No Bound for the First Fixed Point

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Abstract

Our aim is to show that it is impossible to find a bound for the power of the first fixed point of the aleph function.

0 Introduction

$\aleph_\alpha$ is called a fixed point of the $\aleph$-function if $\aleph_\alpha = \alpha$. It is called the first fixed point if α is the least ordinal such that $\aleph_\alpha = \alpha$. More constructive, the first repeat point of the $\aleph$-function is the limit of the following sequence

$$\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \aleph_{\aleph_{\aleph_{\aleph_0}}}, \ldots$$

The following are corner stones results of cardinal arithmetic:

Galvin-Hajnal [Gal-Haj]:

Suppose that $\delta < \aleph_\delta$ and $cf\delta > \aleph_0$. If $\forall \mu < \aleph_\delta (2^\mu < \aleph_\delta)$ then $2^{\aleph_\delta} < \aleph_{(2^{|\delta|})^+}$.

Shelah [She1]:

(a) The same is true also for δ's of cofinality ω.

(b) It is possible to replace $(2^{|\delta|})^+$ by $|\delta|^{+4}$.

Now suppose that $\aleph_\delta = \delta$ and it is a singular cardinal.

The classical results of Prikry and Silver (see [Jech]) show that there are no bounds on the power of a fixed point $\aleph_\delta$ provided that δ is very big (there are a lot of inaccessibles below it, etc.). But are there bounds for small fixed points? For uncountable cofinality the following provides an answer:

Shelah [She1]:

Let $\aleph_\sigma$ be the ω_1-th fixed point of the $\aleph$-function. If $\forall \mu < \aleph_\delta$ $(2^\mu < \aleph_\delta)$ then $2^{\aleph_\sigma} <$ min $((2^{\omega_1})^+$-fixed point, ω_4-th fixed point).

For countable cofinality Shelah ([She2]) showed the following:

The power of the first point of *order* ω is unbounded below the first inaccessible, where fixed points of order ω are elements of the class $C_\omega = \bigcap_{n<\omega} C_n$, with

$$
\begin{aligned}
&C_0 = \{\kappa | \kappa \quad \text{is a cardinal}\} \\
&C_1 = \{\kappa | \quad |C_0 \cap \kappa| = \kappa\} \\
&\vdots \\
&C_{n+1} = \{\kappa | \quad |C_n \cap \kappa| = \kappa\} \\
&\vdots
\end{aligned}
$$

A remaining natural question, explicitly asked in [She1,14.7(γ)] was about a bound of the first fixed point.

Our aim here will be to show that there are no bounds. More precisely the following holds.

Theorem *Suppose that κ is a cardinal of cofinality ω such that for every $\tau < \kappa$ the set $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\tau}\}$ is unbounded in κ. Then for every $\lambda > \kappa$ there is a forcing extension satisfying the following:*

(1) *κ is the first fixed point of the $\aleph$-function*

(2) *GCH holds below κ*

(3) *all the cardinals $\geq \kappa$ are preserved*

(4) *$2^\kappa \geq \lambda$.*

By [Git4], the initial assumptions are sharp. However, we do not know what the right initial assumptions are if one removes "GCH below κ" from the conclusion of the theorem.

The ideas and techniques used in the proof spread through various papers, but we tried to make the presentation largely self-contained. Sections 1 to 4 contain the proof of the theorem. Readers familiar with [Git2] and [Git3] may skip some of the material here (like for example Sec. 1).

In the last section (Section 5) a construction of the same (as those of the theorem) flavour is presented. We show the following:

Theorem 5.21 *The following is consistent.*

(a) *κ is a strong limit of cofinality $\aleph_0$*

(b) $2^\kappa = \kappa^{+3}$

(c) $\{\delta < \kappa | \delta^+ \in b_{\kappa^{+3}}\} \cap b_{\kappa^{+2}} = \emptyset$

where b_λ denote pcf-generator corresponding to λ ($\lambda = \kappa^{++}$ or $\lambda = \kappa^{+++}$).

This somewhat clarifies the situation with pcf-generators, since in all previous constructions satisfying (a) and (b) the condition (c) fails. Also for uncountable cofinality the theorem fails by [She1].

At the end of Section 5 we outline a similar construction related to the study of the strength of various gaps between a singular of cofinality $\aleph_0$ and its power. The result is the following:

Theorem 5.22 *Suppose that κ is a cardinal of cofinality ω, $\aleph_1 \leq \delta < \kappa$, $\nu < \aleph_1$ and the set $\{\alpha < \kappa | o(\alpha) \geq \alpha^{+\delta+1} + 1\}$ is unbounded in κ. Then there is cofinalities preserving, not adding new bounded subsets to κ extension satisfying $2^\kappa \geq \kappa^{+\delta\cdot\nu+1}$.*

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1 Preliminary Results

Let κ be a limit of an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ and each κ_n carries an extender E_n. For a cardinal $\lambda > \kappa^+$ we would like to add to κ λ ω-sequences. Measures of the extender E_n are usually used in order to supply n's elements of such sequences, for every $n < \omega$. Thus, if the length of each E_n is at least λ, then we pick $a_n \subseteq \lambda$ of cardinality less than κ_n and having maximal element in the extender order. Denote it by $mc(a_n)$. Now we can use the basic Prikry tree forcing with measures with index $mc(a_n)$ of $E_n (n < \omega)$ (i.e. $X \in \mathcal{U}_{mc(a_n)}$ iff $mc(a_n) \in j_n(X)$, where $j_n : V \to M_n \simeq Ult(V, E_n)$ is the canonical embedding into the ultrapower by E_n) to add an ω-sequence. It will project easily to all the measures in a_n's producing this way more Prikry sequences. Thus for every $\alpha \in \bigcup_{n<\omega} a_n$, assuming that $a_0 \subseteq a_1 \subseteq \cdots \subseteq a_n \subseteq \cdots$, we will have a Prikry sequence. Moreover they will be ordered under eventual dominance according to their indexes. One can argue that the number of sequences added this way is at most κ. But the sets a_n need not be frozen. We can allow to increase them. This way, generically all λ will be covered. So we will be done provided that the cardinals are preserved. Unfortunately they do collapse. In order to overcome this the "true" initial segments of the Prikry sequences are hidden by mixing them with λ Cohen subsets of κ^+ added simultaneously. We refer for details to [Git3, Sec. 1] where the scheme above is realized. The main advantage of this construction is its simplicity. There however are at least two drawbacks. The first, and a less important for us here – is the consistency strength. Thus existence of extenders of the length λ over each κ_n $(n < \omega)$ is too strong. In [Git-Mag] only one extender of length λ was used and in [Git3,4] no extenders of length λ were used, but instead over each κ_n an extender of a length below κ_{n+1}. The second drawback, and it is crucial here, is the impossibility to move down to relatively small cardinals like the first fixed point of the $\aleph$-function. The problem is that elements of Prikry sequences, or indiscernibles as they are referred in the inner model considerations, resist collapsings. Namely, if $\kappa^+ \leq \tau < \mu \leq \lambda$ are regular cardinals and

$$\langle t_\mu(n) \mid n < \omega\rangle \ , \quad \langle t_\tau(n) \mid n < \omega\rangle$$

are corresponding Prikry sequences then making $t_\mu(n), t_\tau(n)(n < \omega)$ of the same cardinality will collapse necessary μ to τ. Basically, since $\mu = cf\left(\prod\limits_{n<\omega} t_\mu(n)/\text{finite}\right)$ and once $cft_\mu(n) = cft_\tau(n)$ for every $n < \omega$ then also

$$cf\left(\prod_{n<\omega} t_\mu(n)/\text{finite}\right) = cf\left(\prod_{n<\omega} t_\tau(n)/\text{finite}\right) \ .$$

Usually, collapses are made in indiscernibles free areas in order to move a configuration achieved over a singular (or a former regular) down. Thus, for example, in [Git-Mag, Sec. 3], in order to make $2^{\aleph_\omega} = \aleph_{\omega+2}$ an extender of the length δ^{++} was used over δ to produce δ^{++} Prikry sequences (changing cofinality of δ to $\aleph_0$) and simultaneously the Levy collapses in intervals $[\rho_n^{+3}, \rho_{n+1})$ were applied, where $\langle \rho_n \mid n < \omega\rangle$ denotes the Prikry sequence of the normal measure of the extender (which is usually the guide sequence for these type of constructions as well as crucial in analyses of indiscernibles). Here all indiscernibles are inside intervals $[\rho_n, \rho_n^{+3})$ $(n < \omega)$ and hence are not effected by the Levy collapses. In the present situation extenders of the length λ are used over each κ_n $(n<\omega)$. This creates indiscernibles for ρ_n unboundedly often below ρ_{n+1} thus preventing the use of collapses, where $\langle \rho_n \mid n<\omega\rangle$ is the Prikry sequence for the normal measures of the extenders.

One can try instead of using extenders of the length λ over κ_n's and then $a_n \subseteq \lambda$, to use for each $n<\omega$ extenders of the length κ_n^{++}, κ_n^{+n+2}, $\kappa_n^{+\delta}$ (for some fixed $\delta<\kappa_0$), $\kappa_n^{+\kappa_{n-1}}$, $\kappa_n^{+\text{ the least Mahlo above }\kappa_n}$ etc. Instead of a_n as a subset of λ just require that a_n is an order preserving function from λ to the length of the extender over κ_n. Doing this naively will ruin cardinals between κ and λ. Analyses of indiscernibles in a fashion of [Git-Mit] provides good reasons for this. Thus, in general, the Mitchell Covering Lemma provides a connection called assignment function between indiscernibles and measures of extenders. The Mitchell Covering Lemma applies locally. Namely to sets of less than κ of Prikry sequences. This in turn provides assignment functions which are also local. Once such functions agree, they can be combined together into total assignment functions. This last one can be used in calculating (or bounding) of the power of singular cardinals, see [Git-Mit] for such applications. In the case under consideration, the total assignment function exists which in turn will bound the power of κ by κ^+, since, basically, in the ground model the number of possibilities for selecting ω-sequences of measures from extenders over κ_n's is κ^+ (certainly we assume GCH in the ground model). Hence λ will be collapsed. By [Git-Mit], the existence of total assignment function is a common phenomena. Thus, it is true for uncountable cofinality assuming there is no overlapping extenders or for countable one assuming that for some $n<\omega$ $\{\alpha<\kappa \mid o(\alpha)\geq\alpha^{+n}\}$ is bounded in κ. It is still unknown for uncountable cofinality if it is possible to have a situation without a total assignment function. We think that this should be the case and models realizing such a situation may throw light on basic problems of cardinal arithmetic. For cofinality $\aleph_0$ a model without a total assignment function was constructed in [Git1]. Further development of the basic idea of [Git1] was made in [Git2,3,4] in order to blow power of κ using short extenders over κ_n's. Let us sketch a construction of [Git 3, Sec. 2]. It contains basic blocks that will be crucial further for the main construction here. Thus we assume that $\kappa=\bigcup_{n<\omega}\kappa_n$, $\kappa_0<\kappa_1<\cdots<\kappa_n<\cdots$, $\lambda\geq\kappa^{++}$ be a regular cardinal and for every $n<\omega$ E_n is an extender over κ_n of the length κ_n^{+n+2}. Let $\langle \mathcal{U}_{n\alpha} \mid \alpha<\kappa_n^{+n+2}\rangle$ be the sequence of measures (ultrafilters) of E_n, i.e. $X\in\mathcal{U}_{n\alpha}$ iff $\alpha\in j_n(X)$, where $j_n: V\to M_n \approx Ult(V,E_n)$ is the canonical embedding.

Definition 1.1 Let $\mathcal{P}$ be the set of sequences $p=\langle p_n \mid n<\omega\rangle$ so that for some $\ell(p)<\omega$ for every $n<\omega$ the following holds:

(1) if $n < \ell(p)$ then p_n is a partial function from λ to κ_n of cardinality at most κ (i.e. its just a condition in the Cohen forcing for adding λ subsets to κ^+).

(2) if $n \geq \ell(p)$, then p_n is a triple of the form $\langle a_n, A_n, f_n\rangle$ so that

  (a) f_n is a partial function from λ to κ_n of cardinality at most κ

  (b) a_n is a partial order preserving function from λ to κ_n^{+n+2} such that

    (i) $|a_n| < \kappa_n$

    (ii) $\mathrm{dom} a_n \cap \mathrm{dom} f_n = \emptyset$

    (iii) $rnga_n$ has a maximal element and it is above all its elements in the Rudin-Kiesler order (RK- order), i.e. for every $\beta \in rnga \backslash \{\max(rnga)\}$

$$\mathcal{U}_{n\beta} <_{RK} \mathcal{U}_{n,\max(rnga)}$$

    (iv) $\mathrm{dom} a_n \subseteq \mathrm{dom} a_{n+1}$

  (c) $A_n \in \mathcal{U}_{n\,\max(rnga)}$

  (d) for every $\alpha, \beta, \gamma \in rnga_n$ if $\mathcal{U}_{n\alpha} \geq_{RK} \mathcal{U}_{n\beta} \geq_{RK} \mathcal{U}_{n\gamma}$ then

$$\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho)$$

  for every $\rho \in \pi''_{\max(mga_n),\alpha} A_n$

  (e) for every $\alpha > \beta$ in $rnga_n$ and $\nu \in A_n$

$$\pi_{\max(rnga_n),\alpha}(\nu) > \pi_{\max(rnga),\beta}(\nu)$$

  where $\pi_{\mu,\rho}$'s are the canonical projections of $\mathcal{U}_{n\mu}$'s to $\mathcal{U}_{n\rho}$'s derived from $j_n : V \longrightarrow M_n \simeq Ult(V, E_n)$.

Cohen parts of conditions p_n's for $n < \ell(p)$ and f_n's for $n \geq \ell(p)$ desired to "hide" initial segments of the Prikry sequences. Sets of measures ones $\langle A_n \mid n \geq \ell(p)\rangle$ are playing the same role as in the usual tree Prikry forcing. The condition (d) above allows to project freely the Prikry sequence from bigger coordinate to smaller one. For those familiar with extender based Prikry forcing of [Git-Mag], notice that the support of a condition $rnga_n$ is small. It is of cardinality $< \kappa_n$ and not κ_n as in this paper. This allows us to use the full commutativity in (d). The last condition is (e) is responsible for the right order between Prikry sequences that are added by $\mathcal{P}$.

Definition 1.2 Let $p = \langle p_n \mid n < \omega \rangle$, $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$. We define $p \geq q$ iff

(1) $\ell(p) \geq \ell(q)$

(2) for every $n < \ell(q)$ $p_n \supseteq q_n$

(3) for every $n \geq \ell(p)$ the following holds, where $p_n = \langle a_n, A_n, f_n \rangle$ and $q_n = \langle b_n, B_n, g_n \rangle$

  (a) $f_n \supseteq g_n$

  (b) $a_n \supseteq b_n$

  (c) $\pi''_{\max(rnga_n), \max(rngb_n)} A_n \subseteq B_n$

(4) for every n, $\ell(p) > n \geq \ell(q)$ the following holds, where $q_n = \langle b_n, B_n, g_n \rangle$

  (a) $p_n \supseteq g_n$

  (b) $\mathrm{dom} p_n \supseteq \mathrm{dom} b_n$

  (c) $p_n(\max b_n) \in B_n$

  (d) for every $\beta \in \mathrm{dom} b_n \quad p_n(\beta) = \pi_{\max(rngb_n), \beta}(p_n(\max b_n))$.

Definition 1.3 Let $p, q \in \mathcal{P}$. We define $p \geq^* q$ iff

(1) $p \geq q$

(2) $\ell(p) = \ell(q)$

Crucial in Definitions 1.2, 1.3 is 1.2(4) which links together Prikry and Cohen parts of conditions.

For $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$ let $p \restriction n = \langle p_m \mid m < n \rangle$ and $p \backslash n = \langle p_m \mid m \geq n \rangle$. Set $\mathcal{P} \restriction n = \{ p \restriction n \mid p \in \mathcal{P} \}$ and $\mathcal{P} \backslash n = \{ p \backslash n \mid p \in \mathcal{P} \}$.

The proofs next to the lemmas are quite straightforward. We refer to [Git3, Sec. 1-2] for details.

Lemma 1.4 $\langle \mathcal{P}, \leq, \leq^* \rangle$ *satisfies the Prikry condition.*

Lemma 1.5 $\mathcal{P} \simeq \mathcal{P} \restriction n \times \mathcal{P} \backslash n$ *for every* $n < \omega$.

Lemma 1.6 $\langle \mathcal{P} \backslash n, \leq^* \rangle$ *is* κ_n*-closed for every* $n < \omega$.

Let $G \subseteq \mathcal{P}$ be $\langle \mathcal{P}, \leq \rangle$-generic. For every $n < \omega$ define a function $F_n : \lambda \longrightarrow \kappa_n$ as follows: $F_n(\alpha) = \nu$ if for some $p = \langle p_m \mid m < \omega \rangle \in G$ we have $\ell(p) > n$ and $p_n(\alpha) = \nu$. Now for every $\alpha < \lambda$ set $t_\alpha = \langle F_n(\alpha) \mid n < \omega \rangle$.

Lemma 1.7 *For every $\beta < \lambda$ there is α, $\beta < \alpha < \lambda$ such that t_α is different from every t_γ with $\gamma \leq \beta$.*

Combining this lemmas we obtain the following

Proposition 1.8 The forcing $\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets to κ and it adds λ new ω-sequences to κ.

Unfortunately, the total assignment function exists here. This causes the cardinals in the interval $(\kappa^+, \lambda]$ to collapse to κ^+. In order to overcome this the set $\mathcal{P}$ was shrunken to $\mathcal{P}^*$ and an equivalence relation "$\longleftrightarrow$" was defined on $\mathcal{P}^*$. The first change is a light one but the second is quite drastic.

Fix $n < \omega$. For every $k \leq n$ we consider a language $\mathcal{L}_{n,k}$ containing two relation symbols, a function symbol, a constant c_α for every $\alpha < \kappa_n^+$ and constants c_{λ_n}, c. Consider a structure $\mathfrak{a}_{n,k} = \langle H(\chi^{+k}), \in, E_n$, the enumeration of $\left[\kappa_n^{+n+2}\right]^{<\kappa_n^{+n+2}}, 0, 1, \ldots, \alpha \ldots, \kappa_n, \chi \mid \alpha < \kappa_n^{+k} \rangle$ in this language, where χ is a regular cardinal large enough. For an ordinal $\xi < \chi$ we denote by $tp_{n,k}(\xi)$ the $\mathcal{L}_{n,k}$-type realized by ξ in $\mathfrak{a}_{n,k}$.

Let $\mathcal{L}'_{n,k}$ be the language obtained from $\mathcal{L}_{n,k}$ by adding a new constant c'. For $\delta < \chi$ let $\mathfrak{a}_{n,k,\delta}$ be the $\mathcal{L}'_{n,k}$-structure obtained from $\mathfrak{a}_{n,k}$ by interpreting c' as δ. The type $tp_{n,k}(\delta, \xi)$ is defined in an obvious fashion. Further, we shall identify types with ordinals corresponding to them in some fixed well-ordering of the power sets of κ_n^{+k}'s.

Definition 1.9 Let $k \leq n$ and $\beta < \lambda_n$. β is called k-good iff

(1) for every $\gamma < \beta$ $\quad tp_{n,k}(\gamma, \beta)$ is realized unboundedly many times below κ_n^{+n+2}

(2) for every $a \subseteq \beta$ if $|a| < \kappa_n$ then there is $\alpha < \beta$ corresponding to a in the enumeration of $\left[\kappa_n^{+n+2}\right]^{<\kappa_n^{+n+2}}$.

β is called good if it is k-good for some $k \leq n$.

Further we will be interested mainly in k-good ordinals for $k > 2$. If $\alpha, \beta < \kappa_n^{+n+2}$ realize the same k-type for $k > 2$, then $U_{n\alpha} = U_{n\beta}$. Since the number of different $U_{n\alpha}$'s is κ_n^{++}.

The following two lemmas are easy, see [Git3, Sec. 2]

Lemma 1.10 *The set* $\{\beta < \kappa_n^{+n+2} \mid \beta$ *is* n*-good*$\} \cup \{\beta < \kappa_n^{+n+2} \mid cf\beta < \kappa_n$ *contains a club.*

Lemma 1.11 *Suppose that* $n \geq k > 0$ *and* β *is* k*-good. Then there are arbitrarily large* $k-1$*-good ordinals below* β*.*

Definition 1.12 The set $\mathcal{P}^*$ is a subset of $\mathcal{P}$ consisting of sequences $p = \langle p_n \mid n < \omega \rangle$ so that for every n, $\ell(p) \leq n < \omega$ and $\beta \in$ dom a_n there is a nondecreasing converging to infinity sequence of natural numbers $\langle k_m \mid n \leq m < \omega\}$ so that for every $m \geq n$ $a_m(\beta)$ is k_m-good, where $p_m = \langle a_m, A_m, f_m \rangle$.

The orders on $\mathcal{P}^*$ are just the restrictions of $\leq$ and $\leq^*$ of $\mathcal{P}$.

Lemmas 1.4-1.8 are valid for $\langle \mathcal{P}^*, \leq, \leq^* \rangle$ as well as the fact that λ collapses to κ^+.

Let us now define an equivalence relation on $\mathcal{P}^*$.

Definition 1.13 Let $p = \langle p_n \mid n < \omega \rangle$, $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}^*$. We call p and q equivalent and denote this by $p \leftrightarrow q$ iff

(1) $\ell(p) = \ell(q)$

(2) for every $n < \ell(p)$ $p_n = q_n$

(3) there is a nondecreasing sequence $\langle k_n \mid \ell(p) \leq n < \omega \rangle$ with $\lim_{n\to\infty} k_n = \infty$ and $k_{\ell(p)} > 2$ such that for every n, $\ell(p) \leq n < \omega$ the following holds:

   (a) $f_n = g_n$

   (b) doma_n = domb_n

   (c) $rnga_n$ and $rngb_n$ are realize the same k_n-type, (i.e. the least ordinals coding $rnga_n$ and $rngb_n$ are such)

   (d) $A_n = B_n$.

   Notice that, in particular the following is also true:

   (e) for every $\delta \in$ doma_n = domb_n $a_n(\delta)$ and $b_n(\delta)$ are realizing the same k_n-type

   (f) for every $\delta \in$ doma_n = domb_n and $\ell \leq k_n$ $a_n(\delta)$ is ℓ-good if $b(\delta)$ is ℓ-good

   (g) for every $\delta \in$ doma_n = domb_n max($rnga_n$) projects to $a_n(\delta)$ the same way as max($rngb_n$) projects to $b_n(\delta)$.

Let us also define a preordering $\rightarrow$ on $\mathcal{P}^*$.

Definition 1.14 Let $p, q \in \mathcal{P}^*$.
Set $p \rightarrow q$ iff there is a sequence of conditions $\langle r_k \mid k < m < \omega \rangle$ so that

(1) $r_0 = p$

(2) $r_{m-1} = q$

(3) for every $k < m - 1$

$$r_k \leq r_{k+1} \quad \text{or} \quad r_k \leftrightarrow r_{k+1} \ .$$

The next two lemmas show that $\langle \mathcal{P}^*, \rightarrow \rangle$ is a nice subforcing of $\langle \mathcal{P}^*, \leq \rangle$.

Lemma 1.15 *Let $p, q, s \in \mathcal{P}^*$. Suppose that $p \leftrightarrow q$ and $s \geq p$. Then there are $s' \geq s$ and $t \geq q$ such that $s' \leftrightarrow t$.*

Lemma 1.16 *For every $p, q \in \mathcal{P}^*$ such that $p \longrightarrow q$ there is $s \geq p$ so that $q \longrightarrow s$.*

We refer to [Git3, Sec. 2] for the proofs. Now using the Δ-system argument one can show the following:

Lemma 1.17 *$\langle \mathcal{P}^*, \rightarrow \rangle$ satisfies λ-c.c.*

Again, we refer to [Git3, Sec. 2] for the detailed proof.

So, the forcing $\langle \mathcal{P}^*, \rightarrow \rangle$ preserves λ. However, it is not hard to see that the rest of cardinals (if any) in the interval $(\kappa^+, \lambda]$ are collapsed to κ^+. But suppose that we like to preserve cardinals between κ and λ. The problem with straightforward generalization of the forcing $\langle \mathcal{P}^*, \rightarrow \rangle$ (even for $\lambda = \kappa^{+++}$) is that the Δ-system argument of 1.17 breaks down. In [Git3], a preparation forcing was introduced to reduce gradually the number of possible connections between ordinals above and below κ. This worked for λ's below $\kappa^{+\delta}$ with $\delta < \kappa$. In [Git4], generalizations dealing with large λ's were suggested. But they do not fit our aim to make eventually κ into the first fixed point of the $\aleph$-function. The problem with the approach of [Git4] is that the extenders used over κ_n's are relatively long. This in turn produces a lot of indiscernibles resisting collapses for turning κ into the first fixed point.

Let us now explain the basic idea of the present construction. Thus, let $\kappa = \bigcup_{n<\omega} \kappa_n$, $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$, each κ_n for $n \geq 1$ carries an extender E_n of the length κ_{n-1} and κ_0 carries extender of the length κ_0^+. Let λ be an inaccessible above κ. Let ρ_0 denote the one element Prikry sequence for the normal measure of E_0. Then ρ_0^+ will correspond to κ_0^+. Now over κ_1 we force with $E_1 \restriction \kappa_1^{+\rho_0^+ + 1}$. Denote by ρ_1 the one element Prikry sequence for the normal measure of E_1. Then $\rho_1^{+\rho_0^+ +1}$ will correspond to $\kappa_1^{+\rho_0^+ +1}$. At level 3 we will use $E_2 \restriction \kappa_2^{+\rho_1^{+\rho_0^+}+1}$ and so on. It will be arranged that $\lambda = tcf\Big(\prod_{n<\omega} \rho_n^*/\text{finite})$ where $\rho_n^* = \rho_n^{+\rho_{n-1}^* +1}$ and $\rho_0^* = \rho_0^+$. The rest of the cardinals between κ and λ will be connected generically to those of the intervals $[\rho_n^+, \rho_n^{+\rho_{n-1}^*})$. The main difficulty here compared with [Git3,4] is that we need to link between κ_n and κ_{n+1} for every $n < \omega$. Thus, in order to determine ρ_{n+1}^* we need to know ρ_n^* in addition to ρ_{n+1}. This requires dealing with names which complicates the arguments.

2 The Basic Forcing

We define here a forcing notion similar to $\mathcal{P}^*$ of Section 1 but with some additions needed for our further purposes. Our main forcing will be a carefully chosen subset of this forcing notion.

Fix an ordinal $\delta > 1$.

Definition 2.1 $\mathcal{P}^*$ consists of sequences $p = \langle p_n \mid n \leq \ell(p)\rangle^\frown \langle \underset{\sim}{p_n} \mid \ell(p) < n < \omega\rangle$ so that

(1) $\ell(p) < \omega$

(2) for every $n < \ell(p)$ p_n is of the form $\langle \rho_n, h_{<n}, h_{>n}, f_n\rangle$ such that

(i) ρ_n is the $n+1$-th member of the increasing sequence of inaccessible cardinals $\rho_0, \rho_1, \ldots, \rho_{\ell(p)-1}$ and $\rho_0 < \kappa_0 < \rho_1 < \cdots < \rho_{\ell(p)-1} < \kappa_{\ell(p)-1}$

(ii) $h_{<n} \in \text{ Col}\Big(\rho_n^{+\kappa_{n-1}+1}, < \kappa_n\Big)$ where $\kappa_{-1} = 1$

(iii) $h_{>n} \in \text{ Col}(\kappa_n, < \rho_{n+1})$ if $n+1 < \ell(p)$ and $h_{>n} \in \text{ Col}(\kappa_n, < \kappa_{n+1})$ if $n+1 = \ell(p)$

(iv) f_n is a partial function of cardinality at most κ form $\kappa^{+\delta+2}$ to κ_n.

The meaning of the condition (2) is as follows: $\langle \rho_0, \ldots, \rho_{\ell(p)-1}\rangle$ is the initial segment of the Prikry sequence for the normal measures of extenders E_n's over κ_n's. $h_{<n}, h_{>n}$ are desired to preserve only about κ_{n-1} – many cardinals between ρ_n and ρ_{n+1}. Collapsing finally ρ_0 to $\aleph_0$ this will turn κ into the first fixed point of the $\aleph$-function. f_n is like p_n below $\ell(p)$ of Section 1 and its role is to hide the connection between measures of E_n and the corresponding one element Prikry sequences.

(3) if $n = \ell(p)$, then p_n is of the form $\langle e_n, a_n, A_n, S_n, h_{>n}, f_n\rangle$ where

(a) f_n is a partial function from $\kappa^{+\delta+2}$ to κ_n of cardinality at most κ and $\mathrm{dom} f_n \cap \mathrm{dom} a_n = \emptyset$

(b) $h_{>n} \in \mathrm{Col}(\kappa_n, < \kappa_{n+1})$

(c) e_n is an order preserving function between less than κ_{n-1} cardinals $\leq \kappa^{+\delta+2}$ and cardinals inside $[\kappa_n^+, \kappa_n^{\kappa_{n-1}}]$ so that

(i) $\kappa^{+\delta+2} \in \mathrm{dom} e_n$ and $e_n\Big(\kappa^{+\delta+2}\Big) = \kappa_n^{+\delta_{n-1}+1}$ for a regular $\delta_{n-1}+1 < \kappa_{n-1}$.

