Some pathological examples of precipitous ideals

Moti Gitik *
School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Science
Tel Aviv University
Ramat Aviv 69978, Israel

Abstract

We construct a model with an indecisive precipitous ideal and a model with a precipitous ideal with a non precipitous normal ideal below it. Such kind of examples were previously given by M. Foreman [1] and R. Laver [4] respectively. The present examples differ in two ways: first- they use only a measurable cardinal and second- the ideals are over a cardinal. Also a precipitous ideal without a normal ideal below it is constructed. It is shown in addition that if there is a precipitous ideal over a cardinal κ such that

- after the forcing with its positive sets the cardinality of κ remains above $\aleph_1$
- there is no a normal precipitous ideal

then there is $0^\dagger$.

1 Indecisive precipitous ideals.

In [1] M. Foreman isolated the following natural and wide class of ideals:

Definition 1.1 (M. Foreman [1]) Let $Z \subset P(X)$ and J be an ideal on Z. Let $X' \subset X$ and I be the projection of J to an ideal on $P(X')$. Then J **decides** I iff there is a set $A \in \breve{I}$ and a well ordering W of A and sets $A', W', O', I' \in V$ such that for all generic $G \subset P(Z)/J$:

1. An initial segment of the ordinals of V^Z/G is well founded and isomorphic to $(|A'|^+)^V$ and

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2. if $j : V \to M$ is the canonical elementary embedding determined by replacing the ultraproduct V^Z/G by an isomorphic model M transitive up to $(|A'|^+)^V$, then

$$j(A) = A', j(W) = W', j''|A| = O', I' = j(I) \cap P(A')^V.$$

J is called **decisive** if J decides itself.

M. Foreman [1] gave an example of indecisive precipitous ideal. He used a Woodin cardinal for this and his ideal is on $[\omega_2]^{\omega_1}$.

We will deal here with normal precipitous ideals I on a cardinal κ. In this case the definition will have only two requirements:

there are κ' and I' such that for all generic $G \subset P(\kappa)/I$, if $j : V \to M \simeq V^\kappa/G$ is the canonical embedding then

1. $j(\kappa) = \kappa'$
2. $I' = j(I) \cap P(\kappa')^V$.

The referee of the paper note that there is a subtle point here. Thus, by the definition of a decisive ideal, we can only find $A \in \breve{I}$ such that $j(A) = A'$ and $j(I) \cap P^V(A') \in V$. How do we replace A by κ? He suggested the following argument. Let us reconstruct I' from $I'' := j(I) \cap P^V(A')$. Denote $\kappa' \setminus A'$ by B. If we show that $P^M(B) \cap V \in V$, then

$$I' = \{X \cup Y \mid X \in I'', Y \in P^M(B) \cap V\}.$$

If B is a bounded subset of κ', then $B = \kappa \setminus A$, since κ is the critical point of j, $B = \kappa' \setminus A' = j(\kappa \setminus A)$ and $\kappa \setminus A$ must be a bounded subset of κ. Hence, $P^M(B) \cap V = P^V(\kappa \setminus A)$ and we are done.

Suppose now that B is unbounded in κ'. Then $P^M(B) \cap V$ will be in V iff $P^M(\kappa') \cap V$ will be in V. If there is $X \in I''$ which is unbounded in κ', then $P^M(X) \cap V \in V$, since $I'' \in V$ and every subset of X which is in V is also in I''. It is easy to reconstruct $P^M(\kappa') \cap V$ from $P^M(X) \cap V$. Set $Y = \{\alpha \in A \mid \sup(A \cap \alpha) < \alpha\}$. Then Y is an unbounded subset of A (and, hence of κ) which must be in I, since it is not stationary. So, $X = j(Y)$ will be as desired.

It easy to violate the condition (1) above. Thus take two normal ultrafilters U_1, U_2 over a measurable cardinal κ which move κ to different places. Consider $W = U_1 \cap U_2$. There are disjoint sets $A \in U_1$ and $B \in U_2$ with $A \cup B = \kappa$. Hence W is precipitous. But A forces that κ is moved according to j_{U_1} and B forces that κ is moved according to j_{U_2}, which give different values to the images of κ. In [3], the nonstationary ideal over $\aleph_1$ has this type of property.

Here we would like to give an example of a normal precipitous ideal over a cardinal κ so that

- $j \restriction \alpha \in V$, for each ordinal α, also $j \restriction On$ is a class in V, but
- the condition (2) above breaks down as follows: $j(I) \cap P(j(\kappa))^V \notin V$

Theorem 1.2 *Assume GCH. Let κ be a measurable cardinal, U a normal measure over κ and $j : V \to M \simeq {}^\kappa V/U$ the canonical elementary embedding. Then there is a cardinal preserving generic extension V^* of V with an indecisive precipitous filter $W \supseteq U$ over κ such that for all generic $G \subset P(\kappa)/W$, if $j' : V^* \to M \simeq (V^*)^\kappa/G$ is the canonical embedding then*

1. $j' \restriction On = j \restriction On$
2. $j'(W) \cap P(j(\kappa))^{V^*} \notin V^*$

Proof.

Force, using the Backward Easton forcing, Cohen functions

$$\langle f_{\alpha i} | i < \alpha^+ \rangle, f_{\alpha i} : \alpha \to 2$$

and

$$h_\alpha : \alpha \to 2,$$

for each regular $\alpha \leq \kappa$. Thus let $P = P_{\kappa+1}$ be the backward Easton iteration

$$\langle P_\alpha, \dot{Q}_\beta | \alpha \leq \kappa + 1, \beta < \kappa + 1 \rangle,$$

where if β is a regular cardinal then in V^{P_β} we have $Q_\beta = Q_{\beta 0} * Q_{\beta 1}$ with $Q_{\beta 0}$ being the Cohen forcing for adding β^+ Cohen functions from β to 2 and $Q_{\beta 1}$ the Cohen forcing for adding a single Cohen function from β to 2. Otherwise $Q_\beta = \emptyset$.

Let G be a generic subset of P. Pick $G^* \subset j(P)$ in $V[G]$ so that

1. $G \subseteq G^* \restriction P_{\kappa+1}$
2. $j''G \subseteq G^*$

3. G^* is M- generic

4. for each $\gamma < \kappa^+$, $f^*_{j(\kappa)j(\gamma)}(\kappa) = 0$, where $f^*_{j(\kappa)j(\gamma)}$ is the $j(\gamma)$'s Cohen function added by $G^* \cap Q_{j(\kappa)0}$.

It is routine to construct such G^*. Thus, briefly, in order to satisfy the item 3, note that it is enough to show that $G^* \cap (Q_{j(\kappa)0} \restriction j(\alpha))$ is M-generic for each $\alpha < \kappa^+$, since the forcing $Q_{j(\kappa)0}$ in M satisfies $j(\kappa^+)$-c.c. and $j''\kappa^+$ is unbounded in $j(\kappa^+)$. So find first some G' satisfying the items 1-3 above and then change it to G^* by replacing the members of G' that do not satisfy the item 4 by those that do satisfy it. Such change will effect basically a single condition inside $Q_{j(\kappa)0} \restriction j(\alpha)$, for each $\alpha < \kappa^+$.

Now, j extends to $j^* : V[G] \to M[G^*]$ and U extends to a normal ultrafilter

$$U^* = \{X \subseteq \kappa | \kappa \in j^*(X)\}$$

in $V[G]$. Note that for each $\gamma < \kappa^+$ the set

$$A_\gamma = \{\nu < \kappa | f_{\kappa\gamma}(\nu) = 0\} \in U^*.$$

Define in $V[G]$ an extension W of U as follows. Fix an increasing sequence

$$\langle \delta_\gamma | \gamma < \kappa^+ \rangle$$

unbounded in $j(\kappa)$ and with $\delta_0 > \kappa$. Let A be a subset of κ.
Then set $A \in W$ iff there is $r \in j(P)$ such that

1. $r \Vdash \kappa \in j(\dot{A})$, in M

2. $r \restriction P_{j(\kappa)} \in G^*$.

   The next five properties are forced in $P_{j(\kappa)}$ by the empty condition.

3. If $(\mu, \tau) \in \mathrm{dom}(r \restriction Q_{j(\kappa)0})$, then
   - $\mu = j(\gamma)$, for some $\gamma < \kappa^+$
   - if $\tau < \kappa$, then $(r \restriction Q_{j(\kappa)0})(\mu, \tau) = f_{\kappa\gamma}(\tau)$
   - if $\tau = \kappa$, then $(r \restriction Q_{j(\kappa)0})(j(\gamma), \tau) = 0$

4. if $\tau \in \mathrm{dom}(r \restriction Q_{j(\kappa)1})$ and $\tau < \kappa$ then $(r \restriction Q_{j(\kappa)1})(\tau) = h_\kappa(\tau)$

5. $\mathrm{dom}(r \restriction Q_{j(\kappa)1}) \setminus \kappa = \{\delta_\gamma \mid \exists \tau > \kappa \quad (j(\gamma), \tau) \in \mathrm{dom}(r \restriction Q_{j(\kappa)0})\}$

6. if there is no $\gamma < \kappa^+, \tau > \kappa$ such that $(j(\gamma), \tau) \in \mathrm{dom}(r \restriction Q_{j(\kappa)0})$, then $\mathrm{dom}(r \restriction Q_{j(\kappa)1}) \subseteq \kappa$

7. if for some $\gamma < \kappa^+, \tau > \kappa$ we have $(j(\gamma), \tau) \in \mathrm{dom}(r \restriction Q_{j(\kappa)0})$, then

   - $\delta_\gamma \in \mathrm{dom}(r \restriction Q_{j(\kappa)1})$
   - $(r \restriction Q_{j(\kappa)1})(\delta_\gamma) = 1$ and $(r \restriction Q_{j(\kappa)0})(j(\gamma), \tau) = f^*_{j(\kappa)j(\gamma)}(\tau)$ where $f^*_{j(\kappa)j(\gamma)}$ the $j(\gamma)$'s function added by G^* over $j(\kappa)$.
     Note that if $\xi < \kappa$, then $f^*_{j(\kappa)j(\gamma)}(\xi) = f_{\kappa\gamma}(\tau)$.

