On decomposability spectra and an extender which overlaps a measure

Moti Gitik

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1 Introduction

C. Chang and J. Keisler introduced the following notion:

Definition 1.1 Let U be an ultrafilter over a set I and let λ be an infinite cardinal.
U is called $\lambda-$*decomposable* iff there is a partition of I into disjoint sets $\langle I_\alpha \mid \alpha < \lambda \rangle$, so that whenever $S \subseteq \lambda$ and $|S| < \lambda$, $\bigcup_{\alpha \in S} I_\alpha \notin U$.

This can be stated in terms of the Rudin-Keisler ordering (further R-K ordering):

Proposition 1.2 *An ultrafilter U over a set I is $\lambda-$decomposable iff it is above a uniform ultrafilter over λ in the Rudin-Keisler ordering.*

The following natural notion was introduced by P. Lipparini [8]:

Definition 1.3 The *decomposability spectrum* K_U, for an ultrafilter U, is the set of all infinite cardinals λ such that U is $\lambda-$decomposable.

Let us state some relevant results related to the decomposability spectrum.

Theorem 1.4 *(K. Kunen-K. Prikry [7]) Let U be an ultrafilter and let λ be an infinite cardinal.*

1. *If λ is regular and $\lambda^+ \in K_U$, then $\lambda \in K_U$, as well.*
2. *If $\kappa = \mathrm{cof}(\lambda) < \lambda$ and $\lambda^+ \in K_U$, then*
   (a) *$\kappa \in K_U$,*
   or

(b) a final segment of regular cardinals below λ is in K_U.

Theorem 1.5 *(P. Lipparini)*

Suppose that U is an ultrafilter and $\langle \lambda_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of elements of K_U. Then there is a cardinal $\delta \in K_U$ with $\bigcup_{\alpha<\eta} \lambda_\alpha \leq \delta \leq |\prod_{\alpha<\eta} \lambda_\alpha|$.

Theorem 1.6 *(P. Lipparini [8])*

Suppose that U is an ultrafilter, λ is a singular cardinal and $\mathfrak{a}$ is a set of elements of $K_U \cap Reg \cap \lambda$ unbounded in λ with $\min(\mathfrak{a}) > |\mathfrak{a}|$. Then there is a cardinal $\delta \in K_U$ with $\lambda \leq \delta \leq \max(\mathrm{pcf}(\mathfrak{a}))$.

It is easy to deduce the following:

Corollary 1.7 *Let U, λ and $\mathfrak{a}$ be as in Theorem 1.6. If there are* pcf *generators which are disjoint mod bounded subsets of λ (for example, if $|\mathrm{pcf}(\mathfrak{a})| = |\mathfrak{a}|$), then $\lambda \in K_U$ or $\mathrm{pcf}(\mathfrak{a}) \subseteq K_U$.*

Proof. Just apply Theorem 1.6 separately to every pcf −generator.
□

Corollary 1.8 *Let U, λ and $\mathfrak{a}$ be as in Theorem 1.6. If $\lambda^+ \in \mathrm{pcf}(\mathfrak{a})$, then $\lambda \in K_U$ or $\lambda^+ \in K_U$.*

Proof. Apply Theorem 1.6 to a generator for λ^+.
□

Corollary 1.9 *Let U and λ as in Theorem 1.6. If K_U contains a final segment of regular cardinals below λ, then $\lambda \in K_U$ or $\lambda^+ \in K_U$.*

Corollary 1.10 *Let U and λ as in Theorem 1.6. If $2^\lambda = \lambda^+$ or even $\mathrm{pp}(\lambda) = \lambda^+$, then $\lambda \in K_U$ or $\lambda^+ \in K_U$.*

G. Goldberg [5] studied a specially interesting case of $\sigma-$complete ultrafilters U. For such ultrafilters, cardinals of countable cofinality cannot be in the spectrum. This allowed Goldberg to sharpen previous results of Lipparini. Namely, he proved the following:

