More on uniform ultrafilters over a singular cardinal.

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Abstract

We would like to show some additional results related to character of uniform ultrafilters over a singular cardinal and the ultrafilter number.

1 Some general observation.

Let us start with few simple well known observation:

Proposition 1.1 *Suppose that* U, W *are two ultrafilters and* $U \geq_{R-K} W$*.*
Then $\mathrm{ch}(U) \geq \mathrm{ch}(W)$*.*

Proof. Let π be a projection of U to W.
Let $\mathcal{U}$ be a generating family for U.
Then

$$\mathcal{W} = \{\pi'' A \mid A \in \mathcal{U}\}$$

will be a generating family for W.
□

The following follows:

Corollary 1.2 *Suppose that* U *is an ultrafilter over* μ*,* $W \leq_{R-K} U$ *and* $\mathrm{ch}(W) = 2^\mu$*.*
Then $\mathrm{ch}(U) = 2^\mu$*, as well.*

Proposition 1.3 *Suppose that* $U = F - \lim_{i \in I} U_i$ *for an ultrafilter* F *over* I *and ultrafilters* $U_i, i \in I$*.*
Suppose that $\langle U_i \mid i \in I\rangle$ *are* $F-$*discreate, i.e. there are* $X \in F$ *and disjoint sets* $\langle A_i \mid i \in X\rangle$

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such that $A_i \in U_i$, for every $i \in X$.
Assume that for almost every (mod F) $i \in I$, $U_i \geq_{R-K} W_i$. Let $W = F - \lim_{i \in I} W_i$.
Then $U \geq_{R-K} W$.

Proof. Let $X \in F$ and disjoint sets $\langle A_i \mid i \in X \rangle$ such that $A_i \in U_i$, for every $i \in X$.
Assume, in addition, that for every $i \in X$, $U_i \geq_{R-K} W_i$.
Set $A = \bigcup_{i \in X} A_i$. Then, clearly, $A \in U$.
For every $i \in X$, fix a projection π_i of U_i to W_i.
Set $\pi = \bigcup_{i \in X} \pi_i$.
Then π projects U to W.
$\square$

In sixties C. Chang and J. Keisler formulated the following notions:

Definition 1.4 Let U be an ultrafilter on a set I.

1. U is called (κ, λ) regular iff there is subset of U of cardinality λ such that any $\kappa-$members of it have empty intersection.

2. U is called $\lambda-$descendingly incomplete iff there are $\{X_\alpha \mid \alpha < \lambda\} \subseteq U$ such that $\alpha < \beta \rightarrow X_\alpha \supseteq X_\beta$ and $\bigcup_{\alpha < \lambda} X_\alpha = \emptyset$.

3. U is $\lambda-$decomposable iff there is a partition of I into disjoint sets $\langle I_\alpha \mid \alpha < \lambda \rangle$, so that whenever $S \subseteq \lambda$ and $|S| < \lambda$, $\bigcup_{\alpha \in S} I_\alpha \notin U$.

This subject was intensively investigated see for example [2],[9],[10],[11]. Let state some known propositions which are relevant for us here:

Proposition 1.5 *U is $\lambda-$decomposable, then U is $\lambda-$descendingly incomplete.*
If λ is regular, then the converse holds as well.

Proposition 1.6 *An ultrafilter U over I is $\lambda-$decomposable iff it Rudin-Keisler above a uniform ultrafilter over λ.*

Proposition 1.7 *If U is $(\kappa, \lambda)-$regular ultrafilter and ν is a regular cardinal so that $\kappa \leq \nu \leq \lambda$, then U is $\nu-$descendingly incomplete, and so, $\nu-$decompossible.*

Proof. Let $\{X_\alpha \mid \alpha < \lambda\} \subseteq U$ be a family such that the intersection of any $\kappa-$members of it is empty.

Set $Y_\gamma = \bigcup\{X_\alpha \mid \gamma \leq \alpha < \nu\}$.
Then each $Y_\gamma \in U$ and $\beta < \gamma < \nu \to Y_\beta \supseteq Y_\gamma$.
We have

$$\bigcap_{\gamma<\nu} Y_\gamma = \bigcap_{\gamma<\nu} \bigcup\{X_\alpha \mid \gamma \leq \alpha < \nu\} = \bigcup\{\bigcap_{\alpha<\nu} X_{f(\alpha)} \mid f : \nu \to \nu \text{ and } \forall \alpha < \nu(f(\alpha) \geq \alpha)\}.$$

The last union is the union of empty sets , by regularity of ν and $\kappa \leq \nu$.
Hence, $\bigcap_{\gamma<\nu} Y_\gamma = \emptyset$.
□

The following corollaries follows now:

Corollary 1.8 *Let U be a $(\kappa, \lambda)-$regular ultrafilter. Then for every regular $\nu, \kappa \leq \nu \leq \lambda$,* $\mathrm{ch}(U) \geq \mathfrak{u}_\nu$.