8. **If** $r' \in j(P)$ is such that

   (a) $r' \restriction P_{j(\kappa)} = r \restriction P_{j(\kappa)}$;
   the properties (b)-(f) below are forced in $P_{j(\kappa)}$ by the empty condition

   (b) $\mathrm{dom}(r' \restriction Q_{j(\kappa)1}) = \mathrm{dom}(r \restriction Q_{j(\kappa)1})$

   (c) $\mathrm{dom}(r' \restriction Q_{j(\kappa)0}) \subseteq \mathrm{dom}(r \restriction Q_{j(\kappa)0})$

   (d) if $(\mu, \tau) \in \mathrm{dom}(r' \restriction Q_{j(\kappa)0})$, then

   - $\mu = j(\gamma)$, for some $\gamma < \kappa^+$
   - if $\tau < \kappa$, then $(r' \restriction Q_{j(\kappa)0})(\mu, \tau) = f_{\kappa\gamma}(\tau)$
   - if $\tau = \kappa$, then $(r' \restriction Q_{j(\kappa)0})(j(\gamma), \tau) = 0$

   (e) if for some $\gamma < \kappa^+, \tau > \kappa$ we have $(j(\gamma), \tau) \in \mathrm{dom}(r' \restriction Q_{j(\kappa)0})$, then

   - $\delta_\gamma \in \mathrm{dom}(r' \restriction Q_{j(\kappa)1}) = \mathrm{dom}(r \restriction Q_{j(\kappa)1})$
   - $(r' \restriction Q_{j(\kappa)1})(\delta_\gamma) = 1$ and $(r' \restriction Q_{j(\kappa)0})(j(\gamma), \tau) = f^*_{j(\kappa)j(\gamma)}(\tau)$ where $f^*_{j(\kappa)j(\gamma)}$ the $j(\gamma)$'s function added by G^* over $j(\kappa)$

   (f) if $\delta_\gamma \in \mathrm{dom}(r' \restriction Q_{j(\kappa)1})$ and $(r' \restriction Q_{j(\kappa)1})(\delta_\gamma) = 1$, for some $\gamma < \kappa^+$, then for each $\tau > \kappa$ we have $(j(\gamma), \tau) \in \mathrm{dom}(r' \restriction Q_{j(\kappa)0})$ iff $(j(\gamma), \tau) \in \mathrm{dom}(r \restriction Q_{j(\kappa)0})$ (hence, by the previous item, $(r' \restriction Q_{j(\kappa)0})(j(\gamma), \tau) = (r \restriction Q_{j(\kappa)0})(j(\gamma), \tau) = f^*_{j(\kappa)j(\gamma)}(\tau)$),
   then $r' \Vdash \kappa \in j(\dot{A})$, in M.

Intuitively, we put into W sets which insure the following: for each $\gamma < \kappa^+$, if $h_{j(\kappa)}(\delta_\gamma) = 1$ (on the M side) then the master condition deciding in $V[G]$ the function $f_{j(\kappa),j(\gamma)}$ is used. Otherwise, i.e., if $h_{j(\kappa)}(\delta_\gamma) = 0$, then no extension is made.
The role of r''s in the definition is to insure the possibility of a free choice of values 0 or 1 at

each δ_γ. Note that $r'(\delta_\gamma)=0$ implies that there is no $\tau>\kappa$ with $(j(\gamma),\tau)\in \mathrm{dom}(r'\restriction Q_{j(\kappa)0})$, by 8(e) above.

It is not hard to see that W is a normal filter over κ which extends U.
Consider the following forcing notion:

$$R=\{(p_0,p_1)\in (Q_{j(\kappa)0}\times Q_{j(\kappa)1})^{M[G^*\restriction P_{j(\kappa)}]}\mid$$

1. $p_1\restriction\kappa=h_\kappa$
2. for each $\gamma<\kappa^+$ we have $p_0(j(\gamma))\restriction\kappa=f_{\kappa\gamma}$ and $p_0(j(\gamma))(\kappa)=0$
3. if $\delta_\gamma\in\mathrm{dom}(p_1)$ and $p_1(\delta_\gamma)=1$, then $p_0(j(\gamma))\subseteq f^*_{j(\kappa)j(\gamma)}\}$.

Claim 1 The forcing with W-positive sets is isomorphic to R.

Proof. Suppose first that $(p_0,p_1)\in R$. Let $A\in W$ witnessed by $r\in j(P)$. It is enough to find $t\in j(P)$ stronger than (p_0,p_1) which forces "$\kappa\in j(\dot{A})$". Consider

$$a=\{\delta_\gamma\mid\gamma<\kappa^+,\delta_\gamma\in\mathrm{dom}(p_1)\cap\mathrm{dom}(r\restriction Q_{j(\kappa)1})\}=\mathrm{dom}(p_1)\cap\mathrm{dom}(r\restriction Q_{j(\kappa)1})\backslash\kappa.$$

and

$$b=\{\xi\in a\mid p_1(\xi)=0\}.$$

Let $r'\in j(P)$ be obtained from r as follows: for each $\xi\in b$, if $r\restriction Q_{j(\kappa)1}(\xi)=1$, then change the value to 0 and remove $r\restriction Q_{j(\kappa)0}(\xi,\tau)$, for each $\tau>\kappa$. Leave all the rest of r unchanged. Now, such r' satisfies the item 8 above. Hence

$$r'\Vdash\kappa\in j(\dot{A}).$$

On the other hand r' is compatible with (p_0,p_1). Pick t to be a common extension of r' and (p_0,p_1).

Let now X be a W-positive set. We need to find $t\in R$ forcing "$\kappa\in j(\dot{X})$". Note that X is a subset of κ and the forcing satisfies κ^+-c.c., so there is $\eta<\kappa^+$ such that X depends only on $G\restriction(P_\kappa*(Q_{\kappa0}\restriction\eta)*Q_{\kappa1})$. Fix such η and a $P_\kappa*(Q_{\kappa0}\restriction\eta)*Q_{\kappa1}$-name $\dot{X}$ of X. Then $j(\dot{X})$ will be a $P_{j(\kappa)}*(Q_{j(\kappa)0}\restriction j(\eta))*Q_{j(\kappa)1}$-name.
Set

$$a=\{(\gamma,\delta_\gamma)\mid\gamma<\eta\}.$$

Then $a\in M$, since M is closed under κ-sequences of its elements. Also, $j''\eta\in M$. Hence, $Q_{j(\kappa)0}\restriction j''\eta\in M[G^*\restriction(P_{j(\kappa)}]$. Denote the forcing $Q_{j(\kappa)0}\restriction j''\eta$ by S and $Q_{j(\kappa)0})\restriction(j(\eta)\backslash j''\eta)$

by T. We deal here with the Cohen forcings, hence $Q_{j(\kappa)0} \restriction j(\eta)$ can be identified with $S \times T$. Work in $M[G^* \restriction (P_{j(\kappa)} * S)]$. If there is $p_1 \in T \times Q_{j(\kappa)1}$ such that $p_1 \restriction \kappa = h_\kappa$ and

$$p_1 \Vdash_{T * Q_{j(\kappa)1}} \kappa \in j(\dot{X})^{G^* \restriction (P_{j(\kappa)} * S)},$$

then there will be some $p_0 \in G^* \restriction (P_{j(\kappa)} * S)$ forcing this and deciding p_1. So,

$$(p_0, p_1) \Vdash \kappa \in j(\dot{X})$$

and, in addition, it is easy to chose such p_0 so that $(p_0 {}^\frown p_1 \restriction T, p_1 \restriction Q_{j(\kappa)1}) \in R$.
Suppose otherwise. Then

$$(\emptyset, h_\kappa) \Vdash_{T * Q_{j(\kappa)1}} \kappa \in j(\kappa \backslash \dot{X})^{G^* \restriction (P_{j(\kappa)} * S)}.$$

Pick $p_0 \in G^* \restriction (P_{j(\kappa)} * S)$ forcing this. Then

$$(p_0, \emptyset, h_\kappa) \Vdash \kappa \in j(\kappa \backslash \dot{X}).$$

Without loss of generality we can assume that for each $\gamma < \eta$ there is $\tau > \kappa$ with $(j(\gamma), \tau) \in \mathrm{dom}(p_0 \restriction S)$.
Extend h_κ to p_1 by adding to it $(\delta_\gamma, 1)$, for each $\gamma < \eta$.
Set $r(0) = (p_0, \emptyset, p_1)$. Then $r(0)$ satisfies the conditions (1)-(7) of the definition of W with $A = \kappa \backslash X$.
Now we shall deal with r' as in the condition (8) and show that either one of them will have an extension in R forcing "$\kappa \in j(\dot{X})$" or all of them force "$\kappa \in j(\kappa \backslash \dot{X})$", which means that $\kappa \backslash X \in W$ and contradicts positivity of X.
Let

$$\langle b_\xi \mid \xi < \kappa^+ \rangle$$

be an enumeration in $M[G^* \restriction P_{j(\kappa)}]$ of all subsets of η. Note that η is an ordinal less than κ^+ and hence all its subsets in $V[G]$ are in $M[G^* \restriction P_{j(\kappa)}]$ as well. Work in $M[G^* \restriction (P_{j(\kappa)} * S)]$ and define by induction a sequence $\langle r_\xi \mid \xi < \xi^* \leq \kappa^{++} \rangle$. Suppose that $\xi < \kappa^{++}$ and r_ρ is already defined, for each $\rho < \xi$. Consider first

$$r'_\xi = \bigcup \{ (j(\gamma), \tau, i) \mid \exists \rho < \xi \quad ((j(\gamma), \tau) \in \mathrm{dom}(r_\rho \restriction S), (r_\rho \restriction S)(j(\gamma), \tau) = i$$

$$\text{and } r_\rho \restriction Q_{j(\kappa)1}(\delta_\gamma) = 1)\},$$

if $\xi > 0$ and $r'_\xi = r(0) \restriction S$, if $\xi = 0$. Using induction we may assume that $r'_\xi \in G^* \restriction S$. Just note that we work in $M[G^* \restriction (P_{j(\kappa)} * S)]$ and so the forcing S is $j(\kappa)$-closed.

