Remarks on non-closure of the preparation forcing of [2] and an off-piste version of it.

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The preparation forcing $\mathcal{P}'$ of [2] Section 1 is κ^{++}-strategically closed by Lemma 1.1.19. We would like to examine the reasons for lack of closure and directed closure of this forcing.

1 The first reason for a non–closure.

Let us first point out that the forcing $\mathcal{P}'$ is ω_1-closed.

Proposition 1.1 *$\mathcal{P}'$ is ω_1-closed.*

Proof. Let $\langle p_n \mid n < \omega \rangle$ be an increasing sequence of conditions in $\mathcal{P}'$. Assume that for each $n < \omega$ we have

$$p_n = \langle \langle A_n^{0\kappa^+}, A_n^{1\kappa^+}, C_n^{\kappa^+} \rangle, A_n^{1\kappa^{++}} \rangle.$$

Arrange by induction that $A_n^{0\kappa^+} \in C_{n+1}^{\kappa^+}(A_{n+1}^{0\kappa^+})$, for every $n < \omega$. Note that at each stage only finitely many switches are needed for this. Now we just take unions. Set

$$B^{0\kappa^+} = \bigcup_{n<\omega} A_n^{0\kappa^+},$$

$$B^{1\kappa^+} = \bigcup_{n<\omega} A_n^{1\kappa^+} \cup \{B^{0\kappa^+}\},$$

$$D^{\kappa^+} = \bigcup_{n<\omega} C_n^{\kappa^+} \cup \{\langle B^{0\kappa^+}, \{B^{0\kappa^+}\} \cup \{C_n^{\kappa^+} \mid n < \omega\} \rangle\}$$

and

$$B^{1\kappa^{++}} = \bigcup_{n<\omega} B_n^{1\kappa^{++}} \cup \{\sup \bigcup_{n<\omega} B_n^{1\kappa^{++}}\}.$$

Pick $A_\omega^{0\kappa^+}$ to be a model of cardinality κ^+ such that

1. ${}^{\kappa}A_\omega^{0\kappa^+} \subseteq A_\omega^{0\kappa^+}$,

2. $B^{0\kappa^+}, B^{1\kappa^+}, B^{1\kappa^{++}}, D^{\kappa^+} \in A_\omega^{0\kappa^+}$.

Set

$$A_\omega^{1\kappa^+} = B^{1\kappa^+} \cup \{A_\omega^{0\kappa^+}\}, C_\omega^{\kappa^+} = D^{\kappa^+} \cup \{\langle A_\omega^{0\kappa^+}, D^{\kappa^+}(B^{0\kappa^+}) \cup \{A_\omega^{0\kappa^+}\}\rangle\}$$

and

$$A_\omega^{1\kappa^{++}} = B^{1\kappa^{++}} \cup \{\sup(A_\omega^{0\kappa^+} \cap \kappa^{+3})\}.$$

Then

$$p_\omega = \langle\langle A_\omega^{0\kappa^+}, A_\omega^{1\kappa^+}, C_\omega^{\kappa^+}\rangle, A_\omega^{1\kappa^{++}}\rangle$$

will a condition in $\mathcal{P}'$ stronger than every p_n.

□

The of the top models is formally required in the definition of $\mathcal{P}'$ in order to have the largest model of cardinality κ^+ to be closed under κ-sequences. It will be convenient, in the next proposition to deal with p_ω having the top model removed. Let us denote $B^{0\kappa^+}$ by $A^{0\kappa^+}(p_\omega)$, $B^{1\kappa^+}$ by $A^{1\kappa^+}(p_\omega)$, D^{κ^+} by $C^{\kappa^+}(p_\omega)$ and $B^{1\kappa^{++}}$ by $A^{1\kappa^{++}}(p_\omega)$.

Proposition 1.2 *$\mathcal{P}'$ is not ω_2-closed.*

Proof. We construct an increasing sequence of conditions $\langle p_\alpha \mid \alpha < \omega_1\rangle$ of length ω_1 without upper bound.