Corollary 1.9 *Let U be an ultrafilter over μ which is a $(\kappa, \lambda)-$regular.*
Suppose that for some regular $\nu, \kappa \leq \nu \leq \lambda$, $\mathfrak{u}_\nu = 2^\mu$.
Then $\mathrm{ch}(U) = 2^\mu$.

2 Strongly uniform ultrafilters.

Let us define some strengthening of uniformity of an ultrafilter over a singular cardinal.

Definition 2.1 Suppose that κ is a singular cardinal of cofinality η and D is a uniform ultrafilter over κ..

(a) Let $\vec{\tau} = \langle \tau_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of regular cardinals converging to κ.
Let F be an uniform ultrafilter over η.
D is called $(\vec{\tau}, F)-$*uniform* iff for every $A \in D$,
$\{\alpha < \eta \mid |A \cap \tau_\alpha| = \tau_\alpha\} \in F$.

(b) D is called *strongly uniform* iff D is $(\vec{\tau}, F)-$uniform for some $(\vec{\tau}, F)$, as in (a).

Define the corresponding ultrafilter numbers:

Definition 2.2 (a) Let $(\vec{\tau}, F)$ be as above.
$\mathfrak{u}(\kappa, \vec{\tau}, F) = \min(\{\mathrm{ch}(D) \mid D \text{ is } (\vec{\tau}, F) - \text{uniform }\})$.

(b) $\mathfrak{u}^{str}(\kappa) = \min(\{\mathrm{ch}(D) \mid D \text{ is strongly uniform ultrafilter over } \kappa\})$.

Clearly, $\mathfrak{u}(\kappa) \leq \mathfrak{u}^{str}(\kappa)$.

Proposition 2.3 *Suppose that κ is a singular cardinal of cofinality η. Let $\langle \kappa_\alpha \mid \alpha < \eta\rangle$ be an increasing sequence of cardinals converging to κ.*
Suppose that δ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$
2. *there is an increasing sequence of regular cardinals $\vec{\delta} = \langle \delta_\alpha \mid \alpha < \eta\rangle$ such that*
   (a) $\kappa_\alpha < \delta_\alpha \leq \kappa_{\alpha+1} < \delta_{\alpha+1}$*, for every $\alpha < \eta$,*
   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$*, for some ultrafilter F on η which extends the filter of co-bounded subsets of η,*

Let D be a $(\vec{\delta}, F)-$uniform ultrafilter over κ.
Then $\mathrm{ch}(D) \geq \delta$.

Proof. Let us argue that $\mathrm{ch}(D) \geq \delta$.
Suppose otherwise. Let $\mathcal{W}$ be a generating family for D of cardinality less than δ.

Let $\langle f_\xi \mid \xi < \delta\rangle$ be a scale witnessing $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$.
For every $\xi < \delta$ and $i < \eta$ set $A_{\xi i} = \delta_i \setminus f_\xi(i)$.
Let $A_\xi = \bigcup_{i<\eta} A_{\xi i}$.
Then, $A_\xi \in D$, since otherwise $B := \kappa \setminus A_\xi \in D$ and, so, by $(\vec{\delta}, F)-$uniformity, the set

$$X := \{i < \eta \mid |B \cap \delta_i| = \delta_i\} \in F.$$

But, each δ_i is a regular cardinal, hence, if $i \in X$, then $B \cap \delta_i$ is unbounded in δ_i. In particular, $(B \cap \delta_i) \cap A_{\xi i} \neq \emptyset$. Which is impossible, since B is a complement of $A_\xi \supseteq A_{\xi i}$.

