# The Galvin property at $\kappa^{++}$ and not at $\kappa^{+}$

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**Abstract**

We construct a $\kappa-$complete ultrafilter $W$ over $\kappa$ such that $\neg\text{Gal}(\kappa, W, \kappa^+)$ and $\text{Gal}(\kappa, W, \kappa^{++})$. This answers a question of T. Benhamou and G. Goldberg [7].

## 1 Introduction

We deal here with the Galvin property of $\kappa$-complete ultrafilters over $\kappa$.

**Definition 1.1 (The Galvin property at $\lambda$)** Let $U$ be a filter over $\kappa$. Let $\lambda$ be a cardinal s.t. $\kappa < \lambda \leqslant 2^\kappa$. We say that $U$ satisfies the Galvin property at $\lambda$ (and denote it by $\text{Gal}(\kappa, U, \lambda)$) iff for every $\{A_\alpha \mid \alpha < \lambda\} \subseteq U$ there exists a sequence $\langle \alpha_i \mid i < \kappa \rangle \subseteq \lambda$ s.t.

$$\bigcap_{i<\kappa} A_{\alpha_i} \in U.$$

It was shown by F. Galvin [2] that if $2^{<\kappa} = \kappa$ then every normal filter $U$ over $\kappa$ satisfies $\text{Gal}(\kappa, U, \kappa^+)$. U. Abraham and S. Shelah [1] showed consistency of $\neg\text{Gal}(\aleph_1, Cub_{\aleph_1}, \aleph_2)$. The consistency of a negation of the Galvin property for $\kappa-$complete ultrafilters over $\kappa$ was first proved by T. Benhamou, S. Garti and S. Shelah in [4]. A supercompact cardinal was used for this. In [5], a different method was suggested and the initial assumption was reduced to a measurable.

Note that if $\text{Gal}(\kappa, U, \lambda)$ holds for some ultrafilter $U$ and cardinal $\lambda$, then $\text{Gal}(\kappa, U, \lambda')$ holds for every cardinal $\lambda' \geqslant \lambda$. Equivalently, if $\neg\text{Gal}(\kappa, U, \lambda)$ holds for some ultrafilter $U$ and cardinal $\lambda$, then $\neg\text{Gal}(\kappa, U, \lambda')$ holds for every $\lambda' \leqslant \lambda$.
In all previously known examples, it always was the case that either $\text{Gal}(\kappa, U, \lambda)$ holds for

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every $\kappa < \lambda \leqslant 2^\kappa$ or $\neg$ Gal$(\kappa, U, \lambda)$ holds for every $\kappa < \lambda \leqslant 2^\kappa$. This leads to the natural question, asked explicitly by T. Benhamou and G. Goldberg [7]:

*Is it consistent to have a $\kappa-$complete ultrafilter $U$ over $\kappa$ such that* Gal$(\kappa, U, 2^\kappa)$ and $\neg$Gal$(\kappa, U, \kappa^+)$?

The purpose of this paper is to provide an affirmative answer to this question. Namely, we prove the following:

**Theorem 1.2** *Assume GCH and suppose that $\kappa$ is 2-strong cardinal. Then there is a cofinality preserving forcing extension $V^*$ such that $V^* \models 2^\kappa = \kappa^{++}$, and, in $V^*$, there is a $\kappa$-complete ultrafilter $W$ over $\kappa$ s.t. $W$ does not satisfy Gal$(\kappa, W, \kappa^+)$, but satisfies Gal$(\kappa, W, \kappa^{++})$.*

Notice that these assumptions are optimal, since in the extension $\kappa$ is a measurable with $2^\kappa = \kappa^{++}$.

The proof follows the lines of Woodin for blowing up the power of a measurable cardinal (as presented by Cummings in [8]), however, the method of [3] will be used in order to avoid an additional forcing over $\kappa^+$. The idea of constructing ultrafilters without Galvin property of [5] will be crucial in the present construction.

Some generalizations will be discussed at the end of the paper.

## 2 The basic construction

Assume GCH. Let $E$ be $(\kappa, \kappa^{++})-$extender and $j_E : V \to ME$ be the corresponding elementary embedding. Denote by $E_\alpha$ the $\kappa-$complete ultrafilter $\{X \subseteq \kappa \mid \alpha \in j_E(X)\}$ over $\kappa$, for every $\alpha < j_E(\kappa)$. The special attention will be to a normal ultrafilter $E_\kappa$.

For each regular cardinal $\delta$, let Cohen$(\delta, \delta^{++})$ be the Cohen forcing for adding $\delta^{++}-$Cohen functions to $\delta$. It consists of partial functions from $\delta \times \delta^{++}$ to $\delta$ of cardinality less than $\delta$. Let

$$\mathbb{P} = \mathbb{P}_{\kappa+1} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta | \alpha \leqslant \kappa + 1, \beta \leqslant \kappa \rangle$$

be an Easton support iteration where

$$\Vdash_{\mathbb{P}_\alpha} \underset{\sim}{\mathbb{Q}}_\alpha = \text{Cohen}(\check{\alpha}, \check{\alpha}^{++})$$

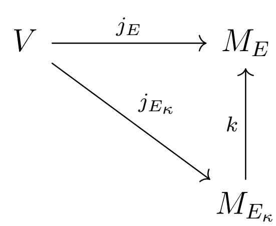
when $\alpha$ is inaccessible and $\underset{\sim}{\mathbb{Q}}_\alpha$ is trivial otherwise.
Let $G = G_{<\kappa} * G_\kappa$ be generic for $\mathbb{P}$, where $G_{<\kappa}$ is a generic for $\mathbb{P}_\kappa$ over $V$ and $G_\kappa$ is a generic for $\mathbb{Q}_\kappa = Cohen(\kappa, \kappa^{++})$ over $V[G]$. Denote by $\langle f_{\kappa,\alpha} | \alpha < \kappa^{++} \rangle$ the generic Cohen functions added by $G_\kappa$. Let us denote $j_E(\kappa)$ by $\kappa_1$ and $j_{E_\kappa}(\kappa)$ by $\kappa_1^{nor}$.

It is standard to check that $V[G_{<\kappa} * G_\kappa] \models 2^\kappa = \kappa^{++}$. Let us deal with measurability and extensions of elementary embeddings. We will have to find a generic $H \in V[G]$ for $j_E(\mathbb{P})$ s.t. $j_E''(G) \subseteq H$ which will allow us to lift the embedding (as in Proposition 9.1 in [8]). In order to do that, we will use the projection of $E$ to its normal ultrafilter $E_\kappa$.

Denote by

$$k : M_{E_\kappa} \to M_E,\ k([f]_{E_\kappa}) = j_E(f)(\kappa).$$

We have the following commutative diagram:




Note that crit$(k) = (\kappa^{++})^{M_{E_\kappa}}$.

Now from elementarity and $\kappa$-closure of $M_{E_\kappa}$ and $M_E$ we get that the iterations $\mathbb{P}$, $j_{E_\kappa}(\mathbb{P})$ and $j_E(\mathbb{P})$ agree up to stage $\kappa$.

The next two lemmas are well known.

**Lemma 2.1** $\mathbb{P} = \mathbb{P}_{\kappa+1} = (j_E(\mathbb{P}))_{\kappa+1}$.

*Proof.* First of all, $(\kappa^{++})^{M_E}=\kappa^{++}$. Since $(j_E(\mathbb{P}))_\kappa=\mathbb{P}_\kappa$, we need to show that if $f$ is a partial function from $\kappa\times\kappa^{++}$, $f\in V[G_{<\kappa}]$, where $G_{<\kappa}$ is $\mathbb{P}_\kappa-$generic and $|f|<\kappa$, then $f\in M_E[G_{<\kappa}]$. This is true since $\mathbb{P}_\kappa$ is $\kappa^+$-c.c. and ${}^\kappa M_E\subseteq M_E$, and then ${}^\kappa M_E[G_{<\kappa}]\subseteq M_E[G_{<\kappa}]$.

