Forcing with finite pistes-4 sizes.

Moti Gitik

December 30, 2019

Our purpose here is to present a special case of the forcing with pistes of [1] when pistes are finite and only three sizes of models ω, ω_1, ω_2 and ω_3 are allowed.

1 Wide pistes.

Assume GCH.
The basic idea behind the structures defined below (Definition 2.1), a finite structure with pistes over $\aleph_3$ is to stay as close as possible to an elementary chain of models. It cannot be literally a chain since models of different sizes are involved and models of bigger cardinality can come before ones of a smaller. The first part (Definition 1.1) describes this "linear" part of conditions in the main forcing. It is called *a wide piste* and incorporates together elementary chains of models of different cardinalities. The main forcing, defined in Section 2.1, will be based on such wide pistes and involves an additional natural but non-linear component called splitting or reflection.

Definition 1.1 A wide piste[1] is a set $\langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in \{\omega, \omega_1, \omega_2, \omega_3\}\rangle$ such that the following hold.

For every $\tau \in s = \{\omega, \omega_1, \omega_2, \omega_3\}$ and $A \in C^\tau$ the following holds:

1. $A \preccurlyeq \langle H(\omega_4), \in, \leq \rangle$, where $\leq$ is some fixed well ordering of $H(\omega_4)$,
2. $|A| = \tau$,
3. $A \supseteq \tau + 1$,
4. $A \cap \tau^+$ is an ordinal,

[1] It is $(\aleph_4, \omega, \omega)-$wide piste of [1] and here we deal with such pistes only.

5. elements of C^τ form a finite $\in -$chain,

6. if $X \in C^\tau$, then ${}^{\tau >}X \subseteq X$,

7. if $X, Y \in C^\tau$, then $X \in Y$ iff $X \subsetneq Y$.

8. (Potentially limit points)
   $C^{\tau lim} \subseteq C^\tau$.
   We refer to its elements as *potentially limit points*.
   The intuition behind this is that it will be possible to add new models unboundedly often below a potentially limit model in interesting cases, and this way it will be turned into a limit one.

   The next condition prevents unneeded appearances of small models between big ones.

9. If $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point and B_0 is its immediate predecessor, then there is no potentially limit point $A \in C^\tau$ with $\tau < \rho$ such that $B_0 \in A \in B_1$.

   The requirement that B_1 is not a potentially limit point is important here. Once dealing with potentially limit points, we would like to allow reflections which may add small intermediate models.
   However, small models which are non-potentially limit points are allowed.

10. Let $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point, B_0 is its immediate predecessor and $A \in C^\tau \cap B_1$, with $\tau < \rho$. If $\sup(A \cap \theta^+) > \sup(B_0 \cap \theta^+)$, then $B_0 \in A$.

    The next condition is of a similar flavor, but deals with smallest models.

11. If $B \in C^\rho$, for some $\rho \in s$, is not a potentially limit point and it is the least element of C^ρ, then there is no potentially limit point $A \in C^\tau$ with $\tau > \rho$ such that $A \in B$[2].

    Both conditions 9 and 11 are designed to allow one to add new models below potentially limit points, which will be essential for properness of the forcing.

    The purpose of the next four conditions is to allow to proceed down the pistes without interruptions at least before reaching a potentially limit point.

12. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^ρ. Then $B' \in A$.

[2] If we drop the requirement $\tau > \rho$, then it may be impossible further to add models of sizes $> \eta$ once a potentially limit point of size η is around.

13. Let $\tau, \rho, \rho^* \in s, \tau < \rho < \rho^*$, $A \in C^\tau, B \in C^{\rho^*}, D \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^{ρ^*}.
Then $B' \in D \in B$ implies $D \in A$.

14. (Linearity) If $\tau, \rho \in s, \tau \leq \rho$, $A \in C^\tau, B \in C^\rho$, then $\sup(A \cap \omega_4) < \sup(B \cap \omega_4)$ implies $A \in B$,

Next few conditions deal with what we call *covering properties*. They are needed in order to show that the forcing is $\omega-$proper.
Suppose that $M \in C^\xi$, for some $\xi \in s, \xi \neq \max(s)$, $D \in C^\rho$, for some $\rho \in s, \rho > \xi$. If $\sup(M \cap \omega_4) < \sup(D \cap \omega_4)$, by the linearity condition, $M \in D$. But suppose that $\sup(M \cap \omega_4) > \sup(D \cap \omega_4)$, i.e., a model of smaller size sits above a model of bigger size on the piste. The simplest situation then will be that just $D \in M$. However it is too much to require this, since as a result the properness will break down and cardinals will collapse, even in the two sizes situation. Weaker requirements should be made. The requirements will insure an existence of so called *covering model* $\tilde{D}$ of D for M. This model should have the following basic properties:

(א) $\tilde{D} \in M$,

(ב) $\tilde{D} \supseteq D$,

(ג) $|\tilde{D}| = |D|$,

(ד) $M \cap \tilde{D} = M \cap D$.

Note such $\tilde{D}$ is unique, if exists. Thus suppose that D', D'', $D' \neq D''$ are two covering models of D for M.
There is $x \in D' \setminus D''$ or $x \in D'' \setminus D'$. Suppose for example that there is $x \in D' \setminus D''$. By elementarity, then there is $x \in D' \setminus D''$ which belongs to M, but this is clearly impossible, since $D' \cap M = D \cap M = D'' \cap M$, by Item (ד) above.
Also note that $\tilde{D} = D$, if $D \in M$.

If $\langle \tilde{D}_i \mid i < \mathrm{cof}(\sup(\tilde{D} \cap \omega_4)) \rangle$ is an increasing continuous sequence of models of cardinality $\leq |\tilde{D}|$ with limit $\tilde{D}$, defined from $\tilde{D}$, then it follows that $D \supseteq \tilde{D}_{\sup(M \cap \mathrm{cof}(\sup(\tilde{D} \cap \omega_4)))}$ and $M \cap D = M \cap \tilde{D} = M \cap \tilde{D}_{\sup(M \cap \mathrm{cof}(\sup(\tilde{D} \cap \omega_4)))}$.

The set of covering models $Covmod(p)$ for $p = \langle \langle C^\tau, C^{\tau lim} \rangle \mid \tau \in \{\omega, \omega_1, \omega_2, \omega_3\} \rangle$ will be the union of sets $\langle Covmod(p)_k \mid k < n(p) \rangle$, $n(p) < \omega$, where

$Covmod(p)_0 = \bigcup_{\tau \in \{\omega_1, \omega_2, \omega_3\}} C^\tau$,

$Covmod(p)_{k+1}$ consists of elements of $Covmod(p)_k$ together with the following models:

(a) $B \cap E$, for some $B, E \in Covmod(p)_k$, $|B| < |E|$ and $\sup(B \cap \omega_4) > \sup(E \cap \omega_4)$,

(b) $B \cap E \cap F$, for some $B, E, F \in Covmod(p)_k$, $|B| < |E| < |F|$ and $\sup(B \cap \omega_4) > \sup(E \cap \omega_4) > F \cap \omega_4$.
Note that in this case $|B| = \aleph_1, |E| = \aleph_2, |F| = \aleph_3$.

(c) $cl(B \cup \aleph_i)$, where $cl(...)$ is the Skolem Hull, $|B| < \omega_i < \omega_4$ and $B \in Covmod(p)_k$ or B is the intersection of models, as in the two previous cases.

Let us state some basic facts that will simplify dealing with relevant models.

The following is a well known:

Fact 1 *Let* $N \prec \langle H(\omega_4), < \rangle$ *and* ρ *is an ordinal in* N.
Then $\sup(cl(N \cup \omega_2) \cap \rho) = \sup(N \cap \rho)$, *if* $\operatorname{cof}(\rho) \leq |N|$ *or* $\operatorname{cof}(\rho) > \aleph_2$
and $cl(N \cup \omega_3) \cap \omega_4 = \sup(N \cap \omega_4)$.
Proof. Let us deal with the first equality, the second is similar.

If $\operatorname{cof}(\rho) \leq |N|$, then $\rho \in N$, and so, by elementarity N will contain a cofinal in ρ sequence. Then $\sup(N \cap \rho) = \rho$, and we are done.
So assume that $\operatorname{cof}(\rho) > |N|$, and then $\operatorname{cof}(\rho) > \aleph_2$.

Let $\eta < \rho$ be in $cl(N \cup \omega_2)$. Then there is a Skolem term t, $a \in N$ and $\alpha < \omega_2$ such that $\eta = t(a, \alpha)$.
Consider $\gamma = \bigcup_{\beta < \omega_2} t(a, \beta)$. Then $\gamma \in N$, by elementarity, and, clearly, $\gamma < \rho$ and $\gamma \geq \eta$.
$\square$ of the fact.