We assumed that $|\mathcal{W}| < \delta$, so there is a single $A \in \mathcal{W}$ such that for $\delta-$many ξ's we have $A \subseteq^* A_\xi$.
Set $A_i = A \cap \delta_i$, for every $i < \eta$.
Without loss of generality, using $(\vec{\delta}, F)-$uniformity, we can assume that $|A_i| = \delta_i$, for every $i < \eta$. Define, for every $i < \eta$, ρ_i to be the κ_i-th element of A_i.
Then there is $\xi^* < \delta$ such that for every $\xi, \xi^* \leq \xi < \delta$, the set

$$\{i < \eta \mid f_\xi(i) > \rho_i\} \in F.$$

Now we pick any $\xi, \xi^* \leq \xi < \delta$ with $A \subseteq^* A\xi$. Then, for most (mod F) i's, $|A_i \setminus A_{\xi i}| \geq \kappa_i$. Hence, $|A \setminus A_\xi| = \kappa$, which is impossible.

Contradiction.

□

Let present an other condition that prevents the character of being too small.

Proposition 2.4 *Suppose that κ is a singular cardinal of cofinality η. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to κ.*
Suppose that δ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$
2. *there is an increasing sequences of regular cardinals $\vec{\tau} = \langle \tau_\alpha \mid \alpha < \eta \rangle$ such that*
   (a) $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, *for every $\alpha < \eta$,*
   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$, *where $\delta_\alpha = 2^{\tau_\alpha}$ and F is an ultrafilter on η which extends the filter of co-bounded subsets of η,*
   (c) $\mathfrak{r}(\tau_\alpha) = \delta_\alpha$ *(non-splitting number), i.e. whenever $S \subseteq [\tau_\alpha]^{\tau_\alpha}$ of cardinality $< \delta_\alpha$, then there is $a \in [\tau_\alpha]^{\tau_\alpha}$ such that for every $s \in S$, $|s \cap a| = |s \setminus a| = \tau_\alpha$. The meaning is that a splits s.*
   In particular, if $2^{\tau_\alpha} = \tau_\alpha^+$, then $\mathfrak{r}(\tau_\alpha) = \tau_\alpha^+ = \delta_\alpha$.

Let D be a $(\vec{\tau}, F)-$uniform ultrafilter over κ.
Then $\mathrm{ch}(D) \geq \delta$.

Proof. Let us argue that $\mathrm{ch}(D) \geq \delta$.
Suppose otherwise. Let $\mathcal{W}$ be a generating family for D of cardinality less than δ.

Let $i < \eta$. Using $\mathfrak{s}(\tau_i) = \delta_i = 2^{\tau_i}$, we define a sequence $\langle A_{i\beta} \mid \beta < \delta_i \rangle$ of subsets of τ_i such that

1. for every $a \in [\tau_i]^{\tau_i}$ there is $\beta < \delta_i$ with $a = A_{i\beta}$,
2. each set $A_{i\beta}$ appears δ_i-many times in the sequence,
3. for every $\beta < \delta_i$ there is $\gamma, \beta \leq \gamma < \delta_i$ such that $A_{i\gamma}$ splits $\langle A_{i\beta'} \mid \beta' < \beta \rangle$.

Let $\langle f_\xi \mid \xi < \delta \rangle$ be a scale witnessing $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$.
Let $\langle B_\zeta \mid \zeta < \rho < \delta \rangle$ be an enumeration of $\mathcal{W}$.
For every $\zeta < \rho$ and $i < \eta$ set $B_{\zeta i} = B_\zeta \cap \tau_i$.
Then there is $X_\zeta \in F$ such that for every $i \in X_\zeta, |B_{\zeta i}| = \tau_i$.

Pick $\alpha_{\zeta i} < \delta_i$ to be such that $B_{\zeta i} = A_{i\alpha_{\zeta i}}$.
Define a function $g_\zeta \in \prod_{i<\eta} \delta_i$ by setting $g_\zeta(i) = \alpha_{\zeta i}$, if $i \in X_\zeta$ and $g_\zeta(i) = 0$, otherwise.
Consider $\langle g_\zeta \mid \zeta < \rho\rangle$. We have $\rho < \delta$ and $\langle f_\xi \mid \xi < \delta\rangle$ a scale in $(\prod_{\alpha<\eta} \delta_\alpha, <_F)$.
Consider $\langle g_\zeta \mid \zeta < \rho\rangle$. We have $\rho < \delta$ and $\langle f_\xi \mid \xi < \delta\rangle$ a scale in $(\prod_{\alpha<\eta} \delta_\alpha, <_F)$.
So, there is $\xi^* < \delta$, such that for every $\zeta < \rho$, the set