Theorem 1.11 *Let U is a $\sigma-$complete ultrafilter, λ is a singular cardinal of cofinality ω. Suppose that $K_U \cap \lambda$ is unbounded in λ. If $2^\lambda = \lambda^+$, then $\lambda^+ \in K_U$. Hence, by 1.4(2), a final segment of regular cardinals below λ is in K_U.*

Theorem 1.12 *Let U is a $\sigma-$complete ultrafilter, λ is a singular cardinal of cofinality ω. Suppose that $K_U \cap Reg \cap \lambda$ is unbounded in λ. If $pp(\lambda) = \lambda^+$, then $\lambda^+ \in K_U$. Hence, by 1.4(2), a final segment of regular cardinals below λ is in K_U.*

Theorem 1.13 *Assume SCH. Let U is a $\sigma-$complete ultrafilter, λ is an infinite cardinal. Suppose that $K_U \cap \lambda$ is unbounded in λ. Then a final segment of regular cardinals below λ is in K_U.*

Let $\eta > \omega$ be the degree of completeness of U. G. Goldberg [5] pointed out that then by J. Ketonen [6], some $\kappa, \eta \leq \kappa < \lambda$ is $(\eta, \lambda^+)-$strongly compact, i.e., there exists a fine $\eta-$complete ultrafilter over $\mathcal{P}_\kappa(\lambda^+)$.

Note that by R. Solovay [11] and J. Bagaria, M. Magidor [1] the Singular Cardinals Hypothesis holds between κ and λ^+.

Our aim here will be to analyze further possibilities for K_U, for a $\sigma-$complete ultrafilter U. In particular we will show that it is impossible to remove SCH assumptions from the Goldberg results above.

Our main result will be the following:

Theorem 1.14 *Assume GCH. Suppose that a cardinal κ carries an extender which overlaps a measurable cardinal λ. Then there is a cardinal preserving generic extension which satisfies the following:*

1. $\mathrm{cof}(\kappa) = \omega$.
2. *No new bounded subsets are added to κ.*
3. $2^\kappa = \lambda^+$.
4. *There is a uniform $\sigma-$complete ultrafilter U over λ such that $K_U = \{\lambda_n \mid n < \omega\} \cup \{\lambda\}$, for some increasing unbounded in κ sequence of regular cardinals $\langle \lambda_n \mid n < \omega \rangle$.*

2 Extender overlapping a measure

Assume GCH. Let E be a $(\kappa, \lambda^+)-$extender overlapping a measurable cardinal λ.

Fix a normal ultrafilter U_λ over λ which belongs to M_E.

Denote by $\mathcal{P}_E$ the extender based forcing with E and G its generic subset.

Let $\langle \lambda_n \mid n < \omega \rangle$ be the Prikry sequence corresponding to λ. Then each $\lambda_n, n < \omega$ is measurable in $V[G]$ and let U_{λ_n} be a normal ultrafilter over λ_n which corresponds to U_λ.

Work in $V[G]$.
Let us take there first the ultrapower M_{λ_0} of V by U_{λ_0}. Denote it by M_0. Note that all inaccessible cardinals $> \lambda_0$ of V, and in particular $\lambda_n, 0 < n < \omega, \kappa$ and λ do not move by $j_{U_{\lambda_0}}$. Also, $j_{U_{\lambda_0}}(U_{\lambda_n}), 0 < n < \omega, j_{U_{\lambda_0}}(U_\lambda)$ are generated by U_{λ_n}'S and U_λ, respectevely.

Continue further. Form M_1 by taking the ultrapower of M_0 with $j_{U_{\lambda_0}}(U_{\lambda_1})$, etc. We form M_n's $n < \omega$ this way. Let M_ω be the direct limit of M_n's and let $j_n : V \to M_n, j_{nm} : M_n \to M_\omega, j_\omega : V \to M_\omega, j_{n\omega} : M_n \to M_\omega$, for every $n \leq m < \omega$, be the corresponding elementary embeddings. Note that $\kappa, \kappa^+, \lambda$ do not move by them, and also, $j_\omega(U_\lambda)$ is generated by U_λ.
Finally apply $j_\omega(U_\lambda)$ to M_ω. Denote by M the resulting ultrapower and by j the corresponding elementary embedding from V to M.
Clearly, M is not closed even under $\omega-$sequences.