Let $\alpha < \omega_1$ and suppose that $\langle p_\beta \mid \beta < \alpha\rangle$ is defined. Define p_α. If α is not a limit of limit ordinals, then we use 1.1 to form p_α for such limit α. Let for a successor α, p_α be an extension of $p_{\alpha-1}$ which has at least ω_1 many splitting points B from its central piste above $\sup(A^{0\kappa^+}(p_{\alpha-1}))$ such that if B_0, B_1 are the immediate predecessors of B with $B_0 \in C^{\kappa^+}(p_\alpha)(A^{0\kappa^+}(p_\alpha))$, then $A^{0\kappa^+}(p_{\alpha-1})$ is in $C^{\kappa^+}(p_\alpha)(B_1)$.

Assume now that α is a limit of limit ordinals.

Let $\langle \alpha_n \mid n < \omega\rangle$ a fixed in advance cofinal sequence in α with $\alpha_0 = 0$ consisting of limit ordinals.

Define p'_α to be the upper bound of $\langle p_\beta \mid \beta < \alpha\rangle$ defined as in 1.1. Let us define p_α by changing $C^{\kappa^+}(p'_\alpha)$ as follows.

We leave all $A^{0\kappa^+}(p_{\alpha_n})$ inside $C^{\kappa^+}(p_\alpha)$.

Pick a splitting point $B \in C^{\kappa^+}(p_{\alpha_0})(A^{0\kappa^+}(p_{\alpha_0}))$. Let B_0, B_1 be its immediate predecessors with $B_0 \in C^{\kappa^+}(p_{\alpha_0})(A^{0\kappa^+}(p_{\alpha_0}))$. Define $C^{\kappa^+}(p_\alpha)(A^{0\kappa^+}(p_{\alpha_0}))$ by switching from B_0 to B_1.

Let now $n, 0 < n < \omega$. Consider $C^{\kappa^+}(p_{\alpha_n})(A^{0\kappa^+}(p_{\alpha_n}))$ in the interval between $A^{0\kappa^+}(p_{\alpha_{n-1}})$ and $A^{0\kappa^+}(p_{\alpha_n})$. Pick a splitting point $B \in C^{\kappa^+}(p_{\alpha_n})(A^{0\kappa^+}(p_{\alpha_n}))$ in this interval with immediate predecessors B_0, B_1 such that

1. $B_0 \in C^{\kappa^+}(p_{\alpha_n})(A^{0\kappa^+}(p_{\alpha_n}))$
2. $B_1 \notin C^{\kappa^+}(p_\beta)(A^{0\kappa^+}(p_\beta))$, for every $\beta \leq \alpha_n$.

Note that this is possible since we required to have at least $\aleph_1$ many splitting points at each successor stage and α_n is countable.
Define $C^{\kappa^+}(p_\alpha)(A^{0\kappa^+}(p_{\alpha_n}))$ by switching from B_0 to B_1.

This completes the definition of the sequence $\langle p_\alpha \mid \alpha < \omega_1 \rangle$.

Let us argue that there is no $p \in \mathcal{P}'$ such that $p \geq p_\alpha$, for every $\alpha < \omega_1$.
Suppose otherwise. Let p be such a condition. Set

$$C := \{\alpha < \omega_1 \mid \alpha \text{ is a limit of limit ordinals}\}.$$

For every $\alpha \in C$ let $f(\alpha)$ be the least $\beta < \alpha$ such that $C^{\kappa^+}(p)(A^{0\kappa^+}(p_\alpha))$ transforms into $C^{\kappa^+}(p_\alpha)(A^{0\kappa^+}(p_\alpha))$ by switches below $A^{0\kappa^+}(p_\beta)$. Recall that only finitely many switches are required to transform $C^{\kappa^+}(p)(A^{0\kappa^+}(p_\alpha))$ into $C^{\kappa^+}(p_\alpha)(A^{0\kappa^+}(p_\alpha))$, by the definition of the order on $\mathcal{P}'$, and hence there must be such β.
Find a stationary $S \subseteq C$ and $\beta^* < \omega_1$ such that $f(\alpha) = \beta^*$, for every $\beta \in S$. Pick $\alpha \in S$ which is a limit point of S. Let $\langle \gamma_n \mid n < \omega \rangle$ be cofinal in α sequence of elements of S. Then $C^{\kappa^+}(p_\alpha)(A^{0\kappa^+}(p_{\gamma_n}))$ and $C^{\kappa^+}(p_{\gamma_n})(A^{0\kappa^+}(p_{\gamma_n}))$ agree on the final segment from $A^{0\kappa^+}(p_{\beta^*})$ up, for each $n < \omega$. But this contradicts the choice of $C^{\kappa^+}(p_\alpha)(A^{0\kappa^+}(p_\alpha))$.
$\square$