Now, the following follows from the previous fact:

Fact 2 *Let* K_0, K_1 *be models of* q *of cardinality* $\aleph_1$, B_0, B_1 *be models of* q *of cardinality* $\aleph_2$ *and* F_0, F_1 *models of* q *of cardinality* $\aleph_3$. *Then either*

(a) $cl((K_0 \cap B_0 \cap F_0) \cup \aleph_3) = cl((K_1 \cap B_1 \cap F_1) \cup \aleph_3)$,
or

(b) $cl((K_0 \cap B_0 \cap F_0) \cup \aleph_3) \in cl((K_1 \cap B_1 \cap F_1) \cup \aleph_3)$,
or

(c) $cl((K_1 \cap B_1 \cap F_1) \cup \aleph_3) \in cl((K_0 \cap B_0 \cap F_0) \cup \aleph_3)$.

Proof. Just compare $\sup(K_0 \cap B_0 \cap F_0 \cap \omega_4)$ with $\sup(K_1 \cap B_1 \cap F_1 \cap \omega_4)$ and apply the fact above.
$\square$ of the fact.

Fact 3 *Suppose* X, Y *are models such that* $\sup(X \cap \aleph_i) = \sup(X \cap \aleph_i)$, *for every* $i \leq 4$. *Then* $X = Y$.
Proof. Just proceed by induction on $i \leq 4$ and show that $X \cap \aleph_i = Y \cap \aleph_i$.

$\square$ of the fact.

Fact 4 *Suppose* $X \in Y$ *are models,* $\mu' \leq \mu$ *are cardinals and* $\mu \in Y$, *then*

$$cl((Y \cap cl(X \cup \mu')) \cup \mu) = cl((Y \cap X) \cup \mu).$$

Proof. Clearly, $cl((Y \cap cl(X \cup \mu')) \cup \mu) \supseteq cl((Y \cap X) \cup \mu)$. Let us show the opposite direction.
So, let $z \in cl((Y \cap cl(X \cup \mu')) \cup \mu)$. Then there are $a \in Y \cap cl(X \cup \mu')$ and $\alpha < \mu$ such that $z = h(a, \alpha)$, for some Skolem function h. Using the assumption that $X, \mu \in Y$, we can find $b \in Y \cap X$ and $\beta \in Y \cap \mu$ such that $a = g(b, \beta)$, for some Skolem function g.
Then $z = h(g(b, \beta), \alpha)$. Using the closure of Skolem functions under the composition and the pairing function on ordinals below μ, we obtain $z = f(b, \gamma)$, for some $\gamma < \mu$ which codes (α, β) and a Skolem function f.
So, $z \in cl((Y \cap X) \cup \mu)$, and we are done.
$\square$ of the fact.

The next fact is a special case of the previous one.

Fact 5 *Suppose* X *is a model,* $\mu' \leq \mu$ *are cardinals, then*

$$cl((cl(X \cup \mu')) \cup \mu) = cl(X \cup \mu).$$

Proof. Just take Y to be big enough to include X and $\mu + 1$.

$\square$ of the fact.

Fact 6 *Suppose* $Y \in X$ *are models,* $\mu' \leq \mu$ *are cardinals and* $\mu' \geq |Y|$, *then*

$$cl((Y \cap cl(X \cup \mu')) \cup \mu) = cl(Y \cup \mu).$$

Proof. Just $Y \in X$ and $\mu' \geq |Y|$ imply that $cl(X \cup \mu') \supseteq Y$. Hence $Y \cap cl(X \cup \mu') = Y$.

$\square$ of the fact.

Note that models in $Covmod(p)$ are completely determined by supremums of their intersections with $\aleph_2, \aleph_3, \aleph_4$.
Let us argue that the process terminates after finitely many steps.

Claim 1 *Suppose that* $F \in Covmod(p), |F| = \aleph_3$ *and* $\mathrm{cof}(F \cap \omega_4) = \omega_3$.
Then $F \in C^{\omega_3}$.

Proof. We claim that F must be already in $Covmod(p)_0$. Suppose otherwise. Pick $k < \omega$ to be the least such that $F \in Covmod(p)_{k+1}$. Then, by Fact 1, F cannot be of the form $cl(B \cup \omega_3)$, for some $B \in Covmod(p)_k$ of cardinality $< \aleph_3$, since $\mathrm{cof}(F \cap \omega_4) = \omega_3$. If such B has cardinality $\aleph_3$, then $B \supseteq \omega_3$, and so, $B = cl(B \cup \omega_3) = F$. But then $F \in Covmod(p)_k$, which is impossible by the choice of k.
The cases (a), (b) of the definition of $Covmod(p)_{k+1}$ are impossible as well by Fact 1 and the assumption $\mathrm{cof}(F \cap \omega_4) = \omega_3$.
Contradiction.
$\square$ of the claim.

The next two claims is similar.

Claim 2 *Suppose that* $F \in Covmod(p), |F| = \aleph_2$ *and* $\mathrm{cof}(F \cap \omega_3) = \omega_2$.
Then there is $E \in C^{\omega_2}$ *such that* $F \cap \omega_3 = E \cap \omega_3$ *and* E *appears in the process of constructing* F.

Claim 3 *Suppose that* $F \in Covmod(p), |F| = \aleph_1$ *and* $\mathrm{cof}(F \cap \omega_3) = \omega_1$.
Then there is $B \in C^{\omega_1}$ *such that* $F \cap \omega_2 = B \cap \omega_2$ *and* B *appears in the process of constructing* F.

Further let us call such E and B *the leading models* of F.

Claim 4 *Suppose that* $F \in Covmod(p), |F| = \aleph_3$ *and* $\mathrm{cof}(F \cap \omega_4) = \omega_2$.
Then $F \in Covmod(p)_1$, *and then* $F = cl(E \cup \omega_3)$, *or* $F \in Covmod(p)_2$, *and then* $F = cl((E \cap G) \cup \omega_3)$, *for some* $E \in C^{\omega_2}, G \in C^{\omega_3}, \sup(E \cap \omega_4) > G \cap \omega_4$.

Let us deal for a while with intersection of models to $H(\omega_3)$. So, only sizes $\aleph_1, \aleph_2$ are relevant.

Consider $Covmod(p)_1$. Its elements of cardinality $\aleph_1$ are of one of the following forms:

- $B \in C^{\omega_1}$,
- $B \cap E$, $B \in C^{\omega_1}, E \in C^{\omega_2}, \sup(B \cap \omega_3) > E \cap \omega_3$.

The elements of cardinality $\aleph_2$ are of one of the following forms:

- $E \in C^{\omega_2}$,
- $cl(Z \cup \omega_2)$, for some $Z \in Covmod(p)_1$ of cardinality $\aleph_1$.

In the second case there is a leading model $B \in C^{\omega_1}$ such that $Z = B$ or $Z = B \cap E$.

Consider now the next stage $Covmod(p)_2$.
Let B' be a model of cardinality $\aleph_1$ and E' of cardinality $\aleph_2$, with $\sup(B' \cap \omega_3) > E' \cap \omega_3$, from the previous stage.
For example, $B' = B \cap E$ and $E' = cl(Z \cup \omega_2)$. Then

$$B' \cap E' = (B \cap E) \cap cl(Z \cup \omega_2) = B \cap (E \cap cl(Z \cup \omega_2)) = B \cap cl(Z \cup \omega_2),$$

since $E \cap \omega_3 \geq \sup(B \cap E \cap \omega_3) > E' \cap \omega_3 = cl(Z \cup \omega_2) \cap \omega_3$.

Also, note that $cl((B \cap cl(Z \cup \omega_2)) \cup \omega_2) = cl(Z \cup \omega_2)$, if the leading model B' of $cl(Z \cup \omega_2)$ is below B,
and $cl((B \cap cl(Z \cup \omega_2)) \cup \omega_2) = cl(B \cap E' \cup \omega_2)$, if the leading model B' of $cl(Z \cup \omega_2)$ is above B, where $Z = B' \cap E'$ or if $Z = B'$, then $E' = \omega_3$.
This means that no new models of cardinality $\aleph_2$ are produced at the stage 2.

Let $Z \in Covmod(p)$ has cardinality $\aleph_3$.
If $\mathrm{cof}(Z \cap \omega_4) = \omega_3$, then Z must be in C^{ω_3}.
If $\mathrm{cof}(Z \cap \omega_4) = \omega_2$, then
$Z = cl(E \cup \omega_3)$, for some $E \in C^{\omega_2}$, or
$Z = cl((E \cap F) \cup \omega_3)$, for some $E \in C^{\omega_2}$,$F \in C^{\omega_3}$ and E above it.
If $\mathrm{cof}(Z \cap \omega_4) = \omega_1$, then
$Z = cl((B \cup \omega_3)$, for some $B \in C^{\omega_1}$, or
$Z = cl(((B \cap E) \cup \omega_3)$, for some $B \in C^{\omega_1}$ and E below B of cardinality $\aleph_2$, or

$Z = cl((B \cap E \cap F) \cup \omega_3)$, for some $B \in C^{\omega_1}$, $F \in C^{\omega_3}$, E above it of cardinality $\aleph_2$ and B above E of cardinality $\aleph_1$.

If $Z \in Covmod(p)$ has cardinality $\aleph_2$ and $\mathrm{cof}(Z \cap \omega_3) = \omega_2$, then Z must be of one of the following forms:
E, for some $E \in C^{\omega_2}$,
or
$E \cap cl((E_1 \cap F_1) \cup \omega_3)$,
for some $E, E_1 \in C^{\omega_2}, F_1 \in C^{\omega_3}$, $E_1 \subseteq E$.