Our prime goal will be to construct G^* such that j extends to $j^* : V[G] \to M[G^*]$ and $M[G^*]$ is closed under λ_0-sequences.
Suppose for a while that we have such G^*.
Define an extension U^* of U as follows:

$$X \in U^* \text{ iff } \lambda \in j^*(X).$$

Clearly, U^* is a uniform ultrafilter over λ, since it extends U.

We have a natural elementary embedding

$$k : M_{U^*} \to M[G^*]$$

defined by setting $k([f]_{U^*}) = j^*(f)(\lambda)$. Clearly, $j^* = k \circ j_{U^*}$.

Lemma 2.1 *The only regular cardinals which are not continuity points of j^* are $\lambda_n, n < \omega$, and λ.*

Proof. $j^* \restriction On = j \restriction On$ and, by the definition of j, its only not continuity points are $\lambda_n, n < \omega$, and λ.
□

Lemma 2.2 *The set of regular cardinals which are not continuity points of j_{U^*} is a subset $\{\lambda_n \mid n < \omega\} \cup \{\lambda\}$.*

Proof. Let δ be a regular cardinal which is not a continuity points of j_{U^*}. Then $\cup(j_{U^*}{''}\delta) < j_{U^*}(\delta)$. Apply k. Then we will have $\cup(j^{*''}\delta) < j^*(\delta)$, by the elementarity. So, we are done.
□

The following is a well known consequence of an uniformity:

Lemma 2.3 *A regular cardinal δ is in K_{U^*} iff it is a non-continuity point of j_{U^*}.*

The next lemma follows from the previous lemma and the fact that U^* is a uniform ultrafilter over λ:

Lemma 2.4 $\lambda \in K_{U^*}$.

Lemma 2.5 *A final segment of λ_n's is in K_{U^*}.*

Proof. We have $j_{U^*} : V[G] \to M_{U^*}$. By elementarity, M_{U^*} is of the form $M'[G']$ such that $G' \in M_{U^*}$ is $M'-$generic for the forcing $j_{U^*}(\mathcal{P}_E)$ and k maps M' to M, $k(G') = G^*$.
G can be viewed as a set $\langle t_\alpha \mid \alpha \in \lambda^+ \setminus \kappa\rangle$ of Prikry sequences of its measures.
Note that $\langle t_\alpha \mid \alpha \in \lambda \setminus \kappa\rangle$ for a scale in $\prod_{n<\omega} t_\lambda(n)$ mod finite, and so t_λ is the exact upper bound of $\langle t_\alpha \mid \alpha \in \lambda \setminus \kappa\rangle$. Remember that $t_\lambda = \langle \lambda_n \mid n < \omega\rangle$.
Denote $j_{U^*}(\langle t_\alpha \mid \alpha \in \lambda^+ \setminus \kappa\rangle)$ by $\langle t'_\alpha \mid \alpha \in j_{U^*}(\lambda^+ \setminus \kappa)\rangle$.
Note that for every inaccessible (in V) $\alpha \in \lambda \setminus \kappa$, $t_\alpha(n)$ is an inaccessible cardinal $< \lambda_n$, for all but finitely many n's. So, for such α's, we have

- $j(\alpha) = \alpha$,
- $j(t_\alpha(n)) = t_\alpha(n)$, for all but finitely many $n < \omega$.

Hence, $j^*(t_\alpha) = t_\alpha$ mod finite, and so, $t'_\alpha = t_\alpha$ mod finite.
Now, inside M_{U^*}, we have that $t'\lambda$ is the exact upper bound of $\langle t'_\alpha \mid \alpha \in \lambda \setminus \kappa\rangle$. Also, $t_\lambda \in M_{U^*}$ due to the closure under $\omega-$sequences.
Hence, $t'_\lambda = t_\lambda$ mod finite.
Consider $j_{U^*}(t_\lambda) = t'_{j_{U^*}\lambda)}$. We have that $t'_{j_{U^*}\lambda)}$ is above t'_λ almost everywhere, since $j_{U^*}\lambda > \lambda$.
This is possible only if a final segment of λ_n's is moved by j_{U^*}.
Note that for every $n < \omega, \bigcup(j''\lambda_n) = \lambda_n$, and hence, using k, the same is true for j_{U^*}.
Combining together, it follows that a final segment of λ_n's are non continuity points of j_{U^*}, and hence, by Lemma 2.3, are in K_{U^*}.
□

Let us turn to the construction of G^*.