$$Z = \{i < \eta \mid g_\zeta(i) < f_{\xi^*}(i)\} \in F.$$

Suppose for simplicity that $Z = \eta$. Let $i < \eta$. Consider the sequence $\langle A_{i\beta} \mid \beta < f_{\xi^*}(i)\rangle$. We have $\mathfrak{s}(\tau_i) = \delta_i > f_{\xi^*}(i)$, so there is $\gamma_i < \delta_i$ such that $A_{i\gamma_i}$ splits $\langle A_{i\beta} \mid \beta < f_{\xi^*}(i)\rangle$.
Let $\bar{A}_{i\gamma_i}$ denotes $\kappa_i \setminus (A_{i\gamma_i} \cup \delta_{i-1})$.
Set $A = \bigcup_{i<\eta} A_{i\gamma_i}$ and $\bar{A} = \bigcup_{i<\eta} \bar{A}_{i\gamma_i}$.
D is an ultrafilter, hence $A \in D$ or $\bar{A} \in D$.
Suppose, for example, that $A \in D$. Then there is $\zeta < \rho$ such that $B_\zeta \subseteq^* A$.
We have $A \cap B_\zeta \in D$, and so, by $(\vec{\tau}, F)-$uniformity, the set

$$X = \{i < \omega \mid A \cap B_\zeta \cap \tau_i \text{ is unbounded in } \tau_i\}$$

is infinite. Clearly, $X \subseteq X_\zeta$.
Now, $|B_\zeta \setminus A| < \kappa$ will imply that for all but boundedly many $i \in X$, $B_{\zeta i} = B_\zeta \cap \tau_i \subseteq^* A \cap \tau_i$.
This is impossible, since $B_{\zeta i}$ appears in $\langle A_{i\beta} \mid \beta < f_{\xi^*}(i)\rangle$ and $A_{i\gamma_i}$ splits this family, for every $i < \eta$.
Contradiction.
□

3 On character of uniform ultrafilters of the form $F - \lim_{\alpha<\eta} U_\alpha$.

Let us combine now regularity properties with the results of the previous section in order to produce lower bounds on the characters of ultrafilters of the form $F - \lim_{\alpha<\eta} U_\alpha$ over singular cardinals.

Proposition 3.1 *Suppose that κ is a singular cardinal of cofinality η. Let $\langle \kappa_\alpha \mid \alpha < \eta\rangle$ be an increasing sequence of cardinals converging to κ.*
Suppose that δ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$

2. *there is an increasing sequence of regular cardinals* $\langle \delta_\alpha \mid \alpha < \eta \rangle$ *such that*

   (a) $\kappa_\alpha < \delta_\alpha \leq \kappa_{\alpha+1}$, *for every* $\alpha < \eta$,

   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$, *for some ultrafilter* F *on* η *which extends the filter of co-bounded subsets of* η,

Suppose that $U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle$ *is such that for every* $\alpha < \eta$

1. U_α *is a uniform ultrafilter over a cardinal* μ_α,

2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$,

3. U_α *is* $(\delta_\alpha, \mu_\alpha)-$*regular or just* $\delta_\alpha-$*decompossible.*

Then U *is a uniform ultrafilter over* κ *and* $\mathrm{ch}(U) \geq \delta$.

Proof. Let $\alpha < \eta$. By Proposition 1.7, U_α is $\delta_\alpha-$decompossible. Then, by Proposition 1.6, $U_\alpha \geq_{R-K} D_\alpha$, for some uniform ultrafilter D_α over δ_α.
Set $D = F - \lim \langle D_\alpha \mid \alpha < \eta \rangle$. Then, by Proposition 1.3, $U \geq_{R-K} D$ and by Proposition 2.3, $\mathrm{ch}(D) \geq \delta$. Now, by Proposition 1.1, $\mathrm{ch}(U) \geq \delta$.
□

The next proposition is similar:

Proposition 3.2 *Suppose that* κ *is a singular cardinal of cofinality* η. *Let* $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ *be an increasing sequence of cardinals converging to* κ.
Suppose that δ *is a regular cardinal such that*

1. $\kappa < \delta \leq 2^\kappa$

2. *there is an increasing sequences of regular cardinals* $\langle \tau_\alpha \mid \alpha < \eta \rangle$ *such that*

   (a) $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, *for every* $\alpha < \eta$,

   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$, *where* $\delta_\alpha = 2^{\tau_\alpha}$ *and* F *is an ultrafilter on* η *which extends the filter of co-bounded subsets of* η,

   (c) $\mathfrak{r}(\tau_\alpha) = \delta_\alpha$.
   In particular, if $2^{\tau_\alpha} = \tau_\alpha^+$, *then* $\mathfrak{r}(\tau_\alpha) = \tau_\alpha^+ = \delta_\alpha$.

