$

Fix $n < \omega$.

Claim 1 $(\mathrm{cof}_{\mathcal{P}}(A_n))^V \leq \aleph_{k^*}$.

Proof. For each $s \in S$ set

$$A_n^s = \{a \in A_n \mid a \preceq s\}.$$

Clearly, every A_n^s is in V and

$$A_n = \bigcup_{s \in S} A_n^s,$$

since S is cofinal for A_n. But note that the sequence $\langle A_n^s \mid s \in S \rangle$ need not be in V. We have

$$Z := \{A_n^s \mid s \in S\} \subseteq \mathcal{P}^V(A_n).$$

It is a subset of $\mathcal{P}^V(A_n)$ consisting of at most $\aleph_{k^*}$ elements. But $|A_n| \leq \aleph_n$.
Let us argue that there is $Y \subseteq \mathcal{P}^V(A_n), Y \in V, |Y| \leq \aleph_{k^*}$ such that $Y \supseteq \{A_n^s \mid s \in S\}$.
Thus, $|\mathcal{P}^V(A_n)|^V < \aleph_\omega$, since $\aleph_\omega$ is a strong limit cardinal in V. Let $|\mathcal{P}^V(A_n)|^V = \aleph_t$, for some $t < \omega$. Fix

$$F : \mathcal{P}^V(A_n) \longleftrightarrow \aleph_t, F \in V.$$

Set $Z_0 = F''Z$. Clearly, it's enough to find some $Z^* \in V, |Z^*|^V \leq \aleph_{k^*}$ such that $Z^* \supseteq Z_0$. If $t \leq k^*$, then just take $Z^* = \aleph_t$. Suppose that $t > k^*$. Then there is $\eta_0 < \aleph_t$ such that $Z_0 \subseteq \eta_0$, since $\aleph_t$ is a regular cardinal in W. Find $t_1 < t$ such that $|\eta_0| = \aleph_{t_1}$. Let $F_1 : \eta_0 \longleftrightarrow \aleph_{t_1}$ be a witnessing function in V. Set $Z_1 = F_1''Z_0$. Clearly, it's enough to find some $Z^* \in V, |Z^*|^V \leq \aleph_{k^*}$ such that $Z^* \supseteq Z_1$, since then $F_1^{-1}{}''Z^* \in V$ will be a desired cover of Z_0. If $t_1 \leq k^*$, then just take $Z^* = \aleph_{t_1}$. If $t_1 > k^*$, then there is $\eta_1 < \aleph_{t_1}$ such that $Z_1 \subseteq \eta_1$, since $\aleph_{t_1}$ is a regular cardinal in W. Continue the process. After finitely many steps we must drop below k^* which in turn will provide a cover.

By shrink Y in V further if necessary we can assume that for every $X \in Y$ there is $t \in P$ such that $x \preceq t$, for every $x \in X$. Working in V, we pick for each $X \in Y$ an element $a_X \in P$ such that $x \preceq a_X$, for every $x \in X$.
Consider now the set

$$\{a_X \mid X \in Y\}.$$

Clearly, it is in V, is cofinal for A_n and has cardinality at most $\aleph_{k^*}$. So we are done.
$\square$ of the claim.

Now, work in V and for each $n < \omega$ pick E_n to be a cofinal for A_n subset of P of cardinality at most $\aleph_{k^*}$. Then

$$E = \bigcup_{n<\omega} E_n$$

is in V, is cofinal for A and $|E| \leq \aleph_{k^*}$. Contradiction.
$\square$

Actually, the proof above provides a bit more information. Thus, the following holds:

Proposition 2.5 *Let W be a cofinality preserving extension of V and κ a cardinal which is singular strong limit in V. Assume that for some $\delta, \mathrm{cof}(\kappa) \leq \delta < \kappa$ the following form of covering holds between V and W:*

$$\forall \mu, \delta \leq \mu < \kappa \quad ([\mu]^{\leq \delta})^V \textit{ is unbounded in } ([\mu]^{\leq \delta})^W.$$

Then for every poset $\mathcal{P} = \langle P, \preceq \rangle \in V$ and $A \subseteq P$ of cardinality at most κ we must have

$$(\mathrm{cof}_{\mathcal{P}}(A))^V \geq \delta \textit{ implies } (\mathrm{cof}_{\mathcal{P}}(A))^W \geq \delta.$$

The proposition 2.6 allows to gain an additional strength in a strong limit case.

Proposition 2.6 *Let κ be a strong limit singular cardinal. Suppose that $\mathcal{P}$ is an unboundedly outer cofinality changeable poset of cardinality κ. Then, in $\mathcal{K}$, κ is a limit of measurable cardinals.*

Proof. Suppose that measurable cardinals of $\mathcal{K}$ are bounded in κ by some $\delta < \kappa$. Then for any W with $\mathcal{K}^W = \mathcal{K}$ we will have by [1],[4]

$$\forall \mu, \delta \leq \mu < \kappa \quad ([\mu]^{\leq \delta})^{\mathcal{K}} \text{ is unbounded in } ([\mu]^{\leq \delta})^W.$$

Now the previous proposition applies.
□

Theorem 2.7 *Suppose that GCH holds (or even for each λ $2^\lambda < \lambda^{+\omega}$). Then there is no a cofinality changeable poset.*

Proof. Suppose otherwise. Let κ be the least cardinal on which there is a cofinality changeable poset. Let $\mathcal{P} = \langle \kappa, \preceq \rangle$ be such poset. Then, by 2.3, κ is singular and $\mathrm{cof}(\mathcal{P}) = \kappa$. Suppose that W is a cofinality preserving extension of V with $(\mathrm{cof}(\mathcal{P}))^W = \eta < \kappa$. Pick some $S = \{\eta_i \mid i < \eta\} \in W$ witnessing the cofinality.

Lemma 2.8 *Let $A \subseteq \kappa$ be a set with $\mathrm{cof}(|A|) > \eta$. Then, in V, $\mathrm{cof}_{\mathcal{P}}(A) < |A|$.*

Proof. For each $i < \eta$ we consider a set

$$A_i = \{\tau \in A \mid \tau \preceq \eta_i\}.$$

Note that each A_i is in V just by its definition. So, $X = \{A_i \mid i < \eta\} \subseteq \mathcal{P}(A)^V$. By the assumption, in V we have $|\mathcal{P}(A)| = |A|^{+n}$, for some $n < \omega$. Now, using regularity of each of the cardinals $|A|^+, |A|^{++}, ..., |A|^{+n}$ and $\mathrm{cof}(|A|) > \eta$ it is easy to find in V a set Y such that

1. $|Y| < |A|$
2. $Y \supseteq X$
3. $Y \subseteq \mathcal{P}(A)$.

Consider (again in V) the following set:

$$Y^* = \{B \in Y \mid \exists \nu \forall \tau \in B \quad \tau \preceq \nu\}.$$

Then Y^* still includes X, since each element of X has such property. Remember that S is cofinal. In particular, for each $\tau \in A$ there is $i < \eta$ such that $\tau \preceq \eta_i$ and, hence $\tau \in A_i$. Then, in V, the following holds:

(*) for each $\tau \in A$ there is $B \in Y^*$ with $\tau \in B$.

Now working in V we pick for each $B \in Y^*$ some $\nu(B)$ such that $\tau \preceq \nu(B)$, for each $\tau \in B$. Set

$$T = \{\nu(B) \mid B \in Y^*\}.$$

Then $T \in V$, $|T| \leq |Y^*| < |A|$ and by (*) we have that for each $\tau \in A$ there is $\nu \in T$ with $\tau \preceq \nu$. Hence, T witnesses $\mathrm{cof}_{\mathcal{P}}(A) < |A|$.

$\square$ of the lemma.

Work in V. Let us prove that for each $A \subseteq \kappa \quad (\mathrm{cof}_{\mathcal{P}}(A))^V \leq \eta$, by induction on $|A|$. If $|A| \leq \eta$ then this is trivial. Suppose that the statement is true for each cardinal less than ρ. Let us prove it for ρ. Let $A \subseteq \kappa$ of cardinality ρ. If $\rho = \mu^+$, for some μ, then by Lemma 2.8 we have $\mathrm{cof}_{\mathcal{P}}(A) < |A| = \rho$. Suppose that ρ is a limit cardinal. If $\mathrm{cof}(\rho) > \eta$, then again by Lemma 2.8 we have $\mathrm{cof}_{\mathcal{P}}(A) < |A| = \rho$. Let finally ρ be a limit cardinal of cofinality at most η. Pick a cofinal in ρ sequence $\langle \rho_i \mid i < \eta \rangle$. We present A as a union of sets A_i, $i < \eta$ such that $|A_i| = \rho_i$. Apply the induction to each of A_i's. We find T_i of cardinality η witnessing $\mathrm{cof}_{\mathcal{P}}(A_i) \leq \eta$. Set $T = \bigcup_{i<\eta} T_i$. Then $|T| = \eta$ and T witnesses $\mathrm{cof}_{\mathcal{P}}(A) \leq \eta$. This completes the induction.

In particular, we obtain that in V, $\mathrm{cof}(\mathcal{P}) \leq \eta$. Contradiction.

$\square$

3 Consistency results on outer cofinality.

Our aim will be to show that it is possible to change outer cofinality of many subsets of a poset in a cofinality preserving extension.