Set

$$p(\xi) = \{(\delta_\gamma, 1) \mid \gamma \in b_\xi\} \cup \{(\delta_\gamma, 0) \mid \gamma \in \eta \backslash b_\xi\}$$

and

$$r''_\xi = \{(j(\gamma), \tau, i) \in r'_\xi \mid \gamma \in b_\xi\}.$$

Turn for a moment to $M[G^* \restriction (P_{j(\kappa)} * (S \restriction j''b_\xi))]$. If there is $p_1^\xi \in (S \restriction j''(\eta \backslash b_\xi)) * T * Q_{j(\kappa)1}$ such that $p_1^\xi \restriction Q_{j(\kappa)1} \geq p(\xi)$ and

$$p_1^\xi \Vdash_{S \restriction j''(\eta \backslash b_\xi) * T * Q_{j(\kappa)1}} \kappa \in j(\dot{X})^{G^* \restriction (P_{j(\kappa)} * S \restriction j''b_\xi)},$$

then there will be some $p_0^\xi \in G^* \restriction (P_{j(\kappa)} * S \restriction j''b_\xi)$ forcing this and deciding p_1^ξ. So,

$$(p_0^\xi, p_1^\xi) \Vdash \kappa \in j(\dot{X})$$

and, in addition, it is easy to chose p_0^ξ such that $(p_0^\xi {}^\frown (p_1^\xi \restriction (S \restriction j''(\eta \backslash b_\xi) * T)), p_1^\xi \restriction Q_{j(\kappa)1}) \in R$. We set $\xi^* = \xi$ and stop the process.
Suppose otherwise. Then

$$(\emptyset, \emptyset, p(\xi)) \Vdash_{S \restriction j''(\eta \backslash b_\xi) * T * Q_{j(\kappa)1}} \kappa \in j(\kappa \backslash \dot{X})^{G^* \restriction (P_{j(\kappa)} * S)}.$$

Pick $p_0^\xi \in G^* \restriction (P_{j(\kappa)} * S \restriction j''b_\xi)$ above r''_ξ forcing this. Then

$$(p_0^\xi, \emptyset, \emptyset, p(\xi)) \Vdash \kappa \in j(\kappa \backslash \dot{X}).$$

Set $r_\xi = (p_0^\xi, \emptyset, \emptyset, p(\xi))$.
This completes the construction. Suppose that the construction never stops, i.e. $\xi^* = \kappa^{++}$. Set, as above

$$r'_{\kappa^{++}} = \bigcup \{(j(\gamma), \tau, i) \mid \exists \rho < \kappa^{++} \quad ((j(\gamma), \tau) \in \mathrm{dom}(r_\rho \restriction S), (r_\rho \restriction S)(j(\gamma), \tau) = i$$

$$\text{and } r_\rho \restriction Q_{j(\kappa)1}(\delta_\gamma) = 1)\},$$

Again, we are in $M[G^* \restriction (P_{j(\kappa)} * S)]$ and so the forcing S is $j(\kappa)$-closed. Hence $r'_{\kappa^{++}} \in G^* \restriction S$. For each $\xi < \kappa^{++}$, set

$$r'_{\kappa^{++}\xi} = \{(j(\gamma), \tau, i) \in r'_{\kappa^{++}} \mid \gamma \in b_\xi\}.$$

Then, for each $\xi < \kappa^{++}$ we have $r'_{\kappa^{++}\xi} \geq p_0^\xi \restriction S$, and hence

$$(r'_{\kappa^{++}\xi}, \emptyset, \emptyset, p(\xi)) \Vdash_{S \restriction j''(\eta \backslash b_\xi) * T * Q_{j(\kappa)1}} \kappa \in j(\kappa \backslash \dot{X})^{G^* \restriction P_{j(\kappa)}}.$$

Finally we pick $q \in G^* \restriction P_{j(\kappa)}$ forcing the above. Set $r = (q, r'_{\kappa^{++}}, \emptyset, p_1)$. Then r witnesses $\kappa \backslash X \in W$, which contradicts the positivity of X.
$\square$ of the claim.

Claim 2 W is a precipitous filter over κ.

Proof. Force with R over $V[G]$. Let $G(R)$ be a generic object. Now, $(G^* \restriction P_{j(\kappa)}) * G(R)$ will be M-generic for $j(P)$. Thus for each $\alpha < j(\kappa^+)$, if we restrict $G(R)$ to its α first Cohen functions (say, including $h_{j(\kappa)}$ as the first one), then we will have mutually generic Cohen functions over $M[G^* \restriction P_{j(\kappa)}]$, since it is a product and so, the order of components does not matter. The forcing satisfies $j(\kappa)^+$-c.c. in $M[G^* \restriction P_{j(\kappa)}]$, so the full $G(R)$ will be generic. Now, $j''G \subseteq G^* \restriction P_{j(\kappa)} * G(R)$, by the definition of R. Hence we can in $V[G * G(R)]$ extend the embedding $j : V \to M$ to $i : V[G] \to M[G^* \restriction P_{j(\kappa)} * G(R)]$. It is not hard to see that the generic ultrapower of W according to $G(R)$ is isomorphic to $M[G^* \restriction P_{j(\kappa)} * G(R)]$.
$\square$ of the claim.

Now let i be a generic embedding obtained by forcing with W-positive sets. Let

$$\vec{B} = \langle B_\gamma | \gamma < j(\kappa^+)\rangle = i(\langle A_\gamma | \gamma < \kappa^+\rangle).$$

The crucial observation would be that

$$B_{j(\gamma)} \in V[G] \text{ iff } h_{j(\kappa)}(\delta_\gamma) = 1,$$

for each $\gamma < \kappa^+$. Note that for every $\gamma < \kappa^+$

$$B_{j(\gamma)} = \{\nu < j(\kappa) | i(f_{\kappa\gamma})(\nu) = 0\}$$

and that $i(f_{\kappa\gamma})$ is in $V[G]$ iff $B_{j(\gamma)}$ is in $V[G]$. Now by definition of R, if $h_{j(\kappa)}(\delta_\gamma) = 0$, then $i(f_{\kappa\gamma}) \notin V[G]$ as it is generic over $V[G]$ and if $h_{j(\kappa)}(\delta_\gamma) = 1$, then $i(f_{\kappa\gamma}) = j^*(f_{\kappa\gamma}) \in V[G]$.

Let us denote by $f_{j(\kappa)\xi}$ the ξ's Cohen function over $j(\kappa)$ of $i(G)$.

Claim 3 $i(W) \cap V[G] \notin V[G]$.

Proof. Suppose otherwise. Let

$$X = i(W) \cap V[G] \in V[G].$$

We have $G^* \in V[G]$, hence the set of the Cohen function over $j(\kappa)$ from G^* is in $V[G]$ as well, i.e.

$$\langle f^*_{j(\kappa)\rho} | \rho < j(\kappa^+)\rangle \in V[G].$$

Hence, also

$$\langle f^*_{j(\kappa)j(\gamma)} | \gamma < \kappa^+\rangle \in V[G].$$

Let

$$j^*(\langle A_\gamma | \gamma < \kappa^+\rangle) = \langle B^*_\gamma | \gamma < j(\kappa^+)\rangle.$$

Then

$$B^*_{j(\gamma)} = \{\nu < j(\kappa) | f^*_{j(\kappa)j(\gamma)}(\nu) = 0\},$$

for each $\gamma < \kappa^+$. Also,

$$\langle B^*_{j(\gamma)} | \gamma < \kappa^+\rangle \in V[G].$$

Consider the set

$$Y = \{\gamma < \kappa^+ | B^*_{j(\gamma)} \in X\}.$$

Clearly, $Y \in V[G]$. Now,

$$B^*_{j(\gamma)} \in i(W) \text{ (or even is in } M[i(G)]) \text{ iff } f^*_{j(\kappa)j(\gamma)} = i(f_{\kappa,\gamma}).$$

One direction is clear. Thus if $f^*_{j(\kappa)j(\gamma)} = i(f_{\kappa,\gamma})$, then $B^*_{j(\gamma)} = B_{j(\gamma)}$ which is in $i(W)$. Suppose now that $B^*_{j(\gamma)} \in i(W)$. Then the function $f^*_{j(\kappa)j(\gamma)} \in M[i(G)]$. Recall that we showed in the previous claim that $M[i(G)] = M[G^* \upharpoonright P_{j(\kappa)} * G(R)]$. Also $G(R)$ is a generic set over $M[G^* \upharpoonright P_{j(\kappa)}]$ for adding $j(\kappa^+)$ Cohen function from $j(\kappa)$ to 2. Suppose that $f^*_{j(\kappa)j(\gamma)} \neq i(f_{\kappa,\gamma})$. Then $G(R)$ will be generic over $M[G^* \upharpoonright P_{j(\kappa)}, f^*_{j(\kappa)j(\gamma)}]$, since for each $\xi < j(\kappa)^+$, if $\xi \notin j''\kappa^+$ or $\xi = j(\tau)$ for some $\tau < \kappa^+$ with $h_{j(\kappa)}(\delta_\tau) = 0$, then $f_{j(\kappa),\xi}$ is generic over $V[G] \supset M[G^* \upharpoonright P_{j(\kappa)}, f^*_{j(\kappa)j(\gamma)}]$. If $\xi = j(\tau)$ for some $\tau < \kappa^+$ with $h_{j(\kappa)}(\delta_\tau) = 1$, then $f_{j(\kappa),\xi} = f^*_{j(\kappa)\xi}$ which is generic over $M[G^* \upharpoonright P_{j(\kappa)}, f^*_{j(\kappa)j(\gamma)}]$. But this means that $f^*_{j(\kappa)j(\gamma)}$ is Cohen generic over $M[G^* \upharpoonright P_{j(\kappa)} * G(R)] = M[i(G)]$ contradicting to the assumption that $f^*_{j(\kappa)j(\gamma)} \in M[i(G)]$. Hence $f^*_{j(\kappa)j(\gamma)}$ must be equal to $i(f_{\kappa,\gamma})$.