We will use the Merimovich Genericity Criterion for an extender based Prikry forcing [9]. In order to formulate it, let state two definitions from [9]:

Definition 2.6 Let $F : \lambda^+ \setminus \kappa \to {}^\omega\kappa$.
Let $N \prec H_\chi$, for χ large enough such that $|N| = \kappa, N \supseteq {}^{\kappa>}N, \mathcal{P}_E \in N$.

N is called $F-$*happy* iff there are $f : d \to {}^{\omega >}\kappa \in \mathcal{P}^*_E$ and an increasing sequence $\langle \tau_n \mid n < \omega \rangle$ such that

1. $d = N \cap \lambda^+ \setminus \kappa$,
2. f is $(N, \mathcal{P}^*_E)-$generic,
3. for every $E(d)-$tree T there is k such that for every n, $k \le n < \omega$, $\langle \tau_k, ..., \tau_n \rangle \in T$,
4. for every $\alpha \in d$, $F(\alpha) = \bigcup \{ f_{\langle \tau_0, ..., \tau_n \rangle}(\alpha) \mid n < \omega \}$.

Definition 2.7 Let $G \subseteq \mathcal{P}_E$. Define $F_G : \lambda^+ \setminus \kappa \to {}^{\omega}\kappa$ by setting for every $\alpha \in \lambda^+ \setminus \kappa$, $F_G(\alpha) = \bigcup \{ f^p(\alpha) \mid p \in G \}$.

Theorem 2.8 *(Merimovich Genericity Criterion) A subset G of $\mathcal{P}_E$ is $\mathcal{P}_E-$generic iff the set of F_G-happy models $N \prec H_\chi$ is unbounded.*

We would like to define G^* in $V[G]$ and then to apply the Merimovich Genericity Criterion in order to argue that G^* is $M-$generic for $j(\mathcal{P}_E)$.

The criterion has basically two parts: namely (2) of 2.6 connects with the Cohen forcing and (3) with the Prikry forcing.

Let us deal first with the Prikry part.
For every $\nu, \kappa \le \nu < \lambda^+$, denote, as before, by t_ν the Prikry sequence in G for the measure E_ν of E. Then $\langle \lambda_n \mid n < \omega \rangle = t_\lambda$.

First, the $j(\nu)-$Prikry sequence of G^*will be $j(t_\nu)$. There are many places which are not of the form $j(\nu)$, for some ν. Still Prikry sequence at them should be defined.

We use $j_{U_{\lambda_n}}$'s to stretch the original sequence $\langle t_\nu \mid \kappa \le \nu < \lambda^+ \rangle$ first and then U to stretch it further. The use of U creates gaps that we will need to fill.

Let $\gamma < j(\lambda^+)$. Define an $\omega-$sequence t'_γ. If γ has a pre-image, then we use the Prikry sequence of it to be t'_γ.

Suppose that this is not the case. Pick then a function $f_\gamma : \lambda \to \lambda^+$ which represents γ in the ultrapower of M_ω by U_λ.
f_γ is in M_ω, so there is $n < \omega$ and $f' \in M_n$ such that $f_\gamma = j_{n\omega}(f')(\lambda_0, ..., \lambda_n)$.
Suppose for simplicity that f' is just in V, the general case is similar.
Recall that E is $(\kappa, \lambda^+)-$extender and $2^\lambda = \lambda^+$. So, $f' \in M_E$.
Then there is a finite $a \subseteq \lambda^+$ and a function $g : [\kappa]^{|a|} \to V_\kappa$ such that $j_E(g)(a) = f'$.
Pick some $\eta \ge_E a$ and replace g by $g' : \kappa \to V_\kappa$, i.e., $j_E(g')(\eta) = f'$.
Now let us use the Prikry sequence t_η. Then $g'(t_\eta(n)) := f'_n : \lambda_n \to \lambda^+_n$, for almost every