Suppose that $U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle$ *is such that for every* $\alpha < \eta$

1. U_α *is a uniform ultrafilter over a cardinal* μ_α,

2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$,

3. U_α *is* $(\tau_\alpha, \mu_\alpha)-$*regular or just* $\tau_\alpha-$*decompossible.*

Then U *is a uniform ultrafilter over* κ *and* $\mathrm{ch}(U) \geq \delta$.

Proof. Let $\alpha < \eta$. By Proposition 1.7, U_α is $\delta_\alpha-$decompossible. Then, by Proposition 1.6, $U_\alpha \geq_{R-K} D_\alpha$, for some uniform ultrafilter D_α over τ_α.
Set $D = F - \lim \langle D_\alpha \mid \alpha < \eta\rangle$. Then, by Proposition 1.3, $U \geq_{R-K} D$ and by Proposition 2.4, $\mathrm{ch}(D) \geq \delta$. Now, by Proposition 1.1, $\mathrm{ch}(U) \geq \delta$.
□

Corollary 3.3 *Let* κ, U, δ *be as in Propositions 3.1 or 3.2. Suppose that* $\delta = 2^\kappa$.
Then $\mathrm{ch}(U) = 2^\kappa$.

Assume as above that κ is a singular cardinal of cofinality η. Define now a cardinal invariant of κ which corresponds to ultrafilters of the form $F - \lim \langle U_\alpha \mid \alpha < \eta\rangle$.

Definition 3.4 Let $\mathfrak{u}'(\kappa)$ be the smallest possible cardinality of $\mathrm{ch}(U)$,
such that U is a uniform ultrafilter over κ of a form $F - \lim \langle U_\alpha \mid \alpha < \eta\rangle$,
where F is a uniform ultrafilter over η and U_α is a uniform ultrafilter over a regular cardinal $< \kappa$, for every $\alpha < \eta$.

Clearly, $\mathfrak{u}(\kappa) \leq \mathfrak{u}^{str}(\kappa) \leq \mathfrak{u}'(\kappa)$. Note that in models of [3], [4], $\mathfrak{u}(\kappa) = \mathfrak{u}^{str}(\kappa) = \mathfrak{u}'(\kappa) = \kappa^+$. However, κ in this models is limit of measurables. In [5], a model with $\mathfrak{u}(\aleph_\omega) = \aleph_{\omega+1} < 2^{\aleph_\omega}$ was constructed. It turns out that $\mathfrak{u}(\kappa) = \mathfrak{u}^{str}(\kappa) < \mathfrak{u}'(\kappa)$ in this model. Namely, the following always holds:

Proposition 3.5 *Assume that* $\aleph_\omega$ *is a strong limit cardinal and* $2^{\aleph_\omega} < \aleph_{\omega_1}$.
Then $\mathfrak{u}'(\aleph_\omega) = 2^{\aleph_\omega}$.

Proof. If $2^{\aleph_\omega} = \aleph_{\omega+1}$, then the statement is obvious.
So, suppose that $2^{\aleph_\omega} > \aleph_{\omega+1}$.
Then $2^{\aleph_\omega}$ is a regular cardinal, since $2^{\aleph_\omega} < \aleph_{\omega_4}$, by S. Shelah [13] and by König, $\mathrm{cof}(2^{\aleph_\omega}) > \aleph_\omega$.
Again, by S. Shelah [13], Ch.IX, 1.8,1.9 there is an increasing sequence $\langle n_i \mid i < \omega\rangle$ such that

$$\mathrm{tcf}(\prod_{i<\omega} \aleph_{n_i}, <_{co-finite}) = 2^{\aleph_\omega}.$$

Let now $U = F - \lim \langle U_i \mid i < \omega \rangle$ be as in Definition 3.4.
Suppose that U_i is a uniform ultrafilter over $\aleph_{m_i}$, for every $i < \omega$.
Let $i < \omega$. By K. Kunen and K. Prikry [10], U_i is $\aleph_k-$descendingly incomplete for every $k \leq m_i$. Hence, it is $\aleph_k-$decompossible,for every $k \leq m_i$.
Now we can apply Proposition 3.1 and to conclude that $\mathfrak{u}'(\aleph_\omega) = 2^{\aleph_\omega}$.
□

Remark 3.6 It is possible to strengthen 3.5 a bit and to relax the requirement on $\aleph_\omega$ being a strong limit, since here $U = F - \lim \langle U_i \mid i < \omega \rangle$ implies that $U \geq_{R-K} F$, and so, by 1.1, $\mathrm{ch}(U) \geq \mathrm{ch}(F)$.