We would like to use now the above reason of non-closure in order to construct a square like principle which is inconsistent with a supercompact cardinal.

Theorem 1.3 *The gap 3 preparation forcing $\mathcal{P}'$ of [2],chapter 1 adds a weak form of $\square^{\leq\kappa^+}_{\kappa^{++}}$.*

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.
Introduce few notions.

Definition 1.4 A limit ordinal $\xi < \kappa^{+3}$ is called *good* iff there is $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ such that

1. $\xi \in A^{0\kappa^+}$,

2. $\xi \in A^{1\kappa^{++}}$,

3. $\mathrm{cof}(\xi) \leq \kappa^+$,

4. there is $A \in A^{1\kappa^+}$ such that

   (a) $\xi \in A$,

   (b) A is an immediate successor of a limit model in $C^{\kappa^+}(A)$.
   Denote this model by A^-.

   (c) For every $E \in C^{\kappa^+}(A) \setminus \{A\}$, $\xi \notin A$,

   (d) $A^- \cap \xi$ is unbounded in ξ,

   (e) $E \cap \xi$ is bounded in ξ, for every $E \in C^{\kappa^+}(A) \setminus \{A, A^-\}$.

Lemma 1.5 *Let ξ be a good ordinal and $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ be a condition witnessing this. Let $A \in A^{1\kappa^+}$ be such that*

1. $\xi \in A$,

2. for every $B \in A^{1\kappa^+}$ with $B \subsetneqq A$, $\xi \notin B$.

Then A satisfies (4) of Definition 1.4. In addition, the sequence $\langle \xi \cap E \mid E \in C^{\kappa^+}(A) \setminus \{A, A^-\}$ does not depend on A.

Proof. Clearly, A is a successor model. Let A^* be a model witnessing (4) of Definition 1.4.

Claim 1 There is no $B \in A^{1\kappa^+} \cap A^*$ with $\xi \in B$.

Proof. Suppose otherwise. Then there is a piste from A^* to B. But A^* is the immediate successor of A^{*-} in $C^{\kappa^+}(A^*)$. Hence it should go via A^{*-}. Which is impossible since $\xi \in B \setminus A^{*-}$.
$\square$ of the claim.
Use now the intersection property for A, A^*. Then, by the claim and the property (2) of A, for some $\eta \in A, \eta^* \in A^*$,

$$A \cap A^* = A \cap \eta = A^* \cap \eta^*.$$

Then $otp(A) = otp(A^*)$ and hence $C^{\kappa^+}(A)$ and $C^{\kappa^+}(A^*)$ have the same order type. In particular, A is an immediate successor of a limit model. Also structures

$$\langle A, C^{\kappa^+}(A), \eta, \in, \subseteq \rangle, \langle A^*, C^{\kappa^+}(A^*), \eta^*, \in, \subseteq \rangle$$

are isomorphic with the isomorphism which is identity over the common part. Then A satisfies (4) of Definition 1.4. In addition we obtain that the sequences $\langle E \cap \xi \mid E \in C^{\kappa^+}(A) \setminus \{A\}\rangle$ and $\langle E \cap \xi \mid E \in C^{\kappa^+}(A^*) \setminus \{A^*\}\rangle$ are the same.
□