(ii) every $\tau \in rnge_n \backslash \{\kappa_n^{+\delta_{n-1}+1}\}$ is a regular cardinal between κ_n^+ and $\kappa_n^{+\delta_{n-1}}$.

The purpose of e_n is to provide a link between values for cardinals determined at level $n-1$ and the level n. Usually, δ_{n-1} will be ρ^*_{n-1}, where $\rho^*_{k+1} = \rho_{k+1}^{+\rho^*_k+1}$ and ρ_k's are from the Prikry sequence of the normal measure.

Also we use $\kappa^{+\delta+2}$ only in order to make the notation more homogeneous. One can use instead some regular $\lambda > \kappa$ as well.

(d) a_n is a function so that

(i) $|a_n| < \kappa_n$

(ii) $\mathrm{dom} a_n \subseteq \kappa^{+\delta+2} \cup \{A \mid |A| \in \mathrm{dom}(e_n)$ and $A \prec \langle H(\kappa^{+\delta+8}), \in, \kappa, \langle \kappa_m \mid m < \omega\rangle, \langle E_m \mid m < \omega\rangle\rangle\}$

(iii) $a_n \restriction On$ is order preserving partial map from $\kappa^{+\delta+2}$ into the interval $\Big(\kappa_n^{+\delta_{n-1}}, \kappa_n^{+\delta_{n-1}+1}\Big)$

(iv) $rng(a_n \backslash On) \subseteq \{B | B \prec \langle H(\kappa_n^{+\delta_{n-1}+k}), \in, \kappa_n, E_n \restriction \kappa_n^{+\delta_{n-1}}\rangle$ for some $k, 2 < k < \omega$ and ${}^{|B|>}B \subseteq B\}$

(v) if $A \in (\mathrm{dom} a_n)\backslash On$ then $|a_n(A)| = e_n(|A|)$

(vi) if $A, B \in \mathrm{dom} a_n \backslash On$ then $A \subset B$ iff $a_n(A) \subset a_n(B)$

(vii) if $A \in (\mathrm{dom} a_n)\backslash On$ and $\alpha \in (\mathrm{dom} a_n) \cap On$ then $\alpha \in A$ iff $a_n(\alpha) \in a_n(A)$

(viii) $\mathrm{dom} a_n \cap \mathrm{dom} f_n = \emptyset$

(ix) $rng(a_n \restriction On)$ has a maximal element and it is above all the rest of the elements of $rng(a_n \restriction On)$ in the Rudin-Kiesler order, i.e. for every $\beta \in rng(a_n \restriction On) \backslash \{max(rnga_n)\}$ $\quad \mathcal{U}_{n\beta} <_{RK} \mathcal{U}_{n \max(rnga)}$.

(x) $rng(a_n \backslash O_n)$ has a maximal under the inclusion model. Denote it further by $\max^1(p_n)$ or $\max^1(a_n)$.

The purpose of a_n, as in the corresponding definition of $\mathcal{P}^*$ in Section 1, is to connect between ordinals above κ and those at level n. We added here submodels to a_n. The role of them will be crucial for proving chain conditions of the main forcing. Notice that in [Git3] submodels does not appear at stage of $\mathcal{P}^*$ explicitly but rather implicitly via coding by ordinals. The conditions (iii) and (v) are technical and will allow further an interplay between levels $n-1$ and n.

(e) $A_n \in \mathcal{U}_{n\max(rng(a_n\ \ On))}$ and $\min A_n^0 > \sup(rngh_{>n-1})$ if $n>0$, where A_n^0 is element by element projection of A_n to the normal measure of E_n, $\mathcal{U}_{n\kappa}$, i.e.

$$A_n^0 = \{\nu^0 \mid \nu \in A_n\}\ , \quad \nu^0 = \pi_{\max(rng(a_n\ \ On)),0}(\nu)\ .$$

(f) S_n is function on A_n^0 so that for every $\rho \in A_n^0 \quad S_n(\rho) \in \mathrm{Col}(\rho^{+\kappa_{n-1}+1}, <\kappa_n)$, where $\kappa_{-1}=1$.

Here, as usual, in such matters S_n provides information about potential collapses. Thus, once one element Prikry sequence ρ_n for the normal measure is picked, $S_n(\rho_n)$ turns into condition of the actual collapse used below κ_n. Notice also that S_n depends only on the normal measure and no indiscernibles are collapsed. This allows to use S_n's freely without restrictions of the type $[S_n]_{\mathcal{U}_{n,\kappa}}$ is in a certain generic set in the $Ult(V,\mathcal{U}_{n,\kappa})$.

(g) for every $\alpha,\beta,\gamma \in rnga_n$ if $\mathcal{U}_{n\alpha} \geq_{RK} \mathcal{U}_{n\beta} \geq_{RK} \mathcal{U}_{n\gamma}$ then

$$\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$$

for every $\rho \in \pi''_{\max(rnga_n\ \ On),\alpha} A_n$

(h) for every $\alpha>\beta$ in $rnga_n$ and $\nu \in A_n$

$$\pi_{\max(rnga_n\ \ On),\alpha}(\nu) > \pi_{\max(rnga_n\ \ On),\beta}(\nu)\ .$$

(4) if $n>\ell(p)$ then

$$\underset{\sim}{p_n} = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, h_{>n}, f_n\rangle$$

is so that

(a) f_n is a partial function from λ to κ_n of cardinality at most κ.

(b) $h_{>n} \in \mathrm{Col}(\kappa_n, <\kappa_{n+1})$.

Once $a_{n-1} \upharpoonright On$ and one element Prikry sequence $\nu \in A_{n-1}$ are decided, e_n, a_n, A_n and S_n are also determined and satisfy the following:

(c) the same as (3)(c) but with the following addition:

(iii) $\delta_{n-1} = (\nu^0)^{+\rho^*_{n-2}+1}$, where $\rho^*_{-1} = 1$ and if $n-2 \geq 0$ then $\rho^*_{n-2} = \rho_{n-2}^{+\rho^*_{n-3}+1}$ which is defined by induction using elements of the Prikry sequence for normal measures $\rho_0, \ldots, \rho_{n-2}$.

(iv) $\mathrm{dom} e_n = (\mathrm{dom} a^*_{n-1}) \cap \mathrm{Card} \cup \{\kappa^{+\delta+2}\}$ and for every $\alpha \in \mathrm{dom} a^*_{n-1} \cap \mathrm{Card}$
$e_n(\alpha) = \kappa_n^{+a^*_{n-1}(\alpha)+1}$
were a^*_{n-1} is the function with domain as those of $a_{n-1} \restriction On$
and $a^*_{n-1}(\alpha) = \pi_{\max(rnga_{n-1}\ On), a_{n-1}(\alpha)}(\nu)$ for every $\alpha \in \mathrm{dom} a_{n-1} \restriction On$.

Notice that $E_{n-1} \restriction \kappa_{n-1}^{+\rho^*_{n-2}+1}$ is used over κ_{n-1}. Hence each $a^*_{n-1}(\alpha)$ will be below $(\nu^0)^{+\rho^*_{n-2}+1} = \delta_{n-1}$.

The rest of the requirements are exactly as (3)(d)-(h).

Let $n = \ell(p)$. For every $t \in \mathrm{dom} a_n$ (either an ordinal or a submodel) there is a sequence $\langle k_m \mid m < \omega \rangle$ nondecreasing and converging to infinity so that the following holds:

(i) For every $m > \ell(p)$ once $\langle \underset{\sim}{p_i} | i < m \rangle$ are decided (and does not matter either way) $t \in \mathrm{dom} a_m$ and $a_m(t)$ realizes k_n-good type.

This is a reformulation of conditions on monotonicity of $\mathrm{dom} a_n$'s of Section 1. Only here we have names instead of actual sets in Section 1.

Definition 2.2 Let $p, q \in \mathcal{P}^*$. Set $p \geq q$ iff

(1) $\ell(p) \geq \ell(q)$

(2) for every $n < \ell(q)$ let $p_n = \langle \rho_n, h_{>n}, h_{<n}, f_n \rangle$ and $q_n = \langle \xi_n, t_{>n}, t_{<n}, g_n \rangle$. Then the following holds:

(a) $\rho_n = \xi_n$

(b) $t_{>n} \subseteq h_{>n}$

(c) $t_{<n} \subseteq h_{<n}$

(d) $g_n \subseteq f_n$

(3) if $n = \ell(q) < \ell(p)$ then the following holds, where $p_n = \langle \rho_n, h_{<n}, h_{>n}, f_n \rangle$ and $q_n = \langle e_n, a_n, A_n, S_n, t_{>n}, g_n \rangle$:

(a) $f_n \supseteq g_n$

(b) $\mathrm{dom} f_n \supseteq \mathrm{dom} a_n \restriction On$

(c) $f_n(\max(\mathrm{dom}(a_n \restriction On)) \in A_n$.

Denote this ordinal by ρ.

(d) for every $\beta \in \mathrm{dom} a_n \restriction On$

$$f_n(\beta) = \pi_{\max(rng(a_n\ On)),a_n(\beta)}(\rho)$$

(e) $\rho_n = \rho^0$

(f) $h_{<n} \supseteq S_n(\rho^0)$

(g) $h_{>n} \supseteq t_{>n}$

(4) if $\ell(q) < n < \ell(p)$ then the following holds where $p_n = \langle \rho_n, h_{<n}, h_{>n}, f_n \rangle$ and $\underset{\sim}{q_n} =$

$\langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, t_{>n}, g_n \rangle$

(a) $f_n \supseteq g_n$ and $h_{>n} \supseteq t_{>n}$

(b) $\langle p_k \mid k < n \rangle$ decides e_n, a_n, A_n and S_n

(c) the condition (3)(b)-(d) hold for $\langle e_n, a_n, A_n, S_n, f_n \rangle$ and p_n.

(5) if $n \geq \ell(p) > \ell(q)$ or $n > \ell(p)$ then the following holds, where $\underset{\sim}{q_n} = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, h_{>n}, f_n \rangle$

and $\underset{\sim}{p_n} = \langle \underset{\sim}{d_n}, \underset{\sim}{b_n}, \underset{\sim}{B_n}, \underset{\sim}{T_n}, t_{>n}, g_n \rangle$

(a) $f_n \subseteq g_n$ and $h_{>n} \subseteq t_{>n}$

(b) it is forced in the simple fashion by only deciding $\underset{\sim}{p_m}$'s $(m < n)$ that

(i) $\underset{\sim}{d_n} \supseteq \underset{\sim}{e_n}$

(ii) $\underset{\sim}{b_n} \supseteq \underset{\sim}{a_n}$

(iii) $\pi''_{\max(rng\underset{\sim}{b_n}\ On),\max(rng\underset{\sim}{a_n}\ On)} \underset{\sim}{B_n} \subseteq \underset{\sim}{A_n}$

(iv) for every $\nu \in \underset{\sim}{B_n^0}$

$$\underset{\sim}{S_n}(\nu) \subseteq \underset{\sim}{T_n}(\nu)$$

(6) if $n = \ell(p) = \ell(q)$ then the following holds, where $q_n = \langle e_n, a_n, A_n, S_n, h_{>n}, f_n\rangle$ and $p_n = \langle d_n, b_n, B_n, T_n, t_{>n}, g_n\rangle$:

(a) $f_n \subseteq g_n$ and $h_{>n} \subseteq t_{>n}$

(b) $e_n = d_n$

Here is where it differs from the previous case. We are not allowed to change e_n once we got to the level $n = \ell(q)$.

(c) $b_n \supseteq a_n$

(d) $\pi''_{\max(rngb_n\ On),\max(rnga_n\ On)} B_n \subseteq A_n$

(e) for every $\nu \in B_n^0$

$$T_n(\nu) \supseteq S_n(\nu)$$

Definition 2.3 Let $p, q \in \mathcal{P}^*$. Set $p \geq^* q$ iff

(1) $p \geq q$

(2) $\ell(p) = \ell(q)$

Definition 2.4 Let p and q be in $\mathcal{P}^*$. We call p and q equivalent and denote this by $p \leftrightarrow q$ iff

(1) $\ell(p) = \ell(q)$

(2) for every $n < \ell(p)$ $p_n = q_n$

(3) there is a nondecreasing sequence $\langle k_n \mid \ell(p) \leq n < \omega\rangle$ with $\lim_{n\to\infty} k_n = \infty$ and $k_{\ell(p)} > 2$ such that the following holds for every n, $\ell(p) \leq n < \omega$ where $\underset{\sim}{p_n} = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h_{>n}}, \underset{\sim}{f_n}\rangle$ and $\underset{\sim}{q_n} = \langle \underset{\sim}{d_n}, \underset{\sim}{b_n}, \underset{\sim}{B_n}, \underset{\sim}{T_n}, \underset{\sim}{t_{>n}}, \underset{\sim}{g_n}\rangle$

(a) if $n = \ell(p)$, then

(i) $f_n = g_n$

(ii) $e_n = d_n$

(iii) $h_{>n} = t_{>n}$

(iv) $\mathrm{dom} a_n = \mathrm{dom} b_n$

(v) $rnga_n$ and $rngb_n$ are realizing the same k_n-type

(vi) $A_n = B_n$

(vii) $S_n = T_n$

(b) if $n > \ell(p)$, then every common extension $\langle r_m \mid m < n\rangle$ of $\langle \underset{\sim}{p}_m \mid m < n\rangle$ and $\langle \underset{\sim}{q}_m \mid m < n\rangle$ which decides the first n elements of the Prikry sequence for the normal measures decides $\underset{\sim}{e}_n, \underset{\sim}{a}_n, \underset{\sim}{A}_n, \underset{\sim}{S}_n$ and $\underset{\sim}{d}_n, \underset{\sim}{b}_n, \underset{\sim}{B}_n, \underset{\sim}{T}_n$ so that they satisfy the conditions (i)-(vii) of (a) above.

Definition 2.5 Let $p, q \in \mathcal{P}^*$ we set $p \to q$ iff there is a sequence $\langle r_k \mid k < m < \omega\rangle$ of elements of $\mathcal{P}^*$ so that

(1) $r_0 = p$

(2) $r_{m-1} = q$

(3) for every $k < m - 1$

$$r_k \leq r_{k+1} \quad \text{or} \quad r_k \leftrightarrow r_{k+1} \ .$$

As in Section 1, the following two lemmas showing that $\langle \mathcal{P}^*, \to\rangle$ is a nice subforcing of $\langle \mathcal{P}^*, \leq\rangle$ are valid.

Lemma 2.6 *Let $p, q, s \in \mathcal{P}^*$. Suppose that $p \longleftrightarrow q$ and $s \geq p$. Then there are $s' \geq s$ and $t \geq q$ such that $s' \longleftrightarrow t$.*

Lemma 2.7 *For every $p, q \in \mathcal{P}^*$ such that $p \to q$ there is $s \geq p$ so that $q \to s$.*

3 The Preparation Forcing

We define first a part of the preparation forcing above κ. The definition follows the lines of [Git4]. It is desired to reduce the number of possible choices gradually to κ^+.

Fix an ordinal $\delta > 1$.

Definition 3.1 The set $\mathcal{P}'$ consists of pairs $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle$ so that the following holds:

(1) for every $\tau \leq \delta$ $A^{0\tau}$ is an elementary submodel of $\langle H(\kappa^{+\delta+2}), \epsilon, \langle \kappa^{+i} \mid i \leq \delta + 2\rangle\rangle$ such that

(a) $|A^{0\tau}| = \kappa^{+\tau+1}$ and $A^{0\tau} \supseteq \kappa^{+\tau+1}$ unless for some $n < \omega$ and an inaccessible τ', $\tau = \tau' + n$ and then $|A^{0\tau}| = \kappa^{+\tau}$ and $A^{0\tau} \supseteq \kappa^{+\tau}$

(b) ${}^{|A^{0\tau}|>}A^{0\tau} \subseteq A^{0\tau}$

(2) for every $\tau < \tau' \leq \delta$, $A^{0\tau} \subseteq A^{0\tau'}$

(3) for every $\tau \leq \delta$, $A^{1\tau}$ is a set of at most $\kappa^{+\tau+1}$ elementary submodels of $A^{0\tau}$ so that

(a) $A^{0\tau} \in A^{1\tau}$

(b) if $B, C \in A^{1\tau}$ and $B \subsetneqq C$ then $B \in C$

(c) if $B \in A^{1\tau}$ is a successor point of $A^{1\tau}$ then B has at most two immediate predecessors under the inclusion and is closed under $\kappa^{+\tau}$-sequences.

(d) let $B \in A^{1\tau}$ then either B is a successor point of $A^{1\tau}$ or B is a limit element and then there is a closed chain of elements of $B \cap A^{1\tau}$ unbounded in $B \cap A^{1\tau}$ and with limit B.

(e) for every $\tau', \tau \leq \tau' \leq \delta$, $A \in A^{1\tau}$ and $B \in A^{1\tau'}$ either $B \supseteq A$ or there are $\ell < \omega$ and $\tau'_1, \tau'_2, \ldots, \tau'_\ell$, $\tau \leq \tau'_1 \leq \cdots \leq \tau'_\ell \leq \delta$, $B_1 \in A \cap A^{1\tau'_1}, \ldots, B_\ell \in A \cap A^{1\tau'_\ell}$ such that

$$B \cap A = B_1 \cap \cdots \cap B_\ell \cap A\ ,$$

if $\tau = \tau'$, then we can pick τ'_1 (and hence all the rest) above τ.

(f) let A be an elementary submodel of $H(\kappa^{+\delta+2})$ of cardinality $|A^{0\tau}|$, closed under $< |A^{0\tau}|$-sequences, $|A^{0\tau}| \in A$ and including $\langle\langle A^{0\tau'}, A^{1\tau'}\rangle \mid \tau' \leq \delta\rangle$ as an element, for some $\tau \leq \delta$. Then for every τ', $\tau \leq \tau' \leq \delta$ and $B \in A^{1\tau'}$ either $B \supseteq A$ or there are $\tau'_1, \ldots, \tau'_\ell$, $\tau \leq \tau'_1 \leq \cdots \leq \tau'_\ell \leq \delta$, $B_1 \in A \cap A^{1\tau'_1}, \ldots, B_\ell \in A \cap A^{1\tau'_\ell}$ such that

$$B \cap A = B_1 \cap \cdots \cap B_\ell \cap A\ .$$

Let for $\tau \leq \delta$ $A^{1\tau}_{in}$ be the set $\{B \cap B_1 \cap \cdots \cap B_n \mid B \in A^{1\tau}, n < \omega$ and $B_i \in A^{1\rho_i}$ for some $\rho_i, \tau < \rho_i \leq \delta$ for every $i, 1 \leq i \leq n\}$.

Definition 3.2 Let $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle$ and $\langle\langle B^{0\tau}, B^{1\tau}\rangle \mid \tau \leq \delta\rangle$ be elements of $\mathcal{P}'$. Then $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle \geq \langle\langle B^{0\tau}, B^{1\tau}\rangle \mid \tau \leq \delta\rangle$ iff for every $\tau \leq \delta$

(1) $A^{1\tau} \supseteq B^{1\tau}$

(2) for every $A \in A^{1\tau}$ either

(a) $A \supseteq B^{0\tau}$

or

(b) $A \subset B^{0\tau}$ and then $A \in B^{1\tau}$

or

(c) $A \not\supseteq B^{0\tau}$, $B^{0\tau} \not\supseteq A$ and then $A \cap B^{0\tau} \in B^{1\tau}_{in}$.

Definition 3.3 Let $\tau \leq \delta$. Set $\mathcal{P}'_{\geq\tau} = \{\langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \mid \exists\langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle$ $\langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle^\frown\langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}'\}$.

Let $G(\mathcal{P}'_{\geq\tau}) \subseteq \mathcal{P}'_{\geq\tau}$ be generic. Define $\mathcal{P}'_{<\tau} = \{\langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle \mid \exists\langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in G(\mathcal{P}'_{\geq\tau})$ $\langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle^\frown\langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}'\}$.

The following lemma is obvious

Lemma 3.4 $\mathcal{P}' \simeq \mathcal{P}'_{\geq\tau} * \underset{\sim}{\mathcal{P}'}_{<\tau} \quad (\tau \leq \delta)$.

Now we are ready to define the main preparation forcing. There is a clear structural parallel between this forcing and the main preparation forcings of [Git3, Sec. 4] and [Git4].

Definition 3.5 The set $\mathcal{P}$ consists of sequences of triples $\langle\langle A^{0\tau}, A^{1\tau}, F^\tau\rangle \mid \tau \leq \delta\rangle$ so that the following holds:

(0) $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle \in \mathcal{P}'$

(1) for every $\tau_1 \leq \tau_2 \leq \delta$
$F^{\tau_1} \subseteq F^{\tau_2} \subseteq \mathcal{P}^*$ ($\mathcal{P}^*$ of the previous section)

(2) for every $\tau \leq \delta$, F^τ is as follows:

(a) $|F^\tau| = |A^{0\tau}|$

(b) for every $p = \langle p_n \mid n \leq \ell(p)\rangle^\cap\langle \underset{\sim}{p_n} \mid \omega > n > \ell(p)\rangle \in F^\tau$ the following holds:

(i) each ordinal mentioned in p_n for $n < \ell(p)$ is in $(A^{0\tau} \cap \kappa^{+\delta+2}) \cap \{|A^{0\tau}|\}$

(ii) for every $n \geq \ell(p)$, for every extension $\langle r_m \mid m < n\rangle$ of $\langle p_m \mid m < \ell(p)\rangle^\cap\langle \underset{\sim}{p_m} \mid m < n\rangle$ deciding first n elements of the Prikry sequence for the normal measure (iii)$_1$ every ordinal appearing in p_n, as it is decided by $\langle r_m \mid m < n\rangle$, is in $(A^{0\tau} \cap \kappa^{+\delta+2}) \cup \{|A^{0\tau}|\}$.

(iii)$_2$ every submodel in the domain of correspondence function a_n of p_n, as it is decided by $\langle r_m \mid m < n\rangle$ belongs to one of the following sets:

$$\{A \subseteq A^{0\tau} \mid \kappa^+ \leq |A| < |A^{0\tau}| \ ,$$

A is an elementary submodel, and for every τ', $\tau \leq \tau' \leq \delta$ and $B \in A^{1\tau'}$ either $B \supseteq A$ or there are $\ell < \omega$ and $\tau'_1, \ldots, \tau'_\ell$, $\tau \leq \tau'_1 \leq \cdots \leq \tau'_\ell \leq \delta$, $B_1 \in A \cap A^{1\tau'_1}, \ldots, B_\ell \in A \cap A^{1\tau'_\ell}$ such that

$$B \cap A = B_1 \cap \cdots \cap B_\ell \cap A\} \ ,$$

$A^{1\tau}$ and $A^{1\tau}_{in}$

such that the picked elements of the last two sets are required to be closed under $< |A^{0\tau}|$-sequences of its elements. If $\tau = 0$, then the first set is empty.

(c) if $p \in F^\tau$ and $q \in \mathcal{P}^*$ is equivalent to p (i.e. $p \leftrightarrow q$) with witnessing sequence $\langle k_n \mid n < \omega\rangle$ starting with $k_0 \geq 4$ then $q \in F^\tau$. This condition as well as the next one provide a closure of F^τ under certain changes of its elements.

(d) if $p = \langle p_n \mid n \leq \ell(p)\rangle^\frown\langle \underset{\sim}{p_n} \mid \omega > n \geq \ell(p)\rangle \in \mathcal{P}^*$ and $q = \langle q_n \mid n < \ell(q)\rangle^\frown\langle \underset{\sim}{q_n} \mid \omega > n \geq \ell(q)\rangle \in F^\tau$, $\underset{\sim}{p_n} = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h_{\geq n}}, \underset{\sim}{f_n}\rangle$ and $\underset{\sim}{q_n} = \langle \underset{\sim}{d_n}, \underset{\sim}{b_n}, \underset{\sim}{B_n}, \underset{\sim}{T_n}, \underset{\sim}{t_{\geq n}}, \underset{\sim}{g_n}\rangle$ for $n \geq \ell(p)$ or $n \geq \ell(q)$ respectively, *then* $p \in F^\tau$ provided

(i) $p \geq q$ (in the order of $\mathcal{P}^*$)

(ii) for every $n < \ell(p)$ every ordinal appearing in p_n is in $A^{0\tau}$

(iii) $a_{\ell(p)} = b_{\ell(p)}$

(iv) for every $n > \ell(p)$ for every $\langle r_m \mid m \leq n-1\rangle$ extending $\langle \underset{\sim}{p_m} \mid m \leq n-1\rangle$ and deciding first $n-1$ elements of the Prikry sequence for the normal measures and so also $\langle e_n, a_n, A_n, S_n\rangle$ and $\langle d_n, b_n, B_n, T_n\rangle$ we require that $a_n = b_n$.

(v) for every $n \geq \ell(p)$ every ordinal appearing in f_n is in $A^{0\tau}$.

The meaning is that we are free to make changes in all the components of an element of F^τ except a_n's (and hence also e_n's). There we should be more careful.

The next two condition allow adding ordinals and submodels.

(e) for every $q \in F^\tau$ and $\alpha \in A^{0\tau}$ there is $p \in F^\tau$ $p = \langle p_n \mid n \leq \ell(p)\rangle^\frown\langle \underset{\sim}{p_n} \mid n > \ell(p)\rangle, \underset{\sim}{p_n} = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h_{\geq n}}, \underset{\sim}{f_n}\rangle$ $(n \geq \ell(p))$ such that $p \geq^* q$ and starting with some $n_0 \geq \ell(p)$ for every extension of $\langle \underset{\sim}{p_m} \mid m < n\rangle$ deciding elements of the Prikry sequence for the normal measures (and so also $\underset{\sim}{a_n}$) we have that $\alpha \in \mathrm{dom} a_n$.

(f) for every $q \in F^\tau$ and $B \in A^{1\tau} \cup A^{1\tau}_{in}$ $^{|B|>}B \subseteq B$, there is $p \in F^\tau$ $p = \langle p_n \mid n \leq \ell(p)\rangle^\frown \langle \underset{\sim}{p_n} \mid \omega > n > \ell(p)\rangle$, $p_n = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h_{\geq n}}, \underset{\sim}{f_n}\rangle$ $(n \geq \ell(p))$ such that $p \geq^* q$ and starting with some $n_0 \geq \ell(p)$ for every extension of $\langle \underset{\sim}{p_m} \mid m < n\rangle$ deciding n elements of the Prikry sequence for the normal measures we have $B \in \mathrm{dom} a_n$. We require also that p is obtained from q by adding only B and probably the intersections of it with other models appearing in q and needed to be added after adding B.

The next condition allows us to put together certain elements of F^τ remaining inside F^τ.

(g) Let $p, q \in F^\tau$ be so that

(i) $\ell(p) = \ell(q)$

(ii) $p_n = q_n$ for every $n < \ell(p)$

(iii) f_n, g_n are compatible (i.e. $f \cup g$ is a function) and also $h_{>n}$, $t_{>n}$ are compatible for every $n \geq \ell(p)$, where $p_n = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h_{\geq n}}, \underset{\sim}{f_n}\rangle$ and $q_n = \langle \underset{\sim}{d_n}, \underset{\sim}{b_n}, \underset{\sim}{B_n}, \underset{\sim}{T_n}, \underset{\sim}{t_{\geq n}}, g_n\rangle$

(iv) $\max^1(q_{\ell(p)}) \in \mathrm{dom} a_{\ell(p)}$ and $a_{\ell(p)} \restriction \max^1(q_{\ell(p)}) \subseteq b_{\ell(p)}$, where $\max^1(q_{\ell(p)})$ denotes the maximal model of $\mathrm{dom} b_{\ell(p)}$ and

$$a_n \restriction B = \{\langle t \cap B, s \cap a_n(B)\rangle \mid \langle t, s\rangle \in a_n\}$$

(v) $e_{\ell(p)} = d_{\ell(p)}$

(vi) $S_{\ell(p)}$ and $T_{\ell(p)}$ are compatible via obvious projection.

(vii) for every $n > \ell(p)$ there is a common extension of $\langle \underset{\sim}{p_m} \mid m < n\rangle$ and $\langle \underset{\sim}{q_m} \mid m < n\rangle$ deciding first n elements of the Prikry sequence for the normal measures.