Theorem 3.1 *1. Suppose GCH and there are ω measurable cardinals then there is an outer cofinality changeable poset.*

2. Suppose GCH and there are ω^2 measurable cardinals then there is an unboundedly outer cofinality changeable poset.

Proof. Let us prove (2). Suppose that $\kappa_0 < \kappa_1 < ... < \kappa_n ...$ $(n < \omega)$ is an increasing sequence of cardinals such that for each $n < \omega$

- there is an increasing sequence of measurable cardinals $\langle \kappa_{nm} \mid m < \omega \rangle$ with limit κ_n

Assume that $\kappa_n < \kappa_{n+1,0}$ for each $n < \omega$. Let $\kappa = \bigcup_{n<\omega} \kappa_n$. We fix a normal ultrafilter U_{nm} over κ_{nm} for each $n, m < \omega$. Fix in addition a scale $\langle f_{n\alpha} \mid \kappa_n \leq \alpha < \kappa_n^+ \rangle$ of functions in $\prod_{m<\omega} \kappa_{nm}$ (mod finite) such that for every $g \in \prod_{m<\omega} \kappa_{nm}$ there is $\alpha, \kappa_n \leq \alpha < \kappa_n^+$ with $f_{n\alpha}(m) > g(m)$, for all $m < \omega$.

Let $0 < n < \omega$, $\nu, \nu' \in [\kappa_{n-1}^+, \kappa_{n0})$ and $\alpha \in [\kappa_n, \kappa_n^+)$. Set

$$\alpha \preceq_n \nu \text{ iff } f_{n\alpha}(0) \leq \nu$$

and

$$\nu \preceq_n \nu' \text{ iff } \nu \leq \nu'.$$

Let $n, m < \omega, 0 < m$, $\nu, \nu' \in [\kappa_{nm-1}, \kappa_{nm})$ and $\alpha \in [\kappa_n, \kappa_n^+)$. Set

$$\alpha \preceq_n \nu \text{ iff } f_{n\alpha}(m) \leq \nu$$

and

$$\nu \preceq_n \nu' \text{ iff } \nu \leq \nu'.$$

Set

$$P = \bigcup_{n<\omega} [\kappa_n, \kappa_n^+) \cup \bigcup_{n,m<\omega} (\kappa_{nm}, \kappa_{n,m+1}) = [\kappa_{00}, \kappa)$$

and

$$\preceq = \bigcup_{n<\omega} \preceq_n .$$

Let $\mathcal{P} = \langle P, \preceq \rangle$.

Lemma 3.2 $\operatorname{cof}(\langle P, \preceq \rangle) = \kappa$.

Proof. Just note that inside each of the intervals $(\kappa_{nm}, \kappa_{n,m+1})$ we have $\prec = <$ and $\operatorname{cof}_{\mathcal{P}}((\kappa_{nm}, \kappa_{n,m+1})) = \kappa_{n,m+1}$.
$\square$

The next lemma follows from the definition of the partial order $\preceq$.

Lemma 3.3 *For each $n < \omega$,*

$\operatorname{cof}_{\mathcal{P}}([\kappa_n, \kappa_n^+)) = \kappa_{n0}$

Force now a new ω sequence to each of $\prod_{m<\omega}\kappa_{nm}$ using the Magidor iteration of the length ω the Diagonal Prikry Forcings with $\langle U_{nm} \mid m<\omega\rangle$, $n<\omega$. We refer to [5], [2] for the relevant stuff on Prikry type forcings.

Let W be a resulting extension. Then V and W agree about cofinalities of all ordinals. Denote by b_n the generic Prikry sequence in $\prod_{m<\omega}\kappa_{nm}$.

Lemma 3.4 *For every $n<\omega$, the set b_n witnesses $(\mathrm{cof}_{\mathcal{P}}([\kappa_n,\kappa_n^+)))^W=\aleph_0$*

Proof. Fix $n<\omega$. For each $\alpha\in[\kappa_n,\kappa_{n+1})$, we have

$$f_{n\alpha}(m)<b_n(m)$$

for all but finitely many m's. Pick some such m. Then $\alpha\prec b_n(m)$.
$\square$ of the lemma.
So, for each $n<\omega$, the set $[\kappa_n,\kappa_n^+))$ changes its outer cofinality from κ_{n0} to ω. The outer cofinality of every interval $(\kappa_{nm},\kappa_{nm+1})$ and hence those of $\mathcal{P}$ remains unchanged.
$\square$

If $V=\mathcal{K}$ or at least every measurable cardinal of $\mathcal{K}$ is regular in V, then the assumptions of the theorem above are optimal by 2.2.
Let us show now how to construct outer cofinality changeable posets allowing V to differ essentially from $\mathcal{K}$. The assumptions used below will be optimal by 2.1.

Theorem 3.5 *Suppose GCH. Let κ be a measurable cardinal. Then in a cardinals and GCH preserving extension there is a poset of cardinality κ^+ which is outer cofinality changeable.*

Proof. Let U be a normal ultrafilter over κ. Force with $\mathcal{P}_U$- the Prikry with U. Let $G\subseteq\mathcal{P}_U$ be generic and $\langle\kappa_n \mid n<\omega\rangle$ be the Prikry sequence derived from G. Assume that $\kappa_0\geq\aleph_1$. Consider $\langle\kappa_{2n} \mid n<\omega\rangle$, i.e. the subsequence consisting of all even members of the original sequence. It is still a Prikry sequence, by the Mathias criterion of genericity for the Prikry forcing.
Set $V_1=V[\langle\kappa_{2n} \mid n<\omega\rangle]$. We define in V_1 a poset $\langle P,\preceq\ \rangle$ and $A\subseteq P$ which will change its outer cofinality to ω in $V[G]$. The construction will be similar to those of 3.1 and $\langle\kappa_{2n+1} \mid n<\omega\rangle$ will be a new cofinal set for A.

Using GCH in V_1, we pick a scale $\langle f_\alpha \mid \alpha<\kappa^+\rangle$ of functions in $\prod_{n<\omega}\kappa_{2n}$ (mod finite) such that for every $g\in\prod_{m<\omega}\kappa_{2n}$ there is $\alpha,\kappa\leq\alpha<\kappa^+$ with $f_\alpha(m)>g(m)$, for all $m<\omega$.

Let $\nu,\nu'<\kappa_0$ and $\alpha\in[\kappa,\kappa^+)$. Set

$$\alpha\preceq\nu \text{ iff } f_\alpha(0)\leq\nu$$

and

$$\nu \preceq \nu' \text{ iff } \nu \leq \nu'.$$

Let $m, 0 < m < \omega$, $\nu, \nu' \in [\kappa_{2(m-1)}, \kappa_{2m})$ and $\alpha \in [\kappa, \kappa^+)$. Set

$$\alpha \preceq \nu \text{ iff } f_\alpha(m) \leq \nu$$

and

$$\nu \preceq \nu' \text{ iff } \nu \leq \nu'.$$

Set $P = \kappa^+$ and $\mathcal{P} = \langle P, \preceq \rangle$.

Lemma 3.6 *In* V_1
$\mathrm{cof}(\langle P, \preceq \rangle) = \kappa$.

Proof. Just note that inside each of the intervals $[\kappa_{2m}, \kappa_{2(m+1)})$ we have $\prec = <$ and $\mathrm{cof}_{\mathcal{P}}([\kappa_m, \kappa_{2(m+1)})) = \kappa_{2(m+1)}$.
□

The next lemma follows from the definition of the partial order $\preceq$.

Lemma 3.7 *In* V_1
$\mathrm{cof}_{\mathcal{P}}([\kappa, \kappa^+)) = \kappa_0$

The following lemma is standard.

Lemma 3.8 *Let* $g \in \prod_{m<\omega} \kappa_{2m}$ *be a function in* V_1. *Then there is* $h_g : [\kappa]^{<\omega} \to \kappa, h_g \in V$ *such that for every* $m < \omega$, *we have* $g(m) = h_g(0, \kappa_0, ..., \kappa_{2(m-1)})$.

Now we turn to $V[G]$.

Lemma 3.9 *Let* $g \in \prod_{m<\omega} \kappa_{2m}$ *be a function in* V_1. *Then* $\langle \kappa_{2m+1} \mid m < \omega \rangle$ *eventually dominates* g.

Proof. Just pick h_g as in Lemma 3.8 and argue that in V, the empty condition in the Prikry forcing forces

$$\text{"}\exists m_0 \forall m \geq m_0 \quad h_g(0, \kappa_0, ..., \kappa_{2(m-1)}) < \kappa_{2m-1}\text{"}.$$

□

Now, for each $\alpha \in [\kappa, \kappa^+)$, we have

$$f_\alpha(m) < \kappa_{2m-1},$$

for all but finitely many $m < \omega$. Pick some such m. Then $\alpha \prec \kappa_{2m-1}$. Hence $A = [\kappa, \kappa^+)$ changes its outer cofinality to ω in $V[G]$.
□

Remark 3.10 It is easy to modify the construction above in order to produce a set A of outer cofinality κ that changes it to ω. Thus, we just split the interval $[\kappa, \kappa^+)$ into ω sets $\langle S_i \mid i < \omega \rangle$ of cardinality κ^+. Set $\alpha \preceq \nu$, for $\alpha \in S_i$, as it was defined above, but only with $m \geq i$.

Similar, combining constructions of 3.5 and 3.1(2), it is possible to show the following.