Lemma 1.6 *A limit of $\leq \kappa^+$ good ordinals is a good ordinal.*

Proof. Let $\langle \xi_i \mid i < \delta \leq \kappa^+\rangle$ be an increasing sequence of good ordinals and $\xi = \bigcup_{i<\delta} \xi_i$. Consider a piste from $A^{0\kappa^+}$ to ξ. Let A be the terminal model of this piste. Then A cannot be a limit model and also it cannot be an immediate successor of a non-limit model by the previous lemma, as ξ is a limit of good ordinals. Denote by A^- the immediate predecessor of A. Consider $C^{\kappa^+}(A) \setminus \{A, A^-\}$. Then ξ is not a member of any of the elements of this set. Moreover, if $E \in C^{\kappa^+}(A) \setminus \{A, A^-\}$, then $E \cap \xi$ is bounded in ξ. Otherwise let $E \cap \xi$ is unbounded in ξ. Let E^+ be the immediate successor of E in $C^{\kappa^+}(A)$. We have $\xi \notin E^+$ but there are ordinals $\geq \xi$ in E^+, for example $\sup(E)$. Let η be the least such ordinal. Then $\mathrm{cof}(\eta) > \kappa^+$, by elementarity of E^+. So $E \cap \eta \subseteq \xi$. But $E \cap \eta \in E^+$, hence also $\xi = \sup(E \cap \eta) \in E^+$. Contradiction.
Now, $C^{\kappa^+}(A) \setminus \{A, A^-\}$ witness goodness of ξ.
□

Corollary 1.7 *The set of good ordinals is a κ^+–club.*

Now we are ready to prove the theorem. Denote by

$$C := \{\alpha < \kappa^{+3} \mid \alpha \text{ is a good ordinal }\}.$$

We will define a partial square sequence $\langle C_\alpha \mid \alpha \in C\rangle$ over C. This by standard argument allows to extend it to

$$\{\alpha < \kappa^{+3} \mid \mathrm{cof}(\alpha) < \kappa^{++}\}.$$

Proceed as follows. If α is a good ordinal then pick a model A witnessing this and set

$$C_\alpha(p) = \{\sup(E \cap \alpha) \mid E \in C^{\kappa^+}(A^-) \setminus \{A^-\}\},$$

where $p \in G(\mathcal{P}')$ and $A \in A^{1\kappa^+}(p)$.

Now if we have $p, q \in G(\mathcal{P}')$ with $A \in A^{1\kappa^+}(p), A^{1\kappa^+}(q)$ then $C^{\kappa^+}(p)(A^-)$ and $C^{\kappa^+}(q)(A^-)$ may differ only on an initial segment and both sets have the same order type, since we can move from $C^{\kappa^+}(p)$ to $C^{\kappa^+}(q)$ using finitely many switches.

Let us pick for every good α a condition $p_\alpha \in G(\mathcal{P}')$ with a witnessing $A_\alpha \in A^{1\kappa^+}(p_\alpha)$ and set

$$C_\alpha := C_\alpha(p_\alpha).$$

Lemma 1.8 *Let α be a good ordinal and β is a limit point of C_α, then we will have a following type of coherency:*

1. *$C_\alpha \cap \beta$ and C_β have a common final segment,*

2. *$otp(C_\alpha \cap \beta) = otp(C_\beta)$.*

Proof. It follows since $C_\beta(p_\alpha) = C_\alpha \cap \beta$ (the coherency for good α's with same p follows by Lemma 1.5) and $C_\beta(p_\alpha), C_\beta(p_\beta) = C_\beta$ have a common final segment and the same order type.
$\square$

Using ideas from Cummings, Foreman, Magidor [1] it is possible to show that this type of a square is weaker than $\square_{\kappa^{++}}$ (at least assuming the consistency of a supercompact cardinal).

If $\mathrm{cof}(\alpha) = \kappa^{++}$ and α is a limit point of $A^{1\kappa^{++}}$, for an element of $G(\mathcal{P}')$ then set

$$C_\alpha = \{\sup(E \cap \alpha) \mid E \in C^{\kappa^+}(A), A \in A^{1\kappa^+}, \alpha \in A \text{ for some } \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')\}.$$

Carmi Merimovich [4] showed that such defined C_α's provide a partial $\square_{\kappa^{++}}^{Cof\kappa^{++}}$. This type of a square lives well with a supercompact cardinals.

2 New definition.

Let us define a new partial order (actually a pre-order) on $\mathcal{P}'$ which will allows to eliminate the first reason of non-closure.