□

**Lemma 2.2** $(j_{E_\kappa}(\mathbb{P}))_{\kappa+1}=\mathbb{P}_\kappa*\underset{\sim}{\mathbb{Q}}^*$ *where* $\underset{\sim}{\mathbb{Q}}^*=(\mathit{Cohen}(\kappa,(\kappa^{++})^{M_{E_\kappa}}))^{V[G]}$.

*Proof.* This is because ${}^\kappa M_{E_\kappa}[G_{<\kappa}]\subseteq M_{E_\kappa}[G_{<\kappa}]$ for any $G_{<\kappa}$ which is $\mathbb{P}_\kappa$ generic.

□

Set $G^{nor}_\kappa=G_\kappa\cap(\mathrm{Cohen}(\kappa,(\kappa^{++})^{M_{E_\kappa}}))^{V[G]}$.

Let us extend first deal with $j_{E_\kappa}$.

Consider the forcing

$$j_{E_\kappa}(\mathbb{P}_\kappa*\mathit{Cohen}(\kappa,\kappa^+))=\mathbb{P}_\kappa*\mathit{Cohen}(\kappa,(\kappa^{++})^{M_{E_\kappa}})*\mathbb{P}_{>\kappa}*\mathit{Cohen}(\kappa_1^{nor},(\kappa_1^{nor})^+)$$

in $M_{E_\kappa}$. Using the $\kappa-$closure of the forcing $\mathbb{P}_{>\kappa}*\mathit{Cohen}(\kappa_1^{nor},(\kappa_1^{nor})^+)$ and the fact that the number of corresponding dense sets is $\kappa^+$, we construct (in $V[G_{<\kappa}*G^{nor}_\kappa]$) a master condition sequence $\langle p_\xi\mid\xi<\kappa^+\rangle$ for $\mathbb{P}_{>\kappa}*\mathit{Cohen}(\kappa_1^{nor},(\kappa_1^{nor})^+)$ over $M_{E_\kappa}[G_{<\kappa}*G^{nor}_\kappa]$. Assume that $\langle p_\xi\mid\xi<\kappa^+\rangle$ is definable via some fixed well ordering. Denote by $G^{nor}_{>\kappa}*\langle f^n_{\kappa_1^{nor},\alpha}or\mid\alpha<j_{E_\kappa}(\kappa^+)\rangle]$ the generic object with Cohen functions over $\kappa_1^{nor}$, generated by $\langle p_\xi\mid\xi<\kappa^+\rangle$.

Now, we use $k$. $\langle k(p_\xi)\mid\xi<\kappa^+\rangle$ will generate a generic over $M_E$. Apply [3] in order to find missing Cohen functions over $\kappa_1$, i.e. those with indexes in $[\kappa_1^+,\kappa_1^{++})^{M_E}$.
Denote the result by $G_{>\kappa}*\langle f_{\kappa_1,\alpha}\mid\alpha<j_E(\kappa^{++})\rangle$.
Note that $G_{>\kappa}*\langle f_{\kappa_1,\alpha}\mid\alpha<j_E(\kappa^+)\rangle$ is fully generated by $\langle k(p_\xi)\mid\xi<\kappa^+\rangle$.
By changing values of Cohen functions $\langle f_{\kappa_1,\alpha}\mid\alpha<j_E(\kappa^{++})\rangle$ if necessary, we can assume that

1. $f_{\kappa_1 j_E(\beta)}(\kappa)=2\dot{\beta}$, for every $\beta<\kappa^+$,
   and

2. $f_{\kappa_1 j_E(\kappa^++\beta)}(\kappa)=\beta$, for every $\beta<\kappa^{++}$.

The first item will be used further for the Galvin property and the second provides a simple representation of ordinals below $\kappa^{++}$ in the ultrapower.

The elementary embeddings $j_{E_\kappa},j_E,k$ extend to

$$j^*_{E_\kappa}:V[G_{<\kappa}*\langle f_{\kappa,\alpha}\mid\alpha<\kappa^+\rangle]\to M_{E_\kappa}[[G_{<\kappa}*G^{nor}_\kappa*G^{nor}_{>\kappa}*\langle f^{nor}_{\kappa_1^{nor},\alpha}\mid\alpha<j_{E_\kappa}(\kappa^+)\rangle],$$

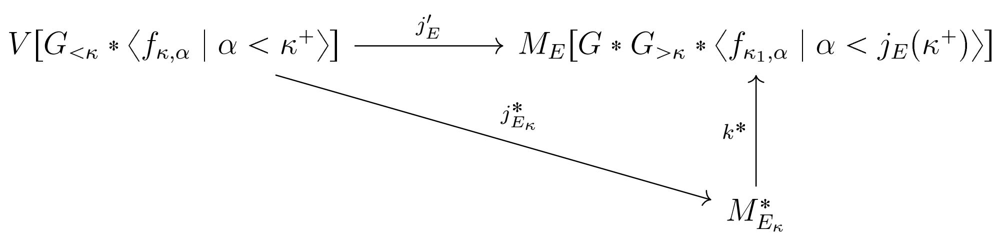
$$j_E^*: V[G] \to M_E[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle],$$

and

$$k^*: M_{E_\kappa}[G_{<\kappa} * G_\kappa^{nor} * G_{>\kappa}^{nor} * \langle f^{nor}_{\kappa_1^{nor},\alpha} \mid \alpha < j_{E_\kappa}(\kappa^+)\rangle] \to M_E[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^+)\rangle.$$

Denote $M_E[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle]$ by $M_E^*$ and $M_{E_\kappa}[G_{<\kappa} * G_\kappa^{nor} * G_{>\kappa}^{nor} * \langle f^{nor}_{\kappa_1^{nor},\alpha} \mid \alpha < j_{E_\kappa}(\kappa^+)\rangle]$ by $M_{E_\kappa}^*$.

The following diagram will be commutative:




where $j_E'$ denotes $j_{E^*} \restriction V[G_{<\kappa} * \langle f_{\kappa,\alpha} \mid \alpha < \kappa^+\rangle]$.

Define, in $V[G]$, a normal ultrafilter $U^*$ over $\kappa$ as follows:

$$U^* = \{X \subseteq \kappa \mid \kappa \in j_E^*(X)\}.$$

The next lemma is well known.

**Lemma 2.3** $j_E^* = j_{U^*}$.

*Proof.* Let $\varphi : M_{U^*} \to M_E^*$ be the elementary embedding defined by $\varphi([f]_{U^*}) = j_E^*(f)(\kappa)$. Let us show that $\varphi$ is onto. So let $x = (\underset{\sim}{x})_{G*G_{>\kappa}*\langle f_{\kappa_1,\alpha}|\alpha<j_E(\kappa^{++})\rangle} \in M_E[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle]]$. Since $\underset{\sim}{x} \in M_E$, there are some

$$g \in V, a = \{\alpha_1, \ldots, \alpha_r\} \in [\kappa^{++}]^{<\omega} \text{ such that } j_E(g)(a) = \underset{\sim}{x}.$$

Define in $V[G]$ a function

$$g^*(\alpha) = (g(\{f_{\kappa,\kappa^+ + \alpha_1}(\alpha), ..., f_{\kappa,\kappa^+ + \alpha_r}(\alpha)\}))_G.$$

Then,

$$\begin{aligned}
\varphi([g^*]_{U^*}) &= j_E^*(g^*)(\kappa) \\
&= (j_E^*(g)(\{j_E^*(f_{\kappa,\kappa^++\alpha_1})(\kappa), ..., j_E^*(f_{\kappa,\kappa^++\alpha_r})(\kappa)\}))_{G*G_{>\kappa}*\langle f_{\kappa_1,\alpha} | \alpha < j_E(\kappa^{++})\rangle} \\
&= (j_E(g)(\{f_{\kappa_1,j_E(\kappa^++\alpha_1)}(\kappa), ..., f_{\kappa_1,j_E(\kappa^++\alpha_r)}(\kappa)\}))_{G*G_{>\kappa}*\langle f_{\kappa_1,\alpha} | \alpha < j_E(\kappa^{++})\rangle} \\
&= (j_E(g)(\{\alpha_1, ..., \alpha_r\}))_{G*G_{>\kappa}*\langle f_{\kappa_1,\alpha} | \alpha < j_E(\kappa^{++})\rangle} \\
&= (j_E(g)(a))_{G*G_{>\kappa}*\langle f_{\kappa_1,\alpha} | \alpha < j_E(\kappa^{++})\rangle} \\
&= (\underset{\sim}{x})_{G*G_{>\kappa}*\langle f_{\kappa_1,\alpha} | \alpha < j_E(\kappa^{++})\rangle} = x.
\end{aligned}$$