(viii) for every $n > \ell(p)$ and every common extension as in (vii) the decided values $\langle e_n, a_n, A_n, S_n\rangle$ of p_n and $\langle d_n, b_n, B_n, T_n\rangle$ of q_n satisfy the following

(viii)$_1$ $\max^1(a_n) = \max^1(a_{\ell(p)})$ and $\max^1(b_n) = \max^1(b_{\ell(p)})$

(viii)$_2$ $\max^1(b_n) \in \mathrm{dom} a_n$

(viii)$_3$ $a_n \restriction \max^1(b_n) \subseteq b_n$

(viii)$_4$ $e_n = d_n$

(viii)$_5$ S_n and T_n are compatible via the obvious projection
then the union of p and q is in F^τ, where the union is defined in obvious fashion taking at each $n \geq \ell(p)$, $a_n \cup b_n$, $f_n \cup q_n$ etc.

(h) there is $F^{\tau *} \subseteq F^\tau$ such that

(i) $F^{\tau *}$ is $\leq^*$-dense in F^τ, i.e. for every $p \in F^\tau$ there is $q \in F^{\tau *}$ with $q \geq^* p$

(ii) $F^{\tau *}$ is closed under unions of $\leq^*$-increasing sequences of its elements, i.e. every $\leq^*$-increasing sequence of elements of $F^{\tau *}$ having union in $\mathcal{P}^*$ has it also in $F^{\tau *}$

(iii) $F^{\tau *}$ is closed under the equivalence relation "$\longleftrightarrow$"

(iv) for every $p \in F^{\tau *}$ $A^{0\tau}$ appears in every $\underset{\sim}{p_n}$ $(n \geq \ell(p))$

(v) for every $p \in F^{\tau *}$, $p = \langle p_n \mid n \leq \ell(p)\rangle^\frown \langle \underset{\sim}{p_n} \mid \omega > n > \ell(p)\rangle$ if $q = \langle q_m \mid m \leq \ell(q)\rangle^\frown \langle \underset{\sim}{q_n} \mid \omega > m\rangle \ell(q)\rangle \geq p$ satisfies the conditions $(\alpha), (\beta)$ below then $q \in F^{\tau *}$

(α) $\langle q_m \mid m < \ell(q)\rangle$ forces (or decides) $\check{q}_{\ell(q)} = \underset{\sim}{p_{\ell(q)}}$

(β) for every $k, \ell(q) < k < \omega$
$\langle q_m \mid m \leq \ell(q)\rangle^\frown \langle \underset{\sim}{q_m} \mid \ell(q) < m < k\rangle$ decides that $\underset{\sim}{q_m} = \underset{\sim}{p_k}$.

(vi) for every $p \in F^{\tau *}$,

$$p = \langle p_n \mid n \leq \ell(p)\rangle^\frown \langle \underset{\sim}{p_n} \mid > n > \ell(p)\rangle\ ,\ \underset{\sim}{p_n} = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{T_n}, \underset{\sim}{h_{\geq n}}, \underset{\sim}{f_n}\rangle$$

if $q = \langle q_n \mid n \leq \ell(q)\rangle^\frown \langle \underset{\sim}{q_n} \mid \omega > n > \ell(q)\rangle$ is such that

(α) $q \in F^{\tau *}$

(β) $q > p$

(γ) $\ell(q) = \ell(p) + 1$

then $p' \in F^\tau$ where

$$p' = \langle q_n \mid n < \ell(p)\rangle^\frown \langle \underset{\sim}{p'_n} \mid \omega > n \geq \ell(p)\rangle$$

is such that

(α) $p'_{\ell(p)}$ is as $p_{\ell(p)}$ the last coordinate (i.e. $f_{\ell(p)}$) is replaced by

$$f_{\ell(p)} \cup q_{\ell(p)} \restriction (On \backslash \mathrm{dom} a_{\ell(p)})\ .$$

(β) for every $n > \ell(p)$ the following holds:

$(\beta)_1$ the last coordinate of p_n is replaced in p'_n by those of q_n

$(\beta)_2$ for every $A \in \left(\bigcup_{\tau \leq \rho \leq \delta} A^{1\rho}\right) \cup \{A \subseteq A^{0\tau} \mid A$ is as it was reqired in (b)(ii)$_2\} \cap \mathrm{dom} a_{\ell(p)}$, for every $\langle r_m \mid m < n\rangle$ extending $\langle \underset{\sim}{q_m} \restriction A \mid m < n\rangle$

$\langle r_m \mid m < n\rangle$ decides that $p'_n \restriction A$ and $q_n \restriction A$ are the same.

The existence of such $F^{\tau *}$'s will be crucial for the proof of the Prikry condition of the final forcing.

The additional (relatively to [Git3]) complication here due to the use of names. During a proof of the Prikry condition different choices from set of measure one should be put together. In order to satisfy the requirement (f) above (which is in turn crucial for the chain condition) we need to do it gently. Thus models should by addable and restrictions to them need to be in F^τ. So we cannot allow extensions of original condition p which have the same Prikry sequence at level $\ell(p)$ for measures in some $A \in \mathrm{dom}(a_{\ell(p)}) \backslash On$ but disagree about elements of A at further levels.

The next condition allows us to restrict or to extend conditions remaining inside F^τ.

(i) Let $p = \langle p_n \mid n \leq \ell(p)\rangle^\frown \langle p_n \mid \ell(p) < n < \omega\rangle \in F^\tau$, $p_n = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h_{\geq n}}, \underset{\sim}{f_n}\rangle (n \geq \ell(p))$, $|B| = \kappa^{+\tau+1}$ or $B \in A^{1\tau'}$ for some $\tau' \leq \tau$. Suppose that for every $n \geq \ell(p)$ every extension of $\langle \underset{\sim}{p_m} \mid m < n\rangle$ deciding the first n elements of the Prikry sequence for the normal measures we have $B \in (\mathrm{dom} a_n) \backslash On$. Then $p \restriction B \in F^{\tau'}$, where $p \restriction B = \langle p_n \restriction B \mid n \leq \ell(p)\rangle^\frown \langle \underset{\sim}{p_n} \restriction B \mid \omega > n \geq \ell(p)\rangle$, for every $n < \ell(p)$ $p_n \restriction B$ is just the usual restriction of the functions of p_n to B; if $n = \ell(p)$ then $p_n \restriction B = \langle e_n \restriction B,\ a_n \restriction B,\ \pi_{\max a_n, B}{}'' A_n, S_n, h_{>n}, f_n \restriction B\rangle$, where $a_n \restriction B$ is defined as in (g)(iv); if $n > \ell(p)$ then $\underset{\sim}{p_n} \restriction B$ is defined as above only dealing with names.

(j) let $p = \langle p_n \mid n \leq \ell(p)\rangle^\frown \langle \underset{\sim}{p_n} \mid \ell(p) < n < \omega\rangle \in F^\tau$, $p_n = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h_{\geq n}}, \underset{\sim}{f_n}\rangle$ $(n \geq \ell(p))$ and for every $n \geq \ell(p)$ every extension of $\langle \underset{\sim}{p_m} \mid m < n\rangle$ deciding the first n elements of the Prikry sequence for the normal measures we have $A^{0\tau} \notin \mathrm{dom} a_n$. Let $\langle \sigma_n \mid \omega > n \geq \ell(p)\rangle$ be so that

(α) $\sigma_n \prec \mathcal{A}_{n,k_n}$ and $|\sigma_n|$ is k_n good for every $n \geq \ell(p)$

(β) $\langle k_n \mid n \geq \ell(p)\rangle$ is increasing

(γ) $k_0 \geq 5$

(δ) ${}^{|\sigma_n|>}\sigma_n \subseteq \sigma_n$ for every $n \geq \ell(p)$

(ξ) for every $n \geq \ell(p)$ every extension $\langle r_m \mid m < n\rangle$ of $\langle \underset{\sim}{p_m} \mid m < n\rangle$ deciding the first n elements of the Prikry sequence for the normal measures, and hence also at $\underset{\sim}{a_n}$, we have $rnga_n \subseteq \sigma_n$.

Then the condition obtained from p by adding $\langle A^{0\tau}, \sigma_n\rangle$ to each a_n with $n \geq \ell(p)$ belongs to F^τ.

(k) if A is an elementary submodel of $H(\kappa^{+\delta+2})$ of a regular cardinality $\kappa^{+\rho}$, closed under $< \kappa^{+\rho}$-sequences and with $\langle\langle A^{0\tau'}, A^{1\tau'}\rangle \mid \tau' < \delta\rangle \in A$, for some $\rho < \tau$, then A is addable to any $p \in F^\tau \cap A$ with the maximal element of $\mathrm{dom}\underset{\sim}{a_n}$'s $A^{0\tau}$, i.e. $A \cap A^{0\tau}$ can be added to p remaining inside F^τ. Also we allowed to correspond A to any sequence of submodels as in (j) only replacing $rnga_n \subseteq \sigma_n$ in (j)(ξ) by $rnga_n \in \sigma_n$ and keeping σ_n of the cardinality corresponding to $\kappa^{+\rho}$.

Definition 3.6 Let $\langle\langle A^{0\tau}, A^{1\tau}, F^\tau\rangle \mid \tau \leq \delta\rangle$ and $\langle\langle B^{0\tau}, B^{1\tau}, F^\tau\rangle \mid \tau \leq \delta\rangle$ be in $\mathcal{P}$. We define

$$\langle\langle A^{0\tau}, A^{1\tau}, F^\tau\rangle \mid \tau \leq \delta\rangle > \langle\langle B^{0\tau}, B^{1\tau}, G^\tau\rangle \mid \tau \leq \delta\rangle$$

iff

(1) $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle > (B^{0\tau}, B^{1\tau}\rangle \mid \tau \leq \delta\rangle$ in $\mathcal{P}$'

(2) for every $\tau \leq \delta$

(a) $F^\tau \supseteq G^\tau$

(b) for every $p \in F^\tau$ and $B \in B^{1\tau} \cup B^{1\tau}_{in}$ if for every $n \geq \ell(p)$ once a_n is decided $B \in \mathrm{dom} a_n$, then $p \restriction B \in G^\tau$, where the restriction is defined as in 3.5 (2g(iv)) and, as usual,

$$p = \langle p_n \mid n \leq \ell(p)\rangle^\frown \langle \underset{\sim}{p_n} \mid \ell(p) \leq n < \omega\rangle \ ,$$

$$p_n = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{h}_{>n} \underset{\sim}{f_n}\rangle$$

for $n \geq \ell(p)$.

Definition 3.7 Let $\tau \leq \delta$. Set $\mathcal{P}_{\geq\tau} = \{\langle A^{0\rho}, A^{1\rho}, F^\rho \mid \tau \leq \rho \leq \delta\rangle \mid \exists\langle\langle A^{0\nu}, A^{1\nu}, F^\nu\rangle \mid \nu < \tau\rangle \ \langle\langle A^{0\nu}, A^{1\nu}, F^\nu\rangle \mid \nu < \tau >^\frown \langle\langle A^{0\rho}, A^{1\rho}, F^\rho\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}\}$.

Let $G(\mathcal{P}_{\geq\tau}) \subseteq \mathcal{P}_{\geq\tau}$ be generic. Define $\mathcal{P}_{<\tau} = \{\langle\langle A^{0\nu}, A^{1\nu}, F^\nu\rangle \mid \nu < \tau\rangle \mid \exists\langle\langle A^{0\rho}, A^{1\rho}, F^\rho\rangle \mid \tau \leq \rho \leq \delta\rangle \in G(\mathcal{P}_{\geq\tau})\langle\langle A^{0\nu}, A^{1\nu}, F^\nu\rangle \mid \nu < \tau\rangle^\frown\langle\langle A^{0\rho}, A^{1\rho}, F^\rho\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}\}$.

The following lemma is obvious:

Lemma 3.8 *$\mathcal{P} \simeq \mathcal{P}_{\geq\tau} * \underset{\sim}{\mathcal{P}_{<\tau}}$ for every $\tau \leq \delta$.*

Let μ be a cardinal. Consider the following game $\mathcal{G}_\mu$:

$$\begin{array}{lllllll} I & s_1 & & s_3 \ \cdots & s_{2\alpha+1} & \cdots & \\ II & & s_2 & & & s_{2\alpha+2} & \cdots \end{array}$$

where $\alpha < \mu$ and the players are picking an increasing sequence of elements of $\mathcal{P}$ i.e. $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_{2\alpha+1} \leq s_{2\alpha+2} \leq \cdots$. The second player plays at even stages (including the limit ones) and the first at odd stages. The first player wins if at some stage $\alpha < \mu$ there is no legal move for II. Otherwise II wins.

$\mathcal{P}$ is called μ-strategically closed if there is a winning strategy for II in the game $\mathcal{G}_\mu$.

Lemma 3.9 *For every $\tau \leq \delta$. $\mathcal{P}_{\geq \tau}$ is $\kappa^{+\tau+1}$-strategically closed. Moreover, if there is no inaccessible $\tau' < \tau$ and $n < \omega$ such that $\tau = \tau' + n$, then $\mathcal{P}_{\geq \tau}$ is $\kappa^{+\tau+2}$-strategically closed.*

Proof. Fix $\tau \leq \delta$. Let $\langle \langle A_i^{0\rho}, A_i^{1\rho}, F_i^{\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle \mid i < i^* \rangle$ be an increasing sequence of conditions in $\mathcal{P}_{\geq \tau}$ already generated by playing the game and we need to define the move $\langle \langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho}, F_{i^*}^{\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle$ of Player I at stage i^*. Define it by induction on ρ. Thus suppose that $\langle \langle A_{i^*}^{0\rho'}, A_{i^*}^{1\rho'}, F_{i^*}^{\rho'} \rangle \mid \tau \leq \rho' < \rho \rangle$ is already defined. We define the triple $\langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho}, F_{i^*}^{\rho} \rangle$. First deal with $\langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho} \rangle$.

If i^* is a limit ordinal and $cfi^* = \kappa(\tau)$, where $\kappa(\widetilde{\rho}) = \kappa^{+\rho}$, if $\widetilde{\rho} = \rho' + n$ for an inaccessible ρ' and $n < \omega$ and $\kappa(\widetilde{\rho}) = \kappa^{+\rho+1}$ otherwise. Then we set $A_{i^*}^{0\rho} = \bigcup_{i<i^*} A_i^{0\rho}$ and $A_{i^*}^{1\rho} = \bigcup_{i<i^*} A_i^{1\rho} \cup \{A_{i^*}^{0\rho}\}$ whenever $\rho = \tau$. Now let $\tau < \rho \leq \delta$. Define $A_{i^*}^{0\rho}$ to be the closure under the Skolem functions and $< \kappa(\rho)$-sequences of $\langle \langle A_i^{j\rho'} \mid i < i^* \rangle \mid \tau \leq \rho' \leq \delta \rangle$ $(j \in 2)$, $\langle A_{i^*}^{i\rho'} \mid \tau \leq \rho' < \rho \rangle$, $\langle F_i^{\rho'} \mid \tau \leq \rho' \leq \delta, i < i^* \rangle$, $\langle F^{\rho'*} \mid \tau \leq \rho' \leq \delta, i < i^* \rangle$, $\langle F^{\rho'*} \mid \tau \leq \rho' \leq \delta, i < i^* \rangle$, $\langle F_{i^*}^{\rho'}, F_{i^*}^{\rho'*} \mid \tau \leq \rho' < \rho \rangle$. We set $A_{i^*}^{1\rho} = \bigcup_{i<i^*} A_i^{1\rho} \cup \{A_{i^*}^{0\rho}\}$.

If i^* is not limit ordinal or it is a limit ordinal but $cfi^* < \kappa(\tau)$, then we define $\langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho} \mid \tau < \rho \leq \delta \rangle$ as above and $\langle A_{i^*}^{0\tau}, A_{i^*}^{1\tau} \rangle$ is defined the same way as $\langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho} \rangle$ was defined above for $\rho > \tau$.

Let us show now that such defined $\langle \langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle$ is in $\mathcal{P}'$. Basically, we need to check the conditions (e) and (f) of Definition 3.1.

We start with (e). Let $\tau \leq \rho \leq \rho' \leq \delta$, $A \in A_{i^*}^{1\rho}$ and $B \in A_{i^*}^{1\rho'}$. If $A \in A_i^{1\rho}$ and $B \in A_{i'}^{1\rho'}$ for some i, $i' < i^*$, then we use (f) for $\langle \langle A_{\bar{i}}^{0\nu}, A_{\bar{i}}^{1\nu} \rangle \mid \tau \leq \nu \leq \delta \rangle$ where $\bar{i} = \max(i, i')$. It provides $\rho \leq \tau_1' \leq \cdots \leq \tau_\ell' \leq \delta$, $B_1 \in A \cap A_{\bar{i}}^{1\tau_1'}, \ldots, B_\ell \in A \cap A_{\bar{i}}^{1\tau_\ell'}$ such that $B \cap A = B_1 \cap \cdots \cap B_\ell \cap A$. Now, since $A_{\bar{i}}^{1\tau_k'} \subseteq A_{i^*}^{1\tau_k'}$ for every $1 \leq k \leq \ell$ we are done.

If $A \in A_i^{1\rho}$ for some $i < i^*$ and $B \in A_{i^*}^{1\rho'} \setminus \bigcup_{i<i^*} A_{i'}^{1\rho'}$ then $B \supseteq \bigcup_{i'<i^*} A_{i'}^{0\rho'}$. In particular, $B \supseteq A_i^{0\rho'} \supseteq A_i^{0\rho}$. If $A \in A_{i^*}^{1\rho} \setminus \bigcup_{i<i^*} A_i^{1\rho}$ and $B \in A_{i'}^{1\rho'}$ for some $i' < i^*$, then we can use 3.1(f) for A, B and $\langle \langle A_{i'}^{0\tau'}, A_{i'}^{1\tau'} \rangle \mid \tau' \leq \delta \rangle \in \mathcal{P}'$. If $A \in A_{i^*}^{1\rho} \setminus \bigcup_{i<i^*} A_i^{1\rho}$ and $B \in A_{i^*}^{1\rho'} \setminus \bigcup_{i<i^*} A_i^{1\rho'}$, then either $B \supseteq A$ or $B \subset A$ and in the last case $\rho' = \rho$ and $B \in A$.

Now let us check the condition (f). Thus let A be an elementary submodel of $H(\kappa^{+\delta+2})$ of cardinality $|A_{i^*}^{0\rho}|$, closed under $< |A_{i^*}^{0\rho}|$-sequences, $|A_{i^*}^{0\rho}| \in A$ and including $\langle \langle A_{i^*}^{0\tau'}, A_{i^*}^{1\tau'} \rangle \mid \tau' \leq \delta \rangle$ as an element, for some $\rho \leq \delta$. Let $\tau' \in [\rho, \delta]$ and $B \in A_{i^*}^{1\tau'}$. Suppose first that $B \in A_{i'}^{1\tau'}$ for some $i' < i^*$. Then, $\langle \langle A_{i'}^{0\nu}, A_{i'}^{1\nu} \rangle \mid \nu \leq \delta \rangle \in A$, since $A_{i^*}^{0\tau} \subseteq A_{i^*}^{0\rho} \subseteq A$ and the sequence $\langle \langle A_{i'}^{0\nu}, A_{i'}^{1\nu} \rangle \mid \nu \leq \delta \rangle \in A_{i^*}^{0\tau}$. So (f) of 3.1 applies to A, B and $\langle \langle A_{i'}^{0\nu}, A_{i'}^{1\nu} \rangle \mid \nu \leq \delta \rangle$ and we are done. Assume now that $B \in A_{i^*}^{1\tau'} \setminus \bigcup_{i<i^*} A_i^{1\tau'}$. Then by the definition of $A_{i^*}^{1\tau'}$, $B = A_{i^*}^{1\tau'}$. If $\tau' = \rho$, then $B \in A$ since $A \supseteq |A_{i^*}^{0\rho}| = \kappa(\rho)$. Hence $A_{i^*}^{0\rho} \in A$ and, also $A_{i^*}^{0\rho} \subseteq A$. Suppose now that $\tau' \notin A$. Set $\widetilde{\tau} = \min((A \setminus \tau') \cap On)$. Then $\widetilde{\tau} \leq \delta$ and $A_{i^*}^{0\tau} \in A$. But

$A \cap B = A \cap A^{0\tilde{\tau}}_{i^*}$, since the chain $\langle A^{0\tau''}_{i^*} \mid \tau'' \leq \delta \rangle \in A$.

Now we turn to the definitions of $F^{\rho}_{i^*}$ and its dense closed subset $F^{\rho *}_{i^*}$. We concentrate on $F^{\rho *}_{i^*}$. $F^{\rho}_{i^*}$ then is defined in a direct fashion satisfying conditions of 3.5.

Suppose first that i^* is a successor ordinal. Then $i^* \geq 2$, since the first player makes the first move. We denote by $p^\frown A$ for p and A as in 3.5(k) a condition obtained by adding A to p. Notice that varying images of A we can have a lot of different conditions. If some B appears in p then we denote by $p \backslash B$ the result of removing all appearances of B inside p. Define $F^{\rho *}_{i^*}$ to be the set including $F^{\rho^*}_{i^*-2}$ (if $i^* = 2$, then just ignore everything with index $i^* - 2$) and all conditions of the form $q^\frown A^{0\rho}_{i^*}$ so that either

(1) $q \in F^{\delta *}_{i^*-1} \cap A^{0\rho}_{i^*}$

(2) $A^{0\delta}_{i^*-2}$ appears in q and $q \restriction A^{0\delta}_{i^*-2} \in F^{\delta *}_{i^*-2}$

(3) $A^{0\rho}_{i^*-2}$ appears in q and $q \restriction A^{0\rho}_{i^*-2} \in F^{\rho *}_{i^*-2}$

or

there are r and t such that

(4) $r \in F^{\delta *}_{i^*-1} \cap A^{0\rho}_{i^*}$

(5) $A^{0\rho}_{i^*-2}$ appears in r and $r \restriction A^{0\delta}_{i^*-2} \in F^{\delta *}_{i^*-2}$

(6) $A^{0\rho}_{i^*-2}$ appears in r and $r \restriction A^{0\rho}_{i^*-2} \in F^{\rho *}_{i^*-2}$

(7) $t \in A^{0\rho}_{i^*} \cap \mathcal{P}^*$ $A^{0\rho}_{i^*-1}$ appears in t and each model appearing in t which does not belong to $A^{0\delta}_{i^*-1}$ is of cardinality less than $\kappa(\rho)$

(8) $r \geq^* t \restriction A^{0\rho}_{i^*-1}$

(9) $q = r \cup t$.

Let us show that the limitations (2),(3) and (5),(6) above are not very restrictive. Thus above every $r' \in F^{\delta}_{i^*-1}$ with $A^{0\delta}_{i^*-2}$ and $A^{0\rho}_{i^*-2}$ inside we find $r \geq^* r'$ in $F^{\delta *}_{i^*-1}$ with $r \restriction A^{0\delta}_{i^*-2} \in F^{\delta *}_{i^*-2}$ and $r \restriction A^{0\rho}_{i^*-2} \in F^{\rho *}_{i^*-2}$. Thus first extend r' to $r_{10} \in F^{\delta^*}_{i^*-1}$. Then consider $r'_{20} = r_{10} \restriction A^{0\delta}_{i^*-2}$. Extend it to r_{20} in $F^{\delta *}_{i^*-2}$. Let $r'_{30} = r_{20} \restriction A^{0\rho}_{i^*-2}$. Extend it to $r_{30} \in F^{\rho *}_{i^*-2}$. Now consider $r'_{11} = r_{10} \cup r_{20} \cup r_{30}$. It belongs to $F^{\rho}_{i^*-1}$ by 3.5(g). Extend it to $r_{11} \in F^{\rho *}_{i^*-1}$. Again consider $r'_{21} = r_{11} \restriction A^{0\delta}_{i^*-2}$ and extend it to $r_{21} \in F^{\delta *}_{i^*-2}$. Let $r'_{31} = r_{21} \restriction A^{0\rho}_{i^*-2}$ and r_{31} be its extension in $F^{\rho *}_{i^*-2}$. Continue by induction and define r_{jk} for every $j = 1, 2, 3$ and $k < \omega$. Then $r = \bigcup_{k<\omega} r_{1k}$ will be as desired, i.e. $r \in F^{\delta *}_{i^*-1}$, $r \restriction A^{0\delta}_{i^*-2} \in F^{\delta *}_{i^*-2}$ and $r \restriction A^{0\rho}_{i^*-2} \in F^{\rho *}_{i^*-2}$.

Let us show that such defined set $F^{\rho *}_{i^*}$ is closed. Thus suppose that $\langle p^\beta \mid \beta < \alpha\rangle$ is a $\leq^*$-increasing sequence of elements of $F^{\rho *}_{i^*}$ with union $p^\alpha \in \mathcal{P}^*$. We need to check that $p^\alpha \in F^{\rho *}_{i^*}$. Consider $\langle (p^\beta \restriction A^{0\delta}_{i^*-1})\backslash (A^{0\delta}_{i^*-1} \cap A^{0\rho}_{i^*}) \mid \beta < \alpha\rangle$ it will be a $\leq^*$-increasing sequence of elements of $F^{\delta *}_{i^*-1}$ with union $(p^\alpha \restriction A^{0\delta}_{i^*-1})\backslash (A^{0\delta}_{i^*-1} \cap A^{0\rho}_{i^*})$. We take $t = p^\alpha \backslash A^{0\rho}_{i^*}$ and $r = (p^\alpha \restriction A^{0\delta}_{i^*-1})\backslash (A^{0\delta}_{i^*-1} \cap A^{0\rho}_{i^*})$. Then $(r \cup t)^\frown A^{0\rho}_{i^*} = p_\alpha$ and it is in $F^{\rho^*}_{i^*}$ by the definition of the last set.

Suppose now that i^* is a limit ordinal. We first include $\bigcup\limits_{\substack{i<i^* \\ i \text{ is even}}} F^{\rho *}_i$ into $F^{\rho *}_{i^*}$.

Assume by induction for every even $i < i^*$ for every $p \in F^{\rho *}_i$ the following holds:

(1) $A^{0\rho}_i$ appears in every component $\underset{\sim}{p_n}$ of p with $n \geq \ell(p)$

(2) if $i' < i$ is even and $A^{0\rho}_{i'}$ appears in every component of $\underset{\sim}{p_n}$ of p with $n \geq \ell(p)$ then $A^{0\delta}_{i'}$ appears as well and
$$(p \restriction A^{0\delta}_{i'})\backslash (A^{0\rho}_i \cap A^{0\delta}_{i'}) \in F^{\delta *}_{i'} \ .$$

A typical element of $F^{\rho *}_{i^*}$ is obtained now in following two fashions. Start with the first one. Let $\langle p^\beta \mid \beta < \alpha < \kappa\rangle$ be a $\leq^*$ – increasing sequence with union p^α in $\mathcal{P}^*$, $p^\beta \in F^{\rho *}_{i_\beta}$ for every $\beta < \alpha$ and $\langle i_\beta \mid \beta \leq \alpha\rangle$ is an increasing sequence of even ordinals with $i_\alpha = i^*$. Extend p^α by adding $A^{0\rho}_{i^*}$ and put the resulting condition into $F^{\rho *}_{i^*}$. Notice that $\langle p^\beta \mid \beta < \alpha\rangle$ as above can be always reorganized as follows. Set $\widetilde{p}^\beta = \bigcup\limits_{\alpha > \beta' \geq \beta} p^{\beta'} \restriction A^{0\delta}_{i_\beta}$. By (2) above $(p^{\beta'} \restriction A^{0\delta}_i)\backslash (A^{0\rho}_{i_{\beta'}} \cap A^{0\delta}_{i_\beta}) \in F^{\delta *}_{i_\beta}$ for every β', $\alpha > \beta' \geq \beta$. By (1) $A^{0\rho}_{i_{\beta'}}$ appears in $p^{\beta'}$, so $A^{0\delta}_{i_\beta} \cap A^{0\rho}_{i_{\beta'}}$ will appear in $p^{\beta'+1}$ and hence in every $p^{\beta''}$ for $\alpha > \beta'' > \beta'$. So,
$$\widetilde{p}^\beta = \bigcup_{\alpha > \beta' \geq \beta} p^{\beta'} \restriction A^{0\delta}_{i_\beta} = \bigcup_{\alpha > \beta' \geq \beta} \Big((p^{\beta'} \restriction A^{0\delta}_{i_{\beta'}})\backslash (A^{0\rho}_{i_{\beta'}} \cap A^{0\delta}_{i_\beta} \Big) \ .$$

The last union is the union of elements of $F^{\delta *}_{i_\beta}$. Hence $\widetilde{p}^\beta$ is in $F^{\delta *}_{i_\beta}$. This way we obtain a new sequence $\langle \widetilde{p}^\beta \mid \beta < \alpha\rangle$ with the same limit but in addition $\widetilde{p}^{\beta'} \restriction A^{0\delta}_{i_\beta} = \widetilde{p}^\beta$ for every $\beta \leq \beta' < \alpha$, as well as $p^\alpha \restriction A^{0\delta}_{i_\beta} \backslash \Big(A^{0\rho}_{i^*} \cap A^{0\delta}_{i_\beta} \Big) = \widetilde{p}^\beta \in F^{\delta *}_{i_\beta}$.