4 On character of uniform ultrafilters of the form $F - \lim_{\alpha<\eta} U_\alpha$, square principles and inner models.

The following crucial observation was made by D. Donder [1]:

Theorem 4.1 *(Donder)*
Let $\kappa > \omega$ be regular and assume that $\square(\kappa)$ holds. Then every uniform ultrafilter U on κ is $(\omega, \tau)-$regular for every $\tau < \kappa$.

Let us combine this with the results of the previous section.

Proposition 4.2 *Suppose that κ is a singular cardinal of cofinality η. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to κ.*
Suppose that δ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$
2. *there is an increasing sequence of regular cardinals $\langle \delta_\alpha \mid \alpha < \eta \rangle$ such that*
   (a) $\kappa_\alpha < \delta_\alpha \leq \kappa_{\alpha+1}$, *for every $\alpha < \eta$,*
   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$, *for some ultrafilter F on η which extends the filter of co-bounded subsets of η,*

Suppose that $U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle$ is such that for every $\alpha < \eta$

1. *U_α is a uniform ultrafilter over a cardinal μ_α,*
2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$,

3. $\square(\mu_\alpha)$ *holds.*

Then U *is a uniform ultrafilter over* κ *and* $\mathrm{ch}(U) \geq \delta$.

Proof. We have μ_α is not weakly compact cardinal in $\mathcal{K}$, so $\square(\mu_\alpha)$ holds in $\mathcal{K}$, by E. Schimmerling and M. Zeman [15].
In addition $(\mu_\alpha^+)^{\mathcal{K}} = \mu_\alpha^+$, hence the sequence which witnesses $\square(\mu_\alpha)$ in $\mathcal{K}$ will witness it in V, as well.
By 4.1, U_α will be (ω, μ_α)-regular. Now, 3.1 applies.
$\square$

Similarly, using 3.2:

Proposition 4.3 *Suppose that* κ *is a singular cardinal of cofinality* η. *Let* $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ *be an increasing sequence of cardinals converging to* κ.
Suppose that δ *is a regular cardinal such that*

1. $\kappa < \delta \leq 2^\kappa$
2. *there is an increasing sequences of regular cardinals* $\langle \tau_\alpha \mid \alpha < \eta \rangle$ *such that*
   - *(a)* $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, *for every* $\alpha < \eta$,
   - *(b)* $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$, *where* $\delta_\alpha = 2^{\tau_\alpha}$ *and* F *is an ultrafilter on* η *which extends the filter of co-bounded subsets of* η,
   - *(c)* $\mathfrak{r}(\tau_\alpha) = \delta_\alpha$.
     In particular, if $2^{\tau_\alpha} = \tau_\alpha^+$, *then* $\mathfrak{r}(\tau_\alpha) = \tau_\alpha^+ = \delta_\alpha$.

Suppose that $U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle$ *is such that for every* $\alpha < \eta$

1. U_α *is a uniform ultrafilter over a cardinal* μ_α,
2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$,
3. $\square(\mu_\alpha)$ *holds.*

Then U *is a uniform ultrafilter over* κ *and* $\mathrm{ch}(U) \geq \delta$.

Corollary 4.4 *Let* κ *be a singular cardinal of cofinality* η.
Suppose thatthere is an increasing sequence of regular cardinals $\langle \delta_\alpha \mid \alpha < \eta \rangle$ *such that*

1. $\kappa = \bigcup_{\alpha<\eta} \delta_\alpha$,

2. $\mathrm{tcf}(\prod_{\alpha<\eta}\delta_\alpha, <_{J^{bd}}) = 2^\kappa$*, where* J^{bd} *is the ideal of all bounded subsets of* η*,*

Suppose that $U = F - \lim \langle U_\alpha \mid \alpha < \eta\rangle$*, for some ultrafilter* F *over* η *which includes all co-bounded subsets of* η*, is such that for every* $\alpha < \eta$

1. U_α *is a uniform ultrafilter over a cardinal* μ_α*,*
2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$*,*
3. $\square(\mu_\alpha)$ *holds.*

Then U *is a uniform ultrafilter over* κ *and* $\mathrm{ch}(U) = 2^\kappa$*.*

Assume now that there is no inner model with a Woodin cardinal and then use the core model $\mathcal{K}$ of R. Jensen and J. Steel [8].
Even under a weaker assumption that there is now inner model with class many strong cardinals, which handled by R. Schindler [12], there are plenty overlapping extenders relevant for consistency results of [3],[4].