Theorem 3.11 *Suppose GCH and there are ω measurable cardinals. Let κ be a limit of ω measurable cardinals. Then in a cardinals and GCH preserving extension there is a poset of cardinality κ which is an unboundedly outer cofinality changeable poset.*

Let us sketch an argument for getting down to $\aleph_{\omega+1}$ for outer cofinality changeable posets. Similar ideas work for unboundedly outer cofinality changeable posets, but with $\aleph_{\omega+1}$ replaced by $\aleph_{\omega^2}$.
Suppose first that κ is a limit of measurable cardinals $\langle \kappa_n \mid n < \omega \rangle$. Let U_n be a normal ultrafilter over κ_n for each $n < \omega$. Use the product of the Levy collapses to turn κ_0 into $\aleph_1$, κ_1 into $\aleph_3$, κ_2 into $\aleph_5$ etc. Then κ will become $\aleph_\omega$. Define a poset $\mathcal{P} = \langle P, \preceq \rangle$ as above (those for 3.1(1)). Finally, use U_n's and the closure of corresponding collapses in order to add a diagonal Prikry sequence preserving all the cardinals. It will witness that the outer cofinality of $[\aleph_\omega, \aleph_{\omega+1})$ is ω.

A construction with a single measurable is a bit less direct. Thus, let κ be a measurable cardinal and U a normal measure over κ. The basic idea will be to turn the even members of a Prikry sequence into $\aleph_n$'s and than to add the odd members to the model as witness for outer cofinality ω of $[\aleph_\omega, \aleph_{\omega+1})$. It should be done accurately in order to avoid further collapses.

Define the forcing as follows.

Definition 3.12 The forcing P_U consists of all sequences

$$\langle \langle \nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n} \rangle, \langle g_0, g_2, ..., g_{2n} \rangle, T, F, f \rangle$$

such that

1. T is tree with the root $\langle \nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n} \rangle$ splitting all the time above the root into sets in U,

2. $g_0 \in Col(\omega, < \nu_0)$,

3. $g_{2k} \in Col(\nu^+_{2(k-1)}, < \nu_{2k})$, for each $k, 1 \leq k \leq n$,

4. $f \in Col(\nu^+_{2n}, < \kappa))$,

5. $F(\vec{\rho}) \in Col(\rho^+_{2m}, < \kappa)$, if for some $m, n < m < \omega$, $\vec{\rho}$ is from the level $2m$ of T, and $F(\vec{\rho}) \in Col(\rho^+_{2m}, < \kappa)$, if $\vec{\rho}$ is from the level $2m+1$ of T,

6. for every $m, n < m < \omega$, if $\vec{\rho}, \vec{\rho'}$ are from the same level $2m$ of T and $\rho_{2m} = \rho'_{2m}$, then $F(\vec{\rho}) = F(\vec{\rho'})$,

7. for every $m, n < m < \omega$, if $\vec{\rho}, \vec{\rho'}$ are from the same level $2m+1$ of T and $\rho_{2m} = \rho'_{2m}, \rho_{2m+1} = \rho'_{2m+1}$, then $F(\vec{\rho}) = F(\vec{\rho'})$.

Define the forcing order $\leq$ and the direct extension order $\leq^*$.

Definition 3.13 Let $p = \langle \langle \nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n} \rangle, \langle g_0, g_2, ..., g_{2n} \rangle, T, F, f \rangle$, $p' = \langle \langle \nu'_0, \nu'_1, ..., \nu'_{2n'-1}, \nu'_{2n'} \rangle, \langle g'_0, g_2, ..., g'_{2n'} \rangle, T', F', f' \rangle \in P_U$. Then $p \geq^* p'$ iff

1. $n = n'$,

2. $\nu_k = \nu'_k$, for every $k \leq 2n$,

3. $g_{2k} \supseteq g_{2k'}$, for every $k \leq n$,

4. $T \subseteq T'$,

5. $f \supseteq f'$,

6. $F(\vec{\rho}) \supseteq F'(\vec{\rho})$, for every $\vec{\rho} \in T$.

Definition 3.14 Let $p = \langle \langle \nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n} \rangle, \langle g_0, g_2, ..., g_{2n} \rangle, T, F, f \rangle$, $p' = \langle \langle \nu'_0, \nu'_1, ..., \nu'_{2n'-1}, \nu'_{2n'} \rangle, \langle g'_0, g_2, ..., g'_{2n'} \rangle, T', F', f' \rangle \in P_U$. Then $p \geq p'$ iff

1. $n \geq n'$,

2. $T \subseteq T'$,

3. the sequence $\langle \nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n} \rangle$ end extends the sequence $\langle \nu'_0, \nu'_1, ..., \nu'_{2n'-1}, \nu'_{2n'} \rangle$ and the new elements come from T',

4. $F(\vec{\rho}) \supseteq F'(\vec{\rho})$, for every $\vec{\rho} \in T$,

5. $g_{2k} \supseteq g'_{2k}$, for every $k \leq n'$,

6. $g_{2n'} \supseteq f'$, if $n > n'$,

7. $g_{2k} \supseteq F'(\langle \nu_0, \nu_1, ..., \nu_{2k} \rangle) \cup F'(\langle \nu_0, \nu_1, ..., \nu_{2k}, \nu_{2k+1} \rangle)$, for each $k, n' < k \leq n$,

8. $f \supseteq F'(\langle \nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n} \rangle)$.

Then $\langle P_U, \leq, \leq^* \rangle$ is a Prikry type forcing notion.
It turns even members of the Prikry sequence into $\aleph_n$'s ($n > 0$), κ will be $\aleph_\omega$ and all the cardinals above κ will be preserved. Thus the cardinals above κ^+ are preserved due to κ^{++}-c.c. of the forcing. κ^+ is preserved since otherwise it would change its cofinality to some $\delta < \kappa$, which is impossible by the standard arguments, see [2] for example.

We define now a projection P_{even} of P_U.

Definition 3.15 Let $p = \langle \langle \nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n} \rangle, \langle g_0, g_2, ..., g_{2n} \rangle, T, F, f \rangle \in P_U$. Set

$$\pi(p) = \langle \langle \nu_0, \nu_2, ..., \nu_{2n} \rangle, \langle g_0, g_2, ..., g_{2n} \rangle, T', F', f' \rangle,$$

where

1. $f' = g_{2n} \cup f$,

2. T' is the subtree of T consisting of points from all even levels $2m, n < m < \omega$ of T which are limits of of ordinals from the level $2m - 1$ of T,

3. F' is the restriction of F to the even levels of T which appear in T'.

Note that $\langle P_{even}, \leq, \leq^* \rangle$ is just the standard forcing which simultaneously changes the cofinality of κ to ω and turns the elements of the Prikry sequence into $\aleph_n$'s.

Lemma 3.16 *The function π defined in 3.15 is a projection of the forcing $\langle P_U, \leq \rangle$ onto $\langle P_{even}, \leq \rangle$.*

Proof. Let $p = \langle\langle\nu_0, \nu_1, ..., \nu_{2n-1}, \nu_{2n}\rangle, \langle g_0, g_2, ..., g_{2n}\rangle, T, F, f\rangle \in P_U$ and $q = \langle\langle\nu_0, \nu_2, ..., \nu_{2i}\rangle, \langle g_0, g_2, ..., g_{2i}\rangle, S, H, h\rangle$ be an extension of $\pi(p)$ in P_{even}. We need to find $r \geq p$ in P whose projection to P_{even} is stronger than q. It is easy using the definition of π to put ordinals from T between elements of $\langle\nu_{2n+2}, .., \nu_{2i}\rangle$. Similar we add levels from T between those of S. Let T_1 be a resulting tree. Shrink it if necessary, such that for any two successive elements $\vec{\rho}^\frown\tau$ and $\vec{\rho}^\frown\tau^\frown\nu$, with $\vec{\rho}^\frown\tau$ from an even level, we have $\nu > \sup(\mathrm{dom}(H(\vec{\rho}^\frown\tau)$. Now, we can put together F and H over such tree. This produces $r \in P_U$ as desired.
□

Let now G be a generic subset of P_U, G_{even} be its projection to P_{even}. Then G_{even} is a generic subset of P_{even} by 3.16. Set $V_1 = V[G_{even}]$. Clearly, κ is $\aleph_\omega$ in V_1 and $W = V[G]$ is a cofinality preserving extension of V_1 by the forcing P_U/G_{even}. Denote by $\langle\kappa_n \mid n < \omega\rangle$ the Prikry sequence added by G. Then $\langle\kappa_{2m} \mid m < \omega\rangle$ is the Prikry sequence produced by G_{even}. The following lemma is analogous to Lemma 3.9.

Lemma 3.17 *Let $g \in \prod_{m<\omega}\kappa_{2m}$ be a function in V_1. Then $\langle\kappa_{2m+1} \mid m < \omega\rangle$ dominates g.*

Work in V_1 and define a poset $\mathcal{P} = \langle P, \preceq\ \rangle$ as in 3.5. Then $A = [\aleph_\omega, \aleph_{\omega+1})$ changes its outer cofinality to ω in W, as witnessed by $\{\kappa_{2m+1} \mid m < \omega\}$.

4 The main consistency result.

In this section our aim will be to construct a model which has a changeable cofinality poset.

Assume GCH. Let $\langle\kappa_n \mid 0 < n < \omega\rangle$ be an increasing sequence of measurable cardinals with limit κ. For each $n, 0 < n < \omega$ fix a normal ultrafilter U_n over κ_n.

We would like first to force a partial order over κ.