Now describe a second way of generating elements of $F^{\rho *}_{i^*}$. Let $\alpha < i^*$ be an even ordinal. We include the following set into $F^{\rho *}_{i^*}$.

$S_\alpha = \{ q^\frown A^{0\rho}_{i^*} \mid q \in F^{\delta *}_\alpha \cap A^{0\rho}_{i^*}$ or there are $t \in A^{0\rho}_{i^*} \cap \mathcal{P}^*$ and $r \in A^{0\rho}_{i^*} \cap F^{\delta *}_\alpha$ such that

(a) $A^{0\delta}_\alpha$ and $A^{0\rho}_\alpha$ are in t and each model appearing in t and not in $A^{0\delta}_\alpha$ is of cardinality $< \kappa(\rho)$

(b) $r \geq^* t \restriction A^{0\delta}_{\alpha}$

(c) $r \restriction A^{0\rho}_{\alpha} \in F^{\rho *}_{\alpha}$

(d) $q = r \cup t\}$

Notice that every $r' \in F^{\delta}_{\alpha}$ with $A^{0\delta}_{\alpha}$ and $A^{0\rho}_{\alpha}$ inside can be extended ($\leq^*$-extension) to $r \in F^{\delta *}_{\alpha}$ with $r \restriction A^{0\rho}_{\alpha} \in F^{\rho *}_{\alpha}$. We just repeat the argument given for the same matter in the case of successor i^*. Thus the requirement (c) above is not really restrictive.

Let us check (2) of the inductive assumption above. Thus, let $i' < i^*$ be even and $A^{0\rho}_{i'}$ appears in every component $\underset{\sim}{p_n}$ with $n \geq \ell(p)$ of $p \in S_{\alpha}$. Then $i' \leq \alpha$. If $i' = \alpha$, then $A^{0\delta}_{\alpha}$ appears in p since $p \in F^{\delta *}_{\alpha}$. Also $(p \restriction A^{0\delta}_{\alpha}) \backslash (A^{0\rho}_{i^*} \cap A^{0\delta}_{\alpha}) \in F^{\delta *}_{\alpha}$ by the choice of S_{α}. Now let $i' < \alpha$. $p \restriction A^{0\rho}_{\alpha} \in F^{\rho *}_{\alpha}$, hence, by induction, $A^{0\delta}_{i'}$ appears in $p \restriction A^{0\rho}_{\alpha} \leq^* (p \restriction A^{0\delta}_{\alpha}) \backslash \Big(A^{0\rho}_{i^*} \cap A^{0\delta}_{\alpha} \Big) \in F^{\delta *}_{\alpha}$.

Apply the induction to $F^{\delta *}_{\alpha}$. We obtain then that

$$\Big((p \restriction A^{0\delta}_{\alpha}) \backslash (A^{0\rho}_{i^*} \cap A^{0\delta}_{\alpha}) \Big) \restriction A^{0\delta}_{i'} \in F^{\delta *}_{i'} \ ,$$

since $A^{0\delta}_{i'} \subset A^{0\delta}_{\alpha}$. Now, $((p \restriction A^{0\delta}_{\alpha}) \backslash (A^{0\rho}_{i} \cap A^{0\delta}_{\alpha})) \restriction A^{0\delta}_{i'} = (p \restriction A^{0\delta}_{i'}) \backslash (A^{0\rho}_{i^*} \cap A^{0\delta}_{i'})$, again since $A^{0\delta}_{i'} \subset A^{0\delta}_{\alpha}$.

This completes the definition of $F^{\rho *}_{i^*}$.

The rest of the proof is just straightforward checking the Definition 3.5. We refer to [Git3, 3.14] for details.

$\square$

The following lemma is a variation of 3.9 having the same proof. It will be used for showing the Prikry condition of the final forcing.

Lemma 3.10 *Let $N \prec H(\chi)$ with χ big enough. Suppose that N is of cardinality κ^+ and is closed under κ-sequences of its elements. Then there are an increasing sequence $\langle \langle A^{0\rho}_{\alpha}, A^{1\rho}_{\alpha}, F^{\rho}_{\alpha} \rangle \mid \rho \leq \delta, \alpha \leq \kappa^+ \rangle$ of elements of $\mathcal{P}$ and an increasing under inclusion sequence $\langle F^{0*}_{\alpha} \mid \alpha \leq \kappa^+ \rangle$ so that*

(a) *$\{ \langle \langle A^{0\rho}_{\alpha}, A^{1\rho}_{\alpha}, F^{\rho}_{\alpha} \rangle \mid \rho \leq \delta \rangle \mid \alpha < \kappa^+ \}$ is N-generic.*

(b) *for every $\alpha \leq \kappa^+$ $F^{0*}_{\alpha} \subseteq F^{0}_{\alpha}$ is a dense and closed subset satisfying 3.5(2(h))*

(c) *$F^{0*}_{\alpha} \in N$ for every $\alpha < \kappa^+$.*

Lemma 3.11 *For every $\tau \le \delta$ $\mathcal{P}_{<\tau}$ satisfies $\kappa^{+\tau+2}$-c.c. in $V^{\mathcal{P}_{\ge\tau}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \underset{\mathcal{P}_{\ge\tau}}{\Vdash} \Big(\langle\langle\langle \underset{\sim}{A}^{0\nu}_{\alpha}, \underset{\sim}{A}^{1\nu}_{\alpha}, \underset{\sim}{F}_{\alpha} \mid \nu < \tau\rangle \mid < \kappa^{+\tau+2}\rangle \quad \text{is an antichain in} \quad \underset{\sim}{\mathcal{P}}_{\le\tau}\Big) \ .$$

We use the winning strategy of the player II defined in 3.9 in order to decide the names of the elements of the antichain. Thus let $\langle\langle A^{1\rho}_{\alpha}, A^{1\rho}_{\alpha}, F^{\rho}_{\alpha}\rangle \mid \rho \le \delta,\ \alpha < \kappa^{+\tau+2}\rangle$ be an increasing sequence of elements of $\mathcal{P}_{\ge\tau}$ so that

(1) for every $\alpha < \kappa^{+\tau+2}$

$$\langle\langle A^{0\rho}_{\alpha+1}, A^{1\rho}_{\alpha+1}, F^{\rho}_{\alpha+1}\rangle \mid \tau \le \rho \le \delta\rangle \underset{\mathcal{P}_{\ge\tau}}{\Vdash} (\forall \alpha' \le \alpha+1 \quad \langle \underset{\sim}{A}^{0\nu}_{\alpha'}, \underset{\sim}{A}^{1\nu}_{\alpha'}, \underset{\sim}{F}^{\nu}_{\alpha'} \mid \nu < \tau\rangle$$
$$= \quad \langle\langle \check{A}^{0\nu}_{\alpha'}, \check{A}^{1\nu}_{\alpha'}, \check{F}^{\nu}_{\alpha'} \mid \nu < \tau\rangle)$$

(2) for every $\alpha < \kappa^{+\tau+2}$ of cofinality $\kappa^{+\tau+1}$

$$A^{0\tau}_{\alpha} = \bigcup_{\beta<\alpha} A^{0\tau}_{\beta} \ .$$

(3) for every $\alpha < \kappa^{+\tau+2}$ and $\nu < \tau$ $\langle\langle A^{0\tau'}_{\beta}, A^{1\tau'}_{\beta}, F^{\tau'}_{\beta}\rangle \mid \tau \le \tau' \le \delta,\ \beta \le \alpha\rangle \in A^{0\nu}_{\alpha+1}$.

Now using Δ-system argument we may assume that the following conditions hold for every $\alpha, \beta < \kappa^{+\tau+2}$ of cofinality $\kappa^{+\tau+1}$ and for every $\nu < \tau$:

(1) $A^{0\nu}_{\alpha+1} \cap \bigcup_{\gamma<\alpha} A^{0\tau} = A^{0\nu}_{\beta+1} \cap \bigcup_{\gamma<\beta} A^{0\tau}_{\gamma} = A^{0\nu}_{\alpha+1} \cap A^{0\nu}_{\beta+1}$

(2) models $A^{0\nu}_{\alpha+1}$ and $A^{0\nu}_{\beta+1}$ are isomorphic over $A^{0\nu}_{\alpha+1} \cap A^{0\nu}_{\beta+1}$

(3) the isomorphic between $A^{0\nu}_{\alpha+1}$ and $A^{0\nu}_{\beta+1}$ induces (in obvious fashion) isomorphisms between $A^{1\nu}_{\alpha+1}$, $A^{1\nu}_{\beta+1}$ and $F^{\nu}_{\alpha+1}$, $F^{\nu}_{\beta+1}$.

Now suppose that $\alpha < \beta < \kappa^{+\tau+2}$ have cofinality $\kappa^{+\tau+1}$. We like to show that $\langle\langle A^{0\rho}_{\alpha+1}, A^{1\rho}_{\alpha+1}, F^{\rho}_{\alpha+1}\rangle \mid \rho \le \delta\rangle$ and $\langle\langle A^{0\rho}_{\beta+1}, A^{1\rho}_{\beta+1}, F^{\rho}_{\beta+1}\rangle \mid \rho \le \delta\rangle$ are compatible. Clearly, there is no problem with ρ's above τ. Define a stronger condition $\langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \rho \le \delta\rangle$. Let $\rho < \tau$ and suppose that for every $\rho' < \rho$ $\langle A^{0\rho'}, A^{1\rho'}, F^{\rho'}, F^{\rho'*}\rangle$ is already defined. Define $\langle A^{0\rho}, A^{1\rho}, F^{\rho}, F^{\rho*}\rangle$.

Set $A^{0\rho}$ to be the closure inside $A^{0\tau}_{\beta+1}$ of $\langle A^{0\rho'}, A^{1\rho'}, F^{\rho'}, F^{\rho'*}\rangle \mid \rho' < \rho\} \cup \{\langle A^{0\rho}_{\alpha+1}, A^{1\rho}_{\alpha+1}, F^{\rho}_{\alpha+1}\rangle\} \cup \{\langle\langle A^{0\nu}_{\beta+1}, A^{1\nu}_{\beta+1}, F^{\nu}_{\beta+1}\rangle \mid \tau \le \nu \le \delta\rangle\} \cup \{\langle A^{0\rho}_{\beta+1}, A^{1\rho}_{\beta+1}, F^{\rho}_{\beta+1}\rangle\}$ under the Skolem functions and $\kappa^{+\rho}$-sequences. Define $A^{1\rho} = A^{1\rho}_{\alpha+1} \cup A^{1\rho}_{\beta+1} \cup \{A^{0\rho}\}$.

Now we turn to definitions of F^ρ and $F^{\rho*}$. Let $F^{\rho^*}_{\alpha+1}$ and $F^{\rho*}_{\beta+1}$ be subsets of $F^\rho_{\alpha+1}$ and $F^\rho_{\beta+1}$ respectively, satisfying 3.5(2(h)). We include first both of them into $F^{\rho*}$. Let us describe how to generate new elements of $F^{\rho*}$.

Let $p^0 = \langle p^0_n \mid n \leq \ell(p^0)\rangle^\frown\langle \underset{\sim}{p}^0_n \mid \omega > n > \ell(p^0)\rangle \in F^\rho_{\alpha+1}$ and $p^1 = \langle p^1_n \mid n \leq \ell(p^1)\rangle^\frown\langle \underset{\sim}{p}^1_n \mid \omega > n \geq \ell(p^1)\rangle \in F^\rho_{\beta+1}$ be such that

(1) $\ell(p^0) = \ell(p^1)$

(2) for every $n < \ell(p^0)$ p^0_n and p^1_n are compatible

(3) for every $n \geq \ell(p_0)$

(a) $A^{0\tau}_\alpha$, $A^{0\rho}_{\alpha+1}$ appear in $\underset{\sim}{p}^0_n$

(b) $A^{0\tau}_\beta$, $A^{0\rho}_{\beta+1}$ appear in $\underset{\sim}{p}^1_n$

(c) $\underset{\sim}{a}^0_n \restriction A^{0\tau}_\alpha = \underset{\sim}{a}^1_n \restriction A^{0\tau}_\beta$, where, as usual, $\underset{\sim}{a}^i_n$ is the correspondence function of $\underset{\sim}{p}^i_n (i \in 2)$

(4) p^0 and p^1 are compatible in $\mathcal{P}^*$, i.e.
they can be combined together without destroying the preservation of order (both "$\in$" and "$\subseteq$").

Now, $F^\nu_{\alpha+1} \subseteq F^\tau_{\alpha+1} \subseteq F^\tau_\beta \subseteq F^\tau_{\beta+1}$ and $F^\nu_{\beta+1} \subseteq F^\tau_{\beta+1}$. Hence, $p^0, p^1 \in F^\tau_{\beta+1} \subseteq F^\delta_{\beta+1}$. Let us combine them together into condition $q \in F^\tau_{\beta+1}$ with $A^{0\tau}_{\beta+1}$ as the maximal set. Thus, we add $A^{0\tau}_\beta$ to p^0 as the maximal element, using $p^0 \in F^\tau_\beta$ and 3.5(2(j)). Let $\widetilde{p}^0$ be the resulting condition. Let $\widetilde{p}^1$ be obtained from p^1 by adding $A^{0\tau}_{\beta+1}$ as the maximal element. By (3(c)) above and 3.5(2(g)) the combination of $\widetilde{p}^0$ and $\widetilde{p}^1$ is in $F^\tau_{\beta+1}$. Notice that for every model $B \in A^{0\rho}_{\beta+1}$ appearing in p^1, either $B \supseteq A^{0\tau}_\beta$ or there are $\tau'_1, \ldots, \tau'_\ell$, $\tau \leq \tau'_1 \leq \cdots \leq \tau_\ell \leq \delta$, $B_1 \in A^{0\tau}_\beta \cap A^{1\tau'_1}_\beta \cap A^{0\rho}_{\beta+1}, \ldots, B_\ell \in A^{0\tau}_\beta \cap A^{1\tau'_\ell}_\beta \cap A^{0\rho}_{\beta+1}$ such that $B \cap A^{0\tau}_\beta = B_1 \cap \cdots \cap B_\ell \cap A^{0\tau}_\beta$. $B_1, \ldots, B_\ell$ can be found inside $A^{0\rho}_{\beta+1}$, since B, $\langle A^{1\tau'}_\beta \mid \tau \leq \tau' \leq \delta\rangle$ are in $A^{0\rho}_{\beta+1}$. By the requirement (1) on the Δ-system, then $B_1, \ldots, B_\ell$ will be in $A^{0\tau}_\alpha \cap A^{0\rho}_{\alpha+1}$.

Finally let q be this combination with addition of $A^{0\tau}_{\beta+1}$ as the maximal element.

Let $F^{\rho*}_{\alpha+1}$, $F^{\rho*}_{\beta+1}$ and $F^{\tau*}_{\beta+1}$ be the fixed dense closed (in the sense of 3.5(2h))) of $F^\rho_{\alpha+1}$, $F^\rho_{\beta+1}$ and $F^\tau_{\beta+1}$ respectively. For each q as constructed above we find $q^* \in F^{\tau*}_{\beta+1}$ such that $q \leq^* q^*$, $q^* \restriction A^{0\rho}_{\alpha+1} \in F^{\rho*}_{\alpha+1}$ and $q^* \restriction A^{0\rho}_{\beta+1} \in F^{\rho*}_{\beta+1}$. Thus, let $q_0 \in F^{\tau*}_{\beta+1}$ be a $\leq^*$-extension of q. Consider $q'_1 = q_0 \restriction A^{0\rho}_{\beta+1}$. Let $q_1 \in F^{\rho*}_{\beta+1}$ be a $\leq^*$-extension of q'_1. Consider $q'_2 = q_1 \restriction$

$(A^{0\rho}_{\beta+1} \cap A^{0\tau}_{\beta})$. By 3.5(2(g)), the combination $\widetilde{q}'_2$ of q'_2 with $q_1 \restriction A^{0\rho}_{\alpha+1}$ is in $F^{\rho}_{\alpha+1}$. Recall that $A^{0\rho}_{\beta+1} \cap A^{0\tau}_{\beta} = A^{0\rho}_{\alpha+1} \cap A^{0\tau}_{\alpha}$. Hence, q'_2 is in $F^{\rho}_{\alpha+1}$. Let $q_2 \in F^{\rho *}_{\alpha+1}$ be a $\leq^*$-extension of $\widetilde{q}'_2$. Using 3.5(2(g)), as in the construction of q, q_2 and q_1 can be combined together. Let q''_1 be the combination. Again, using 3.5(2(g)) we combine q''_1 with q_0 into a condition $q''_0 \in F^{\tau}_{\beta+1}$. At the next stage we pick some $q_3 \in F^{\tau *}_{\beta+1}$ a $\leq^*$-extension of q''_0. Consider $q'_4 = q_3 \restriction A^{0\rho}_{\beta+1}$ and $\leq^*$ – extend it to $q_4 \in F^{\rho *}_{\beta+1}$. Let $q'_5 = q_4 \restriction (A^{0\rho}_{\beta+1} \cap A^{0\tau}_{\beta})$ and $\widetilde{q}'_5$ be the combination of $q_3 \restriction A^{0\rho}_{\alpha+1}$ with q'_5. Find $q_5 \in F^{\rho *}_{\alpha+1}$ a $\leq^*$-extension of $\widetilde{q}'_5$. Continue in the same fashion and define $\langle q_n \mid n < \omega \rangle$ so that for every $n < \omega$

(a) $q_{3n} \in F^{\tau *}_{\beta+1}$

(b) $q_{3n+1} \in F^{\rho *}_{\beta+1}$

(c) $q_{3n+2} \in F^{\rho *}_{\alpha+1}$

(d) $q_{3n+3} {}^{*}\!\geq q_{3n+1}, q_{3n+2}$

(e) $q_{3n+1} {}^{*}\!\geq \restriction q_{3n} \restriction A^{0\rho}_{\beta+1}$

(f) $q_{3n+2} {}^{*}\!\geq q_{3n} \restriction A^{0\rho}_{\alpha+1}$

(g) $q_{3n+2} {}^{*}\!\geq q_{3n+1} \restriction (A^{0\rho}_{\beta+1} \cap A^{0\tau}_{\beta})$

(h) $q_{3(n+1)+j} {}^{*}\!\geq q_{3n+j}$, for every $j < 3$

Now let q^* be the union of $\langle q_n \mid n < \omega \rangle$. By closure properties of $F^{\rho *}_{\alpha+1}$, $F^{\rho *}_{\beta+1}$ and $F^{\tau *}_{\beta+1}$ it will be as desired, i.e. $q^* \in F^{\tau *}_{\beta+1}$, $q^* \restriction A^{0\rho}_{\alpha+1} \in F^{\rho *}_{\alpha+1}$ and $q^* \restriction A^{0\rho}_{\beta+1} \in F^{\rho *}_{\beta+1}$. A typical element of $F^{\rho *}$ is obtained from such q^*'s by adding $A^{0\rho}$ as the maximal element. F^{ρ} is obtained from $F^{\rho *}$ adding everything necessary in order to satisfy the requirement of 3.5. We need to check that such defined F^{ρ} satisfies 3.5(2). Most of the conditions are straightforward. Let us check only 3.5(2(g)). Thus, let $p \in F^{\rho}$ include both $A^{0\rho}_{\alpha+1}$ and $A^{0\rho}_{\beta+1}$. Suppose that $q^* \geq p \restriction A^{0\rho}_{\alpha+1}$ is in $F^{\rho}_{\alpha+1}$. We need to show that then the combination of p and q is in F^{ρ}. $A^{0\tau}_{\beta}$ is in p, by the choice of $F^{\rho *}$ and then F^{ρ}. Then, the choice of the Δ-system implies that $p \restriction A^{0\tau}_{\beta}$ with $A^{0\tau}_{\beta}$ removed is exactly $p \restriction A^{0\rho}_{\alpha+1}$. Since everything inside $A^{0\rho}_{\beta+1}$ intersected with $A^{0\tau}_{\beta}$ is already inside the kernel, i.e. $A^{0\rho}_{0}$. Let $\widetilde{q}$ be obtained from q by adding $A^{0\tau}_{\beta}$ as the maximal element. Then, $\widetilde{q} \in F^{\tau}_{\beta} \subseteq F^{\tau}_{\beta+1}$. Now both $\widetilde{q}$ and p are in $F^{\tau}_{\beta+1}$ and $p \restriction A^{0\tau}_{\beta} \leq^* \widetilde{q}$. So, by 3.5(2(g)) for $F^{\tau}_{\beta+1}$, the combination of p and $\widetilde{q}$ is in $F^{\tau}_{\beta+1}$. Clearly, it is the same as the combination of p and q. So the combination of p and q is in $F^{\tau}_{\beta+1}$ and hence also in F^{ρ}.

This completes the inductive definition of $\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle$ and as well as those of $\langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \rho < \tau\rangle$.

Finally, for ρ, $\tau \leq \rho \leq \delta$ we pick $A^{0\rho}$ to be the closure of $\langle A^{0\nu}_{\beta+1}, A^{1\nu}_{\beta+1}, F^{\nu}_{\beta+1} \mid \tau \leq \nu \leq \delta\rangle$, $\langle\langle A^{0\rho'}, A^{1\rho'}, F^{\rho'}\rangle \mid \rho' < \rho\rangle$ under the Skolem functions and $\kappa(\rho)$-sequences. Let $A^{1\rho} = A^{1\rho}_{\beta+1} \cup \{A^{0\rho}\}$ and let F^{ρ} be defined as it was done at a successor stage in the proof of Lemma 3.9.

Now, $\langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle$ is a condition in $\mathcal{P}_{\geq\tau}$ stronger than $\langle\langle A^{0\rho}_{\beta+1}, A^{1\rho}_{\beta+1}, F^{\rho}_{\beta+1}\rangle \mid \tau \leq \rho \leq \delta\rangle$. It forces that "$\langle\langle \check{A}^{0\rho}, \check{A}^{1\rho}, \check{F}^{\rho}\rangle \mid \rho < \tau\rangle \in \underset{\sim}{\mathcal{P}_{<\tau}}$ and is stronger than both $\langle\langle \check{A}^{0\rho}_{\alpha+1}, \check{A}^{1\rho}_{\alpha+1}, \check{F}^{\rho}_{\alpha+1}\rangle \mid \rho < \tau\rangle$ and $\langle\langle \check{A}^{0\rho}_{\beta+1}, \check{A}^{\rho}_{\beta+1}, \check{F}^{\rho}_{\beta+1}\rangle \mid \rho < \tau\rangle$".

Which contradicts our initial assumptions.

$\square$

If $\tau = \tau' + n$ for some inaccessible $\tau' < \tau$ and $0 < n < \omega$, then repeating the proof of 3.10 we obtain that $\mathcal{P}_{<\tau}$ satisfies $\kappa^{+\tau+1}$ – c.c. The difference here is due entirely to our choice of indexing.

Combining 3.9 and 3.10 together we obtain the following:

Lemma 3.12 *The forcing $\mathcal{P}$ preserves all the cardinals except probably the successors of inaccessibles.*

If one likes to preserve all the cardinals, then instead of the full support taken here, Easton type of support should be used. Thus fix some $\langle\langle \underline{A}^{0\nu}, \underline{A}^{1\nu}, \underline{F}^{\nu}\rangle \mid \nu \leq \delta\rangle \in \mathcal{P}$. Let $\underline{\mathcal{P}}$ consist of elements having Easton support over this fixed condition, i.e.

$$\langle\langle B^{0\nu}, B^{1\nu}, G^{\nu}\rangle \mid \nu \leq \delta\rangle \quad \text{will be in} \quad \underline{\mathcal{P}}$$

iff for every inaccessible $\lambda \leq \delta$,

$$|\{\nu < \lambda | \langle B^{0\nu}, B^{1\nu}, G^{\nu}\rangle \neq \langle \underline{A}^{0\nu}, \underline{A}^{1\nu}, \underline{F}^{\nu}\rangle\}| < \lambda\ .$$

4 The Main Forcing

Let $G \subseteq \mathcal{P}$ be generic. We define our main forcing notion $\mathcal{P}^{**}$ to be

$$\cup\{F^0 \mid \exists A^{00}, A^{10}, \langle\langle A^{0\tau}, A^{1\tau}, F^{\tau}\rangle \mid 0 < \tau \leq \delta\rangle \quad \langle\langle A^{0\nu}, A^{1\nu}, F^{\nu}\rangle \mid \nu \leq \delta\rangle \in G\}\ .$$

The proof of the next lemma is very similar to those of 3.10.

Lemma 4.1 *In $V^{\mathcal{P}}$, $\langle \mathcal{P}^{**}, \rightarrow\ \rangle$ satisfies κ^{++}-c.c.*

Proof. Suppose otherwise. Let us work in V and let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++}\rangle$ be a name of an antichain of the length κ^{++}. Using the strategy of Player II defined in 3.9 we find an increasing sequence

$$\langle\langle A^{0\rho}_\alpha, A^{1\rho}_\alpha, F^\rho_\alpha\rangle \mid \rho \leq \delta, \alpha < \kappa^{++}\rangle$$

of elements of $\mathcal{P}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ so that for every $\alpha < \kappa^{++}$ the following holds:

(1) $\langle\langle A^{0\rho}_{\alpha+1}, A^{1\rho}_{\alpha+1}, F^\rho_{\alpha+1}\rangle \mid \rho \leq \delta\rangle \Vdash_{\mathcal{P}} (\forall \alpha' \leq \alpha+1 \quad \underset{\sim}{p}_{\alpha'} = \check{p}_{\alpha'})$

(2) $p_\alpha \in F^\rho_\alpha$

(3) if $cf\alpha = \kappa^+$ then
$$A^{00}_\alpha = \bigcup_{\beta<\alpha} A^{00}_\beta$$

(4) $\langle\langle A^{0\rho}_\beta, A^{1\rho}_\beta, F^\rho_\beta\rangle \mid \rho \leq \delta,\, \beta \leq \alpha\rangle \in A^{00}_{\alpha+1}$

(5) A^{00}_α and $A^{00}_{\alpha+1}$ appear in every $\underset{\sim}{p}_{\alpha n}$ with $n \geq \ell(p_\alpha)$ where $p_\alpha = \langle p_{\alpha n} \mid n \leq \ell(p_\alpha)\rangle^\frown \langle \underset{\sim}{p}_{\alpha n} \mid n > \ell(p_\alpha)\rangle$.

Now we use the Δ-system argument to insure for every $\alpha, \beta < \kappa^{++}$ of cofinality κ^+ the following:

(1) $\ell(p_{\alpha+1}) = \ell(p_{\beta+1})$

(2) for every $n < \ell(p_\alpha)$ $p_{\alpha+1n}$ and $p_{\beta+1n}$ are compatible.

(3) $p_{\alpha+1} \restriction A^{00}_\alpha$ with A^{00}_α removed is the same as $p_{\beta+1} \restriction A^{00}_\beta$ with A^{00}_β removed.

This means that for every B (ordinal or submodel)
$B \in A^{00}_\alpha$ and appears in $p_{\alpha+1}$ iff $B \in A^{00}_\beta$ and appears in $p_{\beta+1}$.
Also p_α and p_β agrees about such B's.

(4) the values of A^{00}_α in $p_{\alpha+1} \restriction A^{00}_\alpha$ and A^{00}_β in $p_{\beta+1} \restriction A^{00}_\beta$ are decided always to be the same.