Definition 2.1 Let $p = \langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), C^{\kappa^+}(p)\rangle, A^{1\kappa^{++}}(p)\rangle$,
$q = \langle\langle A^{0\kappa^+}(q), A^{1\kappa^+}(q), C^{\kappa^+}(q)\rangle, A^{1\kappa^{++}}(q)\rangle$ be conditions in $\mathcal{P}'$. Define $p \geq_{new} q$ iff there is D such that

1. $\langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), D\rangle, A^{1\kappa^{++}}(p)\rangle \in \mathcal{P}'$,

2. $A^{0\kappa^+}(q) \in D(A^{0\kappa^+}(p))$,

3. $D(A^{0\kappa^+}(q)) = C^{\kappa^+}(q)(A^{0\kappa^+}(q))$.

Remark 2.2 Note that any two conditions $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ and $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, D^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ are $\leq_{new}$–equivalent. They were equivalent according to the order (pre-order) $\leq$ only if it was possible to change C^{κ^+} to D^{κ^+} by finitely many switches. With $\leq_{new}$ infinitely many of them may be applied.

Proposition 2.3 *Let $\eta < \kappa^{++}$ and $\langle p_\alpha \mid \alpha \leq \eta < \kappa^{++}\rangle$ be a $\leq_{new}$–increasing sequence of elements of $\mathcal{P}'$. Suppose that for each limit $\alpha < \eta$ the set $\bigcup_{\beta<\alpha} A^{0\kappa^+}(p_\beta)$ is in $A^{1\kappa^+}(p_\alpha)$. Then there is $p \in \mathcal{P}'$, $p \geq_{new} p_\alpha$, for every $\alpha < \eta$.*

Proof. Use $\langle A^{0\kappa^+}(p_\alpha) \mid \alpha < \eta\rangle$ together with $\langle \bigcup_{\beta<\alpha} A^{0\kappa^+}(p_\beta) \mid \alpha < \eta, \alpha \text{ is a limit ordinal } \rangle$ in order to form $C^{\kappa^+}(p)$, where p is the obvious upper bound of p_α's without the pistes.
$\square$

3 Additional reason for a non-closure.

There is one more reason for non-closure. It has to do with chains of models inside a condition with their union not inside.
Let us describe this type of situation.

Let $\langle p_n \mid n < \omega\rangle$ be an increasing sequence of conditions of $\mathcal{P}'$. There are potentially two ways to extend it. The first (and one which is always available, and which was used above in 1.1) is to take the union of $A^{0\kappa^+}(p_n)$'s and then to extend this to a condition. The second (which is not always possible) is like this: there is $p \in \mathcal{P}'$ such that

1. $p \geq p_n$, for all $n < \omega$,
2. $\bigcup_{n<\omega} A^{0\kappa^+}(p_n) \notin A^{1\kappa^+}(p)$.

Proposition 3.1 *Let $\langle p_\alpha \mid \alpha < \omega_1\rangle$ be an increasing sequence of elements of $\mathcal{P}'$ such that for every limit $\alpha < \omega_1$, $\bigcup_{\beta<\alpha} A^{0\kappa^+}(p_\beta) \notin A^{1\kappa^+}(p_\alpha)$. Then there is no $p \in \mathcal{P}'$ with $p \geq p_\alpha$, for every $\alpha < \omega_1$ and $\bigcup_{\alpha<\omega_1} A^{0\kappa^+}(p_\alpha) \in A^{1\kappa^+}(p)$.*

Proof. Suppose otherwise. Let p be an upper bound and $\bigcup_{\alpha<\omega_1} A^{0\kappa^+}(p_\alpha) \in A^{1\kappa^+}(p)$. Denote $\bigcup_{\alpha<\omega_1} A^{0\kappa^+}(p_\alpha)$ by A. Let us argue that for every $X \in A \cap A^{1\kappa^+}(p)$ there is $\alpha < \omega_1$ with $X \in A^{0\kappa^+}(p_\alpha)$. Consider $\eta = \sup(X \cap \kappa^{+3})$. Then $\eta \in A$, and hence for some α, $\eta \in A^{0\kappa^+}(p_\alpha)$. But $\mathrm{cof}(\eta) \leq \kappa^+$. So $A^{0\kappa^+}(p_\alpha)$ is unbounded in η. By intersection property, then $X \subseteq A^{0\kappa^+}(p_\alpha)$. But $\eta \in A^{0\kappa^+}(p_\alpha) \setminus X$, hence $X \in A^{0\kappa^+}(p_\alpha)$.