The rest of the proof follows easily now. Thus

$$\gamma \in Y \text{ iff } B^*_{j(\gamma)} \in i(W) \text{ iff } f^*_{j(\kappa)j(\gamma)} = i(f_{\kappa,\gamma}) \text{ iff } h_{j(\kappa)}(\delta_\gamma) = 1.$$

But both Y and $\langle \delta_\gamma | \gamma < \kappa^+\rangle$ are in $V[G]$. Hence also $h_{j(\kappa)} \upharpoonright \{\delta_\gamma | \gamma < \kappa^+\}$ is in $V[G]$, which is impossible. Contradiction.

$\square$ of the claim.

$\square$

2 Precipitous ideal without normal ideal below it.

In this section we give an example of a precipitous ideal I over a cardinal κ such that there is no normal ideal below it in the Rudin - Keisler order, i.e. for any function $f : \kappa \to \kappa$ the ideal

$$f_* I = \{A \subseteq \kappa | f^{-1\prime\prime} A \in I\}$$

is not a normal ideal.

Theorem 2.1 *Assume GCH. Let κ be a measurable cardinal. Then there is a precipitous filter without a normal filter below it in the Rudin- Keisler order.*

Proof. Let U be a normal ultrafilter over κ and $j : V \to M \simeq V^\kappa / U$ the canonical elementary embedding. Note that $|j(\kappa)| = |j(\kappa^+)| = \kappa^+$. So, we can find in V an enumeration $\langle Y_i \mid i < \kappa^+ \rangle$ of $j(U)$. Note that ${}^\kappa M \subseteq M$. So every initial segment of this sequence is in M.

Define now by induction a sequence of ordinals $\langle \alpha_i \mid i < \kappa^+ \rangle$ and a sequence of functions $\langle f_i \mid i < \kappa^+ \rangle$ so that

1. $\kappa < \alpha_i < j(\kappa)$, for each $i < \kappa^+$
2. $\alpha_i < \alpha_{i'}$ whenever $i < i'$
3. $\alpha_i \in Y_{i'}$ whenever $i \leq i'$
4. $f_i : \kappa \to \kappa$ is one to one ,increasing and $f(\nu) > \nu$, for each $\nu < \kappa$
5. $j(f_i)(\kappa) = \alpha_i$
6. $\alpha_{i'} \notin \mathrm{rng}(j(f_i))$ whenever $i < i'$.

In order to construct such sequences note that each ordinal μ in the interval $(\kappa, j(\kappa))$ can be represented by a one to one increasing function f from κ to κ such that $f(\nu) > \nu$, for each $\nu < \kappa$. Then the range of such a function will be non stationary. So, in M, $\mathrm{rng}(j(f))$ will be a non stationary subset of $j(\kappa)$. But $j(U)$ is a normal ultrafilter. In particular, each $Y \in j(U)$ is stationary. Hence there is no problem to find $\alpha \in Y \setminus \mathrm{rng}(j(f))$. Also ${}^\kappa M \subseteq M$. Hence we can proceed all the way to κ^+.

Now, set for each $i < \kappa^+$

$$U_i = \{A \subseteq \kappa \mid \alpha_i \in j(A)\}.$$

It is a κ-complete ultrafilter over κ. Clearly, $\mathrm{rng}(f_i) \in U_i$.

Claim 4 For every $i < i' < \kappa^+$, $\quad \mathrm{rng}(f_i) \notin U_{i'}$.

Proof. Just otherwise we will have

$$\alpha_{i'} \in \mathrm{rng}(j(f_i)),$$

which is impossible by 6.

$\square$

Claim 5 For every $i < \kappa^+$ there is a set $B_i \in U_i$ such that $B_i \notin U_{i'}$, for every $i' \neq i$.

Proof. Fix $i < \kappa^+$. By the previous claim it is enough to deal only with $i' < i$. Let $\langle i_\xi \mid \xi < \kappa \rangle$ be an enumeration of i. For each $\xi < \kappa$ we pick a set $A_\xi \in U_i \backslash U_{i_\xi}$. Set

$$B = \{\nu < \kappa \mid \forall \xi < f_i^{-1}(\nu) \quad \nu \in A_\xi\}.$$

Note that f_i^{-1} projects U_i to the normal ultrafilter U. Hence, $j(f^{-1})(\alpha_i) = \kappa$. But each A_ξ is in U_i, for $\xi < \kappa$. So, $\alpha_i \in j(A_\xi)$, for each $\xi < \kappa$. Hence $\alpha_i \in j(B)$ which implies $B \in U_i$. Now for every $\xi < \kappa$, we have $B \backslash A_\xi \subseteq f_i(\xi) < \kappa$ and hence $B \notin U_{i_\xi}$.
Finally take $B_i = B \cap \mathrm{rng}(f_i)$. It is as desired.

$\square$

Now we set $W = \bigcap_{i<\kappa^+} U_i$.

Claim 6 W is a precipitous ideal over κ.

Proof. Let $A \in W^+$. Then for some $i < \kappa^+$, $\quad A \in U_i$. Then $A \cap B_i \in U_i \subseteq W^+$. But Claim 5 implies that W^+ below B_i is U_i and U_i is trivially precipitous. So W is densely often precipitous and hence precipitous.

$\square$

Claim 7 There is no normal filter below W in the Rudin - Keisler order.

Proof. Suppose otherwise. So there is a function $f : \kappa \to \kappa$ such that $f_* W$ is a normal filter. Then f must project each of U_i's to a normal filter as well. Hence, for each $i < \kappa^+$, $j(f)(\alpha_i) = \kappa$. Turn to M. Note first that the set $A = \{\nu < j(\kappa) \mid j(f)(\nu) < \nu\} \in j(U)$. Thus, if $A \notin j(U)$, then $B = j(\kappa) \backslash A \in j(U)$. This implies that $B = Y_{i^*}$ for some $i^* < \kappa^+$. Then, by the construction of α_i's, we will have that every α_i with $i \geq i^*$ is in B. But $j(f)(\alpha_i) = \kappa < \alpha_i$. Which is impossible. So $A \in j(U)$.

Now we use the normality of $j(U)$. There will be $A' \subseteq A$ in $j(U)$ and $\delta < j(\kappa)$ such that $j(f)''A' = \{\delta\}$. So M satisfies the following statement:

$$\exists Z \in j(U) \quad |j(f)''Z| = 1.$$

By elementarity, in V, we will have the following:

$$\exists Z \in U \quad |f''Z| = 1.$$

So, pick such $E \in U$ and some η such that $f''Z = \{\eta\}$. But here η must be below κ and hence it does not move by j. Back in M, we will have $j(E) \in j(U)$ and for each $\rho \in j(E)$ $j(f)(\rho) = \eta$. But $j(E) \in j(U)$, so a final segment of α_i's is in $j(E)$. Which is impossible since $j(f)(\alpha_i) = \kappa > \eta$.

$\square$ of the claim.

$\square$

3 Precipitous ideal with a non precipitous normal ideal below it.

R. Laver [4] starting with a supercompact cardinal gave an example of precipitous ideal on $[\omega_2]^{<\omega_1}$ whose projection to ω_1 is not precipitous. The purpose of this section is to give an example of a precipitous ideal over a cardinal κ such that the normal ideal below it exists but is not precipitous. Only a measurable cardinal will be used for this construction.

Assume GCH. Let κ be a measurable cardinal, U a normal ultrafilter over κ, $j_1 : V \to M_1 \simeq {}^{\kappa}V/U$ the corresponding elementary embedding, $j(\kappa) = \kappa_1$ and $j_2 : V \to M_2 \simeq {}^{[\kappa]^2}V/U^2$ the corresponding elementary embedding into the second ultrapower. It will be useful to view M_2 as the ultrapower of M_1 by $j_1(U)$. Denote by k_{12} the corresponding elementary embedding of M_1 into M_2. The critical point of k_{12} is κ_1.

Force using the Backward Easton iteration ν^+- Cohen functions $f_{\nu\xi} : \nu \to \nu$, $\xi < \nu^+$ for every regular $\nu \leq \kappa$. Let

$$\langle P_\alpha, Q_\beta | \alpha \leq \kappa + 1, \beta < \kappa + 1\rangle$$

be such an iteration. Let $G \subset P_{\kappa+1}$ be generic. Denote by $f_{\alpha\xi} : \alpha \to \alpha$ the ξ-th Cohen function in G added over α.