(5) if $n = \ell(p_{\alpha+1})$, $p_{\alpha+1,n} = \langle e_{\alpha+1,n}, a_{\alpha+1,n}, A_{\alpha+1,n}, S_{\alpha+1,n}, f_{\alpha+1,n}\rangle$ and $p_{\beta+1,n} = \langle e_{\beta+1,n}, a_{\beta+1,n}, A_{\beta+1,n}, S_{\beta+1,n}, f_{\beta+1,n}\rangle$ then the following holds:

(i) $e_{\alpha+1,n} \restriction A^{00}_{\alpha,n} = e_{\beta+1,n} \restriction A^{00}_{\beta,n}, rnge_{\alpha+1,n} = rnge_{\beta+1,n}$ and $e_{\alpha+1,n}, e_{\beta+1,n}$ are order isomorphic over the common part $e_{\alpha+1,n} \restriction A^{00}_{\alpha,n}$

(ii) $a_{\alpha+1,n} \restriction A^{00}_{\alpha,n} = a_{\beta+1,n} \restriction A^{00}_{\beta,n}$, $rnga_{\alpha+1,n} = rnga_{\alpha+1,n}$ and $a_{\alpha+1,n}$, $a_{\beta+1,n}$ are isomorphic over the common part $a_{\alpha+1,n} \restriction A^{00}_{\alpha,n}$ in the language $\{\in, <, \subseteq\}$

(iii) $A_{\alpha+1,n} = A_{\beta+1,n}$

(iv) $S_{\alpha+1,n} = S_{\beta+1,n}$

(v) $f_{\alpha+1,n} \restriction A^{00}_{\alpha,n} = f_{\beta+1,n} \restriction A^{00}_{\beta,n}$, $rngf_{\alpha+1,n} = rngf_{\beta+1,n}$ and $f_{\alpha+1,n}, f_{\beta+1,n}$ are order isomorphic over the common part $f_{\alpha+1,n} \restriction A^{00}_{\alpha,n}$

(6) if $n = \ell(p_{\alpha+1}) + 1$, $r_{\gamma,n-1}$ is an extension of $p_{\gamma,n-1}$ by picking an element of $A_{\gamma,n-1}$ only, $\gamma \in \{\alpha{+}1, \beta{+}1\}$ and the picked element is the same for $\alpha{+}1$ and $\beta{+}1$ (which is possible by (5)(iii)) *then* (5) above holds for the decided by $\langle p_{\alpha+1,m} \mid m < n-1\rangle^\frown\langle r_{\alpha+1,n-1}\rangle$ and $\langle p_{\beta+1,m} \mid m < n-1\rangle^\frown\langle r_{\beta+1,n-1}\rangle$ values of $\underset{\sim\alpha+1,n}{p}$ and $\underset{\sim\beta+1,n}{p}$

(7) if $n > \ell(p_{\alpha+1}) + 1$, $\langle r_{\gamma,k} \mid \ell(p_{\alpha+1}) \leq k < n\rangle$ is defined level by level as in (6) by picking elements of $A_{\gamma,k}$'s ($\ell(p_{\alpha+1}) \leq k < n$) only ($\gamma \in \{\alpha+1, \beta+1\}$) the same way for $\alpha+1$ and $\beta+1$, *then* (5) holds for the decided by $\langle p_{\alpha+1,m} \mid m < \ell(p_{\alpha+1})\rangle^\frown\langle r_{\alpha+1,k} \mid \ell(p_{\alpha+1}) \leq k < n\rangle$ and $\langle p_{\beta+1,m} \mid m < \ell(p_{\alpha+1})\rangle^\frown\langle r_{\beta+1,k} \mid \ell(p_{\alpha+1}) \leq k < n\rangle$ values of $\underset{\sim\alpha+1,n}{p}$ and $\underset{\sim\beta+1,n}{p}$.

The conditions (5)-(7) insure that we always can extend trunks of $p_{\alpha+1}$ and $p_{\beta+1}$ the same (compatible) way any finite number of times.

Let $\alpha < \beta < \kappa^{++}$ be ordinals of cofinality κ^+. We claim that it is possible to find $p^*_{\alpha+1}$ equivalent to $p_{\alpha+1}$ which is forced by $\langle\langle A^{0\rho}_{\beta+1}, A^{1\rho}_{\beta+1}, F^{\rho}_{\beta+1}\rangle \mid \rho \leq \delta\rangle$ to be compatible with $p_{\beta+1}$ in $\langle \mathcal{P}^{**}, \leq^* \rangle$. Consider $p_{\beta+1} \restriction A^{00}_{\beta}$. It is an element of $F^0_\beta \subseteq F^0_{\beta+1}$. Also note that $A^{00}_\alpha \subseteq A^{00}_{\alpha+1} \subseteq A^{00}_{\alpha+2} \subseteq A^{00}_\beta$ are all in A^{01}_β. So $p_{\beta+1} \restriction A^{00}_\beta$ can be extended by adding $A^{00}_{\alpha+2}$ to it using 3.5(2(f)). Let $(p_{\beta+1} \restriction A^{00}_\beta)^\frown A^{00}_{\alpha+2}$ denotes the resulting condition. By the requirement (3) on the Δ-system, $A^{00}_{\alpha+2}$ is added alone without producing additional submodels, i.e. $(p_{\beta+1} \restriction A^{00}_\beta)^\frown A^{00}_{\alpha+2}$ with $A^{00}_{\alpha+2}$ and A^{00}_β removed is the same as $p_{\alpha+1} \restriction A^{00}_\alpha$ with A^{00}_α removed.

Again, use 3.5(2(j)) and extend $(p_{\beta+1} \restriction A^{00}_\beta)^\frown A^{00}_{\alpha+2}$ by adding $A^{00}_{\alpha+1}$. Let

$$q = ((p_{\beta+1} \restriction A^{00}_\beta)^\frown A^{00\frown}_{\alpha+2} A^{00}_{\alpha+1}) \restriction A^{00}_{\alpha+1} \ .$$

Then $q \in F^0_{\alpha+1}$ and if we remove $A^{00}_{\alpha+1}$ from it then it will be the same as $p_{\alpha+1} \restriction A^{00}_\alpha$ with

A^{00}_α removed. Let $q = \langle q_n \mid n \leq \ell(q)\rangle \cap \langle \underset{\sim}{q_n} \mid \omega > n > \ell(q)\rangle$ and for every $n \geq \ell(q)$ $\underset{\sim}{q_n} = \langle \underset{\sim}{e_n}, \underset{\sim}{a_n}, \underset{\sim}{A_n}, \underset{\sim}{S_n}, \underset{\sim}{f_n}\rangle$. Find $n^* \geq \ell(q)$ to be large enough such that for every $n \geq n^*$

(a) $A^{00}_{\alpha+1} \in \mathrm{dom}\underset{\sim}{a_n}$

(b) $\underset{\sim}{a_n}(A^{00}_{\alpha+1})$ is an elementary submodel of $\mathfrak{a}_{n,k_n}$ with $k_n \geq 5$.

Now extend the trunk of q in order to make it of the length n^*. Let r be the resulting condition. By 3.5(2(c)), $r \in F^0_{\alpha+1}$. Extend also the trunk of $p_{\alpha+1}$ to the same length by adding to it $\langle r_n \mid n < n^*\rangle$. Denote the result by $p^*_{\alpha+1}$. Let

$$p^*_{\alpha+1} = \langle p^*_n \mid n \leq n^*\rangle^\cap \langle \underset{\sim}{p^*_n} \mid n > n^*\rangle$$

and $\underset{\sim}{p^*_n} = \langle \underset{\sim}{e^*_n}, \underset{\sim}{a^*_n}, \underset{\sim}{A^*_n}, \underset{\sim}{S^*_n}, \underset{\sim}{f^*_n}\rangle$ for $n \geq n^*$.

For every $n \geq n^*$, we consider $a_n(A^{00}_{\alpha+1})$ and $a^*_n(A^{00}_{\alpha+1})$ as they decided by common extension of trunks to the level n. Pick some $\sigma_n \prec \mathfrak{a}_{n,k_n-1}$ inside $a_n(A^{00}_{\alpha+1})$ realizing the same $k_n - 1$ – type over $rng(a_n)\backslash\{a_n(A^{00}_{\alpha+1})\}$ as those of $a^*_n(A^{00}_{\alpha+1})$, where k_n is as in the requirement (b) above. Let b_n be a function with the same domain as a^*_n and satisfying the following:

(i) $b_n(A^{00}_{\alpha+1}) = \sigma_n$

(ii) $b_n \restriction (\mathrm{dom} a_n\backslash\{A^{00}_{\alpha+1}\}) = a_n \restriction ((\mathrm{dom} a_n)\backslash\{A^{00}_{\alpha+1}\}) = a^*_n \restriction ((\mathrm{dom} a_n)\backslash\{A^{00}_{\alpha+1}\})$

(iii) $rng b_n$ realizes the same $k_n - 1$-type over $rng a^*_n \restriction ((\mathrm{dom} a_n)\backslash\{A^{00}_{\alpha+1}\})$ inside σ_n as those of $rng a^*_n$.

Define $t_n = \langle e^*_n, b_n, A^*_n, S^*_n, f^*_n\rangle$. Finally let $t = \langle p^*_n \mid n < n^*\rangle^\cap \langle \underset{\sim}{t_n} \mid n \geq n^*\rangle$. By its definition, $t \leftrightarrow p^*_{\alpha+1}$. Hence $t \in F^0_{\alpha+1}$.

Now using 3.5(2(j)), we add to t the set $A^{00}_{\alpha+2}$ at the same places as in $(p_{\beta+1} \restriction A^{00}_\beta)^\cap A^{00}_{\alpha+2}$. It is possible by the construction of t. Denote the result by $t^\cap A^{00}_{\alpha+2}$. Finally, we use 3.5(2(g)) to put $(p_{\beta+1} \restriction A^{00}_\beta)^\cap A^{00}_{\alpha+2}$ and $t^\cap A^{00}_{\alpha+2}$ together (extending if necessary the trunk of the first condition using the requirements (5)-(7) on the Δ-system) and then the resulting condition with $p_{\beta+1}$. Thus we obtain an element of $F^0_{\beta+1}$ above t and $p_{\beta+1}$ in the $\leq$-ordering but $t \leftrightarrow p^* \geq p_{\alpha+1}$. Hence $p_{\alpha+1}$ and $p_{\beta+1}$ are compatible. Contradiction.

$\square$

The next lemma is almost standard. We concentrate only on a few points.

Lemma 4.2 $\langle \mathcal{P}^{**}, \leq, \leq^* \rangle$ *satisfies the Prikry condition.*

Proof. Let σ be a statement of the forcing language and $p \in \mathcal{P}^{**}$. We work in V. Find an elementary submodel N of $H(\chi)$, with χ big enough, of cardinality κ^+, closed under κ-sequences of its elements and including $\mathcal{P}$ – names for σ and p. By 3.10, there are an increasing sequence $\langle\langle A^{0\rho}_\alpha, A^{1\rho}_\alpha, F^\rho_\alpha\rangle \mid \rho \leq \delta,\ \alpha \leq \kappa^+\rangle$ of elements of $\mathcal{P}$ and an increasing under inclusion sequence $\langle F^{0*}_\alpha \mid \alpha \leq \kappa^+\rangle$ so that

(a) $\{\langle\langle A^{0\rho}_\alpha, A^{1\rho}_\alpha, F^\rho_\alpha\rangle \mid \rho \leq \delta\rangle \mid \alpha < \kappa^+\}$ is N-generic for the forcing $\mathcal{P}$.

(b) for every $\alpha \leq \kappa^+$ $F^{0*}_\alpha \subseteq F^0_\alpha$ is dense and the closed subset satisfying 3.5(2(h)).

(c) for every $\alpha < \kappa^+$ $F^{0*}_\alpha \in N$.

¿From here let us work inside $N^* = N[\langle\langle A^{0\rho}_\alpha, A^{1\rho}_\alpha, F^\rho_\alpha\rangle \mid \rho \leq \delta,\ \alpha < \kappa^+\rangle]$.

We need to construct $p^* \geq^* p$ deciding σ. The construction is rather standard. We extend every condition generated in the process to an element of $\bigcup_{\alpha<\kappa^+} F^{0*}_\alpha$ (recall that each F^{0*}_α has cardinality κ^+ and belongs to N, so $\bigcup_{\alpha<\kappa^+} F^{0*}_\alpha \subseteq N$). We use the closure properties of F^{0*}_α's 3.5(2(h)) to insure that the conditions generated at intermediate stages as well as the final one p^* are in $\bigcup_{\alpha<\kappa^+} F^{0*}_\alpha$. Let us concentrate here only on one new point due to 2.2(6). The typical situation is as follows: $p^* \geq^* p$ is constructed, there is some $q > p^*$, $q \Vdash \sigma$ and $\ell(q) = \ell(p) + 1$. Assume for simplicity that $\ell(p) = 0$. The problem is with $e_1(q)$, where $q_1 = \langle e_1(q), a_1(q), A_1(q), S_1(q), h_{>1}(q), f_1(q)\rangle$. Thus $e_1(q)$ may be bigger than $e_1(p^*)$, as decided by q_0, where $p^*_1 = \langle \underset{\sim}{e_1(p^*)}, \underset{\sim}{a_1(p^*)}, \underset{\sim}{A_1(p^*)}, \underset{\sim}{S_1(p^*)}, \underset{\sim}{f_1(p^*)}\rangle$. So, formally, such q was not considered during the construction. But let us show that implicitly it actually was. We extend first p_0 by replacing it by q_0. Then we extend $a_1(p^*) \restriction On$ to $a_1(q) \restriction On$. Note that only models of cardinalities in $e_1(q) \backslash e_1(p^*)$ cannot be added to $a_1(p^*)$, in contrast to ordinals. Also the maximal cardinality $\kappa^{+\delta+1}$ and the minimal κ^+ are always inside e_n's. Now, the above extension will make e_2's the same. We extend $\langle A_1(p^*), S_1(p^*), h_{>1}(p^*), f_1(p^*)\rangle$ and then p^*_n for $n \geq 2$ according to $\langle A_1(q), S_1(q), f_1(q)\rangle$ and $\underset{\sim}{q_n}$ for $n \geq 2$. Denote the result by p^{**}. The difference between p^{**} and q is only in $e_1(p^{**})$, which is the same as $e_1(p^*)$, and in $a_1(p^{**}) \backslash On$. We claim that still $p^{**} \Vdash \sigma$. Otherwise, there will be $r \geq p^{**}$ with $\ell(r) > 1$ forcing the negation. But by the definition of the order, $r \geq q$, which is impossible. Thus,

$p^{**} \Vdash \sigma$. But p^{**} was explicitly considered during the construction of p^*. Hence, also $p^* \Vdash \sigma$.

$\square$

Lemma 4.3 *κ is the first fixed point of the $\aleph$-function in* $(V^{\mathcal{P}*\langle\mathcal{P},\leq\rangle})^{\mathrm{Col}(\omega,\kappa_0)}$.

Proof. Let G be a generic subset of $\langle \mathcal{P}^{**}, \leq \rangle$.

Let $\langle \rho_n \mid n < \omega \rangle$ denotes the generic Prikry sequence for the normal measures of the extenders produced by G, i.e. for every $n < \omega$ ρ_n is so that for some $p \in G$ with $\ell(p) > n$ there are $\langle h_{<n}, h_{>n}, f_n \rangle$ such that $p_n = \langle \rho_n, h_{<n}, h_{>n}, f_n \rangle$.

Fix $m < \omega$. Consider
$H_{<m} = \cup\{h_{<m} \mid \exists p \in G\ \ell(p) > m$ and for some $\langle \rho_m, h_{>m}, f_m \rangle$ $p_m = \langle \rho_m, h_{<m}, h_{>m}, f_m \rangle\}$ and
$H_{>m} = \{h_{>m} \mid \exists p \in G \quad \ell(p) > m$ and for some $\langle \rho_m, h_{<m}, f_m \rangle$ $p_m = \langle \rho_m, h_{<m}, h_{>m}, f_m \rangle\}$.

Then $H_{<m}$ will be a generic over V subset of the Levy collapse $\mathrm{Col}(\rho_m^{+\kappa_{m-1}+1}, < \kappa_m)$ and $H_{>m}$ will be a generic over V subset of the Levy collapse $\mathrm{Col}(\kappa_m, < \rho_{m+1})$. So, in $V^{\langle \mathcal{P}^{**}, \leq \rangle}$, the only cardinals between ρ_m and ρ_{m+1} will be ρ_m^{+i} $(i \leq \kappa_{m-1}+1)$ and κ_m. Then the total number of cardinals between ρ_m and ρ_{m+1} will be $\kappa_{m-1}+2$ which is clearly below ρ_m. Hence $\rho_{m-1} < \aleph_{\rho_m}$ which is in turn below ρ_{m+2} since we keep ρ_{m+1}^{+i} as cardinal for every $i \leq \kappa_m + 1$ and $\rho_m < \kappa_m$. So, by induction,

$$\rho_0 < \aleph_{\rho_0} < \rho_1 < \aleph_{\rho_1} < \cdots < \rho_n < \aleph_{\rho_n} < \cdots .$$

Then, obviously, collapsing $\kappa_0 > \rho_0$ to $\aleph_0$ we obtain that $\kappa = \bigcup_{n<\omega} \rho_n$ will be the first repeat point. $\square$

Notice that we used only elements of p_n for p's in G with $n < \ell(p)$. Such elements does not change under the equivalence relation $\leftrightarrow$. Hence, the analog of 4.3 will be true with $\langle \mathcal{P}^{**}, \leq \rangle$ replaced by $\langle \mathcal{P}^{**}, \rightarrow \rangle$.

Lemma 4.4 *κ is the first fixed point of the $\aleph$-function in* $(V^{\mathcal{P}*\langle\mathcal{P}^{**},\rightarrow\rangle})^{\mathrm{Col}(\omega,\kappa_0)}$.

Let G be a generic subset of $\langle \mathcal{P}^{**}, \leq \rangle$. For every $n < \omega$ define a function $F_n : \kappa^{+\delta+1} \rightarrow \kappa_n$ as follows:

$F_n(\alpha) = \nu$, if for some $p \in G$ with $\ell(p) > n$ $f_n(\alpha) = \nu$, where

$$p_n = \langle \rho_n, h_{<n}, h_{>n}, f_n \rangle .$$

Now for every $\alpha < \kappa^{+\delta+1}$ set $t_\alpha = \langle F_n(\alpha) \mid n < \omega \rangle$. Let us show that the set $\{t_\alpha | \alpha < \kappa^{+\delta+1}\}$ has cardinality $\kappa^{+\delta+1}$ in $V^{\mathcal{P}}[G/\leftrightarrow]$. As it was pointed out before 4.4, t_α's does not change by $\leftrightarrow$ and so they are in $V^{\mathcal{P}*\langle\mathcal{P}^{**},\rightarrow\rangle}$. Also, by 4.1, $\kappa^{+\delta+1}$ as well as every cardinal above κ is preserved in $V^{\mathcal{P}*\langle\mathcal{P}^{**},\rightarrow\rangle}$.

Lemma 4.5 *For every $\beta < \kappa^{+\delta+1}$ there is α, $\beta < \alpha < \kappa^{+\delta+1}$ such that for every $\gamma \leq \beta$ $t_\alpha(k)$ is different from $t_\gamma(k)$ for all but finitely many k's.*

Proof. Suppose otherwise. Then there are $p \in G$ and $\beta < \kappa^{+\delta+1}$ such that

$$p \underset{\langle\mathcal{P}^{**},\leq\rangle}{\Vdash} \forall\alpha(\beta < \alpha < \kappa^{+\delta+1} \rightarrow \exists\gamma \leq \beta\ \underset{\sim}{t_\alpha} = \underset{\sim}{t_\gamma})$$

Pick some $\alpha \in \kappa^{+\delta+1}$ which is above every ordinal less than $\kappa^{+\delta+1}$ mentioned in p. Using a simple density argument on $\langle\mathcal{P}, \leq\rangle$ and then 3.5(2(e)) we can find q so that $q \geq^* p$ and for every n large enough α always appears in $\underset{\sim}{q_n}$, i.e. does not matter what is the decided value of $\underset{\sim}{q_n}$, α is inside $\mathrm{dom}(a_n(q))$, where $a_n(q)$, as usual, is the second coordinate of $\underset{\sim}{q_n}$. Then q will force

$$(*) \qquad (\forall\gamma \neq \alpha) \qquad (\exists k_0 < \omega\forall k \geq k_0 \quad t_\alpha(k) \neq t_\gamma(k))\ .$$

This leads to the contradiction. Thus, let $\gamma < \alpha$ and assume that q belongs to a generic subset of $\mathcal{P}^{**}$. Then either $t_\gamma \in V$ or it is a new ω-sequence. If $t_\gamma \in V$ then $(*)$ is clear. If t_γ is new then for some $r \geq q$ in the generic set γ appears in $\mathrm{dom}(\underset{\sim}{a_n}(r))$ for all $n \geq \ell(r)$ where, again $\underset{\sim}{a_n}(r))$ is the second coordinate of $\underset{\sim}{r_n}$. But also α is there and $\underset{\sim}{a_n}(r)$ is order preserving. Hence $F_n(\alpha) \neq F_n(\gamma)$ for every $n \geq \ell(r)$ and $(*)$ holds as well. $\square$

The proof of 4.5 provides more. Thus let $\langle\rho_n \mid n < \omega\rangle$ be the Prikry sequence of the normal measures of the extenders in $V^{\mathcal{P}}[G/\leftrightarrow]$. Again, $\leftrightarrow$ has no influence on it by its definition. Set $\rho^*_{-1} = 1$ and $\rho^*_n = \rho_n^{+\rho^*_{n-1}+1}$ if $n > 0$. For every $\alpha < \kappa^{+\delta+1}$ and $k < \omega$ we define

$$t^*_\alpha(k) = \begin{cases} t_\alpha(k), & \text{if} \quad t_\alpha(k) < \rho^*_k \\ 0, & \text{otherwise} \end{cases}$$

Consider $S = \langle t^*_\alpha \mid \alpha < \kappa^{+\delta+1} \quad \text{and} \quad t_\alpha \notin V\rangle$. By the proof of 4.5 the following holds:

Lemma 4.6 *$V^{\mathcal{P}*\langle\mathcal{P}^{**},\rightarrow\rangle}$ satisfies the following:*

(a) $|S| = \kappa^{+\delta+1}$

(b) S witness $tcf\left(\prod\limits_{n<\omega} \rho^*_n/\text{finite}\right) = \kappa^{+\delta+1}$.

5 A Note on PCF Generators

In this section we construct a model satisfying the following

(a) κ is a strong limit cofinality $\aleph_0$

(b) $2^\kappa = \kappa^{+3}$

(c) $\{\delta < \kappa \mid \delta^+ \in b_{\kappa^{+3}}\} \cap b_{\kappa^{++}} = \emptyset$ where b_λ denotes the pcf generator corresponding to $\lambda(\lambda = \kappa^{++}$ or $\kappa^{+++})$.

In all the previous constructions satisfying (a) and (b) the condition (c) fails. So, this suggested that may be in ZFC (a) + (b) $\rightarrow \neg$ (c).

Our aim will be to show that it is not the case. At the end of the section we outline extensions build on same ideas that can be used to show that the results of [Git4] an ordinal gaps are sharp. Suppose that κ, $\langle \kappa_n \mid n < \omega\rangle$ and $\langle \lambda_n \mid n < \omega\rangle$ are so that

(1) $\kappa = \bigcup_{n<\omega} \kappa_n$

(2) for every $n < \omega$

(i) $\lambda_n < \kappa_n < \lambda_{n+1} < \kappa_{n+1}$

(ii) λ_n carries an extender E_{λ_n} of the length λ^{+n+2}.

(iii) κ_n carries an extender E_{κ_n} of the length κ_n^{+n+2}.

We will use E_{λ_n}'s to generate Prikry sequences witnessing $tcf\Big(\prod\limits_{n<\omega} \rho^{+n+2}/\text{finite}\Big) = \kappa^{++}$, where $\langle \rho_n \mid n < \omega\rangle$ denotes the Prikry sequence for the normal measures of E_{λ_n}'s. E_{κ_n}'s will generate Prikry sequences witnessing

$$tcf\Big(\prod_{n<\omega} \xi_n^{+n+2}/\text{finite}\Big) = \kappa^{+++}$$

where $\langle \xi_n \mid n < \omega\rangle$ denotes the Prikry sequence for the normal measures of E_{κ_n}'s. The Prikry sequences for ξ_n^{+n+2} $(n < \omega)$ will depend essentially on choices that were made for ρ_n^{+n+2}'s. Thus as in the previous construction and in contrast [Git2,3] we shall work with names.

Let $\mathcal{P}'(0)$ denote $\mathcal{P}'$ of 3.1 with $\delta = 0$ and $\mathcal{P}'(1)$ denotes $\mathcal{P}'_{\geq 0}$ of 3.3. with $\delta = 1$. For such δ's $\mathcal{P}'$ is actually very simple. Thus $\mathcal{P}'(0)$ produces a chain of submodels of the length

κ^{++} of $H(\kappa^{++})$ each of cardinality κ^+. $\mathcal{P}'(1)$ adds a chain of the length κ^{+++} of submodels of $H(\kappa^{+++})$ each of cardinality κ^{++}. We combine $\mathcal{P}'(1)$ with the forcing for adding $\square_{\kappa^{++}}$ by initial segments. Denote this forcing by Box (κ^{++}). Every $p \in$ Box(κ^{++}) is of the form $\langle c_\alpha \mid \alpha \leq \delta \rangle$ such that

(1) $\delta < \kappa^{+++}$

(2) for every $\alpha \leq \delta$

(a) $c_\alpha \subseteq \alpha$ is closed unbounded

(b) $otpc_\alpha \leq \kappa^{++}$ and if $cf\alpha < \kappa^{++}$, then $otpc_\alpha < \kappa^{++}$

(c) if β is a limit point of c_α then $c_\beta = c_\alpha \cap \beta$.

(d) if β is a successor point of c_α then $cf\beta = \kappa^{++}$.

For $p, q \in$ Box(κ^{++}) $p \geq q$ iff q is an initial segment of p.

This forcing was introduced by R. Jensen [Dev-Jen] and it is κ^{++}-strategically closed.

We shall use the following variation Box$'(\kappa^{++})$ of Box(κ^{++}) which forces a club into κ^{+++} and a box sequence on it simultaneously.

Definition 5.1 $p = \langle c, \langle c_\alpha \mid \alpha \in \lim(c)\rangle\rangle \in$ Box$'(\kappa^{++})$ iff

(1) $c \subseteq \kappa^{+++}$ is a closed subset of κ^{+++} of cardinality κ^{++}

(2) for every $\alpha \in \lim(c)$ the following holds:

(a) $c_\alpha \subseteq \alpha \cap c$ is closed unbounded

(b) $otpc_\alpha \leq \kappa^{++}$ and if $cf\alpha < \kappa^{++}$ then $otpc_\alpha < \kappa^{++}$

(c) if β is a limit point of c_α then $c_\beta = c_\alpha \cap \beta$

(d) if β is a successor point of c_α then $cf\beta = \kappa^{++}$.

We implement Box$'(\kappa^{++})$ into $\mathcal{P}'(1)$ as follows:

Definition 5.2 $\mathcal{P}''(1)$ consists of $\langle\langle A^{00}, A^{10}\rangle, \langle c_\alpha \mid \alpha \in \lim(\{B \cap \kappa^{+++} \mid B \in A^{10}\})\rangle$ such that

(1) $\langle A^{00}, A^{10}\rangle \in \mathcal{P}'(1)$

(2) $\langle c_\alpha \mid \alpha \in \lim(\{B \cap \kappa^{+++} \mid B \in A^{10}\})\rangle \in$ Box$'(\kappa^{++})$.

Define the ordering in the obvious fashion.

Denote further the set $\lim(\{B \cap \kappa^{+++} \mid B \in A^{10}\})$ by $\lim(A^{10})$. We shall use $\mathcal{P}''(1) \times \mathcal{P}'(0)$. Note that $\mathcal{P}'(0)$ is of cardinality κ^{++} and $\mathcal{P}''(1)$ is κ^{++}-strategically closed.

We will need certain simple and likely known facts about Todorcevic walks [Tod] between ordinals using a fixed box sequence.

Thus let τ be a cardinal and $\langle C_\nu \mid \nu < \tau^+, \nu \text{ limit}\rangle$ a $\square_\tau$-box sequence.

Definition 5.3 Let $\tau^+ > \alpha \geq \beta$. The Todorcevic walk $w(\alpha, \beta)$ from α to β via $\langle C_\nu \mid \nu < \tau^+$ and ν limit$\rangle$ is defined as follows by induction on α:

(a) if $\alpha = \beta$ then it is just $w(\alpha, \beta) = \{\alpha\}$

(b) if $\alpha > \beta$ and α is a successor ordinal, then let $\alpha = \alpha^* + n^*$ for a limit α^* and $0 < n^* < \omega$. If $\beta = \alpha^* + k^*$ for some $k^* < n^*$ then set $w(\alpha, \beta) = \{\alpha^* + \ell \mid \ell \leq n^*\}$

(c) if $\alpha > \beta$ and α is a limit ordinal then consider C_α.

(c1) if $\beta \in C_\alpha$ then pick β^* to be the largest limit element of $C_\alpha \cap (\beta + 1)$ if it exists or 0 otherwise. Set $w(\alpha, \beta) = \{\alpha, \beta\} \cup \{\gamma \in C_\alpha \mid \beta^* \leq \gamma \leq \beta\}$

(c2) if $\beta \notin C_\alpha$ then let $\alpha^>(\beta) = \min(C_\alpha \backslash \beta)$. If $C_\alpha \cap \beta = \emptyset$ (i.e. $\alpha^>(\beta)$ is the least element of C_α) then set $w(\alpha, \beta) = \{\alpha\} \cup w(\alpha^>(\beta), \beta)$. Otherwise define $\alpha^<(\beta)$ to be $\max(C_\alpha \cap \beta)$. Let $\alpha^<(\beta)^*$ be the largest limit element of $C_\alpha \cap (\alpha^<(\beta) + 1)$ if it exists or 0 otherwise. Set $w(\alpha, \beta) = \{\alpha\} \cup w(\alpha^>(\beta), \beta) \cup \{\gamma \in C_\alpha \mid \alpha^<(\beta)^* \leq \gamma \leq \alpha^<(\beta)\}$.

Definition 5.4 A set $E \subseteq \tau^+$ is called walks closed iff

(a) E is a closed set of ordinals

(b) if $\alpha, \beta \in E$ and β is a successor point of C_α then it predecessor in C_α is in E

(c) if $\alpha, \beta \in E$ and $\alpha \geq \beta$ then the walk from α to β is contained in E, i.e. all the ordinals appearing in the walk from α to β via the box sequence $\langle C_\nu \mid \nu < \tau^+\rangle$ are in E.