If $Z \in Covmod(p)$ has cardinality $\aleph_2$ and $\mathrm{cof}(Z \cap \omega_3) = \omega_1$, then Z must be of the following form:
$cl((B \cap E \cap cl((E_1 \cap F_1) \cup \omega_3)) \cup \omega_2)$,
for some $B \in C^{\omega_1}, E, E_1 \in C^{\omega_2}, F_1 \in C^{\omega_3}$, $E_1 \subseteq E$.

If $Z \in Covmod(p)$ has cardinality $\aleph_1$, then Z must be of one of the following forms:
B, for some $B \in C^{\omega_1}$,
or
$B \cap cl((B_1 \cap E_1 \cap F_1) \cup \omega_2) \cap cl((B_2 \cap E_2 \cap F_2) \cup \omega_3)$, for some $B, B_1, B_2 \in C^{\omega_1}$, $B_2 \subseteq B_1 \subseteq B$.

Note that if $\tilde{D} \in Covmod(p)$ and $\mathrm{cof}(\sup(\tilde{D} \cap \omega_4)) = \omega_3$, then $\tilde{D} \in C^{\omega_3}$, since intersections with models of smaller sizes reduces the cofinality of such sup.
Similar, if $\tilde{D} \in Covmod(p)$ and $\mathrm{cof}(\sup(\tilde{D} \cap \omega_4)) = \omega_2$, then only models from $C^{\omega_2} \cup C^{\omega_3}$ where involved in constructing $\tilde{D}$ by taking intersections and cl, as above.

Let us state now the requirements on covering models.

Start with the simplest one.

15. (Covering 1)
If $M \in C^{\omega_2}, D \in C^{\omega_3}$ and $\sup(M \cap \omega_4) > \sup(D \cap \omega_4)$, then there is a covering model $\tilde{D}$ of D for M inside C^{ω_3}.[3]

This the only requirement, if only two sizes of models are considered.
Already dealing with three sizes, an additional requirement is needed:

[3]Note that
(a) if $D \notin M$, then such $\tilde{D}$ must be a potentially limit point by Item 12 above. Thus, it cannot be a successor non-potentially limit point, by Item 12, since its immediate predecessor $\tilde{D}'$ will be in M, and then, $\sup(\tilde{D}' \cap \omega_4) < \sup(D \cap \omega_4)$, and so $D \supseteq \tilde{D}'$.
(b) such $\tilde{D}$ is the least model $D' \in M \cap C^{\omega_3}$ such that $D' \supseteq D$.

16. (Covering 2) If $M \in C^\tau, D \in C^\rho, \tau < \rho$, $\sup(M \cap \omega_4) > \sup(D \cap \omega_4)$, then there is $\tilde{D} \in Covmod(p)$ which is a covering model of D for M.

17. (Strong Covering 1) If $M \in C^{\omega_1}, D \in C^{\omega_3}$, $\sup(M \cap \omega_4) > \sup(D \cap \omega_4)$. Let $\tilde{D} \in M$ be a covering model of D for M. Then either

    (a) $\tilde{D} \in C^{\omega_3}$;

    or

    (b) $\mathrm{cof}(\tilde{D} \cap \omega_4) = \omega_2$ [4] .

    Let then $S \in C^{\omega_2} \cap M$ be its leading model.

    Then either

    i. $D \in S$,[5]

    or

    ii. there is a covering model $D' \in S \cap C^{\omega_3}$ of D for S such that $D \supseteq {D'}_{\sup(M \cap \omega_3)}$, where $\langle {D'}_i \mid i < \omega_3 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_2$ with limit D', defined from D'.

18. (Strong Covering 2) If $M \in C^{\omega}, D \in C^{\omega_2}$, $\sup(M \cap \omega_4) > \sup(D \cap \omega_4)$. Let $\tilde{D} \in M$ be a covering model of D for M. Then either

    (a) $\tilde{D} \in C^{\omega_2}$;

    or

    (b) $\mathrm{cof}(\sup(\tilde{D} \cap \omega_4)) = \omega_1$.

    Let then $S \in C^{\omega_1} \cap M$ be its leading model.

    Then either

    i. $D \in S$,[6]

    or

[4]Note that $D \supseteq \tilde{D}_{M \cap \omega_2}$, where $\langle \tilde{D}_\nu \mid \nu < \omega_2 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_2$ with limit $\tilde{D}$, defined from $\tilde{D}$. Just otherwise there will be $\nu < M \cap \omega_2$ such that $\tilde{D}_\nu \notin D$, but such $\tilde{D}_\nu \in M \cap \tilde{D}$. Contradiction to covering.

[5]Note that this implies that $D \supseteq cl((S_{M \cap \omega_2} \cap \tilde{D}) \cup \omega_3)$, where $\langle S_\nu \mid \nu < \omega_2 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_1$ with limit S, defined from S. Otherwise, there is $x \in cl((S_{M \cap \omega_2} \cap \tilde{D}) \cup \omega_3) \setminus D$. Then $x = h(a, \alpha)$, for some $a \in S_{M \cap \omega_2} \cap \tilde{D}, \alpha < \omega_3$ and a Skolem term h. Such $a \notin D$, since $D \supseteq \omega_3$. Pick $\gamma \in M \cap \omega_2$ such that $a \in S_\gamma$. Then $S_\gamma \cap \tilde{D} \notin D$, since otherwise, we will have $S_\gamma \cap \tilde{D} \subseteq D$, and so $a \in D$. But now, $S_\gamma \cap \tilde{D} \in M \cap \tilde{D} = M \cap D$, which is impossible. Contradiction.

[6]As in the previous condition, then $D \supseteq S_{M \cap \omega_1} \cap \tilde{D}$.

ii. there is a covering model $\tilde{\tilde{D}} \in S \cap C^{\omega_2}$ of D for S such that $D \supseteq \tilde{\tilde{D}}_{\sup(M \cap \omega_2)}$, where $\langle \tilde{\tilde{D}}_i \mid i < \omega_2 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_1$ with limit $\tilde{\tilde{D}}$, defined from $\tilde{\tilde{D}}$.

Let us deal with covering properties when the gap between cardinalities of models involved is at least two. In the present situation - four sizes, it is only ω and ω_3.

19. (Strong Covering 3) If $M \in C^{\omega}, D \in C^{\omega_3}$, $\sup(M \cap \omega_4) > D \cap \omega_4$. Let $\tilde{D} \in M$ be a covering model of D for M. Then either

   (a) $\tilde{D} \in C^{\omega_3}$;
   or

   (b) $\mathrm{cof}(\tilde{D} \cap \omega_4) = \omega_2$.
   Let then $E \in C^{\omega_2} \cap M$ be its leading model.
   Then either

      i. $D \in E$,
      or

      ii. there is a covering model $\tilde{\tilde{D}} \in C^{\omega_3}$ of D for E such that $D \supseteq \tilde{\tilde{D}}_{\sup(M \cap \omega_2)}$, where $\langle \tilde{\tilde{D}}_i \mid i < \omega_2 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_2$ with limit $\tilde{\tilde{D}}$, defined from $\tilde{\tilde{D}}$.

   Or

   (c) $\mathrm{cof}(\tilde{D} \cap \omega_4) = \omega_1$.
   Let then $S \in C^{\omega_1} \cap M$ be its leading model.
   Then either

      i. $D \in S$,
      or

      ii. $D \notin S$ and let then $\tilde{\tilde{D}}$ be a covering model of D for S.
      If $\mathrm{cof}(\tilde{\tilde{D}} \cap \omega_4) = \omega_3$, then $D \supseteq \tilde{\tilde{D}}_{\sup(M \cap \omega_3)}$, where $\langle \tilde{\tilde{D}}_i \mid i < \omega_3 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_2$ with limit $\tilde{D}$, defined from $\tilde{D}$.
      If $\mathrm{cof}(\tilde{\tilde{D}} \cap \omega_4) = \omega_2$, then let $T \in S \cap C^{\omega_2}$ be its leading model.
      Reqire that $D \supseteq T_{\sup(M \cap \omega_2)}$, where $\langle T_i \mid i < \omega_2 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_1$ with limit T, defined from T, and either

A. $D \in T$,

or

B. there is a cover $\tilde{T} \in T \cap C^{\omega_3}$ of D such that $D \supseteq \tilde{T}_{\sup(M \cap \omega_2)}$, where $\langle \tilde{T}_i \mid i < \omega_2 \rangle$ is an increasing continuous sequence of models of cardinality $\aleph_2$ with limit $\tilde{T}$, defined from $\tilde{T}$.

We will state further in the definition of structures with pistes when such non-trivial possibilities of coverings may occur.

20. Let $\tau, \rho, \xi \in s, \tau < \rho < \xi$, $A \in C^\tau, M \in C^\rho, D \in C^\xi$, $M, D \in A$ and $\sup(M \cap \omega_4) > \sup(D \cap \omega_4)$. Then the covering model $\tilde{D}$ of D for M belongs to A.