Consider the following set in $V[G]$:

$$R = \{p \in j_2(P_{\kappa+1}) \mid p \restriction P_{\kappa+1} \in G \text{ and if for some } \xi < \kappa^+ \text{ we have } \langle j_2(\kappa), j_2(\xi)\rangle \in \mathrm{dom}(p)$$

$$\text{then } p(j_2(\kappa), j_2(\xi)) \restriction \kappa \subseteq f_{\kappa\xi} \text{ and } p(j_2(\kappa), j_2(\xi))(\kappa) \geq \kappa_1\}.$$

For each $p \in R$ we pick in $V[G]$ an M_2-generic subset G'_p of $j_2(P_{\kappa+1})$ such that $p \in G'_p$ and $G'_p \restriction \kappa + 1 = G$.

Now we would like to change G'_p a bit. Let $q \in j_2(P_{\kappa+1})$. Transform it into condition $q^* \in R$ as follows:

1. $\mathrm{dom}(q^*) = \mathrm{dom}(q)$
2. for each $\langle \alpha, \beta\rangle \in \mathrm{dom}(q)$ we require $\mathrm{dom}(q(\alpha, \beta)) = \mathrm{dom}(q^*(\alpha, \beta))$
3. if $\langle \alpha, \beta\rangle \in \mathrm{dom}(q)$ and $\alpha \neq j_2(\kappa)$, or $\alpha = j_2(\kappa)$, but $\beta \notin j_2{}''\kappa^+$, then $q^*(\alpha, \beta) = q(\alpha, \beta)$
4. if for some $\xi < \kappa^+$ we have $\langle j_2(\kappa), j_2(\xi)\rangle \in \mathrm{dom}(q)$, then $q^*(j_2(\kappa), j_2(\xi)) \restriction \kappa \subseteq f_{\kappa\xi}$
5. if for some $\xi < \kappa^+$ we have $\langle j_2(\kappa), j_2(\xi)\rangle \in \mathrm{dom}(q), \kappa \in \mathrm{dom}(q(j_2(\kappa), j_2(\xi)))$ then $q^*(j_2(\kappa), j_2(\xi))(\kappa) = \kappa_1$, unless $p(j_2(\kappa), j_2(\xi))(\kappa)$ is defined. In this case we require $q^*(j_2(\kappa), j_2(\xi))(\kappa) = p(j_2(\kappa), j_2(\xi))(\kappa)$.

Set

$$G_p = \{q^* \mid q \in G'_p\}.$$

Clearly, $G_p \subseteq R$.
Note that for each $\alpha < \kappa^+$ the set $G_p \restriction P_{j_2(\kappa)} * (Q_{j_2(\kappa)} \restriction j_2(\alpha))$ is M_2- generic for the forcing $P_{j_2(\kappa)} * (Q_{j_2(\kappa)} \restriction j_2(\alpha))$, since the set $j_2''\alpha$ has cardinality κ and so our change effects here basically a single condition. The forcing $Q_{j_2(\kappa)}$ over $M_2^{P_{j_2(\kappa)}}$ satisfies $j_2(\kappa)$-c.c. and $j_2''\kappa^+$ is unbounded in $j_2(\kappa^+)$. Hence, G_p is M_2- generic for the forcing $j_2(P_{\kappa+1}) = P_{j_2(\kappa)} * Q_{j_2(\kappa)}$.

Extend j_2 to the elementary embedding $j_p : V[G] \to M_2[G_p]$ (note that $j_2''G \subseteq G_p$, so it is possible). Let

$$U_p = \{X \subseteq \kappa \mid \kappa \in j_p(X)\}.$$

The following lemma is routine:

Lemma 3.1 *For every $p \in R$ we have $M_2[G_p] \simeq {}^\kappa V[G]/U_p$ and j_p is the canonical embedding of this ultrapower.*

Proof. Let $p \in R$. Denote the transitive collapse of ${}^\kappa V[G]/U_p$ by M_p. Let be $k : M_p \to M[G_p]$ the elementary embedding defined by $k([g]_{U_p}) = j_p(g)(\kappa)$.
It is enough to show that k does not move ordinals. Let δ be any ordinal. There is $h : [\kappa]^2 \to On$ in V such that

$$j_2(h)(\kappa, \kappa_1) = \delta.$$

j_p extends j_2, hence, also,

$$j_p(h)(\kappa, \kappa_1) = \delta.$$

Pick now $\xi < \kappa^+$ such that $\langle j_2(\kappa), j_2(\xi)\rangle \notin \mathrm{dom}(p)$. Consider $f_{\kappa\xi}$. By the choice of G_p (mainly by the item 5 above), we have $j_p(f_{\kappa\xi})(\kappa) = \kappa_1$. Consider a function $g : \kappa \to On$ defined as follows:

$$g(\nu) = h(\nu, f_{\kappa\xi}(\nu)).$$

Then,

$$k([g]_{U_p}) = j_p(g)(\kappa) = j_p(h)(\kappa, j_p(f_{\kappa\xi})(\kappa)) = j_p(h)(\kappa, \kappa_1) = \delta.$$

□

Set

$$U^* = \bigcap_{p \in R} U_p.$$

It is possible to show that U^* is the normal filter generated in $V[G]$ by U together with the following sets, for each $g : \kappa \to \kappa$, $g \in V$ and $\xi < \kappa^+$:

- $\{\nu < \kappa | f_{\kappa\xi} \restriction \nu = f_{\nu h_\xi(\nu)}\}$, where $h_\xi : \kappa \to \kappa$ denotes the ξ-th canonical function (in particular $[h_\xi]_U = \xi$)
- $\{\nu < \kappa | f_{\kappa\xi}(\nu) > g(\nu)\}$.

We will not use this characterization, but rather deal directly with U^*-positive sets. Note that $A \in (U^*)^+$ iff there is $p \in R$ such that $A \in U_p$. This is immediate since each of U_p's is an ultrafilter.

The following is the crucial observation:

Lemma 3.2 *Let $G(U^*)$ be a generic ultrafilter extending U^*, i.e. a generic subset of $(U^*)^+$. Then for each $\alpha < \kappa^+$ there is $\beta < \kappa^+$ such that*

$$[f_{\kappa\beta}]_{G(U^*)} < [f_{\kappa\alpha}]_{G(U^*)}.$$

Proof.

We work in $V[G]$ and show that for each $A \in (U^*)^+$ and $\alpha < \kappa^+$ there is $\beta < \kappa^+$ such that the set

$$\{\nu \in A | f_{\kappa\alpha}(\nu) > f_{\kappa\beta}(\nu)\}$$

is U^*- positive.

Fix such A and α. Pick $p \in R$ with $A \in U_p$. Suppose for simplicity that

$$p \Vdash \kappa \in j_2(\dot{A})$$

and $\langle j_2(\kappa), j_2(\alpha)\rangle \in \mathrm{dom}(p)$. Otherwise just extend it to such a condition in G_p.
Let us pick $\gamma < \kappa^+$ big enough such that $p \in P_{j_2(\kappa)} * Q_{j_2(\kappa)} \restriction \gamma$. Then the set

$$a = \{\beta < \kappa^+ \mid \langle j_2(\kappa), j_2(\beta)\rangle \in \mathrm{dom}(p)\}$$

is a subset of γ and, hence has cardinality at most κ.
Now we turn to M_1 and view M_2 as its ultrapower. Pick a function $h : \kappa_1 \to P_{j_1(\kappa)+1}$ in M_1 which represents p. Then $k_{12}(h)(\kappa_1) = p$. By elementarity, we will have

$$\{\xi < \kappa_1 \mid h(\xi) \in P_{\kappa_1} * Q_{j_1(\kappa)} \restriction \gamma \text{ and } h(\xi) \Vdash \kappa \in j_1(\dot{A})\} \in j_1(U).$$

Note that the critical point of k_{12} is $\kappa_1 > \kappa$ and both M_1, M_2 are closed under κ sequences of its elements. So, $j_1 \restriction \gamma \in M_1, j_2 \restriction \gamma \in M_2, k_{12}(j_1 \restriction \gamma) = j_2 \restriction \gamma$ and $j_2''a = k_{12}(j_1''a)$. We have, in M_2,

$$\{\beta < \gamma \mid \langle j_2(\kappa), j_2(\beta)\rangle \in \mathrm{dom}(p)\} = a.$$

Now, by elementarity, for most (mod $j_1(U)$) ξ's,

$$\{\beta < \gamma \mid \langle \kappa_1, j_1(\beta)\rangle \in \mathrm{dom}(h(\xi))\} = a.$$

Note that for every $\beta \in a$, we have $p(j_2(\kappa), j_2(\beta))(\kappa) \geq \kappa_1$. Hence, for most (mod $j_1(U)$) ξ's, for each $\beta \in a$ we will have

$$h(\xi)(\kappa_1, j_1(\beta)) \geq \xi.$$

The above shows that the following set in $j_1(U)$:

$$Z = \{\xi < \kappa_1 \mid h(\xi) \in P_{\kappa_1} * Q_{j_1(\kappa)} \restriction \gamma \text{ and } h(\xi) \Vdash \kappa \in j_1(\dot{A}), \{\beta < \gamma \mid \langle \kappa_1, j_1(\beta)\rangle \in \mathrm{dom}(h(\xi))\} = a,$$

$$\forall \beta \in a \quad h(\xi)(\kappa_1, j_1(\beta)) \geq \xi\}.$$

Consider $k_{12}(Z)$. Clearly, it is an unbounded subset of $j_2(\kappa)$. Pick any $\eta \in k_{12}(Z)$ above κ_1. Denote $k_{12}(h)(\eta)$ by q. Then the following will hold by the elementarity:

1. $q \in P_{j_2(\kappa)} * Q_{j_2(\gamma)}$
2. $q \Vdash \kappa \in j_2(\dot{A})$
3. $\{\beta < \gamma \mid \langle j_2(\kappa), j_2(\beta)\rangle \in \mathrm{dom}(q)\} = a$
4. $\forall \beta \in a \quad q(j_2(\kappa), j_2(\beta)) \geq \eta$.