Notation 5.5 For $E \subseteq \tau^+$ we denote by *clw* (E) the least walks closed set including E.

Clearly such a set exists since an intersection of walk closed sets is walk closed.

Lemma 5.6 *Suppose that $E \subseteq \tau^+$ is walk closed. Let $a \subseteq \tau^+$ be finite. Then*

$$|clw(E \cup a) \backslash E| < \aleph_0 \ .$$

Proof. We prove the statement by induction on sup E. Let $\delta = \sup E$. Suppose that for every walks closed set D with $\sup D < \delta$ and every finite $a \subseteq \tau^+$ the set $clw(D \cup a) \backslash D$ is finite.

Now let $a \subseteq \tau^+$ be finite. We like to show that $clw(E \cup a) \backslash E$ is finite as well. Assume as an inductive assumption that for every finite $a' \subseteq \tau^+$ with $\max a' < \max a$ the statement is true.

Using induction on size of a we can assume without loss of generality that $a = \{\alpha\}$ for some $\alpha < \tau^+$.

Case 1. $\alpha > \delta$.
Consider C_α. let $\alpha^{\geq}(\delta)$ be the least element of $C_\alpha \geq \delta$ and $\alpha^{<}(\delta)$ be the last element of C_α below δ. If $\min C_\alpha \geq \delta$ then we just replace α by $\min C_\alpha < \alpha$ and use induction. If there are elements of E below $a^{<}(\delta)$ then let $\delta_1 = \max(E \cap a^{<}(\delta)$. We then define $\alpha^{\geq}(\delta_1)$ and $\alpha^{<}(\delta_1)$ in the same way replacing δ by δ_1 and α by $\alpha^{<}(\delta)$. Again we check if there are elements of E below $\alpha^{<}(\delta_1)$ and if this is the case we define δ_2, $\alpha^{\geq}(\delta_2)$, $\alpha^{<}(\delta_2)$. After finitely many steps there will be δ_k, for some $k < \omega$, so that $\alpha^{<}(\delta_k) \cap E = \emptyset$. Now we consider $a = \{\alpha^{\geq}(\delta_i), \alpha^{<}(\delta_i) \mid i \leq k\}$. Clearly, $\max a = \alpha^{\geq}(\delta) < \alpha$. So we can apply an inductive assumption. Hence, the set $clw(E \cup a) \backslash E$ is finite. But notice that $clw(E \cup \{\alpha\}) = (clw(E \cup a)) \cup \{\alpha\}$. Thus we are done.

Case 2. $\alpha < \delta$.
Let $\delta^* = \min(E \backslash \alpha)$ and $\delta^{**} = \max(E \cap \alpha)$. First notice that if $\delta_1 < \delta_2$ are two successive elements of E then for any $\rho \in E \backslash \delta_2$ and $\xi \in (\delta_1, \delta_2]$ the walk from ρ to ξ necessary passes through δ_2, since E is walks closed.

Consider $E \cap (\delta^{**} + 1)$. It is clearly walks closed. By induction,

$$clw((E \cap (\delta^{**} + 1)) \cup \{\alpha\}) \backslash (E \cap (\delta^{**} + 1)) \ | < \aleph_0 \ .$$

Let $\{\alpha_0, \ldots, \alpha_{k-1}\}$ be the increasing enumeration of this set. For every $i < k$ we pick $\delta_i^* = \min(E \backslash \alpha_i)$ and $\delta_i^{**} = \max(E \cap \alpha_i)$. As it was remarked above for every $i < k$ and $\rho \in E \backslash \delta_i^*$ the walk from ρ to α_i passes via δ_i^*. But the walk from δ_i^* to α_i is finite and depends only on δ_i^* and α_i. Hence $clw(E \cup \{\alpha\}) = (E \backslash \delta^{**} + 1) \cup (clw((E \cap (\delta^* + 1)) \cup \{\alpha\}))$ and we are done. □

Lemma 5.7 *Let $E \subseteq \tau^*$ be walks closed set and $a \subseteq \tau^+$ finite. Then there is a finite $E' \subseteq E$ such that for any $\rho \in$ $clw(E \cup a)$ and $\alpha \in clw(E \cup a) \backslash (E \cup \rho)$ the following holds, where $w(\alpha, \rho)$ is Todorcevic walk from α to ρ:*

(a) *if $\rho \notin E$ then $w(\alpha, \rho) \subseteq E' \cup (clw(E \cup a) \backslash E)$*

(b) *if $\rho \in E$ then there is $\tau \in w(\alpha, \rho) \cap E' \backslash \rho$ so that $w(\alpha, \tau) \subseteq E' \cup (clw(E \cup a) \backslash E)$ and $(w(\alpha, \rho) \backslash w(\alpha, \tau)) \cup \{\tau\} = w(\tau, \rho)$.*

Proof. Let us use induction on $\max(clw(E \cup a))$. Then we can assume that $\max(clw(E \cup a)) = \max(clw(E \cup a) \backslash E)$. Let $\alpha = \max(clw(E \cup a) \backslash E)$.

First note that the set $clw(E \cup a) \cap \alpha$ is bounded in α, since otherwise E will be unbounded in α (by Lemma 5.6, $clw(E \cup a) \backslash E$ is finite) and then $\alpha \in E$ since E is closed.

Denote by α_1 the maximum of $clw(E \cup a) \cap \alpha$. Let $A = w(\alpha, \alpha_1)$ and let $B = clw(A)$. Then, by 5.6, B is finite, since A is such. Consider $E \cap (\alpha_1 + 1)$ and $(B \cup a) \cap (\alpha_1 + 1)$. Now we can apply inductive assumption. Let $E' \subseteq E \cap \alpha$ be a finite set satisfying the conclusion of the lemma for $E \cap \alpha = E \cap (\alpha_1 + 1)$ and $(B \cup a) \cap (\alpha_1 + 1)$. It is easy to check that E' is as required.

$\square$

Lemma 5.8 *Let E be walks closed bounded subset of τ^+ which is an increasing union of walks closed sets E_n $(n < \omega)$ and $a \subseteq \tau^+$ be finite. Then there is $n_0 < \omega$ such that for every $n \geq n_0$*

$$clw(E \cup a) \backslash E = clw(E_n \cup a) \backslash E_n \ .$$

Proof. First note that it is enough to prove the lemma for a set a with $\max(a) > \max E$. Thus for arbitrary a we can just add an ordinal above $\max E$ to it. Let b be such a set. Applying the lemma to b we find $n_0' < \omega$ such that for every $n \geq n_0'$

$$clw(E \cup b) \backslash E = clw(E_n \cup b) \backslash E_n \ .$$

Now we pick $n_0 \geq n_0'$ so that

$$clw(E \cup a) \backslash E = clw(E_{n_0} \cup a) \backslash E \ .$$

This is possible by 5.6. Then for every $n \geq n_0$

$$clw(E \cup a) \backslash E = clw(E_n \cup a) \backslash E \subseteq clw(E_n \cup a) \backslash E_n \ .$$

Let $\rho \in clw(E_n \cup a)\backslash E_n$. We need only to show that $\rho \notin E$. But $\rho \in clw(E_n \cup b)\backslash E_n$, since $b \supseteq a$. Then $\rho \in clw(E \cup b)\backslash E$. In particular $\rho \notin E$.

Hence we can assume without loss of generality that $a \backslash \max E \neq \emptyset$. Consider now the set $clw(E \cup a)\backslash E$. By 5.6 it is finite. For every α in $(clw(E \cup a)\backslash E) \cap \max E$ we set $\widetilde{\alpha} = \min(E\backslash\alpha)$ and $\widetilde{\widetilde{\alpha}} = \max(E \cap \alpha)$, if $E \cap \alpha \neq \emptyset$. Define $A = \{\widetilde{\alpha}, \widetilde{\widetilde{\alpha}} \mid \alpha \in clw(E \cap a)\backslash E\}$. Clearly A is finite. Let E' be a finite subset of E given by 5.7. Set $n_0 < \omega$ to be such that $E' \cup (E \cap clw(a)) \cup A \subseteq E_{n_0}$ and $clw(E \cup a)\backslash E = clw(E_{n_0} \cup a)\backslash E$.

Suppose now that $n \geq n_0$. Clearly, $clw(E \cup a)\backslash E = clw(E_n \cup a)\backslash E \subseteq clw(E_n \cup a)\backslash E_n$. Let $\rho \in clw(E_n \cup a)\backslash E_n$. We need to show that $\rho \in clw(E \cup a)\backslash E$. Suppose otherwise. Then $\rho \in E\backslash E_n$.

Let us show that $clw(E_n \cup a)$ cannot contain such ordinals. Thus, suppose that $\alpha, \beta \in E_n \cup (clw(E \cup a)\backslash E) \supseteq E_n \cup a$, $\alpha > \beta$ and we walk from α to β.

Case 1. $\alpha, \beta \in E_n$.
Then, the walk is included in E_n, since E_n is walks closed.

Case 2. $\alpha \in E_n$, $\beta \in clw(E \cup a)\backslash E$.

Then $\widetilde{\beta}, \widetilde{\widetilde{\beta}}$ are defined. By the choice of n_0, $\widetilde{\beta}$ and $\widetilde{\widetilde{\beta}}$, if defined, are in E_n. The walk from α to β must first get to $\widetilde{\beta}$ remaining completely in E_n (again E_n is walks closed). After this the walk from $\widetilde{\beta}$ to β will be inside $clw(E \cup a)\backslash E$.

Case 3. $\alpha, \beta \in clw(E \cup a)\backslash E$.
Then by 5.7(a) the walk from α to β is contained in $E' \cup (clw(E \cup a)\backslash E)$. Again leaving no space for $\rho \in E\backslash E_n$. Remember that $E' \subseteq E_n$.

Case 4 $\alpha \in clw(E \cup a)\backslash E$, $\beta \in E_n$.
Then 5.7(b) applies. There will be $\tau \in w(\alpha, \beta) \cap E'\backslash\beta$ so that the walk from α to τ is contained in $E' \cup (clw(E \cup a)\backslash E)$ and the rest of the walk is the Todorcevic walk from τ to β. But both τ and β are in E_n. Hence the walk from τ to β is contained in E_n. So, once again there is no place for $\rho \in E\backslash E_n$.

Contradiction. $\square$

The following is an easy consequence of 5.6 and 5.8.

Lemma 5.9 *Let $E, \langle E_n \mid n < \omega\rangle$ and a be as in 5.8. Then there is a finite set $a^* \supseteq a$ and $n_0 < \omega$ such that for every $n \geq n_0$ $E_n \cup a^*$ is walks closed and $E \cup a^*$ is walks closed as well.*

Now we return to the forcing construction. Define the main preparation forcing $\mathcal{P}$.

Definition 5.10 The set $\mathcal{P}$ consists of sequences

$$\langle\langle A^{00}(0), A^{10}(0), F^0(0)\rangle\ , \langle A^{00}(1), A^{10}(1)\rangle,\ \langle c_\nu \mid \nu \in \lim A^{10}(1)\rangle, F\rangle$$

so that

(1) $\langle A^{00}(0), A^{10}(0), F^0(0)\rangle \in \mathcal{P}(0)$.

(2) $\langle\langle A^{00}(1), A^{10}(1)\rangle, \langle c_\nu \mid \in \lim A^{10}(1)\rangle\rangle \in \mathcal{P}''(1)$.

(3) F consists of *all* pairs $p = \langle p^{\vec{\lambda}}, p^{\vec{\kappa}}\rangle$ of sequences $p^{\vec{\lambda}} = \langle p^{\vec{\lambda}}_n \mid n < \omega\rangle$ and $p^{\vec{\kappa}} = \langle \underset{\sim}{p}^{\vec{\kappa}}_n \mid n < \omega\rangle$ so that

(a) $p^{\vec{\lambda}} \in F^0(0)$

(b) $\ell(p^{\vec{\lambda}}) = \ell(p^{\vec{\kappa}}) = \ell(p)$

(c) for every $n < \ell(p)$ $\underset{\sim}{p}^{\vec{\kappa}}_n \in V$

(d) if $n \geq \ell(p)$, then $\underset{\sim}{p}^{\vec{\kappa}}_n = \langle \underset{\sim}{a}^{\vec{\kappa}}_n, \underset{\sim}{A}^{\vec{\kappa}}_n, f^{\vec{\kappa}}_n\rangle$ is so that

(i) $f^{\vec{\kappa}}_n$ is a function of cardinality at most κ from κ^{++} to κ_n

(ii) $\mathrm{dom}\,\underset{\sim}{a}^{\vec{\kappa}}_n \in V$ is as in 2.1(d) but of cardinality $< \lambda_n$ instead of κ_n

(iii) $\langle \underset{\sim}{a}^{\vec{\kappa}}_n, \underset{\sim}{A}^{\vec{\kappa}}_n\rangle$ are names depending on $p^{\vec{\lambda}}_n$ only and in the following way: in order to decide $\langle \underset{\sim}{a}^{\vec{\kappa}}_n, \underset{\sim}{A}^{\vec{\kappa}}_n\rangle$ it is enough to get a value of the one element Prikry sequence of the maximal coordinate of $p^{\vec{\lambda}}_n$ and the projections of it onto the support of $p^{\vec{\lambda}}_n$. Moreover, if $A \in (\mathrm{dom}\,\underset{\sim}{a}^{\vec{\kappa}}_n)\backslash On$ is a limit element of $A^{10}(1)$ and $cf\Big(c_{\sup(A\cap\kappa^{+3})}\Big) = \kappa^{++}$ then for some k_n, $2 < k_n < \omega$ the k_n-type of $a^{\vec{\kappa}}_n(A)$ depends only on the value of one element Prikry sequence corresponding to the normal measure of $a^{\vec{\lambda}}_n$. Also, as usual, we require that $\lim_{n\to\infty} k_n = \infty$.

The above will allow us further to generate equivalent conditions which in turn will be crucial for proving κ^{++}-c.c. of the final forcing.

We now continue to describe the correspondence function $a^{\vec{\kappa}}_k$. Our main attention will be to A's as above, i.e. limit models. Dealing with non-limit models is much easier. For every $\rho < \lambda_n$ a potential element of the Prikry sequence for the normal measure over λ_n, i.e. for example $\rho \in \left(A_n^{\vec{\lambda}}\right)^0$, we reserve a Box (κ_n^{+n+1})-generic box sequence $\square_{\kappa_n^{+n+1}}$ and deal with box sequences $\langle C_\alpha^n \mid \alpha < \kappa_n^{+n+2},\, cf\alpha \leq \rho^{+n+2}\rangle$ defined from it so that $otpC_\alpha^n \leq \rho^{+n+2}$ for every α in its domain. Let $\langle B_\alpha \mid \alpha < \kappa_n^{+n+2}\rangle$ be an increasing continuous sequence of submodels of $\mathfrak{a}_{n,k}(k < \omega)$ of cardinality κ_n^{+n+1}, with $\langle B_\beta \mid \beta \leq \alpha\rangle \in B_{\alpha+1}$ for every α and $\langle C_\alpha^n \mid \alpha < \kappa_n^{+n+2}\rangle \in B_0$. Denote by C the club consisting of $\sup B_\alpha \cap \kappa_n^{+n+2}$ $(\alpha < \kappa_n^{+n+2})$. Now consider

$$\langle C_\alpha^n \cap C \mid \alpha \text{ is a limit point of } C,\ cf\alpha \leq \rho^{+n+2}\rangle\ .$$

Clearly,

(α) $C_\alpha^n \cap C$ is a club in α of order type $\leq \rho^{+n+2}$

(β) if γ is a limit point of $C_\alpha^n \cap C$ then γ is a limit point of C, $cf\gamma \leq \rho^{+n+2}$ and

$$C_\gamma^n \cap C = C_\alpha^n \cap C \cap \gamma\ .$$

Further we shall use different k's as well as different model sequences B_α's.

(e) There is the maximal (under inclusion) model A in $\mathrm{dom}(a_n^{\vec{\kappa}})$. It is a limit element of $A^{10}(1)$ and its intersection with κ^{+3} has cofinality κ^{++}.

We require the following, once the elements of one element Prikry sequences for the support of $a_n^{\vec{\lambda}}$ are decided, where ρ denotes the one for the normal measure and for $\gamma \in \mathrm{dom} a_n^{\vec{\lambda}}$, γ^* denotes the corresponding to γ value of the Prikry sequence then

(e1) $a_n^{\vec{\kappa}}(A)$ is a submodel of $\mathfrak{a}_{n,k_n}$ depending only on the value of ρ (where, as usual, $2 < k_n < \omega$, k_n's are nondecreasing with limit ∞) such that $cf(a_n^{\vec{\kappa}}(A) \cap \kappa_n^{+n+2}) = \rho^{+n+2}$

(e2) for every limit point B of $A^{10}(1)$ which is in $\mathrm{dom} a_n^{\vec{\kappa}}$ we fix the element $C^n_{a_n^{\vec{\kappa}}(B)\cap\kappa^{+n+2}}$ of some box sequence $\vec{C^n} = \langle C_\alpha^n \cap C \mid \alpha \text{ is a limit point of } C \text{ and } cf\alpha \leq \rho^{+n+2}\rangle$, where $\vec{C^n}$ is as described above.

Here we mean that only $C^n_{a_n^{\vec{\kappa}}(B)\cap\kappa^{+n+2}}$'s are fixed for B's as above, but the rest of $\vec{C^n}$ can be further changed. Recall that we have a generic box sequence $\square_{\kappa_n^{+n+2}}$ so there are a lot of possibilities for choosing $\vec{C^n}$'s. Denote further $C^n_{a_n^{\vec{\kappa}}(B)\cap\kappa^{+n+2}}$ by $C^n(B)$. $C^n(B)$ depends on the elements of one element Prikry sequence for the support of $a_n^{\vec{\lambda}}$. It is decided once these elements are decided.

(e3) for every B as in (e2) if $cf(B \cap \kappa^{+++}) < \kappa^{++}$ then $otp(c_{B\cap\kappa^{+++}}) \in \mathrm{dom} a_n^{\vec{\lambda}}$. Let $\xi = a_n^{\vec{\lambda}}(otp(c_{B\cap\kappa^{+++}}))$. Then we require that

$$otp(C^n(B)) = \xi^* \ .$$

(e4) for every B as in (e3) there is $\widetilde{B} \in \mathrm{dom} a_n^{\vec{\kappa}}$ such that

(i) $cf(\widetilde{B} \cap \kappa^{+++}) = \kappa^{++}$

(ii) $B \cap \kappa^{+++}$ is a limit point of $c_{\widetilde{B}\cap\kappa^{+++}}$.

Hence $c_{B\cap\kappa^{+++}} = c_{\widetilde{B}\cap\kappa^{+++}} \cap B \cap \kappa^{+++}$. We require that the same holds below at κ_n. Namely, the following should be true.

(iii) $C^n(B) = C^n(\widetilde{B}) \cap a_n^{\vec{\kappa}}(B) \cap \kappa_n^{+n+2}$.

(e5) let B, B' be limit points of $A^{10}(1)$ so that

(i) $cf(B \cap \kappa^{+3}) = cf(B' \cap \kappa^{+3}) = \kappa^{++}$

(ii) $(B' \cap \kappa^{+3}) \in c_{B\cap\kappa^{+3}}$ (and hence by (i) it is a nonlimit point of $c_{B\cap\kappa^{+3}}$).

Let $\gamma_{B'} < \kappa^{++}$ be so that $B' \cap \kappa^{+3}$ is $\gamma_{B'} + 1$-th element of $c_{B\cap\kappa^{+3}}$. Suppose that $B, B' \in \mathrm{dom} a_n^{\vec{\kappa}}$. Then the following holds:

(α) $\gamma_{B'}, \gamma_{B'} + 1 \in \mathrm{dom} a_n^{\vec{\lambda}}$

(β) $C^n(B')$ depends only on one element Prikry sequences for λ_n needed in order to decide $C^n(B)$ and also those for $a_n^{\vec{\lambda}}(\gamma_{B'}), a_n^{\vec{\lambda}}(\gamma_{B'} + 1)$.

$\mathrm{dom} a_n^{\vec{\kappa}}$ may contain only elements of $c_{A\cap\kappa^{+++}}$, but in general it should not. We would like still to be able to read most of information from $A \cap \kappa^{+++}$ and parameters from κ^{++} only. For this purpose let us use Todorcevic walks via box sequences in order to go down from $A \cap \kappa^{+++}$ to smaller ordinals. Thus let $\alpha = A \cap \kappa^{+++}$ and $\beta = B \cap \kappa^{+++}$ for some $B \in \mathrm{dom} a_n^{\vec{\kappa}}$. Set $\alpha_0^{\geq}(\beta) = \min(c_\alpha \cap \beta)$. If $\alpha_0^{\geq}(\beta) > \beta$ then define $\alpha_0^{<}(\beta) = \sup(c_\alpha \cap \beta)$ and $\alpha_1^{\geq}(\beta) = \min(c_{\alpha_0^{\geq}(\beta)} \cap \beta)$. Continue by induction to define $\alpha_{k-1}^{<}(\beta)$, $\alpha_k^{\geq}(\beta)$ until β is reached. We shall also use elements of $A^{10}(1)$ instead of ordinals. Denote by $A_{k-1}^{<}(B)$ and $A_k^{\geq}(B)$ the models in $A^{10}(1)$ so that $\alpha_{k-1}^{<}(\beta) = A_{k-1}^{<}(B) \cap \kappa^{+++}$ and $\alpha_k^{\geq}(\beta) = A_k^{\geq}(B) \cap \kappa^{+++}$.

The next condition requires that the process can be simulated over κ_n.

(f) for every limit model B of $A^{10}(1)$ which is in $\mathrm{dom}a_n^{\vec{\kappa}}$ the following holds:

(f1) for every $k<\omega$ such that $A_k^{\geq}(B)$ and hence also $A_{k-1}^{<}(B)$ are defined we require that these models are in $\mathrm{dom}a_n^{\vec{\kappa}}$ and the image by $a_n^{\vec{\kappa}}$ of the walk from A to B is exactly the walk from $a_n^{\vec{\kappa}}(A)$ to $a_n^{\vec{\kappa}}(B)$, where at κ_n we use the fixed in (c2) sequences.

(g) if some $D,E\in \mathrm{dom}a_n^{\vec{\kappa}}$ and $D\subseteq E$ then all the models of the walk from E to D are in $\mathrm{dom}a_n^{\vec{\kappa}}$ as well and the image by $a_n^{\vec{\kappa}}$ of the walk from E to D is exactly the walk from $a_n^{\vec{\kappa}}(E)$ to $a_n^{\vec{\kappa}}(D)$.

(h) Let q,r be two extensions of $p_n^{\vec{\lambda}}$ (i.e. at level λ_n) deciding the value of the one element Prikry sequence of the maximal coordinate of $p_n^{\vec{\lambda}}$ together with its projections onto the support of $p_n^{\vec{\lambda}}$. Suppose that $\gamma<\kappa^{++}$ is an element of the support of $p_n^{\vec{\lambda}}$ and $q\restriction\gamma=r\restriction\gamma$, i.e. q and r agree about the values of one element Prikry sequences corresponding to ordinals below γ (in particularly, the one for the normal measure). Then for every $B\in \mathrm{dom}a_n^{\vec{\kappa}}$ with the walks closure of the maximal model of $\mathrm{dom}a_n^{\vec{\kappa}}$ and B involving only models with distances between them which are ordinals below γ the following holds:
q and r forcing the same value for $\underset{\sim}{a}{}_n^{\vec{\kappa}}(B)$.

(i) Suppose that $B\in \mathrm{dom}a_n^{\vec{\kappa}}$ and $cf(B\cap\kappa^{+++})=\kappa^{++}$. Then there are $q=\langle q^{\vec{\lambda}},q^{\vec{\kappa}}\rangle\in F$ and a nondecreasing converging to ∞ sequence $\langle k_n\mid n<\omega\rangle$ of natural numbers with $k_0>4$ so that the following holds:

(i)(a) $q^{\vec{\lambda}}=p^{\vec{\lambda}}$

(i)(b) for every $n\geq\ell(p)$ (or more precisely, starting with n s.t. $B\in \mathrm{dom}a_n^{\vec{\kappa}}$)

(α) B is the maximal model of $a_n^{\vec{\kappa}}(q)$ (i.e. the assignment function of $q_n^{\vec{\kappa}}$)

(β) $\mathrm{dom}a_n^{\vec{\kappa}}(q)=\{C\in \mathrm{dom}a_n^{\vec{\kappa}}\mid C\subseteq B\}$

(γ) $p_n^{\vec{\lambda}}$ forces that $\underset{\sim}{a}{}_n^{\vec{\kappa}}\restriction B$ and $\underset{\sim}{a}{}_n^{\vec{\kappa}}(q)$ are k_n – equivalent.

The intuitive meaning of the condition (i) is that we are able for any B as above turn it into the maximal model.

The order on $\mathcal{P}$ is defined in usual fashion.

Definition 5.11 Let $\langle\langle A^{00}(0),\ A^{10}(0),\ F^0(0)\rangle,\ \langle A^{00}(1),\ A^{10}(1)\rangle,\ \langle c_\nu \mid \nu\in\lim(A^{10}(1))\rangle,\ F\rangle$ and $\langle\langle B^{00}(0),\ B^{10}(0),\ G^0(0)\rangle,\ \langle B^{00}(1),\ B^{10}(1)\rangle,\ \langle d_\nu \mid \nu\in\lim(B^{10}(1))\rangle,\ G\rangle$ be in $\mathcal{P}$. We define

$$\langle\langle A^{00}(0), A^{10}(0), F^0(0)\rangle, \langle A^{00}(1), A^{10}(1)\rangle, \langle c_\nu \mid \nu\in\lim(A^{10}(1)\rangle, F\rangle >$$
$$\langle\langle B^{00}(0), B^{10}(0), G^0(0)\rangle\ ,\ \langle B^{00}(1), B^{10}(1)\rangle, \langle d_\nu \mid \nu\in\lim(B^{10}(1)\rangle, G\rangle$$

iff

(1) $\langle A^{00}(0),\ A^{10}(0),\ F^0(0)\rangle > \langle B^{00}(0), B^{10}(0), F^1(0)\rangle$ in $\mathcal{P}(0)$.

(2) $\langle\langle A^{00}(1),\ A^{10}(1)\rangle,\ \langle c_\nu \mid \nu\in\lim(A^{10}(1))\rangle\rangle > \langle\langle B^{00}(1),\ B^{10}(1)\rangle,\ \langle d_\nu \mid \nu\in\lim(B^{10}(1))\rangle\rangle$ in $\mathcal{P}''(1)$.

(3) let $p=\langle p^{\vec{\lambda}},\ p^{\vec{\kappa}}\rangle\in F$ with $p^{\vec{\kappa}}=\langle \underset{\sim}{p}{}^{\vec{\kappa}}_n \mid n<\omega\rangle$, $p^{\vec{\lambda}}=\langle p^{\vec{\lambda}}_n \mid n<\omega\rangle$,

$\underset{\sim}{p}{}^{\vec{\kappa}}_n=\langle\ \underset{\sim}{a}{}^{\vec{\kappa}}_n, \underset{\sim}{A}{}^{\vec{\kappa}}_n, f^{\vec{\kappa}}_n\rangle$ for $\omega>n\geq\ell(p)$, and $B\in B^{10}(1)$.

Suppose that for every n, $\omega>n\geq\ell(p)$, $\underset{\sim}{a}{}^{\vec{\kappa}}_n(B)$ depends only on the value of one element Prikry sequence for the normal measure over λ_n. Define then $p\restriction B$ in the obvious fashion taking B to play the maximal model. Now we require the following: if $p\restriction B\in F$ then $p\restriction B\in G$.

We shall check now few basic properties of the forcing $\mathcal{P}$ which in the present context require some arguments.

Lemma 5.12 *Let $\langle\langle A^{00}(0),\ A^{10}(0),\ F^0(0)\rangle,\ \langle A^{00}(1),\ A^{10}(1)\rangle,\ \langle c_\nu \mid \nu\in\lim(A^{10}(1))\rangle,\ F\rangle\in\mathcal{P}$, $p=\langle p^{\vec{\lambda}}, p^{\vec{\kappa}}\rangle\in F$, $B\in A^{10}(1)$ is inside the maximal model of p. Then B is addable to p.*

Proof. For every $n\geq\ell(p)$ let $E_n=\mathrm{dom} a^{\vec{\kappa}}_n(p)$, where as usual, $a^{\vec{\kappa}}_n(p)$ is the assignment function of $p^{\vec{\kappa}}_n$. Set $E=\bigcup_{n\geq\ell(p)}E_n$. Then by 5.10(g) E_n's and E are walks closed. Apply 5.9 to $E, \langle E_n \mid \ell(p)\leq n<\omega\rangle$ and $\{B\}$. There will be a finite set of models D and $n_0<\omega$ such that for every $n\geq n_0$ $E_n\cup D$ and $E\cup D$ are walks closed. Now we will extend p by adding to it the elements of D. Note that such extension need not be a direct extension (i.e. $\leq^*$) and $\ell(p)$ may increase as a result. But important thing is that D is finite and the same at each level. So climbing high enough we will be able to add all its members.