It follows that $\{\bigcup_{\beta<\gamma} A^{0\kappa^+}(p_\beta) \mid \gamma < \omega_1\}$ is club in $A \cap A^{1\kappa^+}(p)$. So it must intersect $C^{\kappa^+}(p)$. Contradiction.
$\square$

4 Absence of directed closure.

If we have countably many conditions such that any finite family of them is compatible, then the ω_1-closure (1.1) implies the existence of an upper bound.
But suppose now that we have ω_1 many conditions $\langle p_{n\alpha} \mid n < \omega, \alpha < \omega_1\rangle$ such that for every α,

1. $A^{0\kappa^+}(p_{\alpha n+1}) \supset A^{0\kappa^+}(p_{\alpha n})$,
2. $A^{0\kappa^+}(p_{\alpha+1 n+1}) \supset A^{0\kappa^+}(p_{\alpha n})$,
3. $\bigcup_{n<\omega} A^{0\kappa^+}(p_{\alpha+1 n}) \not\supseteq \bigcup_{n<\omega} A^{0\kappa^+}(p_{\alpha n})$.

It is impossible to find an upper bound for $\langle p_{n\alpha} \mid n < \omega, \alpha < \omega_1\rangle$ without adding $\bigcup_{n<\omega} A^{0\kappa^+}(p_{\alpha n})$ to the central piste for unboundedly many α's, which is not allowed.

5 Off-piste version of the preparation forcing.

In [?] Merimovich used a variation of the Velleman simplified morass forcing [5] as the preparation forcing for gap 3. The advantage of using it is a directed closure of this forcing. Here we would like to present off-piste version of the preparation forcing for higher gaps. Absence of pistes will provide a directed closure. Unfortunately the resulting structure lacks of the intersection property for gaps 4 and above, and so it is unclear how to implement it into the final forcing.

Let us deal with gap 4 case the treatment of higher gaps is similar.

We will have three types of models - of size κ^+, of size κ^{++} and of size κ^{+3} (ordinals). Denote as usual the corresponding sets accumulating this models inside a condition by $A^{1\kappa^+}, A^{1\kappa^{++}}, A^{1\kappa^{+3}}$. $A^{1\kappa^{+3}}$ is a closed set of ordinals of size at most κ^{+3}. $A^{1\kappa^{++}}$ is defined as in [?]. Let us define $A^{1\kappa^+}$.

Definition 5.1 $A^{1\kappa^+}$ is a set of at most κ^+ models of size κ^+ such that the following holds:

1. there is the largest model $A^{0\kappa^+}$,

2. for every $A \in A^{1\kappa^+}$ the following holds:

   (a) either there is a largest $\alpha \in A \cap A^{1\kappa^{+3}}$ or $\sup(A \cap \kappa^{+4})$ is a limit point of $A^{1\kappa^{+3}}$,

   (b) either there is a largest (under the inclusion) model $(A)_{\kappa^{++}} \in A \cap A^{1\kappa^{++}}$ or $A \cap A^{1\kappa^{++}}$ is directed,

   (c) either $A \cap A^{1\kappa^+}$ is directed or A has immediate predecessors and if A' is an immediate predecessor of A then either

      i. there is an immediate predecessor A_0 of A and A, A_0, A' form a Δ-system triple;

         or

      ii. there is $A_0 \in A \cap A^{1\kappa^+}$ (which need not be an immediate predecessor of A) and $A' = \pi_{F_0 F_1}[A_0]$, where $F_0, F_1 \in A \cap A^{1\kappa^{++}}$ are of a same order type.

Let us argue that the intersection property even in its weakest form can break down in the present setting.

Example.

