An equivalence statement for not-too-wide scales

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Abstract

A question of S. Shelah [2] asks about existence of a certain not too wide scale. We show here that existence of a less wide scale is equivalent to a principle $(*)_\kappa$ of [1]. It follows that the answer to the Shelah question is affirmative unless there are rather large cardinals.

1 Introduction

Saharon Shelah in Chapter II of his book [2] proved the following basic fact (2.1, p.55):

Let κ be a singular cardinal of uncountable cofinality. If $\langle \kappa_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ is a strictly increasing continuous sequence with limit κ, then for a club $C \subseteq \kappa$, $(\prod_{i\in C} \kappa_i^+, <_{J_C^{bd}})$ has true cofinality κ^+. So, $\max(\mathrm{pcf}(\{\kappa_i^+ \mid i \in C\}) = \kappa^+$.

He asked the following related question (Question 2.1A, p.55, [2]):

Can we get $\langle f_\xi \mid \xi < \kappa^+\rangle$ witnessing the cofinality such that, for each $i \in C$, $|\{f_\xi \restriction i \mid \xi < \kappa^+\}| < \kappa_{i+1}$?

We consider here a strengthened version:

Can we get $\langle f_\xi \mid \xi < \kappa^+\rangle$ witnessing the cofinality such that, for each $i \in C$, $|\{f_\xi \restriction i \mid \xi < \kappa^+\}| \leq \kappa_i^+$?

Denote by $(**)_\kappa$ the statement:

There exists a scale $\langle f_\xi \mid \xi < \kappa^+\rangle$ witnessing the cofinality such that, for each $i \in C$, $|\{f_\xi \restriction i \mid \xi < \kappa^+\}| \leq \kappa_i^+$.

Let κ be a singular cardinal of uncountable cofinality. The following principle was considered in [1]:

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$(*)_\kappa$: *there exists a strictly increasing continuous sequence* $\langle \kappa_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ *with limit* κ *such that for every limit* $i < \mathrm{cof}(\kappa)$, $\max(\mathrm{pcf}(\{\kappa_j^+ \mid j < i\})) = \kappa_i^+$.

It was shown there that $(*)_\kappa$ implies that

either $\{\alpha < \kappa \mid \mathrm{pp}(\alpha) = \alpha^+\}$ *contains a club, or else* $\{\alpha < \kappa \mid \mathrm{pp}(\alpha) > \alpha^+\}$ *contains a club.*

In particular, if κ is a strong limit, then

either $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ *contains a club, or else* $\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$ *contains a club.*

Note that by Silver's theorem (and by Shelah's generalization of it to pp, see [2], Ch. II,2.4(1),p.59), if $2^\kappa > \kappa^+$ (or $\mathrm{pp}(\kappa) > \kappa^+$), then the second set $\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$ (or $\{\alpha < \kappa \mid \mathrm{pp}(\alpha) > \alpha^+\}$) contains a club.

Our purpose will be to show the following:

Theorem 1.1 $(*)_\kappa$ *and* $(**)_\kappa$ *are equivalent.*

Then, by [1], Theorem 1.3:

Corollary 1.2 $\neg(**)_\kappa$ *implies that* $M_n{}^\sharp(X)$ *exists, for every* $n < \omega$ *and all bounded* $X \subseteq \kappa$. [1].

So, the answer to the Shelah question is affirmative unless there are rather large cardinals in an inner model.

2 $(*)_\kappa \Rightarrow (**)_\kappa$

Assume first that $(*)_\kappa$ holds. Fix a strictly increasing continuous sequence $\langle \kappa_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ with limit κ such that for every limit $i < \mathrm{cof}(\kappa)$, $\max(\mathrm{pcf}(\{\kappa_j^+ \mid j < i\}) = \kappa_i^+$. Find a club $C \subseteq \kappa$ such that $(\prod_{i\in C} \kappa_i^+, <_{J^{bd}_C})$ has true cofinality κ^+. So, $\max(\mathrm{pcf}(\{\kappa_i^+ \mid i \in C\}) = \kappa^+$. Assume for simplicity that $\mathrm{cof}(\kappa) = \omega_1$.

For every limit point α of C (include $\alpha = \omega_1$ as well), we pick a scale $\vec{h}_\alpha = \langle h_{\alpha,\zeta} \mid \zeta < \kappa_\alpha^+\rangle$ in $\prod_{\beta\in C\cap\alpha} \kappa_\beta^+$ mod bounded subsets of α.