Now we turn to a complication due to working with names in the range of $a^{\vec{\kappa}}_n(p)$. The conditions (h) and (i) should be satisfied after adding elements of D to p.

Fix $n, n_0 \le n < \omega$. Let $q_n^{\vec{\lambda}}$ be an extension of $p_n^{\vec{\lambda}}$ deciding $p_n^{\vec{\kappa}}$ and so that all the ordinals $< \kappa^{++}$ needed for walks in $E_n \cup D$ appear in the domain of the assignment function of $q_n^{\vec{\lambda}}$. Below we will use induction on such $q_n^{\vec{\lambda}}$. Assume so that we pick them one by one using some enumeration. We assume that n is large enough in order to be able to add to $p_n^{\vec{\lambda}}$ the missing finite set of ordinals. Let $\{A_i \mid i < k\}$ be the increasing enumeration of D. For every $i < k$ pick $\widetilde{A}_i$ to be the least model of E_n including A_i and $\widetilde{\widetilde{A}}_i$ the last model of E_n included in A_i. By induction we define for every $i < k$ an extension $b_{n,i}^{\vec{\kappa}}$ of the assignment function $a_n^{\vec{\kappa}}(p)$ of $p_n^{\vec{\kappa}}$ including the elements of D of the interval $(\widetilde{\widetilde{A}}_i, \widetilde{A}_i)$. Notice that there may be $i' \neq i'' < k$ such that $\widetilde{\widetilde{A}}_{i'} = \widetilde{\widetilde{A}}_{i''}$ (and then also $\widetilde{A}_{i'} = \widetilde{A}_{i''}$). In this case, we will have $b_{n,i'}^{\vec{\kappa}} = b_{n,i''}^{\vec{\kappa}}$. Let $i < k$ and suppose for every $i' < i$ $b_{n,i'}^{\vec{\kappa}}$ is defined. If there is $i' < i$ such that $\widetilde{\widetilde{A}}_{i'} = \widetilde{\widetilde{A}}_i$ then set $b_{n,i}^{\vec{\kappa}} = b_{n,i'}^{\vec{\kappa}}$. Assume that for every $i' < i$ $\widetilde{\widetilde{A}}_{i'} \neq \widetilde{\widetilde{A}}_i$, i.e. we deal with new intervals. First consider limit points of $c_{\widetilde{A}_i \cap \kappa^{+3}}$ (i.e. the element of the box sequence forced over κ^{+++} corresponding to $\kappa^{+3} \cap \widetilde{A}_i$) between $\widetilde{\widetilde{A}}_i$ and $\widetilde{A}_i$ which are in D, if there are such. Note, that by the choice of $\widetilde{A}_i$ and $\widetilde{\widetilde{A}}_i$, $\widetilde{A}_i$ is a successor point and $\widetilde{\widetilde{A}}_i \cap \kappa^{+++} \in c_{\kappa^{+3} \cap \widetilde{A}_i}$, since E_n is walk closed there is no elements of E_n between $\widetilde{\widetilde{A}}_i$ and $\widetilde{A}_i$. We correspond them to the limit points of the box sequence $C_{a_n^{\vec{\kappa}}(\widetilde{A}_i)}$ over κ_n^{+n+2} according to the values prescribed by $a_n^{\vec{\kappa}}$. Now we turn to the successor points. Let B be the smallest successor element of D between $\widetilde{\widetilde{A}}_i$ and $\widetilde{A}_i$. We consider its box sequence $c_{\kappa^{+3} \cap B}$. Then $\widetilde{\widetilde{A}}_i \cap \kappa^{+3} \in c_{\kappa^{+3} \cap B}$ since, B is the smallest successor point of D and $D \cup E_n$ is walk closed. Let B^* be the largest limit point (if it exists) of $c_{\kappa^{+3} \cap B} \le \kappa^{+3} \cap \widetilde{\widetilde{A}}_i$ and let $B_1^* \subset B_2^* \subset \cdots \subset B_\ell^* \subseteq \widetilde{\widetilde{A}}_i (\ell < \omega)$ be all the successor points of $c_{\kappa^{+3} \cap B}$ between B^* and $\widetilde{\widetilde{A}}_i$, if there any. Notice, that $\langle B_m^* \mid 1 \le m \le \ell \rangle$ and B^* are exactly the elements needed for walks from B to elements of $\mathcal{P}(\widetilde{\widetilde{A}}_i) \cap (D \cup E)$. We now define $b_{n,i}^{\vec{\kappa}}(B)$, (i.e. the value of the extended assignment function on B) to be a model so that

(1) its type is the same as the type of every successor model (with limit points of its box sequence taken into account in the type)

(2) it is above $a_{n,i}^{\vec{\kappa}}(\widetilde{\widetilde{A}}_i)$ as well as all the images of limit points of D (if any) which are below B

(3) it is included into the image of $\widetilde{A}_i$ as well as all the images of limit points of D above B

(4) the distances from it to the images of B^*, $B_1^*, \ldots, B_\ell^*, \approx{A}_i$ and the limit models of D between $\approx{A}_i$ and B are the same as the images under $a_n^{\vec{\lambda}}$ (the assignment function for κ^{++} to λ_n) of the distances from B to B^*, $B_1^*, \ldots, B_\ell^*$, $\approx{A}_i$ and the limit models of D between $\approx{A}_i$ and B respectively.

Our next requirement is needed in order to insure (h) of 5.10 once B is used as a maximal model as in 5.10(i). First fix $B^{**} \in E_n$ to be the element of the walk from B_1^* to B^* if B_1^* is defined or else from $\max(E_n)$ to B^* such that $cf(\kappa^{+3} \cap B^{**}) = \kappa^{++}$ and $B^* \cap \kappa^{+3} \in c_{B^{**} \cap \kappa^{+3}}$. There is such B^{**} since E_n is walks closed (just consider the walk from $\max(E_n)$ to B^*. We will reach such B^{**} one stage before getting to B^*). Let $\beta^* = otpc_{B^* \cap \kappa^{+3}}$.

(5) Split into two cases.

Case 1. In the inductive process before $q_n^{\vec{\lambda}}$ there is no condition which agree with $q_n^{\vec{\lambda}}$ up to β^*.

Then we require $b_{n,i}^{\vec{\kappa}}(B)$ to realize the same type over $\{a_n^{\vec{\kappa}}(S) \mid S \in E_n, S \subseteq B^*\}$ as $a_n^{\vec{\kappa}}(B^{**})$ realizes over this set.

Case 2. In the inductive process before $q_n^{\vec{\lambda}}$ there are conditions that agree with $q_n^{\vec{\lambda}}$ up to β^*.

If $B^* = \approx{A}_i$, then we proceed as in Case 1. Otherwise set $B_{\ell+1}^* = \approx{A}_i$. Consider the images of the walks between $B^*, B_1^*, \ldots, B_{\ell+1}^*$. Find the largest $t \leq \ell + 1$ so that there is a condition ${q'}_n^{\vec{\lambda}}$ appearing before $q_n^{\vec{\lambda}}$ in the inductive process which agrees with $q_n^{\vec{\lambda}}$ up to β^* and also about the distances of the images of the walks between $B^*, B_1^*, \ldots, B_t^*$. We now require that $b_{n,i}^{\vec{\kappa}}(B)$ realizes the same type over a set $T = \{a_n^{\vec{\kappa}}(S) \mid S \in E_n,\ S \subseteq B^* \text{ or } S \in \{B_1^*, \ldots, B_t^*\}\}$ as the type of $b_{n,i}^{\vec{\kappa}}(B)$ over the same set but with $a_n^{\vec{\kappa}}$ and $b_{n,i}^{\vec{\kappa}}(B)$ defined according to ${q'}_n^{\vec{\lambda}}$.

Note that 5.10(h),(i), applied to B_t^* as a maximal model, imply that the type of T is the same under both $q_n^{\vec{\lambda}}$ and ${q'}_n^{\vec{\lambda}}$.

In both cases we require in addition the following:

If there is some ${q'}_n^{\vec{\lambda}}$ appearing before $q_n^{\vec{\lambda}}$ so that ${q'}_n^{\vec{\lambda}}$ and $q_n^{\vec{\lambda}}$ agree about all the distances appearing in $clw(\{\max E_n\}, \{B\})$, then let $b_{n,i}^{\vec{\kappa}}(B)$ be the same (and not only its type) as the model corresponding to B under ${q'}_n^{\vec{\lambda}}$.

Note that here necessary $B^*, B_1^*, \ldots, B_\ell \in clw(\{\max E_n\}, \{B\})$ and so the distances between them are taken into account.

This completes the definition for the model B. We deal with the rest of successor elements of D between $\widetilde{\widetilde{A}}_i$ and $\widetilde{A}_i$ in the same fashion. Thus if B' is such an element, then we assume below it everything is already defined. Now we treat B' exactly as B above only replacing E_n by $E_n \cup \{B'' \in D \mid \widetilde{\widetilde{A}}_i \in B'' \subset B'$ and B'' is a successor model$\}$.

The rest of the induction now follows.

$\square$

The next lemma generalizes 5.12 but actually easily follows from it.

Lemma 5.13 *Let* $t = \langle\langle A^{00}(0), A^{10}(0)\ F^0(0)\rangle,\ \langle A^{00}(1), A^{10}(1)\rangle,\ \langle c_\nu \mid \nu \in \lim(A^{10}(1))\rangle, F\rangle \in \mathcal{P}$, $p = \langle p^{\vec{\lambda}}, p^{\vec{\kappa}}\rangle \in F$, $B \in A^{10}(1)$. *Then* B *is addable to* p.

Proof. We first extend t to $s = \langle\langle B^{00}(0),\ B^{10}(0),\ H^0(0)\rangle,\ \langle B^{00}(1),\ B^{10}(1)\rangle,\ \langle d_\nu \mid \nu \in \lim(B^{10}(1))\rangle,\ H\rangle \in \mathcal{P}$ such that there is a limit $A \in B^{10}(1)$ with $A \supset B$, *otp* $c_{A\cap\kappa^{+3}} = \kappa^{++}$ and the first element of $d_{A\cap\kappa^{+3}}$ is the maximal model of p. Now we add this to p as the maximal model. It is easy because of the triviality of the walk from A to the maximal model of p. Now we use 5.12 in order to add B to the resulting condition.

$\square$

Let us turn now to the closure properties. First we consider $\langle\mathcal{P}, \leq\ \rangle$. In contrast to previous constructions (the one of Section 3 or those of [Git3]) once we have

$$\langle\langle A^{00}(0), A^{10}(0), F^0(0)\ , \langle A^{00}(1), A^{10}(1)\rangle\ , \langle c_\nu \mid \nu \in \lim A^{10}(1)\rangle\rangle$$

the last component F is determined completely. It just includes everything satisfying 5.10(3). Hence, for the forcing $\mathcal{P}$ itself we can just ignore this last component F. Then $\mathcal{P}$, actually splits into $\mathcal{P}''(1)\times\mathcal{P}(0)$. $\mathcal{P}''(1)$ is $\kappa^{++}+1$-strategically closed and $\mathcal{P}(0)$ is $< \kappa^{++}$-strategically closed forcing of cardinality κ^{++}. Hence we have the following:

Lemma 5.14 $\langle\mathcal{P}, \leq\ \rangle$ *preserves all the cardinals and does not add new* κ^+ *– sequences of ordinals.*

Let $G \subseteq \mathcal{P}$ be generic. Define $\mathcal{P}^*$ to be the set of all p's such that for some

$$\langle\langle A^{00}(0), A^{10}(0), F^0(0)\rangle\ , \langle A^{00}(1), A^{10}(1)\rangle\ , \langle c_\nu \mid \nu \in \lim A^{10}(1)\rangle\ , F\rangle \in G$$

we have $p \in F$.

We would like to now show that $\mathcal{P}^*$ has reasonably nice closure properties. This is needed mainly for proving Prikry condition of $\mathcal{P}^*$. We consider first a simpler case.

Lemma 5.15 *Let $\langle p(i) \mid i < \delta\rangle$ be a $\leq^*$-increasing sequence of elements of $\mathcal{P}^*$ with $\delta < \lambda_{\ell(p(0))}$. Suppose that*

(a) *for every $i < \delta$ $p^{\vec{\lambda}}(i)$ is in $(F^0(0))^*$, i.e. in a closed dense subset of $F^0(0)$ with $F^0(0)$ a part of a condition in G*

(b) *$\underset{\sim}{p}^{\vec{\kappa}}(i)$'s have the same maximal model, where $p(i) = \langle p^{\vec{\lambda}}(i), \underset{\sim}{p}^{\vec{\kappa}}(i)\rangle$.*

Then there is $p \in \mathcal{P}^*$ $p \geq^* p(i)$ for every $i < \delta$.

Proof. There is no problem $p^{\vec{\lambda}}(i)$'s since they are in $(F^0(0))^*$ in which $\leq^*$ – unions behave nicely. Now, $\underset{\sim}{p}^{\vec{\kappa}}(i)$'s have the same maximal model. This by 5.10(e) implies that each element of $\mathrm{dom}\left(\underset{\sim}{a}^{\vec{\kappa}}_n(\underset{\sim}{p}^{\vec{\kappa}}(i))\right)$ with $n \geq \ell(p(0))$, as well as its image is controlled by the box sequences from the maximal model and its images, where $\underset{\sim}{a}^{\vec{\kappa}}_n(\underset{\sim}{p}^{\vec{\kappa}}(i))$ is the first coordinate of $\underset{\sim}{p}^{\vec{\kappa}}_n(i)$ i.e. the correspondence function at the n level. But then nothing new can happen at the limit of $\langle \underset{\sim}{p}^{\vec{\kappa}}(j) \mid j < i\rangle$ for a limit $i < \delta$. Since the box sequences (both at κ and κ_n) are already specified. $\square$

The situation is a bit different if we remove the restriction (b) and allow $p^{\vec{\kappa}}(i)$'s with different maximal model.

Let for $n < \omega$ $\mathcal{P}^*_{\geq n}$ denotes all the elements p of $\mathcal{P}^*$ with $\ell(p) \geq n$.

Lemma 5.16 *For every $n < \omega$, $\mathcal{P} * \langle \mathcal{P}^*_{\geq n}, \leq^* \rangle$ is $< \lambda_n$ – strategically closed.*

Proof. Let $\delta < \lambda_n$. We describe a winning strategy for Player I playing at even stages. Thus let $\langle t_0, p_0\rangle$ be his first move such that the set $A^{00}(1)$ of t_0 is the maximal model of p_0. Denote this set by A_0. Let $\langle t_1, p_1\rangle$ be an answer of Player II. If $A_1 =_{df} A^{00}(1)$ of t_1 is equal to A_0 then let Player I play $\langle t_1, p_1\rangle$. Suppose otherwise. Then $A_1 \supset A_0$, by the definition of $\mathcal{P}(1)$. Let $A_1', A_1 \supseteq A_1' \supseteq A_0$ be the maximal model of p_1. We extend t_1 to t_2 so that:

(i) $A_2 =_{df} A^{00}(1)$ of t_2 has the intersection with κ^{+3} of cofinality κ^{++} and

(ii) $A_1' \cap \kappa^{+3}$ is the first element of $c_{A_2 \cap \kappa^{+3}}$.

Now extend p_1 to p_2 by adding A_2 to p_1 as the maximal model and extending the assignment functions $\underset{\sim}{a}^{\vec{\kappa}}_m$'s in the obvious fashion.

We proceed the same way at successor stages. At limit stage $\alpha \leq \delta$ we define

$$c_{\bigcup_{\beta<\alpha}(A_\beta \cap \kappa^{+3})} = \{A_{\beta+2m} \cap \kappa^{+3} \mid m < \omega,\ \beta \text{ limit },\ \beta + 2m < \alpha\}\ .$$

Let A_α be a limit model with $A_\alpha \cap \kappa^{+3}$ of cofinality κ^{++} and $\{A_\beta \mid \beta < \alpha\} \in A_\alpha$. Pick now a club $c_{A_\alpha \cap \kappa^{+3}}$ such that $\bigcup_{\beta<\alpha}(A_\beta \cap \kappa^{+3})$ is its limit point and

$$c_{A_\alpha \cap \kappa^{+3}} \cap \bigcup_{\beta<\alpha}\left(A_\beta \cap \kappa^{+3}\right) = c_{\bigcup_{\beta<\alpha}(A_\beta \cap \kappa^{+3})}\ .$$

Now define t_α in the obvious way extending all t_β's $(\beta < \alpha)$, having $A^{00}(1) = A_\alpha$ and including $c_{A_\alpha \cap \kappa^{+3}}$. Let p_α be extension of p_β's obtained by adding $\bigcup_{\beta<\alpha} A_\beta$, adding A_α as the maximal model and extending the assignment functions $\underset{\sim}{a}{}^{\vec{\kappa}}_m$'s then in the obvious fashion.

□

The straightforward application of 5.16 is the Prikry property of $\mathcal{P} * \mathcal{P}^*$ which in turn insures that no new bounded subsets of κ are added.

Lemma 5.17 *Let $\langle t, p\rangle \in \mathcal{P} * \mathcal{P}^*$ and σ is a statement of the forcing language. Then there is $\langle t^*, p^*\rangle \geq \langle t, p\rangle$ such that $p^* \geq^* p$ and $\langle t^*, p^*\rangle \| \sigma$.*

Lemma 5.18 *The forcing $\mathcal{P} * \mathcal{P}^*$ does not add new bounded subsets to κ.*

Now, as usual, the problem is a chain condition. Working in $V^{\mathcal{P}}$, we define a partial order $\longrightarrow$ on $\mathcal{P}^*$ extending the order $\leq$ of $\mathcal{P}^*$. Then it will be shown that $\langle \mathcal{P}^*, \rightarrow\ \rangle$ is a nice subforcing of $\langle \mathcal{P}^*, \leq\ \rangle$ and that $\langle \mathcal{P}^*, \rightarrow\ \rangle$ satisfies κ^{++}-c.c. The new point in the present situation will be the absence of the equivalent relation $\longleftrightarrow$. Such relations were used in all previous constructions. But here the special role played by the maximal model of a condition cases major difficulties. Thus, if $p, q \in \mathcal{P}^*$ have different maximal sets $A(p)$ and $A(q)$ respectively. Say, for example, $A(p) \in A(q)$ but the connection between $A(q)$ and $A(p)$ via box sequences requires ordinals above κ. It may be impossible to find $q' \leq q$ with maximal model $A(p)$, since in the images of $A(p)$ under the assignment functions $\underset{\sim}{a}{}^{\vec{\kappa}}_n$'s of q are likely to be names depending on values of one element Prikry sequences for λ_n's. Naturally, a condition equivalent to p is supposed to have the same maximal model, i.e. $A(p)$.

Definition 5.19 Let $p, q \in \mathcal{P}^*$ $p = \langle p^{\vec{\lambda}}, \underset{\sim}{p}{}^{\vec{\kappa}}\rangle$ and $q = \langle q^{\vec{\lambda}}, \underset{\sim}{q}{}^{\vec{\kappa}}\rangle$. We set $p \rightarrow q$ iff

(1) $p \leq q$
 or

(2) there is a nondecreasing converging to ∞ sequence $\langle k_n \mid n < \omega\rangle$ of natural numbers with $k_0 > 4$ such that the following holds for every $n < \omega$:

(a) $p_n^{\vec{\lambda}} \longrightarrow_{k_n} q_n^{\vec{\lambda}}$, i.e. in $\mathcal{P}(0)$ $p_n^{\vec{\lambda}}$ is $\longleftrightarrow_{k_n}$ equivalent to some $q' \leq q_n^{\vec{\lambda}}$

(b) $\ell(p) \leq \ell(q)$

(c) for every $n < \ell(q)$

$$\langle p_n^{\vec{\lambda}}, \underset{\sim}{p}_n^{\vec{\kappa}}\rangle \leq \langle q_n^{\vec{\lambda}}, \underset{\sim}{q}_n^{\vec{\kappa}}\rangle$$

Suppose now that $n \geq \ell(q)$ then we require the following:

(d) the maximal model $A(p)$ of $\underset{\sim}{p}^{\vec{\kappa}}$ appears in $\underset{\sim}{q}_n^{\vec{\kappa}}$, i.e. in the domain of the assignment function $\underset{\sim}{a}_n^{\vec{\kappa}}(q)$.

(e) Let $r_n^{\vec{\lambda}}$ be a common nondirect extension of $p_n^{\vec{\lambda}}$ and $q_n^{\vec{\lambda}}$ deciding the values of one element Prikry sequences for λ_n. Such $r_n^{\vec{\lambda}}$ decides completely both $p_n^{\vec{\kappa}}$ and $q_n^{\vec{\kappa}}$. We require then that $p_n^{\vec{\kappa}}$ is k_n – equivalent to some $q_n' \leq q_n^{\vec{\kappa}}$ with $A(p)$ as a maximal model.

The next lemma insures that $\langle \mathcal{P}^*, \rightarrow\rangle$ is a nice subforcing of $\langle \mathcal{P}^*, \leq\rangle$, i.e. every dense open set in $\langle \mathcal{P}^*, \rightarrow\rangle$ generates such a set in $\langle \mathcal{P}^*, \leq\rangle$.

Lemma 5.20 *Suppose that $p \rightarrow q \leq q'$ then there is $p' \geq p$ such that $q' \rightarrow p'$, where $p, q, p', q' \in \mathcal{P}^*$.*

Proof. Denote the maximal models of p, q and q' by $A(p)$, $A(q)$ and $A(q')$ respectively. Pick a model A such that for some element

$$\langle\langle A^{00}(0), A^{10}(0), F^0(0)\rangle, \langle A^{00}(1)\ ,\ A^{10}(1)\rangle,\ \langle\langle c_\nu \mid \nu \in \lim(A^{10}(1))\rangle,\ F\rangle$$

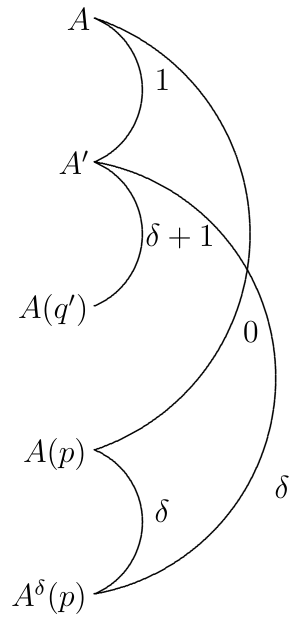
of a generic subset G of $\mathcal{P}$

(a) $A = A^{00}(1)$

(b) $\{A(p), A(q), A(q')\} \subseteq A$

(c) A is a limit point of $A^{10}(1)$

(d) $cf(A \cap \kappa^{+++}) = \kappa^{++}$



(e) $A(p) \cap \kappa^{+++}$ is the first element of $c_{A \cap \kappa^{+3}}$

(f) the walk from A to $A(q')$ proceeds as follows:

first we go down to the model $A' \supset A(q')$ which is the second on $c_{A \cap \kappa^{+3}}$. Then $A(q') \cap \kappa^{+3}$ is the $\delta + 1$-th element of $c_{A' \cap \kappa^{+3}}$. Its δ-th element and all the rest are the same as those of $c_{A(p) \cap \kappa^{+3}}$, where $\delta < \kappa^{++}$ is a limit ordinal above all the distances appearing in the walks between models of q'.

See the diagram on page 55.

Using the density argument, it is easy to find such A and A'. It is obvious that for every $B \supset A(p)$ in q' the walk from A to B goes via A' and then $A(q')$. Hence distances above δ are required. If $B \subset A(p)$ is in q' then the walk from A to B goes via $A(p)$. The walk from A' to $A(p)$ goes via $A(q')$ since $A(q') \cap \kappa^{+3}$ is the least member of $c_{A' \cap \kappa^{+3}}$ above $A(p) \cap \kappa^{+3}$. Again the distance $\delta + 1$ is involved here. The model A will be the maximal model of the condition $p' \geq p$ that we shall define below. We need to satisfy $q' \to p'$. In particular $A(q)$ and $A(q')$ should appear in p'. For every $n < \omega$ let $E_n(q')$ denotes the domain of the assignment function $a_n^{\vec{\kappa}}(q')$ of the condition q'. By the choice of A and A' the set $E_n(q') \cup \{A, A', A^\delta(p)\}$ is walks closed, where $A^\delta(p)$ is the δ-th model of $c_{A(p) \cap \kappa^{+3}}$. Denote it by $E_n(p')$. We define the condition p' with $E_n(p')$ the domain of the assignment of its function $a_n^{\vec{\kappa}}(p')$.

Let us apply 5.10(i) to q' and $A(p)$. We will obtain a condition q^* with $A(p)$ as a maximal set basically agreeing with q' below $A(p)$ or in other words q^* is the restriction of q' to $A(p)$.

More precisely (i)(a) and (i)(b) of 5.10(i) hold for q' and q^*. Now, clearly, $p \longrightarrow q^*$. Also they have the same maximal model $A(p)$. It is routine to find $p^* \geq^* p$ such that $p^* \longleftrightarrow q^*$. We like to extend p^* to p' by adding to it A as the maximal coordinate, A' and all the models of q' between $A(q')$ and $A(p)$. Notice that walks from $A(q')$ and A' to models of q' are the same except for the starting points. Define the p' level by level. Thus fix $n < \omega$ and define p'_n or, basically, $a^{\vec{\kappa}}_n(p')$. Set $\mathrm{dom} a^{\vec{\kappa}}_n(p') = E_n(p') = E_n(q') \cup \{A, A', A^\delta(p)\}$. Let $p'^{\vec{\lambda}}_n \geq^* q'^{\vec{\lambda}}_n$ be an extension including δ in the domain of $a^{\vec{\lambda}}_n(p')$. We will use induction on extensions of $p^{*\vec{\lambda}}_n$ deciding the values of one element Prikry sequences for measures in $\mathrm{dom} a^{\vec{\lambda}}_n(p')$. Suppose that $r^{\vec{\lambda}}_n$ is such an extension and for a smaller one $a^{\vec{\kappa}}_n(p')$ is already defined. Define $a^{\vec{\kappa}}_n(p')$ for $r^{\vec{\lambda}}_n$. Let $a^{\vec{\kappa}}_n(p') \restriction A(p) = a^{\vec{\kappa}}_n(p^*)$. If there is some r appearing before $r^{\vec{\lambda}}_n$ and deciding $a^{\vec{\kappa}}_n(p)(A(p))$ the same way as $r^{\vec{\lambda}}_n$ does, then let $a^{\vec{\kappa}}_n(p')(A)$ be the same as the value of $a^{\vec{\kappa}}_n(p')(A)$ defined with r. Otherwise, we set $a^{\vec{\kappa}}_n(p')(A)$ to be a submodel of a big enough model such that $a^{\vec{\kappa}}_n(p)(A(p))$ is the first element of its fixed box sequence. We require also that its ω-th element of the box sequence (recall that once $a^{\vec{\kappa}}_n(p')(A)$ is fixed also all limit members of some box sequence are fixed as well) includes $a^{\vec{\kappa}}_n(q')(B)$ for every $B \in E_n(q')$ and is a submodel of a large enough model as well. This will leave enough room for elements of $E_n(q')$ that should be added to $\mathrm{dom} a^{\vec{\kappa}}_n(p')$.

Now, if there is some r appearing before $r^{\vec{\lambda}}_n$ which agrees with $r^{\vec{\lambda}}_n$ about the values of ordinals below $\delta+1$, then we define $a^{\vec{\kappa}}_n(p')(A')$, $a^{\vec{\kappa}}_n(p')(A(q'))$ and $a^{\vec{\kappa}}_n(p')(B)$, for every $B \in E_n$ exactly as they are defined according to r.

This will take care of 5.10(h). Now assume that every r appearing before $r^{\vec{\lambda}}_n$ disagree with $r^{\vec{\lambda}}_n$ about ordinals below $\delta+1$. Here we are free of the restriction of 5.10(h). Consider the type realized by $rng\Big(a^{\vec{\kappa}}_n(q')\Big)$ above $rng\Big(a^{\vec{\kappa}}_n(q') \restriction A(p)\Big)$ (where $a^{\vec{\kappa}}_n(q')$ is as decided by $r^{\vec{\lambda}}_n$). Let $rng\Big(a^{\vec{\kappa}}_n(p')\Big) \restriction E_n(q')$ be realizing the same type over $rng\Big(a^{\vec{\kappa}}_n(p^*)\Big)$ inside the model which is the ω-th element of the fixed box sequence for $a^{\vec{\kappa}}_n(p')(A)$. Finally, we define $a^{\vec{\kappa}}_n(p')(A')$ to be a model below the ω-th element of the fixed box sequence for $a^{\vec{\kappa}}_n(p')(A)$ including $a^{\vec{\kappa}}_n(p')(A(q'))$, with $r^{\vec{\lambda}}_n(\delta+1)$-th element of its fixed box sequence equal to $a^{\vec{\kappa}}_n(p')(A(q')) \cap \kappa_n^{+n+2}$ and $r^{\vec{\lambda}}_n(\delta)$-th element equal to $a^{\vec{\kappa}}_n(p')(A^\delta(p)) \cap \kappa_n^{+n+2}$. This completes the definition of $a^{\vec{\kappa}}_n$ and then also those of p'. By the choice of p' we have $p' \geq^* p$. Also, by its definition $q' \to p'$. $\square$

Now we turn to the crucial observation.