By results of E. Schimmerling, M. Zeman [15] and M. Zeman [17], $\square_\kappa$ holds in $\mathcal{K}$ for every κ and $\square(\kappa)$ holds in $\mathcal{K}$ for every regular $\kappa > \omega$ which is not weakly compact.
In particular, if $\kappa^+ = (\kappa^+)^{\mathcal{K}}$, then $\square_\kappa$ holds.
E. Schimmerling proved in [14] that if both $\square(\kappa)$ and $\square_\kappa$ fail and $\kappa \geq 2^{\aleph_0}$, then there is an inner model with Woodin cardinal (and more). He showed also that if κ is a limit cardinal and $\kappa^+ > (\kappa^+)^{\mathcal{K}}$, then $\square(\kappa)$ (see 5.1.1, 4.7 of [14]).

5 A remark on $\mathfrak{r}(\kappa)$.

Note that if U is a uniform ultrafilter over κ and $\mathcal{W}$ is its bases, then $\mathcal{W}$ is a non-splitting family. Namely, if $B \in [\kappa]^\kappa$, then B does not split $\mathcal{W}$, since $B \in U$ or $\kappa \setminus B \in U$, and so contains a member of $\mathcal{W}$.
This implies that $\mathfrak{r}(\kappa) \leq \mathfrak{u}(\kappa)$.

We have seen in the previous section that $\mathfrak{u}'(\kappa)$ is related to $\square(\tau)$'s below κ. Failure of such square principle implies weak compactness in the core model of the corresponding cardinal.
On the other hand T. Suzuki [16] observed that:
a regular uncountable cardinal τ *is a weakly compact iff* $\mathfrak{s}(\tau) \geq \tau^+$*,*
where $\mathfrak{s}(\tau)$ a splitting number of τ is

$$\min\{|S| \mid S \subseteq [\tau]^\tau, \text{for every} x \in [\tau]^\tau \text{ there is } s \in S, |x \cap s| = |x \setminus s| = \tau\}.$$

The next proposition indicates the connection of $\mathfrak{r}(\kappa)$ to weak compactness below.

Proposition 5.1 *Suppose that κ is a singular cardinal of cofinality η. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to κ.*
Suppose that δ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$
2. *there is an increasing sequences of regular cardinals $\langle \tau_\alpha \mid \alpha < \eta \rangle$ such that*
   (a) $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, *for every* $\alpha < \eta$,
   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \tau_\alpha, <_{J^{bd}}) = \delta$,
   (c) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_{J^{bd}}) = \delta$, *where* $\delta_\alpha = 2^{\tau_\alpha}$,
   (d) $\mathfrak{s}(\tau_\alpha) = \delta_\alpha$.
   In particular, τ_α must be at least weakly compact here.
   If $2^{\tau_\alpha} = \tau_\alpha^+$, then we can assume just that τ_α is a weakly compact.[1]

Then $\mathfrak{r}(\kappa) \leq \delta$.

Proof.
Let $\langle f_\xi \mid \xi < \delta \rangle$ be a scale which witnesses $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$ and $\langle h_\zeta \mid \zeta < \delta \rangle$ be a scale which witnesses $\mathrm{tcf}(\prod_{\alpha<\eta} \tau_\alpha, <_F) = \delta$.
Let $i < \eta$. Fix an enumeration $\langle A^i_\beta \mid \beta < \delta_\alpha \rangle$ of all subsets of τ_α of cardinality τ_α.

Define a sequence $\langle A_\alpha \mid \alpha < \delta \rangle$ of subsets of κ of cardinality κ by induction as follows:
Suppose that $\alpha < \delta$ and $A_{\alpha'}$ is defined for every $\alpha' < \alpha$.
Let $i < \eta$. Consider $f_\alpha(i)$. It is an ordinal less than δ_i. So, $\langle A^i_\beta \mid \beta < f_\alpha(i) \rangle$ is not a splitting family, since $\mathfrak{s}(\tau_i)) = \delta_i$. Hence, there is $\beta(\alpha, i)$, $f_\alpha(i) < \beta(\alpha, i) < \delta_i$ such that $A^i_{\beta(\alpha,i)}$ cannot be split by any A^i_β with $\beta < f_\alpha(i)$.
Set $A_\alpha = \bigcup_{i<\eta}(A^i_{\beta(\alpha,i)} \cap (h_\alpha(i), \tau_i))$.
This completes the induction.