2 Structures with pistes - definitions.

Now we are ready to give the main definition.

Definition 2.1 *A structure with pistes*[7] is a set
$p = \langle \langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau \rangle \mid \tau \in s = \{\omega, \omega_1, \omega_2, \omega_3\} \rangle$ such that the following hold:

1. for every $\tau \in s$,
   (a) $A^{0\tau} \preccurlyeq \langle H(\omega_4), \in, \leq \rangle$,
   (b) $|A^{0\tau}| = \tau$,
   (c) $A^{0\tau} \in A^{1\tau}$,
   (d) $A^{1\tau}$ is a finite set of elementary submodels of $A^{0\tau}$,
   (e) each element A of $A^{1\tau}$ has cardinality τ, $A \supseteq \tau + 1$ and $A \cap \tau^+$ is an ordinal.

2. (Potentially limit points) Let $\tau \in s$.
   $A^{1\tau lim} \subseteq A^{1\tau}$. We refer to its elements as *potentially limit points.*
   The intuition behind this is that once extending it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.

[7]It is $(\aleph_3, \omega, \omega)-$structure with pistes of [1] and here we deal with such structures only.

3. (Piste function) The idea behind this is to provide a canonical way to move from a model in the structure to one below.
   Let $\tau \in s$.
   Then, $\mathrm{dom}(C^\tau) = A^{1\tau}$ and
   for every $B \in \mathrm{dom}(C^\tau)$, $C^\tau(B)$ is a finite chain of models in $A^{1\tau} \cap (B \cup \{B\})$ such that the following holds:

   (a) $B \in C^\tau(B)$,

   (b) if $X \in C^\tau(B)$, then $C^\tau(X) = \{Y \in C^\tau(B) \mid Y \in X \cup \{X\}\}$,

   (c) if B has immediate predecessors in $A^{1\tau}$, then one (and only one) of them is in $C^\tau(B)$,

4. (Wide piste) The set

$$\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s\rangle$$

   is a wide piste.

   The next two condition describe the ways of splittings from wide pistes. This describes the structure of $A^{1\tau}$ and the way pistes allow one to move from one of its models to an other.

5. (Splitting points) Let $\tau \in s$. Let $X \in A^{1\tau}$. Then either

   (a) X is minimal under $\in$ or equivalently under $\subsetneq$,
   or

   (b) X has a unique immediate predecessor in $A^{1\tau}$,
   or

   (c) $\tau < \omega_3$, X has exactly two immediate predecessors X_0, X_1 in $A^{1\tau}$, and then the following hold:

      i. (Splitting points of type 1) None of X, X_0, X_1 is a potentially limit point and X, X_0, X_1 form a Δ–*system triple* relative to some $F_0, F_1 \in A^{1\tau^+ lim}$, which means the following:

         A. $F_0 \subsetneq F_1$ and then $F_0 \in C^{\tau^+}(F_1)$, or $F_1 \subsetneq F_0$ and then $F_1 \in C^{\tau^+}(F_0)$,

         B. $X_0 \in F_1$, if $F_0 \subsetneq F_1$ and $X_1 \in F_0$, if $F_1 \subsetneq F_0$,

         C. $F_0 \in X_0$ and $F_1 \in X_1$,

         D. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

E. the structures

$$\langle X_0, \in, \langle X_0 \cap A^{1\rho}, X_0 \cap A^{1\rho lim}, (C^\rho \restriction X_0 \cap A^{1\rho}) \cap X_0 \mid \rho \in s\rangle\rangle$$

and

$$\langle X_1, \in, \langle X_1 \cap A^{1\rho}, X_1 \cap A^{1\rho lim}, (C^\rho \restriction X_1 \cap A^{1\rho}) \cap X_1 \mid \rho \in s\rangle\rangle$$

are isomorphic over $X_0 \cap X_1$. Denote by π_{X_0,X_1} the corresponding isomorphism.

F. $X \in A^{0\tau^+}$.

Or

ii. (Splitting points of type 2) $\tau \in s \cap \omega_2$ and there are $G, G_0, G_1 \in X \cap A^{1\mu}$, $\mu \in s \setminus \tau + 1$, G is a splitting point of types 1 and G_0, G_1 are its immediate predecessors, with witnessing models in X, such that

A. $X_0 \in G_0$,

B. $X_1 \in G_1$,

C. $X_1 = \pi_{G_0 G_1}[X_0]$.

D. X is not a limit or potentially limit point,

E. $X \in A^{0\mu}$,

F. (Pistes go in the same direction) $G_i \in C^\mu(G) \Leftrightarrow X_i \in C^\tau(X), i < 2$.

Further we will refer to such X, i.e. of types 1 or 2, as *splitting points.*

6. Let $\tau, \rho \in s$, $X \in A^{1\tau}, Y \in A^{1\rho}$. Suppose that X is a successor point, but not potentially limit point and $X \in Y$. Then all immediate predecessors of X are in Y, as well as the witnesses, i.e. F_0, F_1 if (5(c)i) holds and G_0, G_1, G if (5(c)ii) holds.

7. Let $\tau \in s$. If $X \in A^{1\tau}$, $Y \in \bigcup_{\rho \in s} A^{1\rho}$ and $Y \in X$, then Y is *a piste-reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n\rangle$ of elements of $A^{1\tau}$ which we call *the piste leading to* Y *from* X such that

   (a) $X = X(0)$,

   (b) for every $i, 0 < i < n$, either

      i. $X(i-1)$ has two immediate predecessors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^\tau(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$,
      or

ii. $X(i) \in C^\tau(X(i-1))$, and then either $i = n$ or $i < n$, $X(i)$ has two immediate predecessors $X(i)_0, X(i)_1$ with $X(i)_0 \in C^\tau(X(i))$, $X(i+1) = X(i)_1$ and $Y \in X(i)_1 \setminus X(i)_0$

(c) $Y = X(n)$, if $Y \in A^{1\tau}$ and if $Y \in A^{1\rho}$, for some $\rho \neq \tau$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\tau}$.

Let us give two examples.

Example 1. *Suppose that $A^{1\tau}$ consists of three models, $Y \in Z \in X$.*
Then the piste from X to Y will be $\langle X, Y\rangle$.

Example 2. *Suppose that $A^{1\tau}$ consists of models $X, Z, T, T_0, T_1, Y_0, Y_1$ such that $Y_0 \in T_0 \in T \in Z \in X$ is $C^\tau(X)$, T is a splitting point with T_0, T_1 its immediate predecessors, $Y_0 \in T_0, Y_1 \in T_1$.*
Then the piste from X to Y_1 goes like this: From X we go down to T, then at T we turn to T_1 and from T_1 we continue to the final destination Y_1.
So the piste from X to Y_1 is $\langle X, T, T_1, Y_1\rangle$.

The sequence $\langle X(i) \mid i \leq n\rangle$ is defined uniquely from X and Y.
In particular, every $Y \in A^{1\tau}$ is piste reachable from $A^{0\tau}$.
In order to formulate further requirements, we will need to describe a simple process of changing the wide pistes. This leads to equivalent forcing conditions once the order will be defined.

Let $X \in A^{1\tau}$. We will define the $X-$wide piste. The definition will be by induction on number of turns (splits) needed in order to reach X by the piste from $A^{0\tau}$.
First, if $X \in C^\tau(A^{0\tau})$, then the $X-$wide piste is just $\langle C^\xi(A^{0\xi}), C^\xi(A^{0\xi}) \cap A^{1\xi lim} \mid \xi \in s\rangle$, i.e. the wide piste of the structure.
Second, if $X \notin C^\tau(A^{0\tau})$, but it is not an immediate predecessor of a splitting point, then pick the least splitting point Y above X. Let Y_0, Y_1 be its immediate predecessors with $Y_0 \in C^\tau(Y)$. Then $X \in Y_i$ for some $i < 2$. Set the $X-$wide piste to be the Y_i-wide piste.
So, in order to complete the definition, it remain to deal with the following principal case:
$X \in A^{1\tau}$ a splitting point of one of the types 1 or 2.

Let X_0, X_1 be its immediate predecessors with $X_0 \in C^\tau(X)$. Assume that the $X-$wide piste $\langle C^\xi_X, C^{\xi lim}_X \mid \xi \in s\rangle$ is defined and assume that $C^\tau(X)$ is an initial segment of C^τ_X. Let the X_0-wide piste be $\langle C^\xi_X, C^{\xi lim}_X \mid \xi \in s\rangle$.

Let us deal with type of splitting separately.

Case 1. X is a splitting point of type 1.