Definition 4.1 Q consists of sequences $q = \langle q_n \mid n < \omega\rangle$ so that

1. $q_0 = \langle a_0(q), \preceq_{q,0}\ \rangle$ and, for each $n, 0 < n < \omega, \quad q_n = \langle a_n(q), \top_{q,n}\rangle$ such that
   (a) $a_n(q) \subseteq \kappa$, for each $n < \omega$,
   (b) $|a_n(q)| \leq \kappa_n$, for each $n, 1 \leq n < \omega$,
   (c) $|a_0(q)| < \aleph_0$,
   (d) $\preceq_{q,0}$ is a partial order on $a_0(q)$,
   (e) for every $n, 0 < n < \omega$ we have $\top_{q,n} \subseteq [a_n(q)]^2$ is a binary relation on $a_n(q)$. We do not require it to be a partial order etc. No limitations are put on $\top_{q,n}$.

2. $n < m$ implies $a_n(q) \subseteq a_m(q)$.

3. Let $\alpha, \beta < \kappa$ and $n, 1 \leq n < \omega$ be the least such that $\alpha < \kappa_n^+$. Then $\alpha \prec_{q,0} \beta$ implies that

   - if $k, 1 \leq k < \omega$ is the least such that $\beta < \kappa_k^+$, then $k \geq n$,
   - $\langle \alpha, \beta \rangle \notin \top_{q,n}$.

Define the forcing order on Q as follows.

Definition 4.2 Let $p, q \in Q$. Set $p \geq_Q q$ iff for each $n < \omega$

1. $a_n(p) \supseteq a_n(q)$
2. $\preceq_{p,0} \cap [a_0(q)]^2 = \preceq_{q,0}$
3. $\top_{p,n} \cap [a_n(q)]^2 = \top_{q,n}$

For each $n < \omega$ let $Q_{>n}$ consists of all $\langle p_m \mid \omega > m > n \rangle$ such that for some $\langle p_k \mid k \leq n \rangle$ we have $\langle p_i \mid i < \omega \rangle \in Q$.
Let $G_{>n}$ be a generic subset of $Q_{>n}$. Define $Q_{\leq n}$ to be the set of all sequences $\langle p_k \mid k \leq n \rangle$ such that for some $\langle p_m \mid \omega > m > n \rangle \in G_{>n}$ we have $\langle p_i \mid i < \omega \rangle \in Q$.

The next lemma is immediate.

Lemma 4.3 *For each $n < \omega$*

1. *the forcing $Q_{>n}$ is κ_{n+1}^+-closed,*
2. *the forcing $Q_{\leq n}$ satisfies κ_n^{++}-c.c. in $V^{Q_{>n}}$,*
3. *$Q \simeq Q_{>n} * Q_{\leq n}$.*

Let G be a generic subset of Q. Work in $V[G]$.
Set

$$\preceq = \bigcup \{ \preceq_{q,0} \mid q \in G \}.$$

Define $\mathcal{P} = \langle \kappa, \preceq \rangle$.

Lemma 4.4 *Let $A \in V$ be a subset of κ_n^+ of cardinality κ_n^+, for some $n, 0 < n < \omega$. Then* $\mathrm{cof}_{\mathcal{P}}(A) = \kappa_n^+$.

Proof. Suppose otherwise. Let B be a set of cardinality κ_n witnessing this. Using 4.3, we can find such $B \in V$.
Work in V. Let $q \in Q$. Extend it to p with $B \subseteq a_n(p)$. Then $|a_n(p)| = \kappa_n$. Pick some $\alpha \in A \setminus (a_n(p) \cup \kappa_n)$. Note that $\alpha \notin a_0(p)$, since $a_0(p) \subseteq a_n(p)$. Extend p to r by adding to it $\langle \alpha, \beta \rangle \in \top_{r,n}$ for each $\beta \in a_n(p)$.
Then, r will force in Q that α is not $\prec$ below any element of B. Thus, otherwise there will be $t \geq r$ and $\beta \in B$ so that $\beta \in a_0(t)$ and $\alpha \prec_{t,0} \beta$. By Definition 4.1(3), then $\langle \alpha, \beta \rangle \notin \top_{t,n}$, which is impossible, since $t \geq r$ and $\langle \alpha, \beta \rangle \in \top_{r,n}$.
$\square$

In particular, the lemma above implies the following:

Lemma 4.5 $\mathrm{cof}(\mathcal{P}) = \kappa$.

Let us turn now to the fixed normal ultrafilters U_n over κ_n's.
Clearly they will not be anymore ultrafilters in $V[G]$ and it is impossible to extend them to normal ultrafilters there since $2^{\aleph_0} > \kappa$. But still U_n's turn to be good enough for our purposes.
Let, for each $n, 1 \leq n < \omega$,

$$j_n : V \longrightarrow M_n \simeq {}^{\kappa_n}V/U_n$$

be the canonical elementary embedding (in V).
We will also consider iterated ultrapowers. Thus, for any $n, 0 < n < \omega$, $U_{\leq n}$ denotes the ultrafilter over $\kappa_1 \times ... \times \kappa_n$ which is the product $U_1 \times ... \times U_n$. Let

$$j_{\leq n} : V \longrightarrow M_{\leq n} \simeq {}^{\kappa_1 \times ... \times \kappa_n}V/U_{\leq n}$$

be the canonical elementary embedding (in V).
For every $n, m, 0 < m < n < \omega$, let $j_{\leq m, \leq n} : M_{\leq m} \longrightarrow M_{\leq n}$ be the induced embedding.
Denote also by $j_{\leq 0}$ the identity map, $j_{\leq n}$ by $j_{\leq 0, \leq n}$ and V by $M_{\leq 0}$.
For each $n, 0 < n < \omega$, and $p \in j_{\leq n}(Q)$ let $f_p : \kappa_1 \times ... \times \kappa_n \longrightarrow Q$ be a function which represents p in $M_{\leq n}$. If $1 \leq m < n$ and $\eta \in \kappa_1 \times ... \times \kappa_m$ then let $f_{p,\eta} : \kappa_{m+1} \times ... \times \kappa_n \rightarrow Q$ be defined as follows:

$$f_{p,\eta}(\nu) = f_p(\eta^\frown \nu).$$

Then $j_{\leq m, \leq n}(f_{p, \langle \kappa_1, ..., \kappa_m \rangle})(\kappa_{m+1}, ..., \kappa_n) = j_{\leq n}(f_p)(\langle \kappa_1, ..., \kappa_n \rangle) = p$.

Fix $n, 1 \leq n < \omega$. Denote by $Q^*_{<n}$ the following set:

$$\{p \in j_{\leq n}(Q) \mid \exists q \in Q_{\geq n} \quad p = \langle p_0, ..., p_{n-1} \rangle^\frown j_{\leq n}(q)\}.$$

I.e. it is the set of all elements p of $j_{\leq n}(Q)$ such that $\langle p_n, p_{n+1}, ...\rangle$ is an image of an element of $Q_{\geq n}$.

Recall that by 4.3(1), the forcing $Q_{\geq n}$ is κ_n^+-closed. Hence, it preserves the measurability of κ_n, as well as all κ_m's with $1 \leq m \leq n$. Moreover, if $G(Q_{\geq n})$ is a generic subset of $Q_{\geq n}$, then

$$G(j_{\leq n}(Q_{\geq n})) = \{t \in j_{\leq n}(Q_{\geq n}) \mid \exists q \in G(Q_{\geq n}) \quad j_{\leq n}(q) \geq t\}$$

will be $M_{\leq n}$-generic subset of $j_{\leq n}(Q_{\geq n})$. Thus, if $D \in M_{\leq n}$ is a dense open subset of $j_{\leq n}(Q_{\geq n})$, then there will be f_D which represents D such that for every $\langle \alpha_1, ..., \alpha_n\rangle \in \kappa_1 \times ... \times \kappa_n$, $f_D(\alpha_1, ..., \alpha_n)$ is a dense open subset of $Q_{\geq n}$. Consider

$$E = \bigcap_{\langle \alpha_1,...,\alpha_n\rangle \in \kappa_1 \times ... \times \kappa_n} f_D(\alpha_1, ..., \alpha_n).$$

It is a dense open subset of $Q_{\geq n}$ by the closure of $Q_{\geq n}$. Pick some $q \in E \cap G(Q_{\geq n})$. Then $j_{\leq n}(q) \in D$ and we are done.

Define now explicitly a projection $\pi_{\leq l, \leq n}$ from the forcing $Q^*_{<n}$ to $j_{\leq l}(Q)$, for each $l < n$. This projection will be defined on a dense subset of $Q^*_{<n}$, rather than on $Q^*_{<n}$.

Definition 4.6 An element p of $Q^*_{<n}$ is called *separated* iff for every $\beta \in a_0(p)$ there is β^* such that

- $j_{\leq n}(\beta^*) \in a_0(p)$,
- let $n(\beta)$ be the least $l, 1 \leq l < \omega$ with $\beta < j_{\leq n}(\kappa_l^+)$. Then $n(\beta) = n(j_{\leq n}(\beta^*))$.
- $(\gamma, \beta) \in \top_{p,m}$ iff $(\gamma, j_{\leq n}(\beta^*)) \in \top_{p,m}$, for every $\gamma \in a_m(p)$, $m, 1 \leq m \leq n(\beta)$,
- $(\beta, \gamma) \in \top_{p,m}$ iff $(j_{\leq n,}(\beta^*), \gamma) \in \top_{p,m}$, for every $\gamma \in a_m(p)$, $m, 1 \leq m \leq n(\beta)$,
- $\gamma \prec_{p,0} \beta$ implies $\gamma \prec_{p,0} j_{\leq n}(\beta^*)$, for every $\gamma \in a_0(p)$,
- $\beta \prec_{p,0} \gamma$ implies $j_{\leq n}(\beta^*) \prec_{p,0} \gamma$, for every $\gamma \in a_0(p)$.