Notice that the fourth equality follows since we have:

$$\forall \alpha < \kappa^{++},\ f_{\kappa_1,j_E(\kappa^++\alpha)}(\kappa) = \alpha.$$

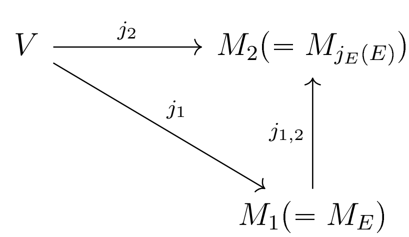
□

# 3 The second ultrapower

We will turn to the second ultrapower by $E$, i.e. $\text{Ult}(M_E, j_{j_E(E)})$ or $M_{E\times E}$, in order to obtain a failure of the Galvin property at $\kappa^+$, as in was done in [5].
To facilitate notations, let us denote:

$$M_1 := M_E,\ M_2 := M^{M_1}_{j_E(E)},\ j_1 := j_E,\ j_{1,2} := j_{j_E(E)},\ j_2 := j_{1,2} \circ j_1.$$

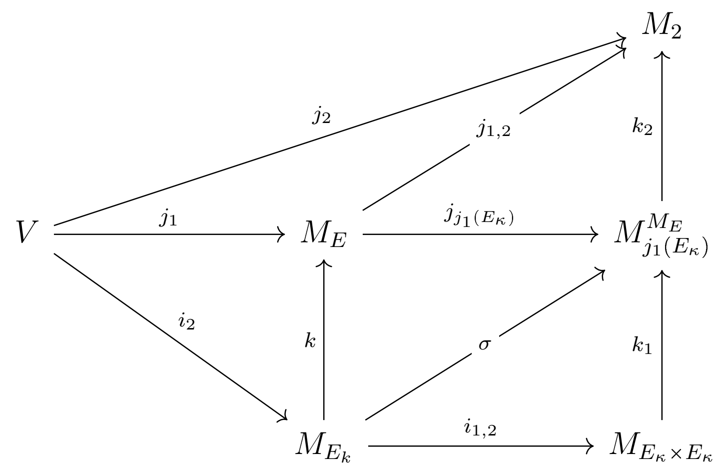
Let also $\kappa_2 := j_2(\kappa) = j_{1,2}(\kappa_1)$. So we have the following commutative diagram:




The second ultrapower of the normal ultrafilter $E_\kappa$ will be essential. Denote

$$M_1^{nor} := M_{E_\kappa},\ M_2^{nor} := M^{M_1^{nor}}_{j_{E_\kappa}(E_\kappa)},\ i_1 := j_{E_\kappa},\ i_{1,2} := j_{j_{E_\kappa}(E_\kappa)},\ i_2 := i_{1,2} \circ i_1.$$

Let also $\kappa_2 := i_2(\kappa) = i_{1,2}(\kappa_1^{nor})$. Consider the following commutative diagram:




Turn to $V[G]$ and consider the second ultrapower $M^{M_{U^*}}_{j_{U^*}} = M_{U^* \times U^*}$ of $U^*$.
By elementarity,

$$M^{M_{U^*}}_{j_{U^*}} = M_2[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle * G_{>\kappa_1} * \langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^{++})\rangle].$$

The part $G_{>\kappa_1} * \langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^{++})\rangle]$ of the generic is constructed in $M_1^*$ exactly the corresponding part, i.e. $G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle$ was constructed in $V[G]$. The pointwise image of the master condition sequence $\langle p_\xi \mid \xi < \kappa^+\rangle$ under $k_2 \circ j_{j_1(E_\kappa)} \circ k$ is used to generate $G_{>\kappa_1} * \langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^+)\rangle$. Note that $j_2{}''\kappa^+$ is unbounded in $j_2(\kappa^+)$, so

$$\langle (k_2 \circ j_{j_1(E_\kappa)} \circ k)(p_\xi) \mid \xi < \kappa^+\rangle$$

generates

$$G_{>\kappa_1} * \langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^+)\rangle.$$

Let us use the idea of [5] and define a $\kappa-$complete ultrafilter $W$ over $\kappa$ as follows.
Change values of Cohen functions in the sequence $\langle f_{\kappa_2,\alpha} \mid \alpha \in j_2{}''\kappa^+\rangle$ on $\kappa_1$.
Thus, for every $\alpha \in j_2{}''\kappa^+$, change $f_{\kappa_2,\alpha}(\kappa_1)$ to be $2 \cdot \beta + 1$, where $j_{1,2}(\beta) = \alpha$.
Denote the changed sequence by $\langle f'_{\kappa_2,\alpha} \mid \alpha \in j_2{}''\kappa^{++}\rangle$.
By the arguments of [5], the sequence $\langle f'_{\kappa_2,\alpha} \mid \alpha \in j_2{}''\kappa^{++}\rangle$ is still Cohen generic and the embedding $j_{E\times E}$ extends. Denote the resulting embedding

$$j'_2 : V[G] \to M'_2 = M_2[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle * G_{>\kappa_1} * \langle f'_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^{++})\rangle].$$

The following holds by the construction:

1. $f'_{\kappa_2, j_{1,2}(\beta)}(\kappa_1)$ is odd if $\beta \in j_1{}''\kappa^+$.

2. $f'_{\kappa_2, j_{1,2}(\beta)}(\kappa_1)$ is even if $\beta \in j_1(\kappa^+) \backslash j_1{}''\kappa^+$.

Define now

$$W = \{X \subseteq [\kappa]^2 | (\kappa, \kappa_1) \in j'_2(X)\}.$$

Note that $W$ is an ultrafilter over $[\kappa]^2$ rather than $\kappa$, however it is Rudin-Kiesler equivalent to an ultrafilter over $\kappa$. We will show that $W$ has the desired properties.

# 4 Failure of the Galvin property for $\kappa^+$

Let us start with the following observation.

**Lemma 4.1** *$j_W = j_2^*$ and $[id]_W = (\kappa, \kappa_1)$.*

*Proof.* Define $\varphi : M_W \to M_2^*$ by setting $\varphi([f]_W) = j_2^*(f)(\kappa, \kappa_1)$. Let us show that $\varphi$ is onto. Denote $G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle * G_{>\kappa_1} * \langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^{++})\rangle$ by $G^*$. Let $(\underset{\sim}{x})_{G^*} \in M_2^*$. There are

$$h \in M_1 \text{ and } a = \{\alpha_1, ..., \alpha_r\} \in [\kappa_1^{++}]^{<\omega}, \text{ such that } \underset{\sim}{x} = j_{1,2}(h)(a),$$

since $j_{1,2}$ is an ultrapower by a $(\kappa_1, \kappa_1^{++})$-extender.
Let us define in $M_1^* = M_E[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle]$ a function $g : \kappa_1 \to ON$ as follows:

$$g(\gamma) = (h(f_{\kappa_1,\kappa_1^+ + \alpha_1}(\gamma), ..., f_{\kappa_1,\kappa_1^+ + \alpha_r}(\gamma)))_{G^*}.$$

Then, by the construction of $g_{\kappa_2}$,

$$\begin{aligned} j_{1,2}^*(g)(\kappa_1) &= (j_{1,2}(h)(f_{\kappa_2, j_{1,2}(\kappa_1^+ + \alpha_1)}(\kappa_1), ..., f_{\kappa_2, j_{1,2}(\kappa_1^+ + \alpha_r)}(\kappa_1)))_{G^*} \\ &= (j_{1,2}(h)(\alpha_1, ..., \alpha_r))_{G^*} \\ &= (j_{1,2}(h)(a))_{G^*} \\ &= (\underset{\sim}{x})_{G^*} = x. \end{aligned}$$

Now, since $M_1^*$ is an ultrapower by a normal ultrafilter, there is a function

$$q : \kappa \to V[G] \text{ such that } g = j_1^*(q)(\kappa).$$

Then

$$x = j_{1,2}^*(j_1^*(q)(\kappa))(\kappa_1) = j_2^*(q)(\kappa)(\kappa_1).$$

We can assume that $q(\beta) : \kappa \to V[G]$ for every $\beta < \kappa$. Define $s : [\kappa]^2 \to V[G]$ by setting $s(\alpha, \beta) = q(\alpha)(\beta)$. Hence,

$$\varphi([s]_W) = j_2^*(s)(\kappa, \kappa_1) = j_2^*(q)(\kappa)(\kappa_1) = x.$$

In particular,
$[id]_W = k([id]_W) = j_2^*(id)(\kappa, \kappa_1) = (\kappa, \kappa_1)$.
□

Let us define, for every $\alpha < \kappa^+$,

$$A_\alpha := \{(\gamma, \beta) \in [\kappa]^2 | f_{\kappa,\alpha}(\beta) \text{ is odd}\}.$$

We changed the Cohen functions over $\kappa_2$ in such a way that these sets will be in $W$.