Define the X_1-wide piste $\langle C^{\xi}_{X_1}, C^{\xi lim}_{X_1} \mid \xi \in s\rangle$ as follows:

- $C^{\xi}_{X_1} = C^{\xi}_{X}$, for every $\xi > \tau$.
  I.e. no changes for models of cardinality $> \tau$.
- $C^{\xi lim}_{X_1} = C^{\xi}_{X_1} \cap A^{1\xi lim}$, for every $\xi \in s$.
  Models that were potentially limit remain such and no new are added.
- $C^{\tau}_{X_1} = (C^{\tau}_{X} \setminus X) \cup C^{\tau}(X_1)$.
  Here we switched the piste from X_0 to X_1.
- $C^{\xi}_{X_1} = \{Z \in C^{\xi}_{X} \mid \sup(Z \cap \omega_4) > \max(\sup(X_0 \cap \omega_4), \sup(X_1 \cap \omega_4))\} \cup \{\pi_{X_0,X_1}(Z) \mid Z \in C^{\xi}_{X} \cap X_0\}$, for every $\xi \in s \cap \tau$.[8]

Note that such defined switch from X_0 to X_1 does not affect at all models of sizes above τ. Models of sizes $\leq \tau$ are effected only if they are contained in X_0 or in X_1.

If X is a splitting point of type 2, then we may need to turn some piste for models of cardinalities $> \tau$ into other directions, in order to satisfy the item 5(c)iiF above.
Proceed as follows.

Case 3. X is a splitting point of type 2.

Let $G, G_0, G_1 \in X \cap A^{1\mu}$ be models which witness that X is a splitting point of type 2 and X_0, X_1 are its immediate predecessors. Now using the induction[9] we can assume that the G_1-wide piste is already defined.
Define the X_1-wide piste to be the G_1-wide piste.

Now we require the following:

8. Let $\tau \in s$ and $X \in A^{1\tau}$. Then the $X-$wide piste is a wide piste, i.e., it satisfies Definition 1.1.

9. (Maximal models are above all the rest) For every $\tau \in s$ and $Z \in \bigcup_{\rho \in s} A^{1\rho}$, if $Z \notin A^{0\tau}$, then there is $\mu \in s$ such that $Z = A^{0\mu}$.

Let us conclude with requirements related to non-trivial cases of covering.

---

[8] In particular, due to this, the next condition implies that for $\xi \in s \cap \tau$, if $Z \in C^{\xi}_{X}, \sup(Z \cap \omega_4) > \max(\sup(X_0 \cap \omega_4), \sup(X_1 \cap \omega_4))$, then $\{\pi_{X_0 X_1}(Z') \mid Z' \in C^{\xi}_{X} \cap X_0\} \subseteq Z$.

[9] The induction is on pairs (n, ζ) ordered lexicographically, where n is the number of turns from the wide piste and ζ is the rank (the usual one as sets) of the model.
We have $G, G_0, G_1 \in X$, so the rank of G, G_0, G_1 is smaller than the rank of X. The number of turns needed to get to G and to X from the top is the same.

10. (Covering possibilities) Suppose that X, Y are on the wide piste, $|X| < |Y|$, X is above Y and $Y \notin X$. Let $\tilde{Y}$ be a covering model of Y for X. Then the following hold:

   (a) if $\tilde{Y} \in Covmod(p)_1$ and $\tilde{Y}$ is not on the wide piste, then there is a splitting point Z on the wide piste between X and Y of cardinality $|X| < |Z| < |Y|$.
   So, $\tilde{Y}$ is of the form $cl(Z \cup |Y|)$ or $cl((Z \cap E) \cup |Y|)$ or $cl((Z \cap E \cap F) \cup |Y|)$, for some E, F on the wide piste, with $|Z| < |E| < |F|$, Z above E and E above F.
   Note that in two last cases, models $R \in X \cap Z$, $|R| = |Z|$ which are above E or below E, but above F, in the last case, will either contain models which are on the wide piste inside $Z \cap E$ ($Z \cap E \cap F$) of cardinality $|Z|$ or will not be on the wide piste.

   (b) If $\tilde{Y} \in Covmod(p)_{n+1} \setminus Covmod(p)_n$, for some $n, 1 \leq n < \omega$, then there is a splitting point Z on the wide piste between X and Y
   of cardinality $|X| < |Z| < |Y|$. The following hold:

      i. $\tilde{Y} = cl(\tilde{Z} \cup |Y|)$, where $\tilde{Z}$ is a covering model of Z for X and it is in $Covmod(p)_n \setminus Covmod(p)_{n-1}$.
      or

      ii. there is E on the wide piste below Z, $|Z| < |E|$ such that $Y \in Z \cap E$, and then, $\tilde{Y} = cl((\tilde{Z} \cap \tilde{E}) \cup |Y|)$, where $\tilde{Z}$ is a covering model of Z for X, $\tilde{E}$ is a covering model of E for X, both of them are in $Covmod(p)_n$ and at least one of them is in $Covmod(p)_n \setminus Covmod(p)_{n-1}$.
      Or

      iii. there are E, F on the wide piste below Z, $|Z| < |E| < |F|$, F below E such that $Y \in Z \cap E \cap F$, and then, $\tilde{Y} = cl((\tilde{Z} \cap \tilde{E} \cap \tilde{F}) \cup |Y|)$, where $\tilde{Z}$ is a covering model of Z for X, $\tilde{E}$ is a covering model of E for X, $\tilde{F}$ is a covering model of F for X, all of them are in $Covmod(p)_n$ and at least one of them is in $Covmod(p)_n \setminus Covmod(p)_{n-1}$.
      We require that in both of the last cases, if $R \in X \cap \tilde{Z}$, $|R| = |Z|$ which are above $\tilde{E}$ or below $\tilde{E}$, but above $\tilde{F}$, in the last case, then R either contain models which are on the wide piste inside $Z \cap E$ ($Z \cap E \cap F$) of cardinality $|Z|$ or R is not on the wide piste.

This completes the definition of a finite structure with pistes.

Denote the set of such defined structures by $\mathcal{P}$ (which corresponds to $\mathcal{P}_{\omega_3 \omega \omega}$ of [1]).

Define an order on $\mathcal{P}$.

Definition 2.2 Let
$p_0 = \langle\langle A_0^{0\tau}, A_0^{1\tau}, A_0^{1\tau lim}, C_0^\tau\rangle \mid \tau \in s\rangle$, $p_1 = \langle\langle A_1^{0\tau}, A_1^{1\tau}, A_1^{1\tau lim}, C_1^\tau\rangle \mid \tau \in s\rangle$ be two elements of $\mathcal{P}$.
Set $p_0 \leq p_1$ (p_1 extends p_0) iff

1. $A_0^{1\tau} \subseteq A_1^{1\tau}$, for every $\tau \in s$,

2. let $A \in A_0^{1\tau}$, for some $\tau \in s$, then $A \in A_0^{1\tau lim}$ iff $A \in A_1^{1\tau lim}$.
   The next item deals with a switching described in Definition 2.1 . It allows to change piste directions.

3. Let $\tau \in s$.
   For every $A \in A_0^{1\tau}$, $C_0^\tau(A) \subseteq C_1^\tau(A)$,
   or
   there are finitely many places below A where pistes change their directions, i.e. there are splitting points $B(0), ..., B(k) \in A_0^{1\tau} \cap (A \cup \{A\})$ with $B(j)', B(j)''$ the immediate predecessors of $B(j)$ $(j \leq k)$ such that

   (a) $B(j)' \in C_0^\tau(B(j))$,

   (b) $B(j)'' \in C_1^\tau(B(j))$.

   If $B \in A_0^{1\tau} \cap (A \cup \{A\})$ is a splitting point different from $B(0), ..., B(k)$ and B', B'' are its immediate predecessors, then
   $B' \in C_0^\tau(B)$ iff $B' \in C_1^\tau(B)$.

4. Let $\tau \in s$.
   If $A \in A_0^{1\tau}$ is a splitting point in p_0, then it remains such in p_1 with the same immediate predecessors.

5. Let $\tau \in s$.
   Let $B \in A_0^{1\tau}$ B not in $A_0^{1\tau\,\mathrm{lim}}$, i.e., it is not a potentially limit, and B a unique immediate predecessor in p_0. Then, in p_1, B has the same unique immediate predecessor.

   This requirement guarantees intervals without models, even after extending a condition.

By 2.2(5), potentially limit points are the only places where non-end-extensions can be made.

3 Properness.

We would like to show that for every $\tau \in s = \{\omega, \omega_1, \omega_2, \omega_3\}$ the forcing $\mathcal{P}$ is $\tau-$proper.

Let us start with ω_3-properness.

Lemma 3.1 *The forcing $\mathcal{P}$ is ω_3-proper.*

Proof.

Let $p \in \mathcal{P}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ large enough such that

1. $|\mathfrak{M}| = \aleph_3$,
2. $\mathfrak{M} \supseteq \aleph_3$,
3. $\mathcal{P}, p \in \mathfrak{M}$,
4. ${}^{\omega_2}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\omega_4)$.

We claim that $p^\frown M$ is $(\mathcal{P}, \mathfrak{M})-$generic. So, let $r \geq p^\frown M$ and $\bar{D} \in \mathfrak{M}$ be a dense open subset of $\mathcal{P}$.
By extending r, if necessary, we can assume that $r \in \bar{D}$.

Let $A_0 \preceq A_1 \preceq A_2 \preceq H(\omega_4)$ be such that

1. $A_0 \in A_1 \in A_2$,
2. $|A_i| = \aleph_i$, for every $i < 3$,
3. $r \in A_0$.

In particular, $M \in A_i$, and so $A_i \cap M \in M$, for every $i < 3$. Set $q = r^\frown A_0{}^\frown A_1{}^\frown A_2$.