Lemma 4.7 *The set of separated conditions is dense in $Q^*_{<n}$.*

Proof. Let $q \in Q^*_{<n}$. We construct a separated condition $p \geq q$.
Let $\beta \in a_0(q)$. Pick some β^* such that $j_{\leq n}(\beta^*) \in j''_{\leq n}\kappa^+_{n(\beta)} \setminus (a_{n(\beta)}(q) \cup j_{\leq n}(\kappa_{n(\beta)}))$. Such β^* exists since the set $j''_{\leq n}\kappa_m^+$ is unbounded in $j_{\leq n}(\kappa_m^+) = (j_{\leq n}(\kappa_m)^+)^{M_{\leq n}}$, for any $m, 1 \leq m < \omega$, and $|a_{n(\beta)}(q)| \leq j_{\leq n}(\kappa_{n(\beta)})$.

Extend q to p by adding β^*'s for each $\beta \in a_0(q)$ as follows.
Fix $\beta \in a_0(q)$. For each $m, n(\beta) < m < \omega$, we add $j_{\leq n}(\beta^*)$ to $a_m(q)$, if it was not already there without making any new commitments about $\top_{q,m}$.
Suppose now that $1 \leq m \leq n(\beta)$. Again we add $j_{\leq n}(\beta^*)$ to $a_m(q)$. Note that $a_m(q) \subseteq a_{n(\beta)}(q)$, and hence $j_{\leq n}(\beta^*) \notin a_m(q)$. Let $\gamma \in a_m(q)$. We set

$$(\gamma, j_{\leq n}(\beta^*)) \in \top_{p,m} \text{ iff } (\gamma, \beta) \in \top_{q,m},$$

and

$$(j_{\leq n,}(\beta^*), \gamma) \in \top_{p,m} \text{ iff } (\beta, \gamma) \in \top_{q,m}.$$

Finally let $m = 0$. We add $j_{\leq n}(\beta^*)$ to $a_0(q)$. Let $\gamma \in a_0(q)$. Set

$$\gamma \prec_{p,0} j_{\leq n}(\beta^*) \text{ iff } \gamma \prec_{q,0} \beta,$$

and

$$j_{\leq n}(\beta^*) \prec_{p,0} \gamma \text{ iff } \beta \prec_{q,0} \gamma.$$

Note that the choice of $\top_{p,m}$ allows us to define $\preceq_{p,0}$ this way.

Now we preform the above construction inductively running on all β's in $a_0(q)$.
$\square$

Denote by $Q^{**}_{<n}$ the set of all separated elements of $Q^*_{<n}$.
Suppose that $p = \langle p_k \mid k < \omega \rangle \in Q^{**}_{<n}$. Set for every $l < n$

$$\pi_{\leq l, \leq n}(p) = \langle j^{-1}_{\leq l, \leq n}{}'' p_m \mid m < \omega \rangle.$$

Lemma 4.8 *For each $l < n$, $\pi_{\leq l, \leq n}$ is a projection map.*

Proof. Let $p \in Q^{**}_{<n}$ and $q = \langle q_k \mid k < \omega \rangle \geq_{j \leq l Q} \pi_{\leq l, \leq n}(p)$. Consider $\langle j_{\leq l, \leq n}{}'' q_k \mid k < \omega \rangle$. We need to extend to a condition $r \in Q^*_{<n}$ stronger than p. Set $r_k = (j_{\leq l, \leq n}{}'' q_k) \cup p_k$, for each $k, 0 < k < \omega$. Set $a_0(r) = (j_{\leq l, \leq n}{}'' a_0(q)) \cup a_0(p)$. Note that $j_{\leq l, \leq n}{}'' a_0(q) = j_{\leq l, \leq n}(a_0(q))$, since $a_0(q)$ is finite. Define $\preceq_{r,0}$ to be the transitive of $(j_{\leq l, \leq n}(\prec_{q,0})) \cup \preceq_{p,0}$, i.e.

$$\alpha \preceq_{r,0} \beta \text{ iff}$$

1. $\alpha, \beta \in a_0(p)$ and $\alpha \preceq_{p,0} \beta$, or
2. $\alpha, \beta \in j_{\leq l, \leq n}(a_0(q))$ and $j^{-1}_{\leq l, \leq n}(\alpha) \preceq_{q,0} j^{-1}_{\leq l, \leq n}(\beta)$, or
3. $\alpha \in j_{\leq l, \leq n}(a_0(q)) \setminus a_0(p), \beta \in a_0(p) \setminus j_{\leq l, \leq n}(a_0(q))$ and there is $\gamma \in a_0(p) \cap j_{\leq l, \leq n}(a_0(q))$ such that $j^{-1}_{\leq l, \leq n}(\alpha) \preceq_{a_0(q)} j^{-1}_{\leq l, \leq n}(\gamma), \gamma \preceq_{a_0(p)} \beta$, or

4. $\beta \in j_{\leq l,\leq n}(a_0(q)) \setminus a_0(p), \alpha \in a_0(p) \setminus j_{\leq l,\leq n}(a_0(q))$ and there is $\gamma \in a_0(p) \cap j_{\leq l,\leq n}(a_0(q))$ such that $j^{-1}_{\leq l,\leq n}(\gamma) \preceq_{a_0(q)} j^{-1}_{\leq l,\leq n}(\beta), \alpha \preceq_{a_0(p)} \gamma$.

Note that the requirement (3) of Definition 4.1 is satisfied. Thus, suppose that $\alpha \preceq_{r,0} \beta$. We need to check that $\langle \alpha, \beta \rangle \notin \top_{r,n(\alpha)}$. We may assume that one of α, β is the image of an element of $a_0(q) \setminus a_0(\pi_{\leq l,\leq n}(p))$ and the other is in $a_0(p) \setminus j''_{\leq l,\leq n} a_0(p)$.
Suppose first that $\alpha \in j''_{\leq l,\leq n} a_0(q) \setminus a_0(p)$ and $\beta \in a_0(p) \setminus j''_{\leq l,\leq n} a_0(p)$. There must be γ such that $j_{\leq l,\leq n}(\gamma) \preceq_{p,0} \beta$ and $j^{-1}_{\leq l,\leq n}(\alpha) \preceq_{q,0} \gamma$. Turn to $j_{\leq n}(\beta^*)$. We must have $j_{\leq l,\leq n}(\gamma) \preceq_{p,0} j_{\leq n}(\beta^*)$, since p is separated. Hence, $j^{-1}_{\leq l,\leq n}(\alpha) \preceq_{q,0} j_{\leq l}(\beta^*)$. Then, $\langle j^{-1}_{\leq l,\leq n}(\alpha), j_{\leq l}(\beta^*) \rangle \notin \top_{q,n(\alpha)}$ and $n(\alpha) \leq n(\beta^*)$. Finally, since p is separated, $\langle \alpha, \beta \rangle \in \top_{p,n(\alpha)}$ would imply that $\langle \alpha, j_{\leq n}(\beta^*) \rangle \in \top_{p,n(\alpha)}$ as well, which is impossible.
Suppose now $\beta \in j''_{\leq l,\leq n} a_0(q) \setminus a_0(p)$ and $\alpha \in a_0(p) \setminus j''_{\leq l,\leq n} a_0(p)$. There must be γ such that $j_{\leq l,\leq n}(\gamma) \succeq_{p,0} \alpha$ and $\gamma \preceq_{q,0} j^{-1}_{\leq l,\leq n}(\beta)$. Consider $j_{\leq n}(\alpha^*)$. We must have $j_{\leq l,\leq n}(\gamma) \succeq_{p,0} j_{\leq n}(\alpha^*)$, since p is separated. Hence, $j^{-1}_{\leq l,\leq n}(\beta) \succeq_{q,0} j_{\leq l}(\alpha^*)$. Then, $\langle j_{\leq l}(\alpha^*), j^{-1}_{\leq l,\leq n}(\beta) \rangle \notin \top_{q,n(\alpha)}$ and $n(\alpha) = n(\alpha^*)$. Finally, since p is separated, $\langle \alpha, \beta \rangle \in \top_{p,n(\alpha)}$ would imply that $\langle j_{\leq n}(\alpha^*), \beta \rangle \in \top_{p,n(\alpha)}$ as well, which is impossible.
□

Lemma 4.9 *Let $n < m < \omega, \alpha < \kappa_n^+$. Then the set*

$$\{\beta < \kappa_m \mid \alpha \prec \beta\}$$

is U_m-positive (in $V[G]$).

Proof. Suppose otherwise. Then there is $Y \in U_m$ such that $\alpha \not\preceq \beta$, for every $\beta \in Y$.
Work in V. Pick $q \in Q$ forcing this statement. Let $k, 0 < k \leq n$ be the least such that $\alpha < \kappa_k^+$.

Consider in M_m the condition $j_m(q)$. Then $\kappa_m \in j_m(\kappa_m) \setminus j_m(\bigcup_{i \leq k} a_{q,i})$, since $|\bigcup_{i \leq k} a_{q,i}| < \kappa_m$. Extend $j_m(q)$ to p as follows. Add $\alpha = j_m(\alpha)$ to $a_{j_m(q),i}$ for each $i \leq k$, if it was not already there. If $\alpha \notin a_{q,0}$, then we set $\alpha \prec_{p,0} \kappa_m$. Note that $\langle \alpha, \kappa_m \rangle \notin \top_{p,k}$, even if $\alpha \in a_k(p)$, since κ_m is new. If $\alpha \in a_{q,0}$, then let us set $\gamma \prec_{p,0} \kappa_m$, for each $\gamma \in a_{q,0}$ such that $\gamma \preceq_{q,0} \alpha$. Note that, if $\gamma \preceq_{q,0} \alpha$, then either $\gamma < \omega_1$ or $\gamma \geq \omega_1$ and then $\gamma < \kappa_{k+1}^+$, by 4.1(3). So $\langle \gamma, \kappa_m \rangle \notin \top_{p,i}$, for each $i \leq k$, even if $\gamma \in a_i(p)$, since κ_m is new.