In particular, we obtain that $q \in R$.
Recall that $\alpha \in a$. Hence, $q(j_2(\kappa), j_2(\alpha)) \geq \eta > \kappa_1$.
Pick now any $\beta, \gamma \leq \beta < \kappa^+$. Extend q to $r = q \cup \{\langle j_2(\kappa), j_2(\beta), \kappa, \kappa_1\rangle\}$, i.e. $r(j_2(\kappa), j_2(\beta))(\kappa) = \kappa_1$.

Then,

$$r \Vdash (\kappa \in j_2(\dot{A}) \text{ and } j_2(\dot{f}_{\kappa\alpha})(\kappa) > j_2(\dot{f}_{\kappa\beta})(\kappa)).$$

Hence the set

$$\{\nu \in A | f_{\kappa\alpha}(\nu) > f_{\kappa\beta}(\nu)\}$$

is in U_r and so it is U^*-positive.
$\square$

The next lemma follows now easily from the previous one:

Lemma 3.3 *U^* is not precipitous.*

Proof. Let $G(U^*)$ be a generic ultrafilter extending U^*, i.e. a generic subset of $(U^*)^+$. Using Lemma 3.2, we can define inductively an increasing sequence of ordinals below κ^+

$$\langle \alpha_n | n < \omega \rangle$$

such that $[f_{\kappa\alpha_n}]_{G(U^*)} < [f_{\kappa\alpha_m}]_{G(U^*)}$, whenever $n > m$. So the ultrapower ${}^{\kappa}V[G]/G(U^*)$ is ill founded.
$\square$

Let us spread a way all the ultrafilters U_p for $p \in R$. We fix (in V) a sequence

$$\langle g_\alpha \mid \kappa \leq \alpha < \kappa^+ \rangle$$

such that for every $\alpha < \kappa^+$

- g_α is one to one
- $\mathrm{dom}(g_\alpha)$ is the set of cardinals below κ
- $\nu \in \mathrm{dom}(g_\alpha) \quad \nu \leq g_\alpha(\nu) < \nu^+$
- $[g_\alpha]_U = \alpha$.

Note that $g_\alpha^{-1}(\tau) = |\tau|$, for each $\tau \in \mathrm{rng}(g_\alpha)$.

For each $\alpha \in [\kappa, \kappa^+)$ let $A_\alpha = \mathrm{rng}(g_\alpha)$. Then, $\alpha \neq \beta$ implies

$$\{|\tau| \mid \tau \in A_\alpha \cap A_\beta\} \notin U.$$

Since, otherwise we will have in M_1, some $\tau, \kappa \leq \tau < \kappa^+$. Then

$$\alpha = j_1(g_\alpha)(\kappa) = \tau = j_1(g_\beta)(\kappa) = \beta,$$

which is impossible, since the functions are one to one.

Fix some enumeration $\langle p_i \mid i < \kappa^+ \rangle$ of R and for each $i < \kappa^+$ use the function g_i in order to move U_{p_i} to A_i. Denote the resulting κ-complete ultrafilter by U_i. Finally, set $F = \bigcap_{i<\kappa^+} U_i$. Then F is precipitous just since $F + A_i$ is U_i. Thus if $Y \in F + A_i$, then $A_i \backslash Y \notin U_i$. But U_i is an ultrafilter, so $A_i \cap Y \in U_i$ and hence $Y \in U_i$. For the opposite direction note that if $Z \subseteq A_i$ and $Z \notin U_i$, then $Z \in \breve{F}$. Just otherwise $Z \in U_\beta$ for some $\beta \neq i$. This implies

$$A_i \cap A_\beta \supseteq Z \cap A_\beta \in U_\beta.$$

But then

$$\{|\nu| \mid \nu \in A_i \cap A_\beta\} \in U,$$

which is impossible.

Consider $H : \kappa \to \kappa$, $H(\tau) = |\tau|$. Now, H is equal to each g_i^{-1} mod U_i, hence it projects U_i's to U_{p_i}'s. So, F is projected by H to U^* which is not precipitous.

4 A remark on the consistency strength of precipitous without normal precipitous.

The long standing open question in this area asks the following:

(T. Jech and K. Prikry) Is it possible to have a precipitous ideal without a normal precipitous?

The previous construction of the paper seem to be irrelevant for this question, since the cardinal remains measurable in all the models above. It is possible to move everything to $\aleph_1$ using the Levy collapse, but still we do not know any effective way to get rid of unwanted filters.
It looks reasonable to try to deduce some strength from the assumption that there is a precipitous ideal without a normal precipitous one. The aim of this section will be to do so under some additional assumptions. Also certain information on a structure of elementary embeddings will be obtained here.

Let us assume that there is no inner with a strong cardinal in order to insure that the core model K exists, is invariant under set forcing extensions and the restrictions of generic embeddings to K are iterated ultrapowers of K by its measures or extenders. We refer to the Mitchel chapter [6] for the relevant material.

Fix a precipitous filter U over a cardinal κ. We will consider the restrictions of its generic embeddings to the core model K. By [6], such restrictions are iterated ultrapowers of K by its measures or extenders.
Note that also the iteration map itself may be new, i.e. not in V. But it is always possible to embed it into an iteration which is defined in K, called the complete iteration. Thus take a regular cardinal χ above all the generators or possible generators of the generic embeddings involved and iterate each measure or extender with index below χ χ- many times.
We pick first a set $X \in U^+$ and a function $c : \kappa \to \kappa$ such that

$$X \|\!\!-_{U^+} \quad \dot{j}(c)([id]_{G(\dot{U}^+)}) = \kappa,$$

where $G(\dot{U}^+)$ is the canonical name of generic ultrafilter and $\dot{j}$ a name of its elementary embedding. Replace further U by $U + X$. Consider $U_{normal} = c_* U$. It is the normal filter Rudin- Keisler below U.
Let $G(U^+)$ be a generic subset of U^+ (i.e. a generic ultrafilter extending U) and $j : V \to M$ the corresponding elementary embedding. Define $G(U^+)_{normal} = \{c''X \mid X \in G(U^+)\}$. Note that $G(U^+)_{normal}$ need not be a generic subset of U_{normal}, as was shown in the previous section. But still we can form ultrapower. Let $i : V \to M_{normal}$ be the corresponding

elementary embedding. There will be also an elementary embedding $k : M_{normal} \to M$ defined by $k([g]_{G(U^+)_{normal}}) = j(g)(\kappa)$.

Consider the restrictions of j and i to K. Denote them by j_K and i_K respectively. Suppose for simplicity that j_K and i_K are in V (or K) otherwise replace them by complete embeddings. We also will have a connecting embedding $k_K : i_K(K) \to j_K(K)$.

Now back in V, we pick $X \in U^+$ deciding both j_K and i_K. Note that if X forces that k_K is the identity, then $U + X$ will be densely often isomorphic to a normal filter and so, there will be a normal precipitous ideal.

We are ready now to state the first result.

Theorem 4.1 *Assume that there is no inner model with a strong cardinal. Suppose that U is precipitous filter over κ and some set X in U^+ forces that i_K has only finitely many generators. Then for some $Y \subseteq X$, $\quad Y \in U^+ \quad (U+Y)_{normal}$ is a normal precipitous filter.*

Proof. Let U and X be as in the statement of the theorem. Assume for simplicity that U_{normal} already exists and X decides both i_K and j_K. We shrink X, if necessary in order to decide the value of $[id]_{\dot{G}}$, i.e. find some set of generators $\kappa = \delta_0 < \delta_1 < ... < \delta_n$ of the decided iterated ultrapower and some $h : [\kappa]^{n+1} \to \kappa, h \in K$ such that

$$\kappa \Vdash_{(U+X)^+} [id]_{\dot{G}} = j(h)(\delta_0, ..., \delta_n).$$

Shrink X again if necessary in order to decide the finitely many generators of i_K. Suppose for simplicity that X already decides this and $\delta_0, ..., \delta_m$ for some $m < n$ are this generators. Note that if we have more than n generator for i_K, then it is possible just to add the missing ones to the list $\langle \delta_0, ..., \delta_n \rangle$.

Also assume that X decides a one to one function $f : \kappa \to [\kappa]^{m+1}$ such that

$$j(f)(\kappa) = \langle \delta_0, ..., \delta_m \rangle.$$

Let us replace $U + X$ by its isomorphic image $h^{-1\prime\prime}U + X$. Denote this precipitous filter over $[\kappa]^{+n+1}$ by W. Then

$$[\kappa]^{n+1} \Vdash_{W^+} [id]_{G(W)} = \langle \delta_0, ..., \delta_n \rangle.$$

Consider the projection

$$W^* = \{A \subseteq [\kappa]^{m+1} | [\kappa]^{n+1} \Vdash_{W^+} \langle \delta_0, ..., \delta_m \rangle \in j(A)\}.$$

Then W^* is isomorphic to a normal filter as witnessed by f^{-1}. Let $pr : [\kappa]^{n+1} \to [\kappa]^{m+1}$ be the projection function to the first $m+1$ coordinates. Then pr projects W onto W^*.