$n < \omega$. Set $t'_\gamma(n) = j_{U_{\lambda_n}}(f'_n)(\lambda_n)$, for every $n < \omega$.
Note that such defined t'_γ depends on the choice of f_γ, etc. However, any other choice will define a sequence which is identical to t'_γ mod finite.
In particular, t'_λ and $\langle \lambda_n \mid n < \lambda \rangle$ agree on a final segment.

Let us argue now that such defined sequences t'_γ's are Prikry sequences for the corresponding measures of $j(E)$ over M.
So, let $B \subseteq \kappa, B \in M$ and $B \in j(E)_\gamma$, for some $\gamma \in j(\lambda^+) \setminus \kappa$. Note that the last ultrapower embedding by $j_\omega(U_\lambda)$ from M_ω to M does not move B. Then there are $n_0 < \omega$ and $A \in M_{n_0}$ such that $B = j_{n_0\omega}(A)$. Suppose for simplicity that A comes already from V.
Similar, with γ, we assume that $\gamma = j_{U_\lambda}(j_\omega(f))(\lambda)$, for some $f : \lambda \to \lambda^+$ in V.
Then, using $j_{U_\lambda} \circ j_\omega = j_\omega \circ j_{U_\lambda}$, we obtain that, in M_{U_λ}, $A \in (j_{U_\lambda}(E))_{\gamma'}$, where $\gamma' = j_{U_\lambda}(f)(\lambda)$.
Set, in V,

$$Z_A = \{\rho < \lambda \mid A \in E_{f(\rho)}\}.$$

Then $Z_A \in U_\lambda$.
Now, in M_E, we have

$$\rho \in Z_A \text{ iff } f(\rho) \in j_E(A).$$

Pick now $\eta < \lambda^+$ large enough such that $\lambda, U_\lambda, Z_A, f$ are in the range of k_η, where $k_\eta : M_{E\eta} \to M_E$ is defined by setting $k_\eta([h]_{E_\eta} = j_E(h)(\eta)$. Denote by $\lambda^*, U^*_\lambda, Z^*_A, f^*$ the pre-images under k_η of $\lambda, U_\lambda, Z_A, f$. Let $\nu \mapsto \lambda_\nu, \nu \mapsto U_\nu, \nu \mapsto Z_\nu, \nu \mapsto f_\nu$ be the functions which represent $\lambda^*, U^*_\lambda, Z^*_A, f^*$ in M_{E_η}.
Then, by elementarity of k_η, there is $C \in E_\eta$ such that for every $\nu \in C$ the following hold:

1. λ_ν is a measurable cardinal,
2. U_ν is a normal ultrafilter over λ_ν,
3. $Z_\nu \in U_{\lambda_\nu}$,
4. $f_\nu : \lambda_\nu \to \lambda^+_\nu$,
5. $\rho \in Z_\nu$ iff $f_\nu(\rho) \in A$.

Consider the sequence t_η. Starting with some $n^* < \omega$, all its members are in C.
Let $n, n^* \leq n < \omega$. Then $\lambda_{t_\eta(n)} = \lambda_n$. We have $Z_\nu \in U_{\lambda_\nu}$, hence $j_{U_{\lambda_n}}(f_{\lambda_n})(\lambda_n) \in j_{U_{\lambda_n}}(A)$.
Note that further ultrapowers with U_{λ_m}'s, $m > n$ will not effect this conclusion.
So,

$$\{j_{U_{\lambda_n}}(f_{\lambda_n})(\lambda_n) \mid n^* \leq n < \omega\} \subseteq j_\omega(A),$$

and we are done.

Let us deal now with Cohen subsets of κ^+.