Then, in M_m,

$$p \geq j_m(q) \text{ and } p \Vdash_{j_m(Q)} \alpha = j_m(\alpha) \prec \kappa_m.$$

But $Y \in U_m$ implies $\kappa_m \in j_m(Y)$. Hence we have in M_m an element of $j_m(Y)$ which is above $j_m(\alpha)$. Then, by the elementarity, there is an element of Y which is above α. Contradiction. $\square$

Similar and unfortunately, the following holds as well.

Lemma 4.10 *Let $n < m < \omega, \alpha < \kappa_n^+$. Then the set*

$$\{\beta < \kappa_m \mid \alpha \perp \beta\}$$

is U_m-positive.

Remark 4.11 Note that the proof of Lemma 4.9 provides a bit more information. Thus, if $r = j_m(q)$ and $\alpha < \kappa_n^+$ does not belong to $a_{q,0}$, then r can be extended to a condition p by either adding $\alpha \prec_{p,0} \kappa_m$ or $\alpha \perp_{p,0} \kappa_m$. Just Definition 4.1 puts no restrictions here.

Let us define now a forcing P similar to the diagonal Prikry forcing. Instead of sets of measure one positive sets will be used. Also a small addition will be made in order to insure that a countable cofinal subset will be added to $\mathcal{P}$.

Definition 4.12 A sequence $\vec{p} = \langle p(0), p(1), ..., p(n), ...\rangle$ will be called *a good sequence* iff

1. $p(0) \in Q$.

   For every $l, n, l < n < \omega$ the following hold:

2. $p(n) \in Q^{**}_{<n}$,
3. $\pi_{\leq l, \leq n}(p(n)) \leq p(l)$,
4. $j_{\leq l, \leq n}(p(l)) \leq p(n)$,
5. there is $n^* < \omega$ such that for every $n \geq n^*$ we have $a_0(p(n)) = j''_{\leq n^*, \leq n} a_0(p(n^*))$,
6. If $\alpha \in a_{n(\alpha)}(p(l))$, then for each $k < \omega$ big enough

   $$j_{\leq l, \leq k}(\alpha) \in a_{n(\alpha)}(p(k)) \text{ implies } \langle j_{\leq l, \leq k}(\alpha), \kappa_k\rangle \notin \top_{p(k), n(\alpha)},$$

   where $n(\alpha)$ is the least $n, 1 \leq n < \omega$ such that $\alpha < \kappa_n^+$.

Lemma 4.13 *Let $\langle p(n) \mid n < \omega\rangle$ be a good sequence. Then here are functions $\langle f_{p(n)} \mid 0 < n < \omega\rangle$ such that*

1. *for every* $n, 0 < n < \omega$, $[f_{p(n)}]_{U_{\leq n}} = p(n)$,

2. *for every* $m, 1 \leq m < \omega$ *and every sequence* $\vec{\nu} = \langle \nu_1, ..., \nu_m \rangle$ *the sequence*

$$\langle f_{p(m)}(\vec{\nu}), [f_{p(m+1),\vec{\nu}}]_{U_{m+1}}, ..., [f_{p(k),\vec{\nu}}]_{U_{m+1} \times ... \times U_k}, ... \mid m < k < \omega \rangle$$

satisfies Definition 4.12 only the second member is in M_{m+1} *instead of* M_1*, etc.*

Proof. Note that for every $m, 1 \leq m < \omega$, the sequence $\langle p(m+1), p(m+2), ..., p(n), ... \mid m+1 \leq n < \omega \rangle \in M_{\leq m}$. Just for each $n, 1 \leq n < \omega$ we have $p(n) \in M_{\leq n}$, and given $m, 1 \leq m < n < \omega$, $M_{\leq n} \simeq M_{\leq m}^{j_{\leq n}(\kappa_{m+1}) \times ... \times j_{\leq n}(\kappa_n)} / j_{\leq n}(U_{m+1} \times ... \times U_n)$.
If $f_{p(n)}$ represents $p(n)$ in $M_{\leq n}$, i.e. $[f_{p(n)}]_{U_1 \times ... \times U_n}$, or equivalently $j_{\leq n}(f_{p(n)})(\kappa_1, ..., \kappa_n) = p(n)$, then

$$j_{\leq m, \leq n}((j_{\leq m}(f_{p(n)}))_{\langle \kappa_1, ..., \kappa_m \rangle})(\kappa_{m+1}, ..., \kappa_n) = p(n).$$

Now the elementarity of the embeddings provides the desired conclusion.
□

Given a good sequence $\vec{p} = \langle p(0), p(1), ..., p(n), ... \rangle$ with $p(0) \in G$. We would like to associate to it a tree $T(\vec{p}) \subseteq [\kappa]^{<\omega}$.
Define it level by level. Thus

$$Suc_{T(\vec{p})}(\langle\rangle) = \{\nu < \kappa_1 \mid f_{p(1)}(\nu) \in G\},$$

where $f_{p(1)} : \kappa_1 \to Q$ is the function given by Lemma 4.13 which represents $p(1)$ in $M_1 = M_{\leq 1}$.
Suppose that the level k of the tree is defined and $\eta = \langle \eta_1, ..., \eta_k \rangle$ is on this level. Set

$$Suc_{T(\vec{p})}(\eta) = \{\nu < \kappa_{k+1} \mid f_{p(k+1)}(\eta^\frown \nu) \in G\},$$

where $f_{p(k+1)} : \kappa_1 \times ... \times \kappa_k \times \kappa_{k+1} \to Q$ is the function given by Lemma 4.13 which represents $p(1)$ in $M_{\leq k+1}$.

The next lemma shows that such trees are wide.

Lemma 4.14 *Let* $\vec{p} = \langle p(0), p(1), ..., p(n), ... \rangle$ *be a good sequence with* $p(0) \in G$ *and* $T(\vec{p})$ *be the associated tree. Then for each* $\eta \in T(\vec{p})$ *the set* $Suc_{T(\vec{p})}(\eta)$ *is* $U_{|\eta|+1}$*-positive.*

Proof. We show the statement by induction on levels.
Let us prove first that $Suc_{T(\vec{p})}(\langle\rangle) \in U_1^+$. Suppose otherwise. Then there is $A \in U_1$ such that for every $\nu \in A$ we have $f_{p(1)}(\nu) \notin G$. Consider (in V) the following set

$$D = \{q \in Q \mid \exists \nu \in A \quad q \geq f_{p(1)}(\nu) \text{ or } \forall \nu \in A \quad q, f_{p(1)}(\nu) \text{ are incompatible } \}.$$

Clearly, D is a dense subset of Q. So, $G \cap D \neq \emptyset$. Pick some $q \in G \cap D, q \geq p(0)$. Then q is incompatible with each $f_{p(1)}(\nu)$, $\nu \in A$. Hence $j_1(q)$ is incompatible with $p(1)$. This contradicts Definition 4.12(3), since the projection of $p(1)$ to Q is weaker than $p(0)$.
Suppose now that $\eta \in T(\vec{p})$, $\eta \neq \langle\rangle$. Let us show that the set $Suc_{T(\vec{p})}(\eta)$ is $U_{|\eta|+1}$-positive. Suppose otherwise. Then there is $A \in U_{|\eta|+1}$ such that for every $\nu \in A$ we have $f_{p(|\eta|+1)}(\eta^\frown \nu) \notin G$. Note that $f_{p(|\eta|)}(\eta) \in G$ by the definition of $T(\vec{p})$. Consider (in V) the following set

$$D = \{q \in Q \mid \exists \nu \in A \quad q \geq f_{p,\eta}(\nu) \text{ or } \forall \nu \in A \quad q, f_{p,\eta}(\nu) \text{ are incompatible } \}.$$

Clearly, D is a dense subset of Q. So, $G \cap D \neq \emptyset$. Pick some $q \in G \cap D, q \geq f_{p(|\eta|)}(\eta)$. Then q is incompatible with each $f_{p,\eta}(\nu)$, $\nu \in A$. Hence $j_{|\eta|+1}(q)$ is incompatible with $[f_{p,\eta}]_{U_{|\eta|+1}}$. This contradicts Lemma 4.13, Definition 4.12(3), since the projection of $[f_{p,\eta}]_{U_{|\eta|+1}}$ to Q is weaker than $f_{p(|\eta|)}(\eta)$.

Definition 4.15 We call a tree $T \subseteq [\kappa]^{<\omega}$ *a good tree* iff there is a good sequence $\vec{p}$ such that $T = T(\vec{p})$.

If T is a good tree and $\eta \in T$, then denote by T_η the set

$$\{\eta' \in T \mid \eta' \geq_T \eta\}.$$

We call such T_η's good trees with trunk η.
Let be T a good tree, as witnessed by a good sequence $\vec{p} = \langle p(0), p(1), ..., p(n), ...\rangle$, and $\eta \in T$ be a point from a level m. Consider the sequence

$$\langle f_{p(m)}(\eta), [f_{p(m+1),\eta}]_{U_{m+1}}, ..., [f_{p(n),\eta}]_{U_{m+1} \times ... \times U_n}, ...\rangle.$$

Denote it by

$$\vec{p}_\eta = \langle f_{p(m)}(\eta), p(m+1)_\eta, ..., p(n)_\eta, ...\rangle.$$

We call it a good sequence for T_η.