Denote A_2 by A.

Let $\delta_M = M \cap \omega_4$ and $\eta_A = \sup(A \cap \delta_M)$. Then η_A has cofinality $< \omega_3$, and so, $\eta_A < \delta_M$. Hence $\eta_A \in M$. Reflect now A, q down to $\mathfrak{M}$ over $A \cap M$ in the language which includes $\bar{D}$. Denote the result by A', q' and let M' be the image of M under this reflection.
Then, $A \cap \eta_A = A' \cap \eta_A$, also,

$$A \cap M = A' \cap M' \text{ and } A \cap M \cap \delta_M = A' \cap M' \cap \delta_M = A' \cap M' \cap \delta_{M'}.$$

Pick some model $\tilde{A}$ of cardinality $\aleph_2$ with A, q, A', q' inside. Pick also an $\in -$increasing sequence of models $\langle \tilde{A}_0, \tilde{A}_1, \tilde{A}_2 \rangle$ with $A, q, A', q', \tilde{A} \in \tilde{A}_0$ and $|\tilde{A}_i| = \aleph_i$.

It is enough to show the following:

Claim 5 $q^\frown q'^\frown \tilde{A}^\frown \langle \tilde{A}_0, \tilde{A}_1, \tilde{A}_2 \rangle \in \mathcal{P}$.

Proof. We need to check that Definition 1.1 is satisfied by the two pistes which form $q^\frown q'^\frown \tilde{A}^\frown \langle \tilde{A}_0, \tilde{A}_1, \tilde{A}_2 \rangle$, i.e., those which are generated by q and by its reflection q'.

Note that each of q, q' is fine. The only problem that may be here - is that new models of cardinality $\aleph_3$ are added to wide pistes of q, q'. For example, M' is added to q and M to q'. Note that only models of size $\aleph_3$ are added, since we reflected into a model M of cardinality $\aleph_3$, so models of smaller sizes reflect and did not remain on wide pistes of the reflected condition.

For example, if there were a model B of cardinality $\aleph_3$ in q on its wide piste with $M \in B$, then B would be reflected to $B' \in M$ and B' will appear on the wide piste of q', and not B.

Basically, we need to check only the covering conditions of Definition 1.1 in the following situation:

$B \in q$ above M on the wide piste of q and D' is a model of cardinality $\aleph_3$ in q' which does not belong to A, i.e., the reflection of some D in A.

But this is easy. Namely, if $B = A$, then M will be such a cover, since due to the reflection, $A \cap D' = A \cap M$.

Suppose that $B \neq A$, then $B \in A$.

If B is countable, then $B \subseteq A$, and again, M will be such a cover, if $M \in B$ or a model $\tilde{M} \in B$ which is the cover of M for B.

If $|B| = \aleph_1$ or $|B| = \aleph_2$, then note that $\sup(B \cap M \cap \omega_4) \in A \cap M$, and so it is below η_A. Hence, if $M \in B$, then $B \cap D' = B \cap M \cap D' = B \cap M$. If $M \notin B$, then the cover of M for B will be as desired.

$\square$ of the claim.

$\square$

Lemma 3.2 *The forcing $\mathcal{P}$ is ω_1-proper and ω_2-proper.*

Proof.

Let $p \in \mathcal{P}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ large enough such that

1. $|\mathfrak{M}| = \aleph_1$, for ω_1-properness, or $|\mathfrak{M}| = \aleph_2$, for ω_2-properness,
2. $\mathfrak{M} \supseteq \aleph_1$ or $\mathfrak{M} \supseteq \aleph_2$, respectively,
3. $\mathcal{P}, p \in \mathfrak{M}$,
4. ${}^{\omega}\mathfrak{M} \subseteq \mathfrak{M}$ or ${}^{\omega_1}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\omega_4)$.

We claim that $p^\frown M$ is $(\mathcal{P}, \mathfrak{M})-$generic. So, let $r \geq p^\frown M$ and $\bar{D} \in \mathfrak{M}$ be a dense open subset of $\mathcal{P}$.

By extending r, if necessary, we can assume that $r \in \bar{D}$.

Let $A_0 \preceq A_1 \preceq H(\omega_4)$ be such that

1. $A_0 \in A_1$,
2. $|A_i| = \aleph_i$, for every $i < 2$,
3. $r \in A_0$.

In particular, $M \in A_i$, and so $A_i \cap M \in M$, for every $i < 2$. Set $q = r^\frown A_0{}^\frown A_1$, if $|M| = \aleph_2$. Denote in this case A_1 by A.

If $|M| = \aleph_1$, then set $q = r^\frown A_0$ and denote A_0 by A.

Reflect now A, q down to $\mathfrak{M}$ over $A \cap M$ and above η_A in the language which includes $\bar{D}$. Denote the result by A', q' and let M' be the image of M under this reflection.

Then, $A \cap M = A' \cap M'$.

Note that $A \cap \omega_1 = A' \cap \omega_1$, since $M \supseteq \omega_1$, and so, $A \cap \omega_1 = A \cap M \cap \omega_1$.

Also, if $|M| = \aleph_2$, then $A \cap \omega_2 = A' \cap \omega_2$, since $M \supseteq \omega_2$, and so, $A \cap \omega_2 = A \cap M \cap \omega_2$.

In addition, if $|M| = \aleph_1$, then $A \cap M \cap \omega_2 \subseteq A'$. However, $M \cap \omega_2 \in A$, but $M \cap \omega_2 \notin A'$, since $A' \subseteq M$.

Similar, if $|M| = \aleph_2$, then $A \cap M \cap \omega_3 \subseteq A'$. However, $M \cap \omega_3 \in A$, but $M \cap \omega_3 \notin A'$, since $A' \subseteq M$.

Pick some model $\tilde{A}$ of cardinality $|A|$ with A, q, A', q' inside. Pick also an $\in -$increasing sequence of models $\langle \tilde{A}_0, \tilde{A}_1 \rangle$ with $A, q, A', q', \tilde{A} \in \tilde{A}_0$ and $|\tilde{A}_i| = \aleph_i$.

If $|A| = \aleph_0$, i.e., for $\aleph_1-$properness, then we can proceed without $\tilde{A}_1$.

It is enough to show the following:

Claim $q^\frown q'^\frown \tilde{A}^\frown\langle \tilde{A}_0, \tilde{A}_1\rangle \in \mathcal{P}$.

Proof. We need to check that Definition 1.1 is satisfied by the two pistes

which form $q^\frown q'^\frown \tilde{A}^\frown\langle \tilde{A}_0, \tilde{A}_1, \tilde{A}_2\rangle$, i.e., those which are generated by q and by its reflection q'.

Note that each of q, q' is fine. The only problem that may to appear - is that new models of cardinalities $\aleph_1, \aleph_2$ and $\aleph_3$ are added to wide pistes of q, q'. For example, M' is added to q and M to q'. Note that only models of sizes $\aleph_1$ and $\aleph_2$ are added, since we reflected into a model M of cardinality $\aleph_1$, so models of countable cardinality reflect and did not remain on wide pistes of the reflected condition.

For example, A reflects to A', but A' is not on the wide piste of A. However, M is on the wide piste of A'.

Basically, we need to check only the covering conditions of Definition 1.1.

Let us deal with ω_1-properness. The argument for ω_2-properness is similar and a bit simpler.

Let us deal first with few typical cases.

Case 1. *There is a new model of cardinality* $\aleph_1$ *above* $\sup(A \cap M \cap \omega_4)$ *which is a reflection of a model with* M *inside.*

Let B' be such a model. Then it is a reflection into M of a model $B \in A$ with $M \in B$. Also, $M' \subseteq B' \subseteq M$. We will have $A \cap B' = A \cap M$, since if $z \in A \cap M$, then $z \in A' \cap M' = A \cap M$ and $M' \subseteq B'$.

Case 2. *There is a new model of cardinality* $\aleph_2$ *or* $\aleph_3$ *above* $\sup(A \cap M \cap \omega_4)$.

Let D' be such a model. Then $A \cap D' = A \cap cl(M \cup |D'|)$.

Namely, $D' \in M$, hence $D' \subseteq cl(M \cup |D'|)$.

Let D be the model that reflects to D'. Then $D \supseteq M$, since $\sup(D' \cap \omega_4) > \sup(A \cap M \cap \omega_4) = \sup(A' \cap M' \cap \omega_4)$, and so, $\sup(D \cap \omega_4) > \sup(A \cap M \cap \omega_4)$. Note that $M, D \in A$, and so, if $\sup(D \cap \omega_4) < \sup(M \cap \omega_4)$, then $\min((M \cap \omega_4) \setminus \sup(D \cap \omega_4)) \in A \cap M \cap \omega_4$.

Hence, $\sup(D \cap \omega_4) > \sup(M \cap \omega_4)$, and so, $D \supseteq M$.

Suppose that $z \in A \cap cl(M \cup |D|)$. Then there are a term t, $a \in M \cap A$ and $\alpha \in A \cap |D|$ such that $z = t(a, \alpha)$. But $D \supseteq M \supseteq M \cap A$ and the reflection does not change $M \cap A$, so $a \in M \cap A$ implies $a \in D'$. Then $z = t(a, \alpha) \in D'$, and we are done.