**Lemma 4.2** $\{A_\alpha \mid \alpha < \kappa^+\} \subseteq W$ *and this sequence witnesses* $\neg Gal(\kappa, W, \kappa^+)$.

*Proof.* The proof repeats the corresponding argument from [5].
□

# 5 Gal$(\kappa, W, \kappa^{++})$

Let $\{B_\alpha \mid \alpha < \kappa^{++}\} \subseteq W$.
Work in $V[G_{<\kappa}]$. Choose a nice name $\underset{\sim}{B}'_\alpha$ for $B_\alpha, \alpha < \kappa^{++}$.

$$\underset{\sim}{B}'_\alpha = \{\{\xi\} \times A^\alpha_\xi | \xi < \kappa\},$$

where each $A^\alpha_\xi$ is a maximal antichain in the Cohen forcing $(Cohen(\kappa, \kappa^{++}))$ which consists of conditions deciding whether $\xi$ is in $B_\alpha$. Note that $|A^\alpha_\xi| \leqslant \kappa$ since the forcing satisfies $\kappa^+$-c.c..
Set

$$a_\alpha = \bigcup_{\xi<\kappa} \bigcup_{p \in A^\alpha_\xi} \{\mu < \kappa^{++} \mid \exists \nu < \kappa \quad \langle \nu, \mu \rangle \in p\}.$$

Then $|a_\alpha| \leqslant \kappa$. Now we use the $\Delta$-system lemma to get a subset $S$ of $\kappa^{++}$ of size $\kappa^{++}$ and $a \subseteq \kappa^{++}$ such that for every $\alpha \neq \beta, \alpha, \beta \in S,\ a_\alpha \cap a_\beta = a$.
Note that

$$\forall \alpha, \beta \in S,\ \text{if } \alpha \neq \beta \text{ then } (a_\alpha \cap a_\beta \cap \kappa^+) = a \cap \kappa^+$$

and there are only $\kappa^+$ options for $a_\alpha \cap \kappa^+$. Hence, by shrinking if necessary, we can assume that for every $\alpha \in S, a_\alpha \cap \kappa^+ = a \cap \kappa^+$.

Let us turn to $V$. Let $\underset{\sim}{B}_\alpha$ be a name in $V$ such that $(\underset{\sim}{B}_\alpha)_{G_{<\kappa}} = \underset{\sim}{B}'_\alpha$ and $(\underset{\sim}{B}_\alpha)_G = B_\alpha$. Now, for every $\alpha < \kappa^{++}$, we would like to pick conditions which force in $M_2$ that $(\check{\kappa}, \check{\kappa_1}) \in j_2(\underset{\sim}{B}_\alpha)$.

Recall that $W$ was defined from

$$j_2' : V[G] \to M_2' = M_2[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle * G_{>\kappa_1} * \langle f'_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^{++})\rangle].$$

$U^* \times U^*$ gives the embedding

$$j_2^* : V[G] \to M_2' = M_2[G * G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^{++})\rangle * G_{>\kappa_1} * \langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^{++})\rangle].$$

Also, $G_{>\kappa} * \langle f_{\kappa_1,\alpha} \mid \alpha < j_E(\kappa^+)\rangle$ was generated by $\langle k(p_\xi) \mid \xi < \kappa^+\rangle$ and $G_{>\kappa_1} * \langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^+)\rangle$ was generated by $\langle k_2(j_{j_1(E_\kappa)}(k(p_\xi))) \mid \xi < \kappa^+\rangle$. Let us further denote the first sequence by $\langle p^1_\xi \mid \xi < \kappa^+\rangle$ and the second by $\langle p^2_\xi \mid \xi < \kappa^+\rangle$.

Recall that

$$\langle f_{\kappa_1,\alpha} \mid j_2(\kappa^+) \leqslant \alpha < j_2(\kappa^{++})\rangle = \langle f'_{\kappa_2,\alpha} \mid j_2(\kappa^+) \leqslant \alpha < j_2(\kappa^{++})\rangle.$$

In addition, $\langle f'_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^+)\rangle$ was obtained from $\langle f_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^+)\rangle$ by changing values at $\kappa_1$ of some of the functions, namely of those with indexes in $j_2{}''\kappa^+$. Denote the sequence of relevant parts of $\langle p^2_\xi \mid \xi < \kappa^+\rangle$, i.e. $p^2_\xi \restriction Cohen(\kappa_2, \kappa_2^+)$, by $\langle r_\xi \mid \xi < \kappa^+\rangle$, and let $\langle r'_\xi \mid \xi < \kappa^+\rangle$ be the corresponding sequence for $\langle f'_{\kappa_2,\alpha} \mid \alpha < j_2(\kappa^+)\rangle$.

For every $\alpha < \kappa^{++}$ we pick $\xi^\alpha < \kappa^+$, $p^\alpha_0 \in G_{<\kappa} * \langle f_{\kappa,\gamma} \mid \gamma < \kappa^+\rangle$, $q^\alpha_0 \in \langle f_{\kappa,\gamma} \mid \kappa^+ \leqslant \gamma < \kappa^{++}\rangle$, $q^\alpha_1$ in the generic set for $\langle f_{\kappa_1,\beta} \mid j_E(\kappa^+) \leqslant \beta < j_E(\kappa^{++})\rangle$, $q^\alpha_2$ in the generic set for $\langle f'_{\kappa_2,\gamma} \mid j_2(\kappa^+) \leqslant \gamma < j_2(\kappa^{++})\rangle = \langle f_{\kappa_2,\gamma} \mid j_2(\kappa^+) \leqslant \gamma < j_2(\kappa^{++})\rangle$ such that

$$p^\alpha_0{}^\frown q^\alpha_0{}^\frown p^1_{\xi^\alpha}{}^\frown q^\alpha_1{}^\frown p^2_{\xi^\alpha} \restriction \kappa_2{}^\frown r'_{\xi^\alpha}{}^\frown q^\alpha_2 \Vdash (\check{\kappa}, \check{\kappa_1}) \in j_2(\underset{\sim}{B}_\alpha).$$

By shrinking if necessary, we can assume that all $\xi^\alpha$'s the same. In order to simplify the notation, denote then $p^1_{\xi^\alpha}$ by $p^1$, $p^2_{\xi^\alpha} \restriction \kappa_2$ by $p^2$ and $r'_{\xi^\alpha}$ by $r$. Also, the number of possibilities for $p^\alpha_0$ is at most $\kappa^+$, hence we can assume that all of $p^\alpha_0$'s are the same $p^0$.

Back in $V$, let us work with names $\underset{\sim}{p}^0, \underset{\sim}{q}^\alpha_0, \underset{\sim}{p}^1, \underset{\sim}{q}^\alpha_1, \underset{\sim}{p}^2, \underset{\sim}{r}, \underset{\sim}{q}^\alpha_2$ of conditions $p^0, q^\alpha_0, p^1, q^\alpha_1, p^2, r, q^\alpha_2$. Let $h_{p^0}, h_{q^\alpha_0}, h_{p^1}, h_{q^\alpha_1}, h_{p^2}, h_r, h_{q^\alpha_2}$ be functions which represent them in $M_2 = M_{E\times E}$.