Note that for any function $g : \kappa \to On$ and any $A \in W^+$ there are $A_g \subseteq A$ in W^+ and $t : [\kappa]^{m+1} \to On$ in K such that

$$A_g \Vdash_{W^+} i(g)(\delta_0) = i_K(t)(\delta_0, ..., \delta_m) = i_K(t)(i(f_0)(\kappa), ..., i(f_m)(\kappa)).$$

It follows since $\delta_0, ..., \delta_m$ are all the generators of $i_K = i \restriction K$.

We claim now that W^* is precipitous. Suppose otherwise. Then there is an $\in$- decreasing sequence of functions $\langle g_l | l < \omega \rangle$ mod G, for some generic ultrafilter $G \subseteq (W^*)^+$. Define by induction for each $l < \omega$ a W^*-positive set A^*_l and a function $t_l \in K$ so that

1. $A^*_l \subseteq \{\langle \nu_0, ..., \nu_m \rangle | g_l(\nu_0) = t_l(\nu_0, ..., \nu_m)\}$
2. $A^*_l \in G$
3. $A^*_l \supseteq A^*_{l+1}$
4. $A^*_{l+1} \subseteq \{\langle \nu_0, ..., \nu_m \rangle | g_l(\nu_0) > g_{l+1}(\nu_0)\}$

Start with g_0. For each $t : [\kappa]^{m+1} \to On$ in K let

$$A^*_{0t} = \{\langle \nu_0, ..., \nu_m \rangle \mid g_0(\nu_0) = t(\nu_0, ..., \nu_m)\}$$

and

$$A_{0t} = \{\langle \nu_0, ..., \nu_n \rangle \mid g_0(\nu_0) = t(\nu_0, ..., \nu_m)\}.$$

Clearly, $A_{0t} = pr^{-1}{}'' A^*_{0t}$.
Note that for any two such functions t, r we have either

$$\{\langle \nu_0, ..., \nu_m \rangle \mid r(\nu_0, ..., \nu_m) = t(\nu_0, ..., \nu_m)\} \in W^*$$

or

$$\{\langle \nu_0, ..., \nu_m \rangle \mid r(\nu_0, ..., \nu_m) \neq t(\nu_0, ..., \nu_m)\} \in W^*.$$

This depends on whether $i_K(r)(\delta_0, ..., \delta_m) = i_K(t)(\delta_0, ..., \delta_m)$ or not.
Consider now a set $T \subset K$ of t's, $t : [\kappa]^{m+1} \to On$ in K, such that

$$\langle A_{0t} \mid t \in T \rangle$$

is a maximal antichain in W^+. Such T exists by the assumption of the claim applied to g_0. We argue that

$$\langle A^*_{0t} \mid t \in T\rangle$$

is also is a maximal antichain but in $(W^*)^+$. Thus let $B \in (W^*)^+$. Consider $A = pr^{-1\prime\prime}B$. Then $A \in W^+$. So, for some $t \in T$ we must have $A \cap A_{0t} \in W^+$. But then $pr''(A \cap A_{0t}) \in (W^*)^+$ and $B \supseteq pr''(A \cap A_{0t}) = pr''A \cap pr''A_{0t} = (pr''A) \cap A^*_{0t}$.
Let now $t, r \in T, t \neq r$. Then

$$A^*_{0t} \cap A^*_{0r} = \{\langle \nu_0, ..., \nu_m\rangle \mid g_0(\nu_0) = r(\nu_0, ..., \nu_m) = t(\nu_0, ..., \nu_m)\}.$$

If $A^*_{0t} \cap A^*_{0r} \in (W^*)^+$, then

$$\{\langle \nu_0, ..., \nu_m\rangle \mid r(\nu_0, ..., \nu_m) = t(\nu_0, ..., \nu_m)\} \in W^*.$$

Just since then we must have $i_K(r)(\delta_0, ..., \delta_m) = i_K(t)(\delta_0, ..., \delta_m)$. Then, also

$$pr^{-1\prime\prime}\{\langle \nu_0, ..., \nu_m\rangle \mid r(\nu_0, ..., \nu_m) = t(\nu_0, ..., \nu_m)\} = \{\langle \nu_0, ..., \nu_n\rangle \mid r(\nu_0, ..., \nu_m) = t(\nu_0, ..., \nu_m)\} \in W.$$

So, A_{0t} and A_{0r} are the same mod W, which is impossible.

Pick now $t_0 \in T$ such that $A^*_{0t_0} \in G$. Let A^*_0 be $A^*_{0t_0}$ and A_0 be A_{0t_0}.

Let us turn to the next stage. Consider

$$Z^* = \{\langle \nu_0, ..., \nu_m\rangle \in A^*_0 \mid g_0(\nu_0) > g_1(\nu_0)\}.$$

Then $Z^* \in G$. Let $Z := pr^{-1\prime\prime}Z^*$. Note that

$$Z = \{\langle \nu_0, ..., \nu_n\rangle \in A_0 \mid g_0(\nu_0) > g_1(\nu_0)\}.$$

For each $t : [\kappa]^{m+1} \to On$ in K let

$$A^*_{1t} = \{\langle \nu_0, ..., \nu_m\rangle \in Z^* \mid g_1(\nu_0) = t(\nu_0, ..., \nu_m)\}$$

and

$$A_{1t} = \{\langle \nu_0, ..., \nu_n\rangle \in Z \mid g_1(\nu_0) = t(\nu_0, ..., \nu_m)\}.$$

Clearly, $A_{1t} = pr^{-1\prime\prime}A^*_{1t}$.
Consider now a set $T \subset K$ of t's, $t : [\kappa]^{m+1} \to On$ in K, such that

$$\langle A_{1t} \mid t \in T\rangle$$

is a maximal antichain in W^+ below Z. Such T exists by the assumption of the claim applied to g_1 and Z.
We argue that

$$\langle A^*_{1t} \mid t \in T\rangle$$

is also is a maximal antichain but in $(W^*)^+$ and below Z^*. The argument is exactly as those for 0.

Pick now $t_1 \in T$ such that $A^*_{1t_1} \in G$. Let A^*_1 be $A^*_{1t_1}$ and A_1 be A_{1t_1}.

The argument for arbitrary $l > 1$ is identical.

Then for each $l < \omega$ we will have

$$\{\langle \nu_0, ..., \nu_n\rangle \mid \langle \nu_{m+1}, ..., \nu_m\rangle \in A^*_l\} \Vdash_{W^+} i(g_l)(\delta_0, ..., \delta_m) = i_K(t_l)(\delta_0, ..., \delta_m)$$

$$= i_K(t_l)(i(f_0)(\kappa), ..., i(f_m)(\kappa)).$$

Now, $i_K(K)$ is well founded, hence there must be $l' < l < \omega$ such that

$$i_K(t_{l'})(\delta_0, ..., \delta_m) \leq i_K(t_l)(\delta_0, ..., \delta_m).$$

So, the set

$$A = \{\langle \nu_0, ..., \nu_n\rangle \in [\kappa]^{n+1} \mid t_{l'}(\nu_0, ..., \nu_m) \leq t_l(\nu_0, ..., \nu_m)\} \in W.$$

Hence,

$$A \Vdash_{(W)^+} \dot{i}(t_{l'})(\delta_0, ..., \delta_m) \leq \dot{i}(t_l)(\delta_0, ..., \delta_m).$$

But the generic embedding i extends i_K and $A \cap \{\langle \nu_0, ..., \nu_n\rangle | \langle \nu_0, ..., \nu_m\rangle \in A^*_l\}$ is W-positive. Contradiction.

□

The next question is how can we guarantee that the number of generators of i_K is finite. The following gives a sufficient condition.

Theorem 4.2 *Suppose that there is a precipitous ideal over a cardinal κ so that after the forcing with its positive sets the cardinality of κ remains above $\aleph_1$. If there is no inner model in which κ is a limit of measurable cardinals and an inner model with a measurable cardinal in the interval $(\kappa, 2^\kappa]$, then there must be a normal precipitous ideal as well.*

Remark 4.3 Note that 2^κ may be very large while 2^κ in the sense of the generic ultrapower may be just κ^+. Only one measurable is needed in order to create such situation. This was done first by J.-P. Levinski [5].

Proof.

We show that the assumptions of the theorem imply that j_K (the restriction of a generic embedding j to K) must have only finitely many generators. Suppose otherwise. Let $G \subseteq U^+$ be generic and assume that j_K has infinitely many generators.
By the assumption of the theorem, κ is not a limit of measurable cardinals in K. Hence, j_K is formed by iterating the normal measure over κ (and its images) infinitely many times. Let

$$\kappa = \kappa_0 < \kappa_1 < .... < \kappa_n...$$

be the critical points of this iteration and $\kappa_\omega = \bigcup_{n<\omega} \kappa_n$. Then $j(\kappa) \geq \kappa_\omega$. Let $K' = j(K)$. Then

$$M \models (K' \text{ is my core model }).$$

In $V[G]$ pick a sequence $\langle f_n | n < \omega \rangle$ of functions from κ to κ , such that for each $n < \omega$ $f_n \in K$ and $\kappa_n = [f_n]_G$.
Now, by the assumption of the theorem, there are no measurable cardinals in K between κ and $(2^\kappa)^V$. So, the Dodd-Jensen Covering Lemma applies and we can find $X \in V \cap {}^\kappa\kappa, |X| \leq \aleph_1$ such that $X \supseteq \{f_n | n < \omega\}$. But, by the assumption of the theorem, we have $|\kappa| > \aleph_1$ in $V[G]$, hence $|X|^V < \kappa$. It follows now that $j''X \in M$ and $|j''X|^M = |X|^V < \kappa$. Consider the set

$$Y = \{j(f)([id]_G) | f \in X\}$$

We have $Y \in M, |Y|^M = |j''X|^M < |\kappa|$. Also,

$$Y \supseteq \{j(f_n)([id]_G) | n < \omega\}.$$

Hence, Y is unbounded in κ_ω. But κ_ω is a regular cardinal in K'. There are no measurable cardinals in K' in the interval $[\kappa, \kappa_\omega)$ (just $j(\kappa) \geq \kappa_\omega$, so if there are such, then κ will be a limit of measurables in K) . Still, in M, κ_ω changed its cofinality to something below $|\kappa| > \aleph_1$. This is impossible by the Dodd-Jensen Covering Lemma. Contradiction.
$\square$

Let us conclude with a bit more general result than those of 4.1.