Case 3. *There is a new model of cardinality* $\aleph_2$ *or* $\aleph_3$ *below* $\sup(A \cap M \cap \omega_4)$.

Let D' be such a model and D its pre-image under the reflection. Then $\sup(D \cap \omega_4) < \sup(A \cap M \cap \omega_4)$, since elements of $A \cap M$ do not move under the reflection. Also, $D \notin M$,

so the is $E \in M$ which is the cover of D for M. Then $E \in A \cap M$. In particular, E does not move under the reflection.
Note that $D' \subset D$. Thus, $D', D \subseteq E$, $M \cap E = M \cap D$ and $D' \in M \cap E$.

Let us argue that $A \cap D' = A \cap cl((M \cap E) \cup |D|)$. Clearly, $A \cap D' \subseteq A \cap cl((M \cap E) \cup |D|)$. We need to show that $A \cap D' \supseteq A \cap cl((M \cap E) \cup |D|)$.

Suppose that $z \in A \cap cl((M \cap E) \cup |D|)$. Then there are a term t, $a \in M \cap E \cap A$ and $\alpha \in A \cap |D|$ such that $z = t(a, \alpha)$. But $D \supseteq M \cap E \supseteq M \cap E \cap A$ and the reflection does not change $M \cap E \cap A$, so $a \in M \cap E \cap A$ implies $a \in D'$. Then $z = t(a, \alpha) \in D'$, and we are done.

Case 4. *There is a new model of cardinality* $\aleph_1$ *below* $\sup(A \cap M \cap \omega_4)$.

Let D' be such a model and D its pre-image under the reflection. Then $\sup(D \cap \omega_4) < \sup(A \cap M \cap \omega_4)$, since elements of $A \cap M$ do not move under the reflection. Also, $D \notin M$, so there is $E \in M$ of cardinality $\aleph_2$ (or $\aleph_3$) which is a part of a $\Delta-$system that produces such D. Then $E \in A \cap M$. In particular, E does not move under the reflection.
Let us argue that $A \cap D' = A \cap M \cap E$.

Assume for simplicity that M, D are from a $\Delta-$ as witnessed by models E and E_0, i.e. $E_0 \in D$ and $M \cap E = D \cap E_0$.
We have $E_0 \subset E$, since D is below M. So, $D \in E$. Then $D' \in E$ and $D' \subset E$, as well, since E does not move under the reflection to M.
Hence, $A \cap D' \subseteq A \cap M \cap E$.
Let us show the opposite direction. So let $z \in A \cap M \cap E$. Then $z \in A \cap D \cap E_0 \subseteq A \cap D \cap E$. So, $z \in A \cap M \cap D$. But elements of $A \cap M$ do not move under the reflection to M. So, z does not move. However D is moved to D'. Hence, $z \in D'$, and we are done.

Turn now to a general situation. Instead of A let us deal with an arbitrary countable model H (in q) which is above M.
We proceed by considering the cases above with A replaced by H.

Case $1'$**.** *There is a new model of cardinality* $\aleph_1$ *above* $\sup(A \cap M)$ *which is a reflection of a model with* M *inside.*

Let B' be such a model. Then it is a reflection into M of a model $B \in A$ with $M \in B$. Also, $M' \subseteq B' \subseteq M$. We will have $H \cap B' = H \cap M$, since if $z \in H \cap M$, then $z \in H' \cap M' = H \cap M$ and $M' \subseteq B'$.
If $M \in H$, then we are finished.
Suppose that $M \notin H$. Then there are $M^*, D^* \in H$ which are in q, $|M^*| = \aleph_1$, $|D^*| = \aleph_2$ and $|F^*| = \aleph_3$ such that $H \cap M = H \cap M^*$ or $H \cap M = H \cap M^* \cap D^*$ or $H \cap M = H \cap M^* \cap D^* \cap F^*$.

So, $H \cap B' = H \cap M = H \cap M^*$ or $H \cap B' = H \cap M = H \cap M^* \cap D^*$ or $H \cap B' = H \cap M = H \cap M^* \cap D^* \cap F^*$.

Case $2'$. *There is a new model of cardinality $\aleph_2$ or $\aleph_3$ above* $\sup(A \cap M \cap \omega_4)$.

Let D' be such a model. Then $A \cap D' = A \cap cl(M \cup |D'|)$, as was shown in Case 2 above. We have

$$H \cap D' = H \cap A \cap D' = H \cap A \cap cl(M \cup |D'|) = H \cap cl(M \cup |D'|).$$

If $M \in H$, then we are done.

Suppose that $M \notin H$.

Assume first that there is $M^* \in H$ which is the cover of M, i.e., $H \cap M^* = H \cap M$. Let us argue that then

$$H \cap D' = H \cap cl(M^* \cup |D'|).$$

Clearly,

$$H \cap D' \subseteq H \cap cl(M^* \cup |D'|),$$

since $H \cap D' = H \cap cl(M \cup |D'|)$ and $M \subseteq M^*$.

Let show the opposite inclusion. So, let $z \in H \cap cl(M^* \cup |D'|)$. Then there are a term t, $\alpha < |D'|$ and $a \in M^*$ such that $z = t(\alpha, a)$. We have $z, M^* \in H$, hence there are $\alpha \in H, a \in H \cap M^*$ such that $z = t(\alpha, a)$.

Recall that $H \cap M^* = H \cap M$. Hence, $a \in H \cap M$. So, $z = t(\alpha, a) \in H \cap cl(M \cup |D'|)$, and we are done.

The remaining possibility is that there are $M^* \in H$ of cardinality $\aleph_1$ or $\aleph_2$ and $F^* \in H$ of cardinality $\aleph_2$ or $\aleph_3$ such that $M^* \cap F^*$ is the cover of M.

We claim that then

$$H \cap D' = H \cap cl((M^* \cap F^*) \cup \aleph_2).$$

The argument is as above, only replace M^* with $M^* \cap F^*$.

Case $2''$. $|M| = \aleph_2$ *and there is a new model of cardinality $\aleph_3$ above* $\sup(A \cap M \cap \omega_4)$.

Let D' be such a model. Then $A \cap D' = A \cap cl(M \cup \aleph_3)$, as was shown in Case 2 above. We have

$$H \cap D' = H \cap A \cap D' = H \cap A \cap cl(M \cup \aleph_3) = H \cap cl(M \cup \aleph_3).$$

If $M \in H$, then we are done.

Suppose that $M \notin H$.

Consider a new possibility here that there is $M^* \in H$ which is the cover of M, i.e., $H \cap M^* = H \cap M$, is of a form $cl(N \cup \aleph_2)$, for some $N \in H$ of cardinality $\aleph_1$ with $M \in N$.

We have $H \cap D' = H \cap cl(M^* \cup \aleph_3)$.
Let us argue that
$$H \cap D' = H \cap cl(N \cup \aleph_3).$$
Clearly, $cl(M^* \cup \aleph_3) \subseteq cl(N \cup \aleph_3)$, and so, $H \cap D' \subseteq H \cap cl(N \cup \aleph_3)$.
Let us show the opposite inclusion.
Thus let $z \in H \cap cl(N \cup \aleph_3)$. Then $z = h(a, \alpha)$, for some $a \in H \cap N$ and $\alpha \in H$. We have $H \cap N \subseteq H \cap cl(N \cup \aleph_2) = H \cap M$. Hence $a \in H \cap M$. So, $z \in H \cap cl(M \cup \aleph_3)$, and we are done.

Case $3'$**.** *There is a new model of cardinality* $\aleph_2$ *or* $\aleph_3$ *below* $\sup(A \cap M \cap \omega_4)$.

Let D' be such a model and D its pre-image under the reflection. Then $D \cap \omega_4 < \sup(A \cap M \cap \omega_4)$, since elements of $A \cap M$ do not move under the reflection. Also, $D \notin M$, so the is $E \in M$ which is the cover of D. Then $E \in A \cap M$. In particular, E does not move under the reflection.
Note that $D' \subset D$. Thus, $D', D \subseteq E$, $M \cap E = M \cap D$ and $D' \in M \cap E$.

It was proved in Case 3 above that
$$A \cap D' = A \cap cl((M \cap E) \cup |D'|).$$
This implies that
$$H \cap D' = H \cap A \cap D' = H \cap A \cap cl((M \cap E) \cup |D'|) = H \cap cl((M \cap E) \cup |D'|).$$

Let now $M^* \in H$ be the cover of M and $E^* \in H$ be the cover of E.

We claim that
$$H \cap D' = H \cap cl((M^* \cap E^*) \cup |D'|).$$
Clearly, $H \cap D' = H \cap cl((M \cap E) \cup |D'|) \subseteq H \cap cl((M^* \cap E^*) \cup |D'|)$.
Let us show the opposite direction. So, let $z \in H \cap cl((M^* \cap E^*) \cup |D'|)$. Then there are a term t, $\alpha < |D'|$ and $a \in M^* \cap E^*$ such that $z = t(\alpha, a)$. We have $z, M^*, E^* \in H$, hence there are $\alpha \in H, a \in H \cap M^* \cap E^*$ such that $z = t(\alpha, a)$.
Recall that $H \cap M^* = H \cap M$ and $H \cap E = H \cap E^*$. Hence, $a \in H \cap M \cap E$. So, $z = t(\alpha, a) \in H \cap cl((M \cap E) \cup |D'|)$, and we are done.