Definition 4.16 Let $\vec{p} = \langle p(n) \mid n < \omega\rangle, \vec{q} = \langle q(n) \mid n < \omega\rangle$ be good sequences. We set $\vec{p} \geq \vec{q}$ iff $p(n) \geq q(n)$, for each $n < \omega$.

Definition 4.17 The forcing notion P consists of all pairs $\langle \eta, T\rangle$ such that

1. $\eta = \langle \eta_1, ..., \eta_n\rangle$ for some $n < \omega$ and $\eta_1 < \kappa_1 < \eta_2 < \kappa_2 < ... < \kappa_{n-1} < \eta_n < \kappa_n$

2. T is a good tree with the trunk η.

Definition 4.18 Let $\langle \eta, T(\vec{p}) \rangle, \langle \eta', T'(\vec{p'}) \rangle \in P$. Set $\langle \eta, T(\vec{p}) \rangle \leq \langle \eta', T'(\vec{p'}) \rangle$ iff

1. $\eta' \in T(\vec{p})$,
2. η' is an end extension of η,
3. $T'(\vec{p'}) \subseteq T(\vec{p})$,
4. $\vec{p}_{\eta'} \leq \vec{p'}$.

Definition 4.19 Let $\langle \eta, T \rangle, \langle \eta', T' \rangle \in P$. Set $\langle \eta, T \rangle \leq^* \langle \eta', T' \rangle$ iff

1. $\langle \eta, T \rangle \leq \langle \eta', T' \rangle$,
2. $\eta = \eta'$.

Lemma 4.20 *Let $\langle p(0), p(1), ..., p(n), ... \mid n < \omega \rangle$ be a good sequence and $q'(0) \geq p(0)$. Then there are $\langle q(0), q(1), ..., q(n), ... \mid n < \omega \rangle$ such that*

1. *$q(0) \geq q'(0)$,*
2. *$\langle q(0), q(1), ..., q(n), ... \mid n < \omega \rangle$ is a good sequence,*
3. *for each $n < \omega$, $q(n) \geq p(n)$,*

Proof. Combine $j_{\leq n}(q'(0))$ with $p(n)$, for each $n < \omega$, and turn the results into separated conditions. Let us argue that (6) of Definition 4.12 can be easily satisfied. Thus, we need to take care of α's in $a_0(q'(0)) \setminus a_0(p(0))$ such that $j_{\leq l}(\alpha)$ does not appear in $a_{n(\alpha)}(p(l))$, for $1 \leq l < \omega$. Given such α, we consider k's above $n(\alpha)$. Then $j_{\leq k}(\alpha) \notin a_{n(\alpha)}(p(k))$. But then we are free to set

$$\langle j_{\leq k}(\alpha), \kappa_k \rangle \notin \top_{q(k), n(\alpha)}.$$

Denote the result for each $n < \omega$, by $q(n)$. By the construction we have $q(n) \geq p(n)$.
$\square$

The following is a slightly more general statement with a similar proof.

Lemma 4.21 *Let $\langle p(0), p(1), ..., p(n), ... \mid n < \omega \rangle$ be a good sequence, $l < \omega$ and $q'(m) \geq p(m)$, for every $m \leq l$. Then there are $\langle q(0), ..., q(n), ... \mid n < \omega \rangle$ such that*

1. *$q(0) = q'(0)$,*
2. *$q(m) \geq q'(m)$, for every $m \leq l$,*

3. $\langle q(0), q(1), ..., q(n), ... \mid n < \omega\rangle$ *is a good sequence,*

4. for each $n < \omega$*,* $q(n) \geq p(n)$*.*

Force with $\langle P, \leq \rangle$. Let $\langle \eta_1, ..., \eta_n, ...\rangle$ be a generic sequence.

Lemma 4.22 *The sequence* $\langle \eta_1, ..., \eta_n, ...\rangle$ *is cofinal in* $\mathcal{P}$*.*

Proof. Work in V. Let $\vec{p} = \langle p(0), p(1), ..., p(n), ... \mid n < \omega\rangle$ be a good sequence and $\alpha < \kappa$. We may assume that $\alpha \in a_0(p(0))$, just otherwise extend $\vec{p}$ and add α.
Recall that $a_0(p(0))$ is finite. So there are only finitely many elements of $a_0(p(0))$ below α in the order $\preceq_{p(0),0}$. Let $\langle \alpha_i \mid i \leq i^*\rangle$ be an enumeration of all these elements with $\alpha_{i^*} = \alpha$. By Definition 4.12 (6) there is $k^* < \omega$ such that for every $k, k^* \leq k < \omega, i \leq i^*$

$$\langle j_{\leq k}(\alpha_i), \kappa_k\rangle \notin \top_{p(k),n(\alpha_i)}.$$

Use now (5) of Definition 4.12 and find $n^* < \omega$ such that for every $n \geq n^*$ we have $a_0(p(n)) = j''_{\leq n^*,\leq n} a_0(p(n^*))$.
Pick any $k, k^* + n^* \leq k < \omega$. Consider $a_0(p(k))$. By Definition 4.12, $j_{\leq k}(\alpha_i) \in a_0(p(k))$, for every $i \leq i^*$, but $\kappa_k \notin a_0(p(k))$, since $\kappa_k \notin j''_{\leq n^*,\leq k}\kappa_k$, as the critical point of j_k with $k > n^*$. On the other hand we have $\langle j_{\leq k}(\alpha_i), \kappa_k\rangle \notin \top_{p(k),n(\alpha_i)}$. Hence it is possible to extend $p(k)$ to some $q(k)$ by adding κ_k to each $a_m(p(k))$ ($m < \omega$) and setting

$$j_{\leq \kappa_k}(\alpha_i) \prec_{q(k),0} \kappa_k,$$

for each $i \leq i^*$. Finally we find a good sequence above $\vec{p}$ which accommodates $q(k)$ using 4.20, 4.21.
□

The main issue now will be to show that the forcing P preserves cofinalities or which is equivalent here- preserves cardinals.

Lemma 4.23 *The forcing* $Q * \langle \underset{\sim}{P}, \leq, \leq^* \rangle$ *satisfies the Prikry Property, i.e. for every statement* σ *and* $\langle p, \langle \eta, \underset{\sim}{T}\rangle\rangle \in Q * \underset{\sim}{P}$ *there are* $q, \underset{\sim}{R}$ *such that*

- $p \leq_Q q$,
- $q \Vdash_Q \langle \eta, \underset{\sim}{T}\rangle \leq^* \langle \eta, \underset{\sim}{R}\rangle$
- $\langle q, \langle \eta, \underset{\sim}{R}\rangle\rangle \| \sigma$

Proof. Let σ be a statement and $\langle\langle p, \langle\eta, \underset{\sim}{T}\rangle\rangle \in Q * \underset{\sim}{P}$. Suppose for simplicity that η is the empty sequence.

Let $\vec{p} = \langle p(0), p(1), ..., p(n), ...\rangle$ be a good sequence with $T(\vec{p}) = T$ and $p(0) \geq p$.

For each $\nu < \kappa_1$ we define by induction a sequence $\vec{r}(\nu) = \langle r(1,\nu), r(2,\nu), ..., r(n,\nu), ... \mid 1 \leq n < \omega\rangle$ by induction as follows.

Suppose that for each $\nu' < \nu$ the sequence $\vec{r}(\nu')$ is defined. Define $\vec{r}(\nu)$.

Case 1. $\nu = \nu' + 1$.

Consider first the sequence $\vec{r}_*(\nu') = \langle r_*(1,\nu'), ..., r_*(n,\nu')... \mid 1 \leq n < \omega\rangle$ which is obtained from $\vec{r}(\nu')$ as follows:

- for each $1 \leq k < \omega$, let $(r_*(k,\nu'))_0 = j_{\leq k}{}'' p(k)_0$;
- for each $k, m < \omega, 1 \leq k, m$ let $(r_*(k,\nu'))_m = (r(k,\nu'))_m$.

Set

$$\vec{r}(\nu) = \vec{r}_*(\nu'),$$

unless there is a good sequence $\langle r'(n) \mid n < \omega\rangle$ such that

- $\langle r'(n) \mid n < \omega\rangle \geq \vec{r}_*(\nu')$,
- $r'(1)_{\langle\nu\rangle} \Vdash_Q (\langle\nu, \underset{\sim}{T}(\langle r'(2)_{\langle\nu\rangle}, ..., r'(n)_{\langle\nu\rangle}, ... \mid 2 \leq n < \omega\rangle)\rangle \| \sigma)$.

In this case let $\vec{r}(\nu)$ be such a sequence.

Case 2. ν is a limit ordinal.

Then we define first a sequence $\vec{r}_*(\nu)$ as follows:

- for each $1 \leq k < \omega$, let $(r_*(k,\nu))_0 = j_{\leq k}{}'' p(k)_0$;
- for each $k, m < \omega, 1 \leq k, m$ let $(r_*(k,\nu))_m = \bigcup_{\nu'<\nu}(r(k,\nu'))_m$.

Set

$$\vec{r}(\nu) = \vec{r}_*(\nu),$$

unless there is a good sequence $\langle r'(n) \mid n < \omega\rangle$ such that

- $\langle r'(n) \mid n < \omega\rangle \geq \vec{r}_*(\nu)$,
- $r'(1)_{\langle\nu\rangle} \Vdash_Q (\langle\nu, \underset{\sim}{T}(\langle r'(2)_{\langle\nu\rangle}, ..., r'(n)_{\langle\nu\rangle}, ... \mid 2 \leq n < \omega\rangle)\rangle \| \sigma)$.