We will define, by induction on $\alpha \in C \cup \{\omega_1\}$, an increasing mod bounded sequence $\langle h'_{\alpha,\xi} \mid \xi < \kappa_\alpha^+\rangle$ in $\prod_{i\in C\cap\alpha} \kappa_i^+$ such that, for every $\xi < \kappa_\alpha^+$,

1. $h_{\alpha,\xi+1} \le h'_{\alpha,\xi+1}$, for every limit point α of C as well as ω_1,

[1] $M_n{}^\sharp(X)$ is the a sharp for $n-$many Woodin cardinals over X, see [3]

2. $h'_{\alpha,\xi} \restriction \beta$ is equal mod bounded to some $h'_{\beta,\zeta}, \zeta < \kappa_\beta^+$, for every point α of C as well as ω_1 and every $\beta \in C \cap \alpha$,

3. if $\mathrm{cof}(\xi) = \omega_1$ and there is a strictly increasing sequence $\langle h''_{\alpha,\rho_i} \mid i < \omega_1, \rho_i < \xi \rangle$ such that each h''_{α,ρ_i} is equal mod bounded to h'_{α,ρ_i}, then $h'_{\alpha,\xi}$ is equal mod bounded to the pointwise union of $\langle h''_{\alpha,\rho_i} \mid i < \omega_1, \rho_i < \xi \rangle$.

Suppose first that such $\langle h'_{\alpha,\xi} \mid \xi < \kappa_\alpha^+ \rangle$ was constructed. Then it is possible to derive the desired conclusion as follows.
Start with a technical definition.

Definition 2.1 Let us call $t \in \prod_{\beta \in C \cap \alpha} \kappa_\beta^+$ **close to** $\langle h'_{\beta,\xi} \mid \beta \leq \alpha, \xi < \kappa_\beta^+ \rangle$ iff for every $\beta \in C \cap \alpha + 1$, there is $\zeta_\beta < \kappa_\beta^+$ such that $t \restriction \beta = h'_{\beta,\zeta_\beta}$ (mod bounded).

Lemma 2.2 *For every $\alpha \in C \cup \{\omega_1\}$, the number of t's in $\prod_{\beta \in C \cap \alpha} \kappa_\beta^+$ which are close to $\langle h'_{\beta,\xi} \mid \beta \leq \alpha, \xi < \kappa_\beta^+ \rangle$ is at most κ_α^+.*

Proof. We prove the lemma by induction on α.

Case 1. *α is a successor point of C.*

It is obvious.

Case 2. *α is a limit point of C.*

Let $t \in \prod_{\beta \in C \cap \alpha} \kappa_\beta^+$ be close to $\langle h'_{\beta,\xi} \mid \beta \leq \alpha, \xi < \kappa_\beta^+ \rangle$.
There is $\xi_0 < \kappa_\alpha^+$ such that $h'_{\alpha,\xi_0}(\gamma) = t(\gamma)$, for all but boundedly many $\gamma \in C \cap \alpha$. Then there is $\beta_1 \in C \cap \alpha$ such that for every $\gamma \in C \cap (\beta_1, \alpha), h'_{\alpha,\xi_0}(\gamma) = t(\gamma)$.
Consider now $t \restriction \beta_1$. Then there is $\xi_1 < \kappa_{\beta_1}^+$ such that $h'_{\beta_1,\xi_1}(\gamma) = t(\gamma)$, for all but boundedly many $\gamma \in C \cap \beta_1$. Then there is $\beta_2 \in C \cap \beta_1$ such that for every $\gamma \in C \cap (\beta_2, \beta_1)$,
$h'_{\beta_1,\xi_1}(\gamma) = t(\gamma)$.
The process terminates after finitely many steps. Set $\beta_0 = \alpha$ and let $\langle \langle \beta_i, \xi_i \rangle \mid i < n \rangle$ be the pairs generated on such steps. We have

$$t = (\bigcup_{i<n-1} h_{\beta_i,\xi_i} \restriction (\beta_{i+1}, \beta_i)) \cup h_{\beta_{n-1},\xi_{n-1}}.$$

Finally, the number of possibilities for such finite sequences $\langle \langle \beta_i, \xi_i \rangle \mid i < n \rangle$ is at most κ_α^+. So, we are done.
□

Lemma 2.3 *For every $\alpha \in C \cup \{\omega_1\}$, for every $\beta \in C \cap \alpha$,*
$|\{h'_{\alpha,\xi} \restriction \beta \mid \xi < \kappa_\alpha^+\}| \leq \kappa_\alpha$.

Proof. Let $\xi < \kappa_\alpha^+$. Then the condition 2 above implies that $h'_{\alpha,\xi} \restriction \beta$ is close to $\langle h'_{\gamma,\zeta} \mid \gamma \leq \beta, \zeta < \kappa_\beta^+\rangle$. Now the result follows from the previous lemma.
□

Let us turn now to the construction of the sequence $\langle h'_{\alpha,\xi} \mid \xi < \kappa_\alpha^+\rangle$.
Proceed by induction on $\alpha \in C$ and ensure the desired properties by changing scales over α's.
It is easy for successor points α of C.