Suppose that $A \in A^{1\kappa^+}$, the largest model $(A)_{\kappa^{++}} \in A \cap A^{1\kappa^{++}}$ exists and it is a limit point of $A^{1\kappa^{++}}$ (limit means here that $(A)_{\kappa^{++}} \cap A^{1\kappa^{++}}$ is directed or equivalently of a limit rank). Assume that there is no increasing sequence of elements of $(A)_{\kappa^{++}} \cap A^{1\kappa^{++}}$ which union is $(A)_{\kappa^{++}}$.

Note that existence of a limit model $X \in (A)_{\kappa^{++}}$ which is not a limit of an increasing sequence of elements of $(A)_{\kappa^{++}}$ is forced upon us, as in 4, if we like to have directed closure and not only a closure.

Assume that the rank of $(A)_{\kappa^{++}}$ is some $\mu < \kappa^{+3}$ of cofinality κ^{++}.

Suppose that the only model in $A \cap A^{1\kappa^{++}}$ that includes B is $(A)_{\kappa^{++}}$.

Consider

$$A \cap (A)_{\kappa^{++}} = \bigcup (A \cap A^{1\kappa^{++}}) \setminus \{(A)_{\kappa^{++}}\}.$$

Set $Z = \bigcup(A \cap A^{1\kappa^{++}})$. If $Z \in A^{1\kappa^{++}}$, then use the intersection property between B and Z. Suppose $Z \notin A^{1\kappa^{++}}$. Pick $Y \in (A)_{\kappa^{++}} \in A \cap A^{1\kappa^{++}}$, $Y \supset Z$ of the smallest rank (it must be $\sup(A \cap \mu)$).

If $B \in (A)_{\kappa^{++}} \in A \cap A^{1\kappa^{++}}$ is a model of rank δ, for some δ, $\mu > \delta \geq \sup(A \cap \mu)$. Consider $Y \cap B$. Then there is $\xi \in Y \cap A^{1\kappa^{+3}}$ such that

$$Y \cap B = Y \cap \xi.$$

Hence

$$A \cap B = A \cap Y \cap B = A \cap (A)_{\kappa^{++}} \cap \xi.$$

But if the rank of B is small, then B can be an element of $Y \setminus Z$ which does not include Z, and even the rank of B may be in A. If in addition no element of Z includes $B \cap Z$, then the intersection property between A and B will break down.

Let us construct an example having such B. Fix a continuous chain of elementary submodels $\langle M_i \mid i \leq \kappa^{++}+1\rangle$ each of size κ^{+3}, $M_i \cap \kappa^{+4} = \mu_i$ and ${}^{\kappa^{++}>}M_{i+1} \subseteq M_{i+1}$. Let X be an elementary submodel of $M_{\kappa^{++}+1}$ of size κ^{++} such that $\langle \mu_i \mid i \leq \kappa^{++}\rangle \in X$.
For each $i < \kappa^{++}$ let $X_i \in M_{i+1}$ be a reflection of X to M_{i+1} over $X \cap M_i$.
Add models of size κ^{++} which include X_0, X and then reflect it down to every $i, 0 < i < \kappa^{++}$. Continue in a similar fashion and extend the family into directed one. Then pick a model which includes it and again reflect down. Proceed $\kappa^{++}+1$ many stages. Let E be the final model of the rank κ^{++}. Through all the models from the constructed family of models of size κ^{++} which have ordinals $\geq \mu_{\kappa^{++}}$ but keep E (in particular, X is out).
The resulting family will be our $A^{1\kappa^{++}}$. Let $A^{1\kappa^{+3}}$ be the set $\{M_i \mid i \leq \kappa^{++}\}$. Let $A = A^{0\kappa^{+}}$ be an elementary submodel of $M_{\kappa^{++}+1}$ of size κ^{+} with $A^{1\kappa^{++}}$ and $A^{1\kappa^{+3}}$ inside. Let $A \cap \kappa^{++} = \eta$. Then $\sup(A \cap \mu_{\kappa^{++}}) = \mu_\eta$. Set $B = X_\eta$. The pair A, B will fail to have the intersection property as it was explained above.

References

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