Case $4'$**.** *There is a new model of cardinality* $\aleph_1$ *below* $\sup(A \cap M \cap \omega_4)$.

Let D' be such a model and D its pre-image under the reflection. Then $\sup(D \cap \omega_4) < \sup(A \cap M \cap \omega_4)$, since elements of $A \cap M$ do not move under the reflection. Also, $D \notin M$, so the is $E \in M$ of cardinality $\aleph_2$ which is a part of a $\Delta-$system that produces such D.

Then $E \in A \cap M$. In particular, E does not move under the reflection.
We already proved that $A \cap D' = A \cap M \cap E$.

Then

$$H \cap D' = A \cap H \cap D' = H \cap A \cap D' = H \cap A \cap M \cap E = (H \cap M) \cap (H \cap E).$$

All the models H, M, E in q. Hence, by Definition 1.1 and the intersection properties, $H \cap M = H \cap N$ and $H \cap E = H \cap L$, for some $N, L \in H$. Here we allow N to be of the form $K \cap cl((S \cap G) \cup \omega_2)$ and L of the form $cl((X \cap Y) \cup \omega_2)$ with all components K, S, G, X, Y in H and in q.
We can conclude, applying the claim, that $H \cap M = K \cap cl((S \cap G) \cup \omega_2)$ or $H \cap M = K \cap cl((X \cap Y) \cup \omega_2)$.
□

Lemma 3.3 *The forcing $\mathcal{P}$ is $\omega-$proper, i.e., proper.*

Proof.

Let $p \in \mathcal{P}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ large enough such that

1. $|\mathfrak{M}| = \aleph_0$,

2. $\mathcal{P}, p \in \mathfrak{M}$,

Set $M = \mathfrak{M} \cap H(\omega_4)$.

We claim that $p^\frown M$ is $(\mathcal{P}, \mathfrak{M})-$generic. So, let $r \geq p^\frown M$ and $\bar{D} \in \mathfrak{M}$ be a dense open subset of $\mathcal{P}$.
By extending r, if necessary, we can assume that $r \in \bar{D}$.

Let $r \restriction M$ be the set of all models of r which belong to M.
Extend then inside M, $r \restriction M$ to a condition $s \in \bar{D}$.
We claim that r and s are compatible.
Moreover $r \cup s$ is almost a condition. In order to turn it into a condition, new (i.e., those not in r) models should be mapped through $\Delta-$systems, when this applies.

The issue is with new models of sizes $\aleph_1, \aleph_2$ and $\aleph_3$.

Deal first with those of size $\aleph_3$.

So, let D be a model in r which is not in M of cardinality $\aleph_3$ and there is a new model E in M of cardinality $\aleph_3$.

Then either the ordinal $E \cap \omega_4$ is above or below $D \cap \omega_4$, which implies $D \in E$ or $E \in D$, and we are done.

Let us turn to models of cardinalities $\aleph_1, \aleph_2$.

Consider first the following situation:

D be a model in r which is not in M of cardinality $\aleph_3$ and B is a new model of cardinality $\aleph_2$ in M.

Assume that we have $F \in M$ and in s of cardinality $\aleph_3$ such that $M \cap F = M \cap D$, i.e. a covering model F of D for M is in $Covmod(r)_0$. Also let $F \in B$.

Then $F = \bigcup_{i<\omega_3} F_i$ where $\langle F_i \mid i < \omega_3 \rangle$ is increasing continuous sequence of models of cardinality $\aleph_2$ with limit F, defined from F.

Set $\sup(M \cap \omega_3) = \eta$. Then $M \cap F = M \cap cl(F_\eta \cup \omega_3)$. Then D cannot be below $cl(F_\eta \cup \omega_3)$, since $M \cap F = M \cap D$. So, $D \cap \omega_4 \geq \eta$.

We have, $i_B = \sup(B \cap \omega_4) \in M$, and hence, $i_B < \eta$. Clearly, $B \cap F = F_{i_B}$. Hence,

$$B \cap D \subseteq B \cap F = F_{i_B} \subseteq B \cap cl(F_{i_B} \cup \omega_3) \subseteq B \cap D.$$

So, $B \cap D = B \cap F$.

Suppose now that a covering model F of D for M is such that $\mathrm{cof}(F \cap \omega_3) = \omega_2$. Let $E \in M, |E| = \aleph_2$ be its leading model on the wide piste.

If B is below E, then 2.1(10) applies.

Assume that B is above E, and so, $E \subseteq B$.

Then by the strong covering property 18 of 1.1, there is $\tilde{D}$ in E and in r which is a cover of D for E. Note that $\tilde{D}$ not in M. Let $\langle \tilde{D}_i \mid i < \omega_3 \rangle$ be an increasing continuous sequence of models of cardinality $\aleph_2$ with limit $\tilde{D}$, defined from $\tilde{D}$. We have, by the strong covering property 18 of 1.1, $D \supseteq \tilde{D}_{\sup(M \cap \omega_3)}$.

Now, $B \in M$, hence $B \cap \omega_3 < \sup(M \cap \omega_3)$. Then,

$$B \cap \tilde{D} = \tilde{D}_{B \cap \omega_3} \subseteq \tilde{D}_{\sup(M \cap \omega_3)}.$$

Hence,

$$B \cap D \subseteq B \cap \tilde{D} = \tilde{D}_{B \cap \omega_3} \subseteq \tilde{D}_{\sup(M \cap \omega_3)} \subseteq D.$$

So, $B \cap D = B \cap \tilde{D}$.

Consider now a new case $|B| = \aleph_1$ and $\mathrm{cof}(F \cap \omega_4) = \omega_1$. Let $S \in M \cap C^{\omega_1}(r)$ be a leading model of F. Assume that $B \supseteq S$, otherwise we apply 2.1(10).

By the strong covering property 19 of 1.1, either $D \in S$, and then we are done, or there is $\tilde{D}$ in S and in r which is a cover of D for S. Note that $\tilde{D}$ not in M.

If $\mathrm{cof}(\tilde{D}\cap\omega_4)=\omega_3$, then Let $\langle \tilde{D}_i \mid i<\omega_3\rangle$ be an increasing continuous sequence of models of cardinality $\aleph_2$ with limit $\tilde{D}$, defined from $\tilde{D}$. Then, by 19 of 1.1, $D\supseteq \tilde{D}_{\sup(M\cap\omega_3)}\supseteq \tilde{D}_{\sup(B\cap\omega_3)}$, since $B\in M$.
Suppose that $\mathrm{cof}(\tilde{D}\cap\omega_4)=\omega_2$, then let $T\in S\cap C^{\omega_2}(r)$ be its leading model. By 19(ii) of 1.1, $D\supseteq T_{\sup(M\cap\omega_2)}$, where $\langle T_i \mid i<\omega_3\rangle$ is an increasing continuous sequence of models of cardinality $\aleph_1$ with limit T, defined from T. We have $B\in M$, so $\sup(M\cap\omega_2)>B\cap\omega_2$, and then $B\cap T\subseteq T_{B\cap\omega_2}\subseteq T_{\sup(M\cap\omega_2)}\subseteq D$. So, if $D\in T$, then we are done.
Suppose that $D\notin T$. Apply 19(ii)(B) of 1.1. There is a covering model $\tilde{T}\in T\cap C^{\omega_3}$ of D for T such that $D\supseteq \tilde{T}_{\sup(M\cap\omega_3)}$, where $\langle \tilde{T}_i \mid i<\omega_3\rangle$ be an increasing continuous sequence of models of cardinality $\aleph_2$ with limit $\tilde{T}$, defined from $\tilde{T}$. If $\tilde{T}\in B$, then it will be a covering model of D for B, since $B\cap\tilde{T}=B\cap\tilde{T}_{\sup(B\cap\omega_3)}=B\cap\tilde{T}_{\sup(M\cap\omega_3)}=B\cap D$.
Suppose that $\tilde{T}\notin B$. We have $\tilde{T}\supseteq T_{\sup(M\cap\omega_2)}$, since $D\supseteq T_{\sup(M\cap\omega_2)}$ and $\tilde{T}$ is a cover of D for T. But then $B\cap\tilde{T}=B\cap cl(T_{B\cap\omega_2}\cup\omega_3)$, since clearly, $\tilde{T}\subseteq cl(T\cup\omega_3)$. So,

$$B\cap D=B\cap T\cap D=B\cap T\cap\tilde{T}=B\cap cl(T\cup\omega_3)\cap\tilde{T}=B\cap cl(T_{B\cap\omega_2}\cup\omega_3)=B\cap cl(T\cup\omega_3).$$

Hence, $cl(T\cup\omega_3)$ is a covering model of D for B in the present case, and we are done.
□

References

[1] M. Gitik, Short extenders forcing without preparation