If α is the least limit point of C, then the size of an equivalence class is κ_α. In order to satisfy (3), we can take a scale having least upper bounds inside whenever they exist.

The same, using the induction, is true for a successor limit point of C.
Let us deal with α which is a limit of limit points of C. Assume the above for every $\beta < \alpha$.
Suppose that $\xi < \kappa_\alpha^+$ and $h'_{\alpha,\rho}$ is defined for every $\rho < \xi$.
Let us define $h'_{\alpha,\xi}$.

Assume first that either ξ is a successor ordinal or $\mathrm{cof}(\xi) \neq \omega_1$ or that $\mathrm{cof}(\xi) = \omega_1$ but the assumption of (3) fails.
Pick first some ξ_0 such that h_{α,ξ_0} is above each $h'_{\alpha,\rho}$, for every $\rho < \xi$ mod bounded. Then pick some $h'_{\alpha,\xi,0} \in \prod_{\beta\in C\cap\alpha} \kappa_\beta^+$ which is everywhere above h_{α,ξ_0} and $h_{\alpha,\xi}$.
Now, for every $\beta \in \alpha \cap C$, pick $\eta(\beta,0) < \kappa_\beta^+$, such that $h'_{\beta,\eta(\beta,0)}$ is above mod bounded $h'_{\alpha,\xi,0} \restriction \kappa_\beta$. Then we pick $h''_{\beta,\eta(\beta,0)}$ which is equal mod bounded to $h'_{\beta,\eta(\beta,0)}$ which is everywhere above $h'_{\alpha,\xi,0} \restriction \kappa_\beta$.
Define $f_{\alpha,\xi,0} \in \prod_{\beta\in C\cap\alpha} \kappa_\beta^+$ by setting

$$f_{\alpha,\xi,0}(\kappa_\gamma) = \sup_{\gamma\leq\beta<\alpha} h''_{\beta,\eta(\beta,0)}(\kappa_\gamma).$$

We define the next $f_{\alpha,\xi,1}$ in a similar fashion only replacing $h'_{\alpha,\xi,0}$ by $f_{\alpha,\xi,0}$.
Thus, for every $\beta \in \alpha \cap C$, pick $\eta(\beta,1), \eta(\beta,0) < \eta(\beta,1) < \kappa_\beta^+$, such that $h'_{\beta,\eta(\beta,1)}$ is above mod bounded $f_{\alpha,\xi,0} \restriction \kappa_\beta$. Then we pick $h''_{\beta,\eta(\beta,1)}$ which is equal mod bounded to $h'_{\beta,\eta(\beta,1)}$ which is everywhere above $f_{\alpha,\xi,1} \restriction \kappa_\beta$.
Define $f_{\alpha,\xi,1} \in \prod_{\beta\in C\cap\alpha} \kappa_\beta^+$ by setting

$$f_{\alpha,\xi,1}(\kappa_\gamma) = f_{\alpha,\xi,0}(\kappa_\gamma) + \sup_{\gamma\leq\beta<\alpha} h''_{\beta,\eta(\beta,1)}(\kappa_\gamma) + 1.$$

Continue this way and define $f_{\alpha,\xi,i}$, for every $i \leq \omega_1$ taking pointwise unions at limit stages. Finaly, set $h'_{\alpha,\xi} = f_{\alpha,\xi,\omega_1}$.

Note that for every $\beta \in C \cap \alpha$, the sequence $\langle \eta(\beta, i) \mid i < \omega_1 \rangle$ is a strictly increasing sequence of ordinals below κ_β^+. Also, the sequence of functions $\langle f_{\alpha,\xi,i} \restriction \kappa_\beta \mid i < \omega_1 \rangle$ is strictly increasing. So, the condition (3) applies. Hence, $h'_{\beta,\eta(\beta,\omega_1)}$ is equal mod bounded to $f_{\alpha,\xi,\omega_1} \restriction \kappa_\beta$, which is $h'_{\alpha,\xi} \restriction \kappa_\beta$, where $\eta(\beta, \omega_1) = \bigcup_{i<\omega_1} \eta(\beta, i)$. It follows that $h'_{\alpha,\xi}$ satisfies (2).

Assume now that $\mathrm{cof}(\xi) = \omega_1$ and the assumption of (3) holds.

Let $\langle h''_{\alpha,\rho_i} \mid i < \omega_1, \rho_i < \xi \rangle$ be a strictly increasing sequence such that each h''_{α,ρ_i} is equal mod bounded to h'_{α,ρ_i}. Then $h'_{\alpha,\xi}$ is defined to be the pointwise union of $\langle h''_{\alpha,\rho_i} \mid i < \omega_1, \rho_i < \xi \rangle$.

Let us argue that the condition (2) above is satisfied.

