# On a question of Pereira

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**Abstract**

Answering a question of Pereira we show that it is possible to have a model violating the Singular Cardinal Hypothesis without a tree-like continuous scale.

## 1 Introduction.

Let us recall some relevant basic definitions of the Shelah PCF -theory. We refer to Shelah's book [5] or to Abraham, Magidor handbook article [1] for detailed presentation.

Let $A$ be a set of regular cardinals with $\min(A) > |A|$ and $I$ an ideal over $A$. A sequence $\overline{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ of functions in $\prod A$ is called *a scale witnessing a true cofinality* $\lambda$ iff

- $\alpha < \beta$ implies $f_\alpha <_I f_\beta$
- for every $f \in \prod A$ there is $\alpha < \lambda$ such that $f <_I f_\alpha$.

A function $f \in \prod A$ is called an *exact upper bound* modulo $I$ of a $<_I$-increasing sequence of functions $\langle f_\alpha \mid \alpha < \delta \rangle$ in $\prod A$ iff

- $f_\alpha <_I f$ for every $\alpha < \delta$
- if $g <_I f$, then for some $\alpha < \delta$ we have $g <_I f_\alpha$.

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A scale $\overline{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ is *continuous at a limit ordinal* $\delta < \lambda$ if when there exists an exact upper bound of $\langle f_\alpha \mid \alpha < \delta \rangle$ then $f_\delta$ is such an exact upper bound. A scale is continuous iff it is continuous at every limit ordinal.

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals. A scale $\overline{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ in $\prod_{n<\omega} \kappa_n$ is called a *tree-like scale* iff for every $n < \omega$ and $\alpha < \beta < \lambda$

- $f_\alpha(n) = f_\beta(n)$ implies $f_\alpha \restriction n = f_\beta \restriction n$.

Luis Pereira [4] asked the following question:

Suppose that $\kappa$ is a strong limit cardinal of cofinality $\omega$, $2^\kappa = \lambda > \kappa^+$. Does it necessary exist a continuous tree-like scale witnessing this?

In [4] Pereira constructed models having such scales. He showed that an existence continuous tree-like implies the PCF conjecture for intervals and suggested that probably always there are continuous tree-like scales.
The purpose of this note will be to give a negative answer to Pereira's question.

# 2 The model

We will show that models constructed in [3] do not have continuous tree-like scales. Namely, the Extender Based Prikry forcing does not add continuous tree-like scales.

Let us briefly review the basic settings and the definition of the Extender Based Prikry forcing. We refer to [3] or to the handbook article [2](Section 3) for a detailed presentation.

Let $V$ be a GCH-model with a cardinal $\kappa$ carrying an extender $E$ such that

- if $j : V \to M \simeq \mathrm{Ult}(V, E)$ is the corresponding elementary embedding, then $M \supseteq V_{\kappa+2}$.

For each $\alpha < \kappa^{++}$ define a $\kappa$-complete ultrafilter $U_\alpha$ over $\kappa$ by setting $X \in U_\alpha$ iff $\alpha \in j(X)$. Clearly, $U_\kappa$ is a normal ultrafilter and each $U_\alpha$ for $\alpha < \kappa$ is trivial.
Define a partial ordering $\leq_E$ on $\kappa^{++}$:

$$\alpha \leq_E \beta \text{ iff } \alpha \leq \beta \text{ and for some } f \in {}^\kappa\kappa, \quad j(f)(\beta) = \alpha.$$

For each $\alpha \leq_E \beta$ a projection $\pi_{\beta\alpha}$ of $U_\beta$ onto $U_\alpha$ was chosen. It satisfies the following:

$$j(\pi_{\beta\alpha})(\beta) = \alpha.$$

The projection $\pi_{\beta\kappa}$ to the normal ultrafilter $U_\kappa$ does not depend on $\beta$ (i.e. $\pi_{\beta\kappa} = \pi_{\beta'\kappa}$, for any $\beta', \kappa \leq \beta' < \kappa^{++}$). For each $\nu < \kappa$ we denote $\pi_{\beta\kappa}(\nu)$ by $\nu^0$. A sequence $\langle \nu_0, ..., \nu_n \rangle$ is called a $\circ$-increasing sequence iff $(\nu_0)^0 < ... < (\nu_n)^0$. We say that an ordinal $\nu$ is permitted to a sequence $\langle \nu_0, ..., \nu_n \rangle$ iff $\nu^0 > (\nu_n)^0$.

Denote by $\mathcal{P}(E)$ the Extender Based Prikry forcing with $E$.
It consists of all elements the elements $p$ of the form
$\{\langle \gamma, p^\gamma \rangle \mid \gamma \in g \backslash \{\max(g)\}\} \cup \{\langle \max(g), p^{\max(g)}, T \rangle\}$, where

(1) $g \subseteq \kappa^{++}$ of cardinality $\leq \kappa$ which has a maximal element in $\leq_E$-ordering and $\kappa \in g$. Further let us denote $g$ by $\mathrm{supp}(p)$, $\max(g)$ by $mc(p)$, $T$ by $T^p$, and $p^{\max(g)}$ by $p^{mc}$ ($mc$ for the maximal coordinate).