Note that in $V[G]$ we have many Cohen generic subsets of κ^+. For example $\langle t_\gamma(0) \mid \gamma \in \lambda^+ \setminus \kappa \rangle$ produces a set of λ^+-many Cohen generic over V subsets of κ^+. Just organize them into blocks in V of size κ^+.

Now we can apply Theorem 3.6 of [4], with a measurables λ_0 and λ in order to get additional Cohen functions over $\mathrm{Ult}(M_0, U_\lambda)$. By Kunen-Paris, it easy to move them further to M taking ultrapowers by U_{λ_n}'s, $0 < n < \omega$.

Finally we correct the constructed above Prikry sequences using such Cohen's.

3 Prikry forcing case

Let deal first with the basic Prikry forcing.

Suppose that U is a normal ultrafilter over a measurable cardinal κ which is a limit of measurables.

For every $\nu < \kappa$ let ν^* denotes the least measurable above ν. Pick a normal ultrafilter $U(\nu^*)$ over ν^*.

Consider the ultrapower M_U. For every $\alpha \in [\kappa, j_U(\kappa))$ we consider a $\kappa-$complete ultrafilter $U_\alpha = \{X \subseteq \kappa \mid \alpha \in j_U(X)\}$. All of them are Rudin-Keisler equivalent to U. Namely, if $[f]_U = \alpha$, then f will be a witness for such equivalence between U_α and U.

A Prikry sequence for U will generate those for U_α's.

Let $\langle \kappa_n \mid n < \omega \rangle$ be a Prikry sequence for U.

Let $\lambda = \kappa^*$. Set $\lambda_n = \kappa_n^*$, for every $n < \omega$. Then $\langle \lambda_n \mid n < \omega \rangle$ be a Prikry sequence for U_λ.

Consider now in M_U the normal ultrafilter $U(\lambda)$ over λ. Take an ultrapower of M_U with $U(\lambda)$. Let $j : V \to M := {M_{U(\lambda)}}^{M_U}$.

For every $\alpha \in [\kappa, j(\kappa))$ we consider a $\kappa-$complete ultrafilter $W_\alpha = \{X \subseteq \kappa \mid \alpha \in j(X)\}$.

Note that inside $V[\langle \kappa_n \mid n < \omega \rangle]$ we will not have a Prikry sequence for W_λ.

However, let us define an iterated ultrapower of V inside $V[\langle \kappa_n \mid n < \omega \rangle]$. Set

$$M_0 = M_{U(\lambda_0)}, M_1 = {M_{U(\lambda_1)}}^{M_{U(\lambda_0)}}, ...$$

Finally, let M_ω be the direct limit of M_n's.

We argue that $\langle \lambda_n \mid n < \omega \rangle$ is a Prikry sequence for W_λ over M_ω. Deal with an equivalent ultrafilter $W_{\{\kappa,\lambda\}}$. Let $A \in W_{\{\kappa,\lambda\}} \cap M_\omega$. Then for some $n < \omega$, A has a preimage in M_n. Assume for simplicity that it has a preimage already in V. Denote it by B. Then, in V,

$$\{\nu < \kappa \mid \{\rho < \nu^* \mid (\nu, \rho) \in B\} \in U(\nu^*)\} \in U.$$

Denote the projection of B to its first coordinate intersected with the set above by B_0 and for every $\nu \in B_0$ let

$$B_{\nu 1} = \{\rho < \nu^* \mid (\nu, \rho) \in B\}.$$

Then, starting with some $n_0 < \omega$, all κ_n's are in B_0.
But then, for every $n \geq n_0$, $\lambda_n \in j_{U(\lambda_n)}(B_{\kappa_n 1})$.
Hence, for every $n \geq n_0$, $(\kappa_n, \lambda_n) \in j_\omega(B) = A$.

Note that $M[\langle \kappa_n \mid n < \omega \rangle, \langle \lambda_n \mid n < \omega \rangle]$ (or the same model $M[\langle \lambda_n \mid n < \omega \rangle]$) is closed under ω-sequences or even λ_0-sequences of its elements. It is not closed under λ_0^+-sequences, since $U(\lambda_0)$ is not inside.

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