For every $X \subseteq \eta, \alpha, \zeta < \delta$ set

$$A(\alpha, X, \zeta) = \bigcup_{i \in X}(A^i_{\beta(\alpha,i)} \cap (h_\zeta(i), \tau_i)).$$

In particular, $A_\alpha = A(\alpha, \eta, \alpha)$.

[1]Note that in [3], [4], measurability was used instead in order to get an upper bound for $\mathfrak{u}'(\kappa)$.

Consider now

$$Z = \{A(\alpha, X, \zeta) \mid \alpha, \zeta < \delta, X \subseteq \eta\}.$$

We claim that Z is an unsplittable family.
Suppose otherwise. Then there is $B \subseteq \kappa, |B| = \kappa$ such that for every $A \in Z$, both $A \cap B$ and $A \setminus B$ have cardinality κ.
Note first that for unboundedly many $i < \eta$, $|B \cap \tau_i| = \tau_i$. Just otherwise, for all but boundedly many i's, there is $\rho_i < \tau_i$ such that $B \cap \tau_i \subseteq \rho_i$.
Then there is $\alpha < \delta$ such that for all but boundedly many i's, $\rho_i < h_\alpha(i)$. Hence, there is $i^* < \eta$ such that for every $i, i^* \leq i < \eta$, $B \cap A_\alpha \cap \tau_i \subseteq \tau_{i^*}$.
This is impossible, since $|B \cap A_\alpha| = \kappa$.

Assume now for simplicity that for every $i < \eta, |B \cap \tau_i| = \tau_i$.
Then for every $i < \eta$, there is $\beta_i < \delta_i$ such that $B \cap \tau_i = A^i_{\beta_i}$.
Find $\alpha < \delta$ such that for all but boundedly many i's, $f_\alpha(i) > \beta_i$.
Again, assume for simplicity that this holds for every $i < \eta$. Recall that by the choice of $A^i_{\beta(\alpha,i)}$, it cannot be split by any A^i_β with $\beta < f_\alpha(i)$. In particular, by $B \cap \tau_i = A^i_{\beta_i}$.
So, either $A^i_{\beta(\alpha,i)} \cap B \cap \tau_i$ is bounded in τ_i or $A^i_{\beta(\alpha,i)} \setminus (B \cap \tau_i)$ is bounded in τ_i .
Suppose for example that the set

$$X = \{i < \eta \mid A^i_{\beta(\alpha,i)} \cap B \cap \tau_i \text{ is bounded in } \tau_i\}$$

has cardinality η.
Let for every $i \in X$, $\gamma_i < \tau_i$ be a bound of $A^i_{\beta(\alpha,i)} \cap B \cap \tau_i$. If $i \in \eta \setminus X$, then set $\gamma_i = 0$.
There is $\zeta < \delta$ and $i^* < \eta$ such that for every $i, i^* \leq i < \eta$, $h_\zeta(i) > \gamma_i$.
Then, for every $i \in X \setminus i^*$, $A^i_{\beta(\alpha,i)} \cap B \cap \tau_i \subseteq h_\zeta(i)$.
But then $A(\alpha, X, \zeta) \cap B \subseteq \tau_{i^*} < \kappa$. Contradiction.
$\square$

Define $\mathfrak{r}^{str}(\kappa)$ to be

$$\min(\{|X| \mid X \text{ is an unsplittable family,}$$

$$\text{such that for some increasing sequence of regular cardinals below } \kappa,$$

$$\vec{\tau} = \langle \tau_\alpha \mid \alpha < \mathrm{cof}(\kappa), \text{ for every } A \in X, \text{ for unboundedly many } \alpha < \mathrm{cof}(\kappa), |A \cap \tau_\alpha| = \tau_\alpha\}).$$

Clearly, $\kappa^+ \leq \mathfrak{r}(\kappa) \leq \mathfrak{r}^{str}(\kappa)$.

The proposition above actually shows that $\mathfrak{r}^{str}(\kappa) \leq \delta$.

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