In this case let $\vec{r}(\nu)$ be such a sequence.

Define $\vec{r}_*(\kappa_1)$ as in **Case 2** above. Let $\nu < \kappa_1$. Denote by $\vec{r}_*(\kappa_1)^\frown\nu$ the sequence $\langle \vec{r}_*(\kappa_1)^\frown\nu(n) \mid 1 \leq n\rangle$ such that

- for each $1 \leq k < \omega$, $\quad (\vec{r}_*(\kappa_1)^\frown\nu(k))_0 = r(k,\nu)_0$;
- for each $k, m < \omega, 1 \leq k, m$ let $(\vec{r}_*(\kappa_1)^\frown\nu(k))_m = (r_*(k,\kappa_1))_m$.

Now we have the following:
for each $\nu < \kappa_1$, if there is a good sequence $\langle r'(n) \mid n < \omega\rangle$ such that

- $\langle r'(n) \mid n < \omega\rangle \geq \vec{r}_*(\kappa_1)^\frown\nu$,
  and
- $r'(1)_{\langle\nu\rangle} \Vdash_Q (\langle \nu, \underset{\sim}{T}(\langle r'(2)_{\langle\nu\rangle}, ..., r'(n)_{\langle\nu\rangle}, ... \mid 2 \leq n < \omega\rangle)\rangle \| \sigma)$,

then

$$\vec{r}_*(\kappa_1)^\frown\nu(1))_{\langle\nu\rangle} \Vdash_Q (\langle \nu, \underset{\sim}{T}(\langle \vec{r}_*(\kappa_1)^\frown\nu(2)_{\langle\nu\rangle}, ..., \vec{r}_*(\kappa_1)^\frown\nu(n)_{\langle\nu\rangle}, ... \mid 2 \leq n < \omega\rangle)\rangle \| \sigma).$$

This defines a splitting of κ_1 into three sets. By shrinking to a set in U_1, if necessary, we assume that every $\nu < \kappa_1$ is in the same part of the partition.
Suppose that we are in the situation in which σ is forced by every $\nu < \kappa_1$. Define then a good sequence $\vec{q} = \langle q(n) \mid n < \omega\rangle$ which is above $\vec{p}$ and forces σ, i.e.

$$q(0) \Vdash_Q (\langle\langle\rangle, T(\vec{q})\rangle \Vdash \sigma).$$

Thus,

- for each $m, 1 \leq m < \omega$ set $q(1)_m = j_1((r_*(1,\kappa_1))_m)$,
- $(q(0))_0 = \pi_{\leq 0, \leq 1}{}''[\langle r(1,\nu)_0 \mid \nu < \kappa_1\rangle]_{U_1}$;
- for each $1 \leq k < \omega$, $\quad (q(k))_0 = \pi_{\leq k, \leq k+1}{}''[\langle r(k+1,\nu)_0 \mid \nu < \kappa_1\rangle]_{U_1}$;
- for each $k, m < \omega, 1 \leq k, m$ let $q(k+1)_m = j_1((r_*(k,\kappa_1))_m)$.

If σ is not decided a set of ν's in U_1, then we proceed similar, but deal with pairs $\langle \nu_1, \nu_2\rangle \in \kappa_1 \times \kappa_2$, triples $\langle \nu_1, \nu_2, \nu_3\rangle \in \kappa_1 \times \kappa_2 \times \kappa_3$ etc., instead of $\nu \in \kappa_1$. At certain level a decision about σ will be made and then we will be able to go back down and the contradiction will be derived.
$\square$

Our final goal will be to show that the forcing with P over $V[G]$ preserves cofinalities. The usual Prikry argument does not work here directly due to the lack of closure (beyond $\aleph_1$). The idea will be to redo the proof of the Prikry property while splitting Q into $Q_{>n}$ and $Q_{\leq n}$, such that $Q_{>n}$ has enough closure and $Q_{\leq n}$ satisfies enough chain condition.

Let us start by showing that all cardinals below κ_1 (and then also κ_1 are preserved.

Lemma 4.24 *$\langle P, \leq \rangle$ preserves all the cardinals below κ_1.*

Proof. Work in V with $Q * P$. Let $\mu < \lambda < \kappa_1$ be cardinals, λ a regular cardinal and let $\underset{\sim}{h}$ be a $Q * P$ name of a function from μ to λ, as forced by the weakest condition.
Let $\vec{p}$ be a good sequence. As in Lemma 4.23 we find a good sequence $\vec{q}^{00} \geq \vec{p}$ and $\delta^{00} < \lambda$ such that

$$q(0)^{00} \Vdash_Q (\langle \langle \rangle, T(\vec{q}^{00} \rangle \Vdash \underset{\sim}{h}(0) = \delta^{00}).$$

Note that $\lambda < \kappa_1$ and so the number of possible values for $\underset{\sim}{h}$ is bounded in κ_1. Hence on the set in U_1 we will have the same value.
One can try now to do the same with $\underset{\sim}{h}(1)$. But as a result $\vec{q}^{00}$ may increase. If we continue further and go through all $\underset{\sim}{h}(n)$, then due to the luck of closure of $Q_{\leq 0}$ (recall that conditions there are just finite) there may be no single condition stronger than all the constructed in the process.
Let us instead continue to deal with $\underset{\sim}{h}(0)$ and find $\vec{q}^0 \geq \vec{p}$ and δ^0 such that

- $q(0)^0 \Vdash_Q (\langle \langle \rangle, T(\vec{q}^0 \rangle \Vdash \underset{\sim}{h}(0) \leq \delta^0)$.
- $q(k)^0_0 = p(k)_0$, for each $k < \omega$

It is not hard to do just using (5) of Definition 4.12 and c.c.c. of $Q_{\leq 0}$ and its images. Just run the argument of 4.23 enough ($< \omega_1$) times.

Now, with $\vec{q}^0$ and δ^0 we continue to $\underset{\sim}{h}(1)$. Define similar $\vec{q}^1$ and δ^1 so that

- $\vec{q}^1 \geq \vec{q}^0$,
- $q(0)^1 \Vdash_Q (\langle \langle \rangle, T(\vec{q}^1 \rangle \Vdash \underset{\sim}{h}(1) \leq \delta^1)$,
- $q(k)^1_0 = p(k)_0$, for each $k < \omega$

Continue and define for each $m < \omega$, $\vec{q}^m$ and δ^m so that

- $\vec{q}^m \geq \vec{q}^{m-1}$,
- $q(0)^m \Vdash_Q (\langle \langle \rangle, T(\vec{q}^m \rangle \Vdash \underset{\sim}{h}(m) \leq \delta^m)$,
- $q(k)^m_0 = p(k)_0$, for each $k < \omega$

Finally, we can now put all this $\vec{q}^m$'s together. There is a good sequence $\vec{q}$ such that

- $\vec{q} \geq \vec{q}^m$, for every $m < \omega$,

- $q(0)\Vdash_Q (\langle\langle\rangle, T(\vec{q}\rangle \Vdash \underset{\sim}{h}(m) \leq \delta^m)$, for every $m < \omega$.

So we constructed a condition above $\vec{p}$ which bounds $\underset{\sim}{h}$ by $\bigcup_{m<\omega} \delta_m < \lambda$.
□

Lemma 4.25 *The forcing $\langle P, \leq \rangle$ preserves all the cofinalities of ordinals.*

Proof. It is enough to show that each regular cardinal $\lambda < \kappa$ is preserved. If $\lambda \leq \kappa_1$, then this is done above in Lemma 4.24. Assume that $\lambda > \kappa_1$. Pick $n, 2 \leq n < \omega$ to be the least such that $\lambda < \kappa_n$.
Let $\mu < \lambda$. Let $\underset{\sim}{h}$ be a $Q * P$ name of a function from μ to λ, as forced by the weakest condition.
We make a non-direct extension first - just pick some $\langle \nu_1, ..., \nu_{n-1} \rangle \in \kappa_1 \times ... \times \kappa_{n-1}$ and work above them. This way we will left only with ultafilters which are at least κ_n-complete.
Now we can repeat the proof of 4.24 only instead of c.c.c. we will use κ_{n-1}^{++}-c.c. of the relevant forcing. Note that $\kappa_{n-1}^{++} < \kappa_n$, so we have enough completeness to run the argument.
□

5 Cofinality changeable poset over $\aleph_\omega$.

It is possible using same ideas to construct a cofinality changeable poset over $\aleph_\omega$. Thus use the product of the Levy collapses to turn κ_1 into $\aleph_3$, κ_2 into $\aleph_6$,...,κ_n into $\aleph_{3m}$,.... Then force with Q as above and add a poset $\mathcal{P}$. The filters generated by U_n's can be used to produce a Prikry sequence cofinal in $\mathcal{P}$. The proof that all the cardinals are preserved is similar to the argument above, only instead of sets in U_n's we shrink now to U_n-positive ones.

References

[1] A. J. Dodd, The Core Model, London Math. Society, LNS 61(1982).

[2] M. Gitik, Prikry-type Forcings, in Handbook of Set Theory

[3] A. Miller, List of problems, http://at.yorku.ca/i/a/a/i/00.dir/index.htm

[4] W. Mitchell, The covering lemma, in Handbook of Set Theory.

[5] K. Prikry, Changing measurable into accessible cardinals, Dissertationes Math.,68:55,1970.