Lemma 5.21 *In $V^{\mathcal{P}}$, $\langle \mathcal{P}^*, \to \rangle$ satisfies κ^{++}-c.c.*

Proof. Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++}\rangle$ be a name of an antichain of the length κ^{++}. Using strategic closure of the forcing $\mathcal{P}$ we define by induction an increasing sequence $\langle t_\alpha \mid \alpha < \kappa^{++}\rangle$ of elements of $\mathcal{P}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ so that for every $\alpha < \kappa^{++}$

$$t_\alpha \Vdash \underset{\sim}{p}_\alpha = \check{p}_\alpha \ .$$

Let t_0 and p_0 be arbitrary such that $t_0 \Vdash \underset{\sim}{p}_0 = \check{p}_0$.

Now suppose that $\alpha < \kappa^{++}$ and for every $\beta < \alpha$ t_β and p_β are already defined. Let

$$t_\beta = \langle\langle A^{00}_\beta(0), A^{10}_\beta(0), F^0_\beta(0)\rangle \ , \ \langle A^{00}_\beta(1), A^{10}_\beta(1)\rangle, \langle c_\nu \mid \nu \in \lim A^{10}_\beta(1)\rangle \ , F_\beta\rangle \ ,$$

$$p_\beta = \langle p^{\vec{\lambda}}, \underset{\sim}{p}^{\vec{\kappa}}\rangle, \ p^{\vec{\lambda}}_\beta = \langle p^{\vec{\lambda}}_{\beta n} \mid n < \omega\rangle \text{ and } \underset{\sim}{p}^{\vec{\kappa}}_\beta = \langle \underset{\sim}{p}^{\vec{\kappa}}_{\beta n} \mid n < \omega\rangle \ .$$

If $\alpha = \alpha' + 1$, then we pick

$$t_\alpha = \langle\langle A^{00}_\alpha(0), A^{10}_\alpha(0), F^0_\alpha(0)\rangle \ , \ \langle A^{00}_\alpha(1), A^{10}_\alpha(1)\rangle \ , \ \langle c_\nu \mid \nu \in \lim A^{10}_\alpha(1)\rangle \ , \ F_\alpha\rangle$$

to be an extension of $t_{\alpha'}$ deciding $\underset{\sim}{p}_\alpha$ so that $\langle\langle A^{00}_\beta(0), A^{10}_\beta(0), F^0_\beta(0)\rangle \mid \beta \le \alpha'\rangle \in A^{00}_\alpha(0)$ and $\langle t_\beta \mid \beta \le \alpha'\rangle \in A^{00}_\alpha(1)$.

If α is a limit ordinal, then we use the strategic closure of $\mathcal{P}$. This way we can obtain t_α stronger than each t_β with $\beta < \alpha$, deciding $\underset{\sim}{p}_\alpha$ and so that $\langle\langle A^{00}_\beta(0), A^{10}_\beta(0), F^0_\beta(0)\rangle \mid \beta < \alpha\rangle \in A^{00}_\alpha(0), \bigcup_{\beta<\alpha} A^{00}_\beta(0) \in A^{00}_\alpha(0) \cap A^{10}_\alpha(0)$, $\langle t_\beta \mid \beta < \alpha\rangle \in A^{00}_\alpha(1), \bigcup_{\beta<\alpha} A^{00}_\beta(1) \in A^{00}_\alpha(1) \cap A^{10}_\alpha(1)$ and $c_{(\bigcup_{\beta<\alpha} A^{00}_\beta(1))\cap\kappa^{+3}} = \{A^{00}_\beta(1) \cap \kappa^{+3} \mid \beta < \alpha\}$.

This completes the inductive definition of $\langle t_\alpha \mid \alpha < \kappa^{++}\rangle$ and $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$.

Now we use $\kappa^{++} + 1$ – strategic closure of $\mathcal{P}''(1)$ in order to extend the part of t_α's over κ^{+++}, i.e.

$$\{\langle\langle A^{00}_\alpha(1), A^{10}_\alpha(1)\rangle, \langle c_\nu \mid \nu \in \lim A^{10}_\alpha(1)\rangle \mid \alpha < \kappa^{++}\} \ .$$

Thus we set $A^{00}(1) = \bigcup_{\alpha<\kappa^{++}} A^{00}_\alpha(1)$, $A^{10}(1) = \bigcup\limits_{\alpha<\kappa^{++}} A^{10}_\alpha(1) \cup \{A^{00}(1)\}$ and

$$c_{A^{00}(1)\cap\kappa^{+3}} = \{A^{00}_\alpha(1) \cap \kappa^{+3} \mid \alpha < \kappa^{++}\}$$

We extend each t_α to t'_α by replacing in it

$$\langle\langle A^{00}_\alpha(1), A^{10}_\alpha(1)\rangle \ , \ \langle c_\nu \mid \nu \in \lim A^{10}_\alpha(1)\rangle\rangle$$

by

$$\langle\langle A^{00}(1), A^{10}(1)\rangle\ ,\ \langle c_\nu \mid \nu \in \lim A^{10}(1)\rangle\rangle$$

and F_α by the set F'_α which includes everything satisfying 5.10(3) (it is determined completely once we have all the rest of the components).

Let $\alpha < \kappa^{++}$ be a limit ordinal. Pick a limit $\alpha^*, \alpha \leq \alpha^* < \kappa^{++}$ such that $\bigcup\limits_{\beta<\alpha^*} A^{00}_\beta(1)$ includes the models appearing in $p^{\vec{\kappa}}_\alpha$.

Now we like to extend each of p_α's, for a limit α, by adding $\bigcup\limits_{\beta<\alpha} A^{00}_\beta(0)$, $A^{00}_\alpha(0)$ to $p^{\vec{\lambda}}_\alpha$ and $\bigcup\limits_{\beta<\alpha} A^{00}_\beta(1), \bigcup_{\beta<\alpha^*} A^{00}_\beta(1) A^{00}_\alpha(1), A^{00}(1)$ to $p^{\vec{\kappa}}_\alpha$. The addition of $\bigcup_{\beta<\alpha} A^{00}_\beta(0)$ and $A^{00}_\alpha(0)$ to $p^{\vec{\lambda}}_\alpha$ does not cause problems. But in order to add to $p^{\vec{\kappa}}_\alpha$ models, we probably need to first pass from t'_α to $t'_{\widetilde{\alpha}}$ for some $\widetilde{\alpha}, \alpha^* \leq \widetilde{\alpha} < \kappa^{++}$, since such additions may introduce new walks and in turn new distances. It means ordinals below κ^{++} that may not be in $A^{00}_\alpha(0)$. Thus we need to first move to a larger $A^{00}_{\widetilde{\alpha}}(0)$ which includes such ordinals and then extend inside $t'_{\widetilde{\alpha}}$. Denote the resulting extension of p_α by q_α. As usual, $q_\alpha = \langle q^{\vec{\lambda}}_\alpha, q^{\vec{\kappa}}_\alpha\rangle$ and $q^{\vec{\lambda}}_\alpha = \langle q^{\vec{\lambda}}_{\alpha n} \mid n < \omega\rangle$, $q^{\vec{\kappa}}_\alpha = \langle q^{\vec{\kappa}}_{\alpha_n} \mid n < \omega\rangle$.

Now we shall use Δ-system arguments. For every limit $\alpha < \kappa^{++}$ let $S_\alpha \subseteq \kappa^{++}$ be the set consisting of all the ordinals appearing in $q^{\vec{\lambda}}_\alpha$ and all the distances of walks between the models appearing in $q^{\vec{\kappa}}_\alpha$. Then, clearly, $|S_\alpha| \leq \kappa$. Find a stationary $T \subseteq \{\alpha < \kappa^{++} \mid cf\alpha = \kappa^+\}$ and $S \subseteq \kappa^{++}$ such that for every $\alpha \in T$ $S_\alpha \cap \alpha = S$. Shrinking T a bit more we may assume that $\langle S_\alpha \mid \alpha \in T\rangle$ is a Δ-system with kernel S. Notice that $\alpha \in S_\alpha$ since the distance from $A^{00}(1)$ to $\bigcup\limits_{\beta<\alpha} A^{00}_\beta(1)$ is exactly α. In other words $\bigcup\limits_{\beta<\alpha} A^{00}_\beta(1) \cap \kappa^{+3}$ is α-th element of $c_{A^{00}(1)\cap\kappa^{+3}}$ and both models are in $q^{\vec{\kappa}}_\alpha$. Let γ be the least limit ordinal bigger or equal than every element of S. In removing if necessary the initial segment from T let us assume that $\min T > \gamma$. Consider $\bigcup\limits_{\beta<\gamma} A^{00}_\beta(1)$.

Claim 5.21.1 For every $\alpha \in T$ there are no models appearing in $q^{\vec{\kappa}}_\alpha$ strictly between $\bigcup\limits_{\beta<\gamma} A^{00}_\beta(1)$ and $\bigcup\limits_{\beta<\alpha} A^{00}(1)$.

Proof. Suppose otherwise.
Consider then the walk form $A^{00}(1)$ to a model B such that $\bigcup\limits_{\beta<\gamma} A^{00}_\beta(1) \subset B \subset \bigcup\limits_{\beta<\alpha} A^{00}_\beta(1)$. Already the first step in this walk should produce a distance strictly between γ and α, since both $\bigcup_{\beta<\gamma} A^{00}_\beta(1)\cap\kappa^{+3}$ and $\bigcup_{\beta<\alpha} A^{00}_\beta(1)\cap\kappa^{+3}$ are limit points of the box sequence $c_{A^{00}(1)\cap\kappa^{+3}}$. Recall that $q^{\vec{\kappa}}_\alpha$ is walks closed. Hence we should have in S_α an ordinal between γ and α. This is impossible by the choice of γ.

$\square$ of the claim.

The following claim is similar to the previous one.

Claim 5.21.2 Let $\alpha \in T$ and B be a model such that $\bigcup_{\beta<\gamma} A^{00}_\beta(1) \subseteq B \subset \bigcup_{\beta<\alpha} A^{00}_\beta(1)$ (which is not in $q^{\vec{\kappa}}_\alpha$ by 5.20.1). Then for every model $C \supseteq B$ appearing in $q^{\vec{\kappa}}_\alpha$ the walk from C to B is the same as the walk from C to $\bigcup_{\beta<\alpha} A^{00}_\beta(1)$ and then the walk from $\bigcup_{\beta<\alpha} A^{00}_\beta(1)$ to B.

Proof. If $\bigcup_{\beta<\alpha} A^{00}_\beta(1) \cap \kappa^{+3}$ is an element of the box sequence for $C \cap \kappa^{+3}$, then it is clear. Suppose that $\bigcup_{\beta<\alpha} A^{00}_\beta(1) \cap \kappa^{+3}$ is not an element of the box sequence of $C \cap \kappa^{+3}$. There are no elements of $c_{C\cap\kappa^{+3}}$ between $\bigcup_{\beta<\gamma} A^{00}_\beta(1)$ and $\bigcup_{\beta<\alpha} A^{00}_\beta(1)$, since otherwise the walk from C to $\bigcup_{\beta<\alpha} A^{00}(1)$ will necessarily produce models between $\bigcup_{\beta<\gamma} A^{00}_\beta(1)$ and $\bigcup_{\beta<\alpha} A^{00}_\beta(1)$. But this is impossible by 5.20.1, since both C and $\bigcup_{\beta<\alpha} A^{00}_\beta(1)$ appear in $q^{\vec{\kappa}}_\alpha$ and $q^{\vec{\kappa}}_\alpha$ is walks closed. Hence the first element of $c_{C\cap\kappa^{+3}}$ above $B \cap \kappa^{+3}$ will be actually the first element of $c_{C\cap\kappa^{+3}}$ above $\bigcup_{\beta<\gamma} A^{00}_\beta(1)$ as well. The same is true about the last element of $c_{C\cap\kappa+3}$ below $B \cap \kappa^{+3}$. Let D denote the model with $D \cap \kappa^{+3}$ being the least element of $c_{C\cap\kappa^{+3}}$ above $\bigcup_{\beta<\alpha} A^{00}_\beta(1)$. Then D appears in $q^{\vec{\kappa}}_\alpha$ since $q^{\vec{\kappa}}_\alpha$ is walks closed. Now we can deal with D exactly the same as we did with C or we can use an appropriate inductive assumption.

$\square$ of the claim.

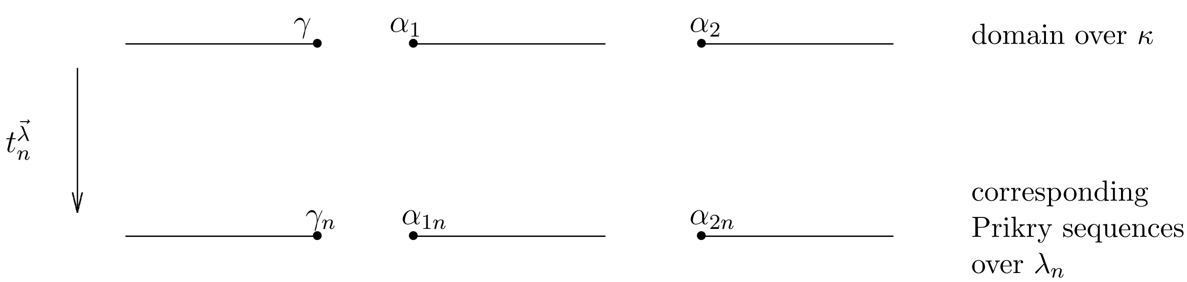
Now let $r_\alpha = \langle r^{\vec{\lambda}}_\alpha, r^{\vec{\kappa}}_\alpha \rangle$ be obtained from q_α by adding $\bigcup_{\beta<\gamma} A^{00}_\beta(1)$ to $q^{\vec{\kappa}}_\alpha$, where $\alpha \in T$. By Claim 5.20.2, this can be done without adding any further models, since models of $q^{\vec{\kappa}}_\alpha$ together with $\bigcup_{\beta<\gamma} A^{00}_\beta(1)$ will still form a walks closed set. We also add γ to S but denote the result by the same letter S. By shrinking T more, if necessary, we can assume without loss of generality that models of $r^{\vec{\kappa}}_{\alpha_1}$ and $r^{\vec{\kappa}}_{\alpha_2}$ with $\alpha_1, \alpha_2 \in T$ have the same configuration with respect inclusions and walks over S. This is possible, since the number of models in each $r^{\vec{\kappa}}_\alpha$ is at most κ and the cardinality of S is as well at most κ. Shrinking more, if necessary, we insure that the assignment functions, sets of measure one, etc. of $r^{\vec{\kappa}}_\alpha$'s behave the same.

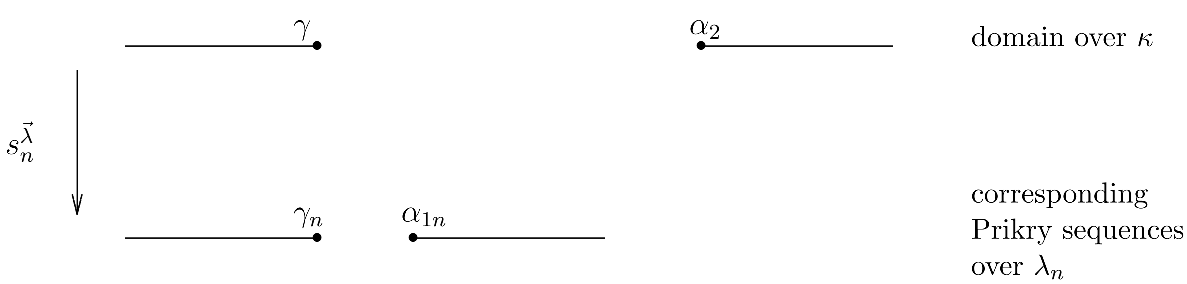
Now let $\alpha_1 < \alpha_2 \in T$. We like to show that r_{α_1} and r_{α_2} are compatible in the order $\rightarrow$. First we deal with $r^{\vec{\lambda}}_{\alpha_1}$ and $r^{\vec{\lambda}}_{\alpha_2}$. By standard arguments (see [Git1] or [Git2 Sec. 2] there is $r'^{\vec{\lambda}}_{\alpha_2}$ equivalent to $r^{\vec{\lambda}}_{\alpha_2}$ so that

(a) $r'^{\vec{\lambda}}_{\alpha_2}$ and $r^{\vec{\lambda}}_{\alpha_2}$ agree about ordinals $\leq \gamma$

(b) $r^{\vec{\lambda}}_{\alpha_1}$ and $r'^{\vec{\lambda}}_{\alpha_2}$ can be combined together into one condition (probably by the cost of increasing their trunks).

Let $r^{\vec{\lambda}}$ be the combination of $r^{\vec{\lambda}}_{\alpha_1}$ with $r'^{\vec{\lambda}}_{\alpha_2}$. Then all three conditions $r^{\vec{\lambda}}_{\alpha_1}$, $r^{\vec{\lambda}}_{\alpha_2}$ and $r^{\vec{\lambda}}$ agree about ordinals $\leq \gamma$. Now we like to use this property and 5.10(3(h)) in order to combine $r^{\vec{\kappa}}_{\alpha_1}$ and $r^{\vec{\kappa}}_{\alpha_2}$ together. Thus we consider conditions $r_1 = \langle r^{\vec{\lambda}}, \underset{\sim}{r}^{\vec{\kappa}}_{\alpha_1} \rangle$ and $r_2 = \langle r^{\vec{\lambda}}, \underset{\sim}{r}^{\vec{\kappa}}_{\alpha_2} \rangle$. In r_1, r_2, as far as we are concerned, with $\underset{\sim}{r}^{\vec{\kappa}}_{\alpha_1}, \underset{\sim}{r}^{\vec{\kappa}}_{\alpha_2}$ nothing has changed. Fix $n < \omega$. Let $t^{\vec{\lambda}}_n$ be an extension of $r^{\vec{\lambda}}_n$ deciding the values of one element Prikry sequences for the ordinals of the domain of the assignment function $a^{\vec{\lambda}}_n(r^{\vec{\lambda}})$ of $r^{\vec{\lambda}}_n$. We now pick the extension $s^{\vec{\lambda}}_n$ of $r^{\vec{\lambda}}_{\alpha_2 n}$ obtained by switching for every $\delta \in \mathrm{dom} a^{\vec{\lambda}}_n(r^{\vec{\lambda}}_{\alpha_2})$ the value $t^{\vec{\lambda}}_n(\delta)$ to $t^{\vec{\lambda}}_n(\delta')$, where $\delta' \in \mathrm{dom}(a^{\vec{\lambda}}_n(r^{\vec{\lambda}}_{\alpha_1}))$ is the element corresponding to δ under the order isomorphism between $\mathrm{dom} a^{\vec{\lambda}}_n(r^{\vec{\lambda}}_{\alpha_2})$ and $\mathrm{dom} a^{\vec{\lambda}}_n(r^{\vec{\lambda}}_{\alpha_1})$. Such defined $s^{\vec{\lambda}}_n$ will be the extension of $r^{\vec{\lambda}}_{\alpha_2 n}$ since $r^{\vec{\lambda}}_{\alpha_2}$ and $r'^{\vec{\lambda}}_{\alpha_2}$ are equivalent. Also, for every $\xi \leq \gamma$ we have $\xi \in \mathrm{dom} a^{\vec{\lambda}}_n(r^{\vec{\lambda}}_{\alpha_2 n}) \cap \mathrm{dom} a^{\vec{\lambda}}_n(r^{\vec{\lambda}}_{\alpha_1})$ and $t^{\vec{\lambda}}_n(\xi) = s^{\vec{\lambda}}_n(\xi)$.





See the diagram on p. 60.

By 5.10(3(h)), then $t^{\vec{\lambda}}_n$ and $s^{\vec{\lambda}}_n$ will force the same value of $\underset{\sim}{a}^{\vec{\kappa}}_n(r_{\alpha_2})(B)$ for every $B \in \mathrm{dom}\underset{\sim}{a}^{\vec{\kappa}}_n(r_{\alpha_2})$ with the walks closure of B and $A^{00}(1)$ involving only models with distances between them at most γ, where as usual $a^{\vec{\kappa}}_n(r_{\alpha_2})$ is the assignment function of $r^{\vec{\kappa}}_{\alpha_2 n}$. In particular, the values of $A^{00}(1)$, $\bigcup\limits_{\beta<\gamma} A^{00}_\beta(1)$, all the models of $c_{\kappa^{+3}\cap \bigcup\limits_{\beta<\gamma} A^{00}_\beta(1)}$ as well as the models at distances at most γ from the above mentioned models do not change if we switch between $t_n^{\vec{\lambda}}$ and $s_n^{\vec{\lambda}}$. Now recall that by the choice of r_{α_1} and r_{α_2}, $a_n^{\vec{\kappa}}(r_{\alpha_1})(B)$ as forced by $t_n^{\vec{\lambda}}$ will be the same as $a_n^{\vec{\kappa}}(r_{\alpha_2})(B')$ forced by $s_n^{\vec{\lambda}}$, where $B \in \mathrm{dom} a_n^{\vec{\kappa}}(r_{\alpha_1})$ and $B' \in \mathrm{dom} a_n^{\vec{\kappa}}(r_{\alpha_2})$ corresponds to it under the order isomorphism. Hence, $t_n^{\vec{\lambda}}$ forces the same values of $a_n^{\vec{\kappa}}(r_{\alpha_1})$ and $a_n^{\vec{\kappa}}(r_{\alpha_2})$ applied to $A^{00}(1)$, $\bigcup_{\beta<\gamma} A^{00}_\beta(1)$, all the models of $c_{\kappa^{+3}\cap \bigcup\limits_{\beta<\gamma} A_\beta(1)}$ as well as all the models of common domain at distances at most γ from the above mentioned models. Also, every common model $B \in \mathrm{dom} a_n^{\vec{\kappa}}(r_{\alpha_1}) \cap \mathrm{dom} a_n^{\vec{\kappa}}(r_{\alpha_2})$ can be reached from $A^{00}(1)$ by the walk in which all the distances are at most γ, since γ was picked this way. Thus, $t_h^{\vec{\lambda}}$ forces that $a_n^{\vec{\kappa}}(r_{\alpha_1})(B) = a_n^{\vec{\kappa}}(r_{\alpha_2})(B)$. Now we can just define $a_n^{\vec{\kappa}} = a_n^{\vec{\kappa}}(r_{\alpha_1}) \cup a_n^{\vec{\kappa}}(r_{\alpha_2})$. It will be an assignment function since $a_n^{\vec{\kappa}}(r_{\alpha_1})$ and $a_n^{\vec{\kappa}}(r_{\alpha_2})$ move walks at level κ to walks at level κ_n preserving "$\subseteq$", by 5.10(3(g)) and $\mathrm{dom} a_n^{\vec{\kappa}}$ will be walks closed by Claim 5.21.2. Since $n < \omega$ and $t_n^{\vec{\lambda}}$ were arbitrary it is easy now to define $\underset{\sim}{r}^{\vec{\kappa}} = \langle \underset{\sim}{r}_n^{\vec{\kappa}} \mid n < \omega\rangle$ with $\underset{\sim}{a}_n^{\vec{\kappa}}$ being the assignment function of $\underset{\sim}{r}_n^{\vec{\kappa}}$. Thus, we finish with $r = \langle r^{\vec{\lambda}}, \underset{\sim}{r}^{\vec{\kappa}}\rangle$ which is stronger than both r_{α_1} and r_{α_2}. Contradiction.

□

Let V_1 be a generic extension of $V^{\mathcal{P}}$ by $\langle \mathcal{P}^*, \longrightarrow\rangle$. Then, by Lemmas above, V and V_1 agree about cofinalities of ordinals and have the same bounded subsets of κ. Denote by $\langle \xi_n \mid n < \omega\rangle$ the Prikry sequences for the normal measure of extenders E_{κ_n} over κ_n's and let $\langle \rho_n \mid n < \omega\rangle$ be the Prikry sequences for normal measure of extenders E_{λ_n} over λ_n's. Now it is routine to deduce the desired result:

Theorem 5.21.

(a) $tcf\Big(\prod\limits_{n<\omega} \xi_n^{+n+2}/\text{finite}\Big) = \kappa^{+3}$.

(b) $tcf\Big(\prod\limits_{n<\omega} \rho_n^{+n+2}/\text{finite}\Big) = \kappa^{++}$.

(c) $b_{\kappa^{++}} = \{\rho_n^{+n+2} \mid n < \omega\}$.

(d) $b_{\kappa^{+++}} = \{\xi_n^{+n+2} \mid n < \omega\}$.

(e) $\{\delta < \kappa \mid \delta^+ \in b_{\kappa^{+3}}\} \cap b_{\kappa^{++}} = \emptyset$

We would like to now sketch the applications of the forcing technique developed above to wider gaps. Thus in the model just constructed, $2^\kappa = \kappa^{+3}$. By [Git3 Sec. 4]it is possible to handle any $\delta < \kappa$ producing a model with $2^\kappa \geq \kappa^{+\delta+1}$. The initial assumption their is "$\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta+1} + 1\}$ is unbounded in κ". Combining both techniques together it is possible to produce wider gaps starting with the same initial assumptions. Thus the following holds:

Theorem 5.21 *Suppose that κ is a cardinal of cofinality ω, $\delta < \kappa, \nu < \aleph_1$ and the set $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta+1} + 1\}$ is unbounded in κ. Then there is cofinalities preserving, not adding new bounded subsets to κ extension satisfying $2^\kappa \geq \kappa^{+\delta\cdot\nu+1}$.*

Remark. The simplest new case is a model of $2^\kappa \geq \kappa^{+\omega_1+2}$ starting from $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\omega_1+1} + 1\}$ unbounded in κ.

This result almost completes (at least assuming GCH below) the study of the strength of various gaps between a singular of cofinality $\aleph_0$ and its power. We refer to [Git4] for detailed discussion of the matter.

Outline of the Construction

Let us deal with $\nu = 2$. The general case of any countable ν is just standard once one can handle $\nu = 2$.

We pick sequences $\langle \kappa_n \mid n < \omega \rangle$ and $\langle \lambda_n \mid n < \omega \rangle$ so that

(1) $\kappa = \bigcup_{n<\omega} \kappa_n$

(2) for every $n < \omega$

(i) $\delta < \lambda_n < \kappa_n < \lambda_{n+1} < \kappa_{n+1}$

(ii) λ_n carries an extender E_{λ_n} of the length $\lambda_n^{+n+\delta+1}$

(iii) κ_n carries an extender E_{κ_n} of the length $\kappa_n^{+n+\delta+1}$.

The extenders E_{λ_n}'s will generate Prikry sequences so that $tcf\Big(\prod\limits_{n<\omega} \rho_n^{+n+\mu+1}/\text{finite}\Big) = \kappa^{+\mu+1}$, for every $\mu \leq \delta$, where $\langle \rho_n \mid n < \omega \rangle$ denotes the Prikry sequence for the normal measures of E_{λ_n}'s. The extenders E_{κ_n}'s will generate Prikry sequences witnessing

$$tcf\Big(\prod_{n<\omega} \xi_n^{+n+\mu+1}/\text{finite}\Big) = \kappa^{+\delta+\mu+1} \ ,$$

for every μ, $1 \leq \mu \leq \delta$, where $\langle \xi_n \mid n < \omega \rangle$ denotes the Prikry sequence for normal measures of E_{κ_n}'s. The preparation forcing $\mathcal{P}$ of 5.10 was combined from two blocks $\mathcal{P}(0)$ and $\mathcal{P}''(1)$.

Here we can use their analogs $\mathcal{P}(\delta)$ and $\mathcal{P}''(\delta+1)$. $\mathcal{P}(\delta)$ was explicitly defined in [Git3, Sec. 4]. The definition of $\mathcal{P}''(\delta+1)$ is very similar to those of $\mathcal{P}''(1)$ but replacing $\mathcal{P}(0)$ by $\mathcal{P}(\delta)$. The connection between these two blocks is via models of cardinality $\kappa^{+\delta+1}$. They are the smallest models of $\mathcal{P}''(\delta)$. The models of $\mathcal{P}(\delta)$ (or more precisely) ordinal parts of them are contained in $\kappa^{+\delta+1}$. The cofinality of $a_n^{\vec{\kappa}}\Big(A \cap \kappa^{+\delta+\delta+1}\Big)$ will be $\rho_n^{+n+\delta+1}$ for every limit model A of cardinality $\kappa^{+\delta+1}$ in $\mathcal{P}''(\delta)$.

Further construction is parallel to one developed above. The proof of κ^{++}-c.c. of the final forcing is a bit more involved and requires redoing of the proof of κ^{++}-c.c. from [Git3, Sec. 4] of the forcing derived from $\mathcal{P}(\delta)$.

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