Then,

$$j_2(h_{p^0})(\kappa) = \underset{\sim}{p}^0, j_2(h_{p^1})(\kappa) = \underset{\sim}{p}^1, j_2(h_{p^2})(\kappa, \kappa_1) = \underset{\sim}{p}^2, j_2(h_r)(\kappa, \kappa_1) = \underset{\sim}{r}.$$

In addition, there is a generator $\rho^\alpha < \kappa^{++}$ of $E$ such that

$$j_E(h_{q_0^\alpha})(\rho^\alpha) = \underset{\sim}{q}{}_0^\alpha, j_E(h_{q_1^\alpha})(\rho^\alpha) = \underset{\sim}{q}{}_1^\alpha \text{ and } j_2(h_{q_2^\alpha})(\rho^\alpha, j_E(\rho_\alpha)) = \underset{\sim}{q}{}_2^\alpha.$$

Set

$$E_{\rho_\alpha} = \{X \subseteq \kappa \mid \rho_\alpha \in j_E(X)\}.$$

Let $\pi : \kappa \to \kappa$ be the function defined by setting $\pi(\nu) =$ the largest inaccessible $\leqslant \nu$, if exists and 0 otherwise.
Then $\pi$ projects $E_{\rho_\alpha}$ on $E_\kappa$.
So, $j_E(\pi)(\rho_\alpha) = \kappa$ and it follows that $j_2(\pi)(j_1(\rho_\alpha)) = \kappa_1$. However, $\pi$ is not one-to-one, unless $\rho_\alpha = \kappa$. Recall that $j_{U^*}(f_{\kappa,\kappa^++\rho_\alpha}) = f_{\kappa_1,j_E(\kappa^++\rho_\alpha)}$ and $f_{\kappa_1,j_E(\kappa^++\rho_\alpha)}(\kappa) = \rho_\alpha$ and $U^*$ is normal. Hence, there is a set $D_\alpha$ in $U^*$ on which the function $f_{\kappa,\kappa^++\rho_\alpha}$ is one-to-one. So, $f^{-1}_{\kappa,\kappa^++\rho_\alpha}$ can be used as one-to-one projection in $V[G]$ at least on the set $f_{\kappa,\kappa^++\rho_\alpha}{}''D_\alpha$ as a projection of $\{X \subseteq \kappa \mid \rho_\alpha \in j_{U^*}(X)\}$ to $U^*$.

For every $\alpha < \kappa^{++}$ consider, in $V$, the following set

$$A_\alpha = \{(\nu_1, \nu_2) \in [\kappa]^2 \mid h_{p^0}(\pi(\nu_1))^\frown h_{q_0^\alpha}(\nu_1)^\frown h_{p^1}(\pi(\nu_1))^\frown h_{q_1^\alpha}(\nu_1)^\frown$$
$$h_{p^2}(\pi(\nu_1), \pi(\nu_2))^\frown h_r(\pi(\nu_1), \pi(\nu_2))^\frown h_{q_2^\alpha}(\nu_1, \nu_2) \Vdash (\pi(\check{\nu}_1), \pi(\check{\nu}_2)) \in \underset{\sim}{B}_\alpha\}.$$

By elementarity, $A_\alpha \in E_{\rho_\alpha} \times E_{\rho_\alpha}$.

Now, in $V[G]$, let

$$A_{p^0} = \{(\nu, \mu) \in [\kappa]^2 \mid h_{p^0}(\nu) \in G_{<\nu} * \langle f_{\nu,\gamma} \mid \gamma < \nu^+\rangle\},$$

$$A_{q_0^\alpha} = \{(\nu_1, \nu_2) \in [\kappa]^2 \mid h_{q_0^\alpha}(\nu_1) \in \langle f_{\pi(\nu_1),\gamma} \mid \pi(\nu_1)^+ \leqslant \gamma < \pi(\nu_1)^{++}\rangle\},$$

$$A_{p^1} = \{(\nu, \mu) \in [\kappa]^2 \mid h_{p^1}(\nu) \in G_{(\nu,\mu)} * \langle f_{\mu\gamma} \mid \gamma < \mu^+\rangle\},$$

$$A_{q_1^\alpha} = \{(\nu_1, \nu_2) \in [\kappa]^2 \mid h_{q_1^\alpha}(\nu_1) \in \langle f_{\pi(\nu_2),\gamma} \mid \pi(\nu_2)^+ \leqslant \gamma < \pi(\nu_2)^{++}\rangle\},$$

$$A_{p^2} = \{(\nu, \mu) \in [\kappa]^2 \mid h_{p^1}(\nu, \mu) \in G_{(\mu,\kappa)}\},$$

$$A_r = \{(\nu, \mu) \in [\kappa]^2 \mid h_r(\nu, \mu) \in \langle f_{\kappa,\gamma} \mid \gamma < \kappa^+\rangle\},$$

$$A_{q_2^\alpha} = \{(\nu_1,\nu_2) \in [\kappa]^2 \mid h_{q_2^\alpha}(\nu_1,\nu_2) \in \langle f_{\kappa,\gamma} \mid \kappa^+ \leqslant \gamma < \kappa^{++}\rangle\}.$$

Actually, all the sets above but $A_r$ are in $V[G_{<\kappa} * \langle f_{\kappa,\gamma} \mid \kappa^+ \leqslant \gamma < \kappa^{++}\rangle]$.

It follows from the definitions of the sets involved that

$$B_\alpha \supseteq \pi''(A_\alpha \cap A_{q_0^\alpha} \cap A_{q_1^\alpha} \cap A_{q_2^\alpha}) \cap (A_{p^0} \cap A_{p^1} \cap A_{p^2} \cap A_r) =$$
$$\{(\pi(\nu_1),\pi(\nu_2)) \in A_{p^0} \cap A_{p^1} \cap A_{p^2} \cap A_r \mid (\nu_1,\nu_2) \in A_\alpha \cap A_{q_0^\alpha} \cap A_{q_1^\alpha} \cap A_{q_2^\alpha}\}.$$

Also, $A_{p^0}, A_{p^1}, A_{p^2}$ and $A_r$ are in $W$.

Denote $\langle f_{\kappa,\gamma} \mid \kappa^+ \leqslant \gamma < \kappa^{++}\rangle$ by $R$.

Let us make first the following simple observation:

**Lemma 5.1** *For every $\alpha < \kappa^{++}$, there is $X_{A_\alpha} \in E_\kappa$ such that $[X_{A_\alpha}]^2 \subseteq \pi''A_\alpha$.*

*Proof.* We have $A_\alpha \in E_{\rho_\alpha} \times E_{\rho_\alpha}$. Then,

$$A'_\alpha = \pi''A_\alpha = \{(\pi(\nu_1),\pi(\nu_2)) \mid (\nu_1,\nu_2) \in A_\alpha\} \in E_\kappa \times E_\kappa.$$

The normality of $E_\kappa$ implies the conclusion.

□

Let us show the following:

**Lemma 5.2** *Suppose that there is a sequence*

$$\langle X_{A_{q_0^\alpha}}, X_{A_{q_1^\alpha}}, X_{A_{q_2^\alpha}} \mid \alpha < \kappa^{++}\rangle$$

*in $V[G_{<\kappa} * R]$ which consists of sets in $U^*$ such that*

$$[X_{A_{q_0^\alpha}}]^2 \subseteq \pi''A_{q_0^\alpha},\ [X_{A_{q_1^\alpha}}]^2 \subseteq \pi''A_{q_1^\alpha},\ [X_{A_{q_2^\alpha}}]^2 \subseteq \pi''A_{q_2^\alpha},$$

*for every $\alpha < \kappa^{++}$.*
*Then there is some $I \subseteq \kappa^{++}$ such that*

$$|I| = \kappa \text{ and } \bigcap_{\alpha \in I} B_\alpha \in W.$$

*Proof.* Let

$$X_\alpha := X_{A_\alpha} \cap X_{A_{t^\alpha}} \cap X_{A_{m^\alpha}} \cap X_{A_{s^\alpha}} \cap D_\alpha \in U^* \cap V[G * R],$$

where $X_{A_\alpha}$ is as in Lemma 5.1 and $D_\alpha$ is a set picked to ensure that the projection is one-to-one.