For every $i < \omega_1$, there is $\alpha(i) < \alpha$ such that

$$h''_{\alpha,\rho_i}(\kappa_\beta) = h'_{\alpha,\rho_i}(\kappa_\beta),$$

for every $\beta, \alpha(i) \leq \beta < \alpha$.

Find a stationary $S \subseteq \omega_1$ and $\alpha' < \alpha$ such that for every $i \in S$,

$$h''_{\alpha,\rho_i}(\kappa_\beta) = h'_{\alpha,\rho_i}(\kappa_\beta),$$

for every $\beta, \alpha' \leq \beta < \alpha$. Pick any $\gamma, \alpha' < \gamma < \alpha$. Then

$$\langle h''_{\alpha,\rho_i} \restriction (\alpha', \gamma) \mid i \in S \rangle$$

satisfies (3) for γ. This means that $h'_{\alpha,\xi} \restriction \kappa_\gamma$ is equal mod bounded to the least upper bound of $\langle h''_{\alpha,\rho_i} \restriction (\alpha', \gamma) \mid i \in S \rangle$ (it is an increasing everywhere sequence) and so $h'_{\alpha,\xi}$ satisfies (2) for κ_γ. In order to deal with γ's which are less or equal than α' we can initially define $h'_{\alpha,\xi} \restriction \kappa_{\alpha'}$ to be $h'_{\alpha',\zeta}$, for some $\zeta < \kappa_{\alpha'}^+$.

Finally, $\langle h'_{\omega_1,\xi} \mid \xi < \kappa^+ \rangle$ in $\prod_{i \in C} \kappa_i^+$ will be as desired, since for every $i \in C$,

$$|\{h'_{\omega_1,\xi} \restriction i \mid \xi < \kappa^+\}| \leq \kappa_i^+,$$

by the conditions (2) and Lemma 2.3 above.

3 $(**)_\kappa \Rightarrow (*)_\kappa$

Assume now that $(**)_\kappa$ holds. Let $\langle f_\xi \mid \xi < \kappa^+ \rangle$ witnessing the cofinality such that, for each $i \in C$, $|\{f_\xi \restriction i \mid \xi < \kappa^+\}| \leq \kappa_i^+$.

Suppose that $(*)_\kappa$ fails for the sequence $\langle \kappa_\alpha \mid \alpha \in C\rangle$. This means that there a stationary set $S \subseteq C$ such that for every $\alpha \in S$,

$$\max(\mathrm{pcf}(\{\kappa_\beta^+ \mid \beta \in C \cap \alpha\}) > \kappa_\alpha^+.$$

Take $\alpha \in S$. Consider

$$Z_\alpha = \{f_\xi \restriction \kappa_\alpha \mid \xi < \kappa^+\}.$$

Then $Z_\alpha \subseteq \prod_{\beta\in C\cap\alpha} \kappa_\beta^+$ and $|Z_\alpha| \leq \kappa_\alpha^+$. $\max(\mathrm{pcf}(\{\kappa_\beta^+ \mid \beta \in C \cap \alpha\}) > \kappa_\alpha^+$ implies that there is an unbounded in α set $A_\alpha \subseteq C \cap \alpha$ and a function $t_\alpha \in \prod_{\beta\in A_\alpha} \kappa_\beta^+$ such that $t_\alpha > f_\xi$ on an unbounded subset of A_α, for every $\xi < \kappa^+$.
Put all such t_α's together now. Namely, let $t \in \prod_{\gamma\in C} \kappa_\gamma^+$ be the pointwise sup over α of all of them. In particular, for every $\alpha \in S$, $t \geq t_\alpha$ on an unbounded subset of α.
Pick $\xi < \kappa^+$ such that $f_\xi > t$ mod bounded. But then, for every $\alpha \in S$ large enough, we will have $f_\xi \restriction A_\alpha < t_\alpha \leq t \restriction A_\alpha$ mod bounded, which is clearly impossible. Contradiction.
□

4 Open problems

In conclusion let us stated few related open questions.

Question 1. *Is the negative answer to the Shelah question consistent?*

A stronger version:

Question 2. *Is $\neg(*)_\kappa$ consistent, for singular cardinals κ of uncountable cofinality?*

The next question seems to be one of the basic questions on behavior of the power function behavior:

Question 3. *Let κ be a singular cardinal of uncountable cardinality. Is it possible that the set $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ is both stationary and co-stationary?*

Note that by the theorem of Silver, $2^\kappa = \kappa^+$.

In the next question pp replaces the power function.

Question 4. *Let κ be a singular cardinal of uncountable cardinality. Is it possible that the set $\{\alpha < \kappa \mid \mathrm{pp}(\alpha) = \alpha^+\}$ is stationary and co-stationary?*

By Shelah's analog of Silver's theorem for pp, $\mathrm{pp}(\kappa) = \kappa^+$.

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