Theorem 4.4 *Let I be a precipitous ideal over a cardinal κ and its projection I_{normal} to a normal ideal exists. Then I_{normal} is precipitous provided the following conditions hold:*

1. *there exists i^* an iterated ultrapower of the core model K such that $\kappa \Vdash_{I^+}$ the embedding $i_K = i \restriction K$ can be completed to i^*, i.e. there is k, $k \circ i_K = i^*$*

2. *there is an I_{normal}-increasing sequence of functions $\langle f_\alpha \mid \alpha < \delta \rangle$ from κ to κ such that*

$$\kappa \Vdash_{I^+} (\forall \tau(\tau \textit{ is a generator of } i_K \Rightarrow (\exists \alpha < \delta \quad \tau = i(f_\alpha)(\kappa))).$$

3. *if for some $Y \in I^+_{normal}$ and $\alpha < \delta$ we have*

$$\pi^{-1} \text{``} Y \Vdash_{I^+} k(i(f_\alpha)(\kappa)) \textit{ is a generator of } i^*,$$

(where π denotes a projection of I to I_{normal}) then there are $Z \subseteq Y$, $Z \in I^+_{normal}$ and an ordinal τ such that

$$\pi^{-1} \text{``} Z \Vdash_{I^+} k(i(f_\alpha)(\kappa)) = \tau.$$

Remark 4.5 1. Note that if i_K has only finitely many generators, then it easy to satisfy the conditions of the theorem. Just shrink to a positive set deciding the order between the generators and their values. In this case i^* will be i_K itself.

2. The meaning of the condition (3) of the theorem is that once $k(i(f_\alpha)(\kappa))$ is forced by I_{normal} to be a generator, then it is possible to decide exactly (again using I_{normal}) which generator $k(i(f_\alpha)(\kappa))$ is.

Proof.

Let $G \subseteq I^+_{normal}$ be generic and $\langle g_n | n < \omega \rangle \in V[G]$ be a sequence of functions from κ to κ each of the functions in V.

Claim 8 Let $g : \kappa \to On$ be a function in V. Then there are $\xi_1 < ... < \xi_n < \delta$ and a function $h : [\kappa]^n \to \kappa$, $h \in K$ such that

1. $\{\nu \mid g(\nu) = h(f_{\xi_1}(\nu), ..., f_{\xi_n}(\nu))\} \in G$

2. for some $Y \in G$

$$\pi^{-1} \text{``} Y \Vdash_{I^+} \forall k, 1 \leq k \leq n, i(f_{\xi_k})(\kappa) \text{ is a generator of } i_K.$$

Proof. Work in V. Let $X \in I^+_{normal}$. Let $G(I)$ be a generic subset of I^+ with $\pi^{-1} \text{``} X \in G(I)$. Consider the corresponding $j : V \to M \simeq V \cap {}^\kappa V/G(I)$ and $i : V \to M_{normal} \simeq V \cap {}^\kappa V/\pi \text{``} G(I)$. Let $i_K = i \restriction K$.
Now, find an ordinal τ with $i(g)(\kappa) = \tau$. There are $\delta_1 < ... < \delta_m \leq \tau$ generators of i_K and a

function $h' : [\kappa]^m \to On$, $h' \in K$ such that $\tau = i_K(h')(\delta_1, ..., \delta_n)$. By the assumption of the theorem, for each $\delta_k, 1 \leq k \leq n$, there is ζ_k such that $\delta_k = i(f_{\zeta_k})(\kappa)$. Hence

$$i(g)(\kappa) = \tau = i_K(h')(i(f_{\zeta_1})(\kappa), ..., i(f_{\zeta_m})(\kappa)).$$

So, back in V, the set

$$Y' = X \cap \{\nu \mid g(\nu) = h'(f_{\zeta_1}(\nu), ..., f_{\zeta_m}(\nu))\}$$

is in I^+_{normal}. By the density argument, then $\zeta_1, ..., \zeta_m$ and h satisfy (1) above.

The condition (2) is more delicate. We shall use here the assumption that the sequence of functions $\langle f_\alpha \mid \alpha < \delta \rangle$ is an I_{normal}-increasing.
It worth to note that the construction of Section 3 (namely the way how the non precipitousness of the normal filter was insured)is based on the "play" with functions $f_\alpha : \kappa \to \kappa$ such that $i(f_\alpha)(\kappa) = \kappa_1$ for one i, but it is possible to find an other i' and $\beta > \alpha$ with $i'(f_\beta)(\kappa) = \kappa_1$ and $i'(f_\beta)(\kappa) < i'(f_\alpha)(\kappa)$.

Let $\zeta \in \{\zeta_1, ..., \zeta_m\}$. Suppose that there is a generic $G(I) \subseteq I^+$ with π^{-1}"Y' inside, such that $i(f_\zeta)(\kappa)$ is not a generator of i_K. Then there are generators of $\mu_1 < ... < \mu_l$ of i_K, all strictly less than $i(f_\zeta)(\kappa)$ and $t : [\kappa]^l \to \kappa$ in K, such that

$$i(f_\zeta)(\kappa) = i_K(t)(\mu_1, ..., \mu_l).$$

By the assumption of the theorem, then there are $\eta_1, ..., \eta_l$ such that $\mu_k = j(f_{\eta_k})(\kappa)$,$1 \leq k \leq l$. The crucial here is that for each $k, 1 \leq k \leq l$, we must have $\eta_k < \zeta$, since $i(f_\eta)(\kappa) < i(f_{\eta'})(\kappa)$ iff $\eta < \eta'$. Also, note that a set witnessing not being a generator (i.e. that there is t as above) is of the form π^{-1}"S for some $S \in I^+_{normal}$.
Now continue similar with f_η's. Since we go down according to indexes of the functions, the process should stop after finitely many steps. The final $\xi_1, ..., \xi_n$ will the finite sequence of indexes which guaranties that the corresponding functions are generators, i.e.

$$\pi^{-1}\text{``}Y \Vdash_{I^+} \forall k, 1 \leq k \leq n, i(f_{\xi_k})(\kappa) \text{ is a generator of } i_K,$$

for some $Y \subseteq Y'$, $Y \in G$. The desired function $h : [\kappa]^n \to On$ will be the composition of h' with t's defined in the process.
$\square$ of the claim.

Now, for each $m < \omega$, repeating the process of 4.1, we pick $\xi_{1n}, ..., \xi_{n_m m}$ and h_m satisfying the claim for g_m. Without loss of generality assume that $m \leq m'$ implies that $\{\xi_{1m}, ..., \xi_{n_m m}\} \subseteq \{\xi_{1m'}, ..., \xi_{n_{m'} m'}\}$.

Fix $m < \omega$. Let $X = \{\nu \mid g_m(\nu) = h_m(f_{\xi_{1m}}(\nu), ..., f_{\xi_{n_m m}}(\nu))\} \in G$.
By the condition (3) of the theorem, there are $\langle \tau_1, ..., \tau_{n_m} \rangle$ generators of i^* and $Z_m \subseteq X, Z_m \in G$ such that

$$\pi^{-1}``Z_m \Vdash_{I^+} \forall k, 1 \leq k \leq n_m, \quad k(i(f_{\xi_{km}})(\kappa)) = \tau_k.$$

Set

$$U_m = \{A \subseteq [\kappa]^m \mid A \in K, \langle \tau_1, ..., \tau_{n_m} \rangle \in i^*(A)\}.$$

Then U_m is a κ-complete ultrafilter in K.
Let $A \in U_m$. We claim that then

$$S = \{\nu \in Z_m \mid \langle f_{\xi_{1m}}(\nu), ..., f_{\xi_{n_m m}}(\nu) \rangle \notin A\} \in I_{normal}.$$

Suppose otherwise. Clearly,

$$\pi^{-1}``S \Vdash_{I^+} \forall k, 1 \leq k \leq n_m, \quad k(i(f_{\xi_{km}})(\kappa)) = \tau_k$$

and

$$\kappa \Vdash_{I^+} \langle \tau_1, ..., \tau_{n_m} \rangle \in k(i(A)).$$

Hence, using the elementarity of k, we obtain that

$$\pi^{-1}``S \Vdash_{I^+} \langle i(f_{\xi_{1m}})(\kappa), ..., i(f_{\xi_{n_m m}})(\kappa) \rangle \in i(A).$$

But, also

$$\pi^{-1}``S \Vdash_{I^+} \kappa \in i(S).$$

Which is impossible together.

Now, in K, there are $m < m' < \omega$ such that the following set

$$\{\langle \eta_1, ..., \eta_{n_{m'}} \rangle \in [\kappa]^{n_{m'}} \mid h_m(\eta_1, ..., \eta_{n_m}) \leq h_{m'}(\eta_1, ..., \eta_{n_{m'}})\}$$

is in the product of corresponding measures, i.e. in $U_{m'}$. This implies

$$\{\nu \mid g_m(\nu) \leq g_{m'}(\nu)\} \in G.$$

□

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