**Claim 1** For every $X \in U^* \cap V[G * R]$, $[X]^2 \in W$.

*Proof.* Denote the embedding $j_2' \restriction V[G * R]$ by $\tilde{j}_2$. Then

$$\tilde{j}_2 : V[G * R] \to M_2[G_{\kappa_2} * R_2] \text{ where } R_2 = j_2^*(R) = \langle f_{\kappa_2,\gamma} \mid j_2(\kappa^+) \leqslant \gamma < j_2(\kappa^{++})\rangle.$$

Note that $j_2^* \restriction V[G * R] = \tilde{j}_2$, since

$$j_2^*(R) = \langle f_{\kappa_2,\gamma} \mid j_2(\kappa^+) \leqslant \gamma < j_2(\kappa^{++})\rangle = R_2.$$

Then it follows that

$$\forall X \in V[G * R],\ \tilde{j}_2(X) = j_2^*(X).$$

Now, if $X \in U^* \cap V[G * R]$, then $\kappa \in j_1^*(X)$ which implies that $(\kappa, \kappa_1) \in j_2^*([X]^2)$. So, $(\kappa, \kappa_1) \in \tilde{j}_2([X]^2) = j_2'([X]^2) = j_W([X]^2)$. Hence, $[X]^2 \in W$.
□ of the claim.

The original proof of Galvin will provide

$$I \subseteq \kappa^+ \text{ such that } |I| = \kappa \text{ and } \bigcap_{\alpha \in I} X_\alpha \in U^*.$$

The proof is done inside $V[G * R]$. In particular, $I$ and $\bigcap_{\alpha \in I} X_\alpha$ will be in $V[G * R]$. Then, by the claim above, $[\bigcap_{\alpha \in I} X_\alpha]^2 \in W$. So, we will obtain the following:

$$\begin{aligned}
\bigcap\nolimits_{\alpha \in I} B_\alpha &\supseteq \bigcap_{\alpha \in I} (\pi''(A_\alpha \cap A_{q_0^\alpha} \cap A_{q_1^\alpha} \cap A_{q_2^\alpha})) \cap (A_{p^0} \cap A_{p^1} \cap A_{p^2} \cap A_r) \\
&\supseteq \bigcap\nolimits_{\alpha \in I} [X_\alpha]^2 \cap (A_{p^0} \cap A_{p^1} \cap A_{p^2} \cap A_r) \\
&= [\bigcap\nolimits_{\alpha \in I} X_\alpha]^2 \cap (A_{p^0} \cap A_{p^1} \cap A_{p^2} \cap A_r) \in W
\end{aligned}$$

For the reader convenience, let us go through the Galvin proof in order to see that we can stay within $V[G * R]$. For every $\alpha < \kappa^+,\ \xi < \kappa$, let

$$H_{\alpha,\xi} = \{\beta < \kappa^+ | X_\alpha \cap \xi = X_\beta \cap \xi\}.$$

Since we have that $\langle X_\alpha | \alpha < \kappa^+\rangle \in V[G * R]$, clearly $H_{\alpha,\xi} \in V[G * R]$.

**Claim 2** There is some $\alpha^* < \kappa^+$ s.t. for every $\xi < \kappa$, $|H_{\alpha^*,\xi}| = \kappa^+$.

*Proof.* Assume otherwise. Then for every $\alpha < \kappa^+$ let $\xi_\alpha < \kappa$ s.t. $|H_{\alpha,\xi_\alpha}| < \kappa^+$. By the pigeonhole principle, there is some $\xi^* < \kappa$ and a set $A \subseteq \kappa^+$ with $|A| = \kappa^+$ s.t. $\forall \alpha \in A$,

$\xi_\alpha = \xi^*$. But since $\kappa$ is measurable, $2^{\xi^*} < \kappa$ so there is some $A' \subseteq A$ of size $\kappa^+$ and $C \subseteq \xi^*$ s.t. $\forall \alpha \in A'$, $X_\alpha \cap \xi^* = C$. And this is a contradiction since if $\alpha \in A'$ then $A' \subseteq H_{\alpha,\xi^*}$ and $|A'| = \kappa^+ > |H_{\alpha,\xi^*}|$.

□

Let $\alpha^* < \kappa^+$ as in the claim. So we can define an increasing sequence $\langle \alpha_i | i < \kappa \rangle \in V[G*R]$ s.t. $\alpha_i \in H_{\alpha^*,i+1}$.

**Claim 3** $X_{\alpha^*} \cap \underset{i<\kappa}{\triangle} X_{\alpha_i} \subseteq \bigcap_{i<\kappa} X_{\alpha_i}$.

*Proof.* Let

$$\beta \in X_{\alpha^*} \cap \underset{i<\kappa}{\triangle} X_{\alpha_i} \text{ and } j < \kappa.$$

If $j < \beta$, then $\beta \in \underset{i<\kappa}{\triangle} X_{\alpha_i}$ and thus $\beta \in X_{\alpha_j}$. If $\beta \leqslant j$, then $\beta \in X^*_\alpha$ and $X_{\alpha^*} \cap (j+1) = X_{\alpha_j} \cap (j+1)$ which means that $\beta \in X_{\alpha_j}$.

□

It is clear from the proof above that $\bigcap_{i<\kappa} X_{\alpha_i} \in V[G*R]$, and we get that $\bigcap_{i<\kappa} B_{\alpha_i} \in W$ as needed.

□

## 5.1 Relevant filters

Instead of dealing with particular sets, let us present a more general approach based on filters.

Recall that $R = \langle f_{\kappa,\gamma} \mid \kappa^+ \leqslant \gamma < \kappa^{++} \rangle$, i.e. the set of generic over $V[G_{<\kappa}]$ Cohen functions with indexes in the interval $[\kappa^+, \kappa^{++})$.

We have master condition sequences $\langle p^1_\xi \mid \xi < \kappa^+ \rangle$ for $P_{(\kappa,\kappa_1)} * Cohen(\kappa_1, \kappa_1^+)$ of $M_1$ and $\langle p^2_\xi \mid \xi < \kappa^+ \rangle$ for $P_{(\kappa,\kappa_1)} * Cohen(\kappa_1, \kappa_1^+)$, $P_{(\kappa_1,\kappa_2)} * Cohen(\kappa_2, \kappa_2^+)$ of $M_2$, inside $V[G]$.

Let us replace the parts which require Cohens from $\langle f_{\kappa,\gamma} \mid \gamma < \kappa^+ \rangle$ by their names. In addition, the method of [3] was used in order to generate Cohen functions $\langle f_{\kappa_1,\gamma} \mid j_E(\kappa^+) \leqslant \gamma < j_E(\kappa^{++}) \rangle$ (and, then $\langle f_{\kappa_2,\gamma} \mid j_2(\kappa^+) \leqslant \gamma < j_2(\kappa^{++}) \rangle$). So, there is a master condition sequence $\langle \underset{\sim}{t}^1_\xi \mid \xi < \kappa^+ \rangle$ for $P_{(\kappa,\kappa_1)} * Cohen(\kappa_1, [\kappa_1^+, \kappa_1^{++}))$ of $M_1$ inside $V[G_{<\kappa} * R]$, where parts which rely on $\langle f_{\kappa,\gamma} \mid \gamma < \kappa^+ \rangle$ are names. Similarly, there is a master condition sequence $\langle \underset{\sim}{t}^2_\xi \mid \xi < \kappa^+ \rangle$ for $P_{(\kappa,\kappa_1)} * Cohen(\kappa_1, [\kappa_1^+, \kappa_1^{++})) * P_{(\kappa_1,\kappa_2)} * Cohen(\kappa_2, [\kappa_2^+, \kappa_2^{++}))$ of $M_2$ inside $V[G_{<\kappa} * R]$, where parts which rely on $\langle f_{\kappa,\gamma} \mid \gamma < \kappa^+ \rangle$ are names. Note that the part of this sequence for $P_{(\kappa_1,\kappa_2)} * Cohen(\kappa_2, [\kappa_2^+, \kappa_2^{++}))$ is build as $\langle \underset{\sim}{t}^1_\xi \mid \xi < \kappa^+ \rangle$ but only replacing $V$ by $M_1$. By elementarity, we can use $\langle j_{12}(t^1_\xi) \mid \xi < \kappa^+\rangle$ for it, and so to use $t^2_\xi = t^1_\xi{}^\frown j_{12}(t^1_\xi), \xi < \kappa^+$, where $j_{12} : M_1 \to M_2$ is the embedding by $j_E(E)$.