(2) for $\gamma \in g$ $p^\gamma$ is a finite $\circ$-increasing sequence of ordinals $< \kappa$.

(3) $T$ is a tree with a trunk $p^{mc}$ consisting of $\circ$-increasing sequences. All the splittings in $T$ are required to be on sets in $U_{mc(p)}$, i.e. for every $\eta \in T$, if $\eta \geq_T p^{mc}$ then the set

$$Suc_T(\eta) = \{\nu < \kappa \mid \eta^\frown \langle \nu \rangle \in T\} \in U_{mc(p)} \ .$$

Also require that for $\eta_1 \geq_T \eta_2 \geq_T p^{mc}$

$$Suc_T(\eta_1) \subseteq Suc_T(\eta_2) \ .$$

(4) For every $\gamma \in g$, $\pi_{mc(p),\gamma}(\max(p^{mc}))$ is not permitted for $p^\gamma$.

(5) For every $\nu \in Suc_T(p^{mc})$

$$|\{\gamma \in g \mid \nu \quad \text{is permitted for} \quad p^\gamma\}| \leq \nu^0$$

(6) $\pi_{mc(p),\kappa}$ projects $p^{mc}$ onto $p^\kappa$,

in particular, $p^{mc}$ and $p^\kappa$ are of the same length.

Let $p, q \in \mathcal{P}(E)$. We say that $p$ extends $q (p \geq q)$ if

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.

(2) for every $\gamma \in \mathrm{supp}(q)$ $p^\gamma$ is an end-extension of $q^\gamma$.

(3) $p^{mc(q)} \in T^q$.

(4) for every $\gamma \in \mathrm{supp}(q)$

$p^\gamma \backslash q^\gamma = \pi_{mc(q),\gamma}$ "$((p^{mc(q)} \backslash q^{mc(q)}) \restriction (length\ (p^{mc}) \backslash (i+1))$ where $i \in \mathrm{dom}(p^{mc(q)})$ is the largest such that $p^{mc(q)}(i)$ is not permitted for $q^\gamma$.

(5) $\pi_{mc(p),mc(q)}$ projects $T^p_{p^{mc}}$ into $T^q_{q^{mc}}$.

(6) for every $\gamma \in \mathrm{supp}(q)$, for every $\nu \in Suc_{T^p}(p^{mc})$ if $\nu$ is permitted for $p^\gamma$, then

$$\pi_{mc(p),\gamma}(\nu) = \pi_{mc(q),\gamma}(\pi_{mc(p),mc(q)}(\nu)) \ .$$

If $t \in T^q$, then we denote by $(q)_t$ the extension of $q$ obtained by adding $t$ to $q^{mc}$, projecting it to the rest of coordinates according to the rules above and replacing $T^q$ by $\{s \in T^q \mid s \text{ extends } t\}$.

Let $p, q \in \mathcal{P}(E)$. We say that $p$ is direct extension of $q (p \geq^* q)$ if

(1) $p \geq q$, and

(2) for every $\gamma \in \mathrm{supp}(q)$ $p^\gamma = q^\gamma$.

Let us warm up with the following statement that appears implicitly in [3], [2] and will be essential further for understanding the way of combining conditions together.

**Proposition 2.1** *Suppose $q \in \mathcal{P}(E), a \subseteq \kappa^{++} \setminus \mathrm{supp}(q), |a| \leq \kappa$ and $\langle d^\gamma \mid \gamma \in a\rangle$ is a sequence of finite $^\circ$-increasing sequences. Let $\alpha < \kappa^{++}$ be $\leq_E$-above every ordinal in $\mathrm{supp}(q) \cup a$. Then there is $p \geq^* q$ such that*

1. *$mc(p) = \alpha$,*

2. *$\mathrm{supp}(p) = \mathrm{supp}(q) \cup a \cup \{\alpha\}$,*

3. *for each $\gamma \in a$, $\quad p^\gamma$ is an end-extension of $d^\gamma$.*