Work in $V[G_{<\kappa}, R]$ and define $U' \subseteq \mathcal{P}(\kappa)$ as follows:

$$X \in U' \text{ iff } \exists p \in G_{<\kappa} * R \quad \exists \xi < \kappa^+ \quad M_1 \models (p^\frown 0_{Cohen(\kappa,\kappa^+)}{}^\frown \underset{\sim}{t}{}^1_\xi \Vdash_{j_1(P_{\kappa+1})} \kappa \in j_1(\underset{\sim}{X})).$$

**Lemma 5.3** *$U'$ is a normal filter over $\kappa$ in $V[G, R]$.*

*Proof.* Clearly, $U'$ is a $\kappa-$complete filter in $V[G, R]$. Let us argue that it is normal. Thus let $\{X_\alpha \mid \alpha < \kappa\} \subseteq U'$ and $X = \Delta_{\alpha<\kappa} X_\alpha$. For every $\alpha < \kappa$ pick $p^\alpha \in G * R$ and $\xi^\alpha < \kappa^+$ witnessing that $X_\alpha \in U'$. Let $\xi^* = \sup_{\alpha<\kappa} \xi^\alpha$. Suppose that there is no $p \in G * R$ such that $p^\frown 0_{Cohen(\kappa,\kappa^+)}{}^\frown \underset{\sim}{t}{}^1_{\xi^*} \Vdash \kappa \in j_1(\underset{\sim}{X})$. Then there are $\beta < \kappa$, $q \in G$ and $\xi \geqslant \xi^*$ such that

$$q^\frown t^1_\xi \Vdash \kappa \notin j_1(\underset{\sim}{X}_\beta).$$

But this is impossible since $q$ is compatible with $p^\beta$ and $t^1_\xi \geqslant t^1_{\xi^\beta}$.
□

The next lemma follows from the definitions.

**Lemma 5.4** *$U' \subseteq U^* \cap V[G_{<\kappa}, R]$.*

**Remark 5.5** *Note that since $U^*$ is an ultrafilter in the full generic extension, $U^* \cap V[G_{<\kappa}, R]$ is an ultrafilter over $V[G, R]$. However, $U^* \cap V[G, R] \notin V[G, R]$, since otherwise each Cohen function $f_{\kappa,\xi}$ will be in $V[G_{<\kappa}, R]$ as well. Namely, for every $\xi < \kappa^+$, $\tau, \rho < \kappa$,*

$$f_{\kappa\xi}(\tau) = \rho \textit{ iff } \{\nu < \kappa \mid f_{\nu h_\xi(\nu)}(\tau) = \rho\} \in U^*,$$

*where $h_\xi \in V$ is the canonical function which represents $\xi$.*

Let us define now a two dimensional version of $\tilde{U}$ of $U'$. Let $X \in V[G_{<\kappa} * R]$ be subset of $[\kappa]^2$. Set

$$X \in \tilde{U} \text{ iff } \exists p \in G_{<\kappa} * R \quad \exists \xi < \kappa^+ \quad M_2 \models$$

$$(p^\frown 0_{Cohen(\kappa,\kappa^+)}{}^\frown \underset{\sim}{t}{}^1_\xi{}^\frown 0_{Cohen(\kappa_1,\kappa_1^+)}{}^\frown j_{12}(\underset{\sim}{t}{}^1_\xi) \Vdash (\kappa, \kappa_1) \in j_2(\underset{\sim}{X})).$$

We have the following analog of 5.4:

**Lemma 5.6** *$\tilde{U} \subseteq (U^*)^2 \cap V[G, R]$.*

Let us denote $(A)_\nu = \{\rho < \kappa \mid (\nu, \rho) \in A\}$, where $A \subseteq [\kappa]^2$ and $\nu < \kappa$.

Let $A \in \tilde{U}$. Then there are $p \in G * R$ and $\xi < \kappa^+$ such that, in $M_2$,

$$(p^\frown 0_{Cohen(\kappa,\kappa^+)}{}^\frown \underset{\sim}{t}{}^1_\xi{}^\frown 0_{Cohen(\kappa_1,\kappa_1^+)}{}^\frown j_{12}(\underset{\sim}{t}{}^1_\xi) \Vdash (\kappa, \kappa_1) \in j_2(\underset{\sim}{A})).$$

Recall that $j_2 = j_{12} \circ j_1$.

Then, in $M_2$,

$$(p^\frown 0_{Cohen(\kappa,\kappa^+)}{}^\frown \underset{\sim}{t}{}^1_\xi{}^\frown 0_{Cohen(\kappa_1,\kappa_1^+)}{}^\frown j_{12}(\underset{\sim}{t}{}^1_\xi) \Vdash \kappa_1 \in j_{12}(j_1(\underset{\sim}{A})_\kappa).$$

Let $h_{t^1_\xi}$ be a function that represents $t^1_\xi$ in $M_1$, i.e. $j_1(h_{t^1_\xi})(\kappa) = t^1_\xi$.

Consider the set

$$X = \{\nu < \kappa \mid \text{in } M_1, p^\frown 0_{Cohen(\nu,\nu^+)}{}^\frown h_{t^1_\xi}(\nu)^\frown 0_{Cohen(\kappa,\kappa^+)}{}^\frown t^1_\xi \Vdash \kappa \in j_1((\underset{\sim}{A})_\nu)\}.$$

Note that for every $\nu < \kappa$, if $h_{t^1_\xi}(\nu) \in G_{<\kappa} * R$, then $(A)_\nu \in U'$.

Also, the set $\{\nu < \kappa \mid h_{t^1_\xi}(\nu) \in G_{<\kappa} * R\}$ is in $U'$, as witnessed by $t^1_\xi$.

Hence,

$$Y = \{\nu < \kappa \mid (A)_\nu \in U'\} \in U'.$$

Set $B = Y \cap \Delta_{\nu \in Y}(A)_\nu$. By normality, $B \in U'$.

**Lemma 5.7** *$[B]^2 \subseteq A$.*

*Proof.* Let $(\nu, \rho) \in [B]^2$. Then $\nu < \rho$, $\nu, \rho \in Y$, so, $\rho \in (A)_\nu$. Hence, $(\nu, \rho) \in A$.
□

It follows now:

**Lemma 5.8** *$(U')^2 = \tilde{U}$.*

*Proof.* By 5.7, $(U')^2 \supseteq \tilde{U}$. The opposite inclusion follows from the definitions of $U'$ and $\tilde{U}$.
□

Finally recall that by 5.6, $\tilde{U} \subseteq (U^*)^2 \cap V[G_{<\kappa}, R]$. Hence,

$$(U')^2 = \tilde{U} \subseteq (U^*)^2 \cap V[G_{<\kappa}, R] = W \cap V[G_{<\kappa}, R].$$

So we can run the Galvin argument with the normal filter $U'$ inside $V[G_{<\kappa}, R]$ and it will give the desired conclusion for $W$, by Lemma 5.2.

# 6 Some possible generalizations

We showed above that it is consistent to have a $\kappa$-complete ultrafilter which satisfies the Galvin property at $\kappa^{++}$ but not at $\kappa^+$. The same method, with minor changes, can be applied to cardinals above $\kappa^{++}$. Namely, building on [9] instead of [3] for higher cardinals, the following holds:

**Theorem 6.1** *Assume GCH. Let $\lambda \geqslant \kappa$ and suppose that $\kappa$ carries a $(\kappa, \lambda^+)-$extender. Then there is a cofinality preserving forcing extension $V^*$ such that $V^* \models 2^\kappa = \lambda^+$, and, in $V^*$, there is a $\kappa$-complete ultrafilter $W$ over $\kappa$ s.t. $W$ does not satisfy Gal($\kappa, W, \lambda$), but satisfies Gal($\kappa, W, \lambda^+$).*

The above however does not cover all the possibilities. Namely, can the first stage where the Galvin property holds be a limit cardinal? It turns out that not everything is possible.

**Proposition 6.2** *Let $\kappa$ be a measurable cardinal, $\lambda, \kappa < \lambda \leqslant 2^\kappa$, be a limit cardinal with* $\mathrm{cof}(\lambda) < \kappa$. *Suppose that $\neg$Gal($\kappa, W, \mu$) holds for every cardinal $\mu, \kappa \leqslant \mu < \lambda$. Then $\neg$Gal($\kappa, W, \lambda$).*

*Proof.* Suppose otherwise that $\mathrm{Gal}(\kappa, W, \lambda)$ holds.
For every cardinal $\mu, \kappa \leqslant \mu < \lambda$, let $\langle A^\mu_i | i < \mu\rangle$ witnesses $\neg\mathrm{Gal}(\kappa, W, \mu)$. Let $\langle \mu_\xi \mid \xi < \mathrm{cof}(\lambda)\rangle$ be a cofinal in $\lambda$ sequence which consists of cardinals above $\kappa$.
Consider a family

$$\{A^{\mu_\xi}_i \mid \xi < \mathrm{cof}(\lambda), i < \mu_\xi\}.$$

It has cardinality $\lambda$, hence there are $\kappa-$many sets in the family with intersection in $W$. But then, due to the fact that $\mathrm{cof}(\lambda) < \kappa$, there is some $\xi^* < \mathrm{cof}(\lambda)$ such that $\kappa-$many of them are from $\langle A^{\mu_{\xi^*}}_i \mid i < \mu\rangle$. Then, their intersection is not in $W$. Contradiction.
□

**Proposition 6.3** *Let $\kappa$ be a measurable cardinal, $\lambda, \kappa < \lambda \leqslant 2^\kappa$, be a limit cardinal with* $\mathrm{cof}(\lambda) = \kappa$. *Suppose that $\neg$Gal($\kappa, W, \mu$) holds for every cardinal $\mu, \kappa \leqslant \mu < \lambda$. Then $\neg$Gal($\kappa, W, \lambda$).*

*Proof.* Suppose otherwise that $\mathrm{Gal}(\kappa, W, \lambda)$ holds.
For every cardinal $\mu, \kappa \leqslant \mu < \lambda$, let $\langle A^\mu_i | i < \mu\rangle$ witnesses $\neg\mathrm{Gal}(\kappa, W, \mu)$. Let $\langle \mu_\xi \mid \xi < \kappa =$

$\mathrm{cof}(\lambda)\rangle$ be a cofinal in $\lambda$ sequence which consists of cardinals above $\kappa$.
Consider a family

$$\langle B_i^{\mu_\xi} \mid \xi < \kappa, i < \mu_\xi\rangle \text{ where } B_i^{\mu_\xi} = A_i^{\mu_\xi}\backslash\xi.$$

Notice that $\langle B_i^{\mu_\xi} \mid i < \mu_\xi\rangle$ still witnesses $\neg\mathrm{Gal}(\kappa, W, \mu_\xi)$.
The family $\langle B_i^{\mu_\xi}|\xi < \kappa, i < \mu_\xi\rangle$ has size $\lambda$. Hence, there is

$$F \subseteq \{B_i^{\mu_\xi}|\xi < \kappa, i < \mu_\xi\},\ |F| = \kappa$$

with $\bigcap F \in W$.
If there exists $\xi^* < \kappa$ such that $\kappa$ many sets from $F$ belong to $\{B_i^{\mu_{\xi^*}}|i < \mu_{\xi^*}\}$, then we will get a contradiction to the fact that $\langle B_i^{\mu_{\xi^*}}|i < \mu_{j^*}\rangle$ witnesses $\neg\mathrm{Gal}(\kappa, W, \mu_{\xi^*})$.
Suppose that there is no such $\xi^*$. Then there will be an increasing sequence $\langle\xi_\delta \mid \delta < \kappa\rangle$ and a sequence $\langle i(\delta) \mid \delta < \kappa\rangle$ such that

$$\{B_{i(\delta)}^{\mu_{\xi_\delta}} \mid \delta < \kappa\} \subseteq F.$$

By $\mathrm{Gal}(\kappa, W, \lambda)$,

$$\bigcap\{B_{i(\delta)}^{\mu_{\xi_\delta}} \mid \delta < \kappa\} \in W.$$

However, $B_{i(\delta)}^{\mu_{\xi_\delta}} \cap \xi_\delta = \emptyset$. So, $\bigcap\{B_{i(\delta)}^{\mu_{\xi_\delta}} \mid \delta < \kappa\} \cap \xi_\delta = \emptyset$, for every $\delta < \kappa$. Hence, $\bigcap\{B_{i(\delta)}^{\mu_{\xi_\delta}} \mid \delta < \kappa\} = \emptyset$. Contradiction.
□

## 6.1 $\aleph_{\kappa^{++}}$ and beyond

We do not know whether $\aleph_{\kappa^+}$ can be a breaking point, i.e., wether it is possible to have a $\kappa-$complete ultrafilter $W$ over $\kappa$ such that

1. $2^\kappa \geqslant \aleph_{\kappa^+}$,
2. $\neg\mathrm{Gal}(\kappa, W, \mu)$, for every cardinal $\mu, \kappa \leqslant \mu < \aleph_{\kappa^+}$,
3. $\mathrm{Gal}(\kappa, W, \aleph_{\kappa^+})$.

However, it is possible at $\aleph_{\kappa^{++}}$ or at any $\lambda$ of cofinality $\geqslant \kappa^{++}$. Let us sketch the argument.

Let $\lambda = \aleph_{\kappa^{++}}$.

Proceed as in the construction for $\kappa^{++}$ above. We have Cohen functions $\langle f_{\kappa,\alpha} \mid \alpha < \lambda \rangle$. Let

$$A_\alpha = \{\nu < \kappa \mid f_{\kappa\alpha}(\nu) \text{ is odd } \}.$$

We define a normal ultrafilter $U^*$ over $\kappa$ as before. Then a $\kappa-$complete ultrafilter $W$ will be defined.

Let us arrange non-Galviness.

For every $\xi < \kappa^{++}$, we change values of functions $f_{\kappa_2,\gamma}(\kappa_1)$ for $\gamma$'s in the interval $(j_2(\kappa)^{+j_2(\xi)}, j_2(\kappa)^{+j_2(\xi+1)})$ if $\gamma$ has a pre-image under $k$ but not under $j_2$. This, by the usual argument, will insure $\neg\text{Gal}(\kappa, \kappa^{+\xi+1})$ using $\{A_\zeta \mid \kappa^{+\xi} < \zeta < \kappa^{+\xi+1}\}$.

Note, and this is crucial, that nothing is done in intervals of the form $(j_2(\kappa)^{+\tau}, j_2(\kappa)^{+\tau+1})$ with $\tau = \kappa$ or cofinality $\kappa$.

In particular, we cannot use $\{A_\zeta \mid \zeta < \kappa^{+\kappa}\}$ to witness $\neg\text{Gal}(\kappa, \kappa^{+\kappa})$. Just, for example, the sequence $\langle A_{\kappa^{+\mu}} \mid \mu < \kappa \rangle$ may be problematic, since no changes are done inside the interval $(j_2(\kappa)^{+\kappa}, j_2(\kappa)^{+j_2(\kappa)})$.

Turn now to $\text{Gal}(\kappa, W, \lambda)$, where $W$ is result of changes above.

Let $\{B_\rho \mid \rho < \lambda\} \subseteq W$. Denote by $b_\rho \subseteq \lambda, |b_\rho| \leqslant \kappa$ the support of $B_\rho$. Shrink to $\rho_\tau$'s with $\bigcup_{\tau<\kappa^{++}} b_{\rho_\tau}$ unbounded in $\lambda$. Now we form a $\Delta-$system with the union still unbounded in $\lambda$, by shrink the family if necessary.

The rest of the argument is similar to $\kappa^{++}$ case.

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