*Proof.* Suppose that $|a| = \kappa$. Let $\langle \gamma_i \mid i < \kappa\rangle$ be an enumeration of $a$. Pick an increasing sequence of inaccessible cardinals $\langle \rho_i \mid i < \kappa\rangle$ such that $\rho_0 > \max(q^{mc})$ and $\rho_i > \max(d^{\gamma_i})$, for each $i < \kappa$. Set $e^{\gamma_i} = d^{\gamma_i \frown} \rho_i$, for each $i < \kappa$.
Now we are ready to define $p$. Set $mc(p) = \alpha$, $\mathrm{supp}(p) = \mathrm{supp}(q) \cup a$ and let $p^{mc}$ be a $^\circ$-increasing sequence which projects onto $q^{mc}$ by $\pi_{\alpha,mc(q)}$. Define $p^\gamma = q^\gamma$, for each $\gamma \in \mathrm{supp}(q)$ and $p^\gamma = e^\gamma$, for each $\gamma \in a$. Set $T_0$ to be the inverse image of $T^q$ by $\pi_{\alpha,mc(q)}$. Then $\{\langle \gamma, p^\gamma\rangle \mid \gamma \in \mathrm{supp}(p) \backslash \{\alpha\}\} \cup \{\langle \alpha, p^\alpha, T_0\rangle\} \in \mathcal{P}(E)$. It is not necessarily an extension of $q$, due to the item 6 of the definition of the extension. In order to repair this, let us shrink the

tree $T_0$ a little.

Denote $\mathrm{Suc}_{T_0}(p^\alpha)$ by $A$. For $\nu \in A$ set

$$B_\nu = \{\gamma \in \mathrm{supp}(q) \mid \gamma \neq mc(q) \text{ and } \nu \text{ is permitted for } q^\gamma\} \ .$$

Then $|B_\nu| \leq \nu^0$, since $\pi_{\alpha,mc(q)}(\nu) \in \mathrm{Suc}_{T^q}(q^{mc}), \nu^0 = \pi_{\alpha\kappa}(\nu) =$
$\pi_{mc(q),\kappa}(\pi_{\alpha mc(q)}(\nu))$, and $q$ being in $\mathcal{P}(E)$ satisfies condition (5) of the definition of $\mathcal{P}(E)$. Clearly, for $\nu, \delta \in A$, if $\nu^0 = \delta^0$ then $B_\nu = B_\delta$, and if $\nu^0 > \delta^0$ then $B_\nu \supseteq B_\delta$. Also, if $\nu \in A$ and $\nu^0$ is a limit point of $\{\delta^0 \mid \delta \in A\}$, then $B_\nu = \bigcup\{B_\delta \mid \delta \in A \text{ and } \delta^0 < \nu^0\}$. So the sequence $\langle B_\nu \mid \nu \in A\rangle$ is increasing and continuous (according to the $\nu^0$'s). Obviously, $\bigcup\{B_\nu \mid \nu \in A\} = \mathrm{supp}(q) \backslash \{mc(q)\}$. Let $\langle \xi_i \mid i < \kappa\rangle$ be an enumeration of $\mathrm{supp}(q)\backslash\{mc(q)\}$ such that for every $\nu \in A$

$$B_\nu \subseteq \{\xi_i \mid i < \nu^0\} \ .$$

Now pick for every $i \in A$ a set $C_i \subseteq A$, with $C_i \in U_\alpha$ so that for every $\nu \in C_i$ $\pi_{\alpha\xi_i}(\nu) = \pi_{mc(q),\xi_i}(\pi_{\alpha,mc(q)}(\nu))$. Let $C = A^\frown \Delta^*_{i<\kappa} C_i := \{\nu \in A \mid \forall i < \nu^0(\nu \in C_i)\}$. Then $C \in U_\alpha$.

Now define $T^p$ to be the tree obtained from $T_0$ by intersecting every level of $T_0$ with $C$. Set $p = \{\langle \gamma, p^\gamma\rangle \mid \gamma \in \mathrm{supp}(p)\backslash\{\alpha\}\} \cup \{\langle \alpha, p^\alpha, T^p\rangle\}$. Let us show that condition (6) of the definition of the order on $\mathcal{P}(E)$ is now satisfied. Suppose $\gamma \in \mathrm{supp}(q)$. If $\gamma = mc(q)$, then everything is trivial. Assume that $\gamma \in \mathrm{supp}(q)\backslash\{mc(q)\}$. Then for some $i_0 < \kappa$ $\gamma = \xi_{i_0}$. Suppose that some $\nu \in C$ is permitted for $q^\gamma$. Then $\xi_{i_0} = \gamma \in B_\nu$. Since $B_\nu \subseteq \{\xi_i \mid i < \nu^0\}$, $i_0 < \nu^0$. Then $\nu \in C_{i_0}$. Hence

$$\pi_{\alpha\xi_{i_0}}(\nu) = \pi_{mc(q),\xi_{i_0}}(\pi_{\alpha,mc(q)}(\nu)) \ .$$

So condition (6) is satisfied by $p$. Hence, $p \geq^* q$.

$\square$

**Remark.** If, for every $\nu \in Suc_{T^p}(p^\alpha)$, $|\{\gamma \in a \mid \nu$ is permitted for $d^\gamma\}| \leq \nu^0$ (i.e. (5) of the definition of $\mathcal{P}(E)$ is satisfied by $d^\gamma$'s), then there is no need to extend $d^\gamma$'s to $e^\gamma$'s.

We will use the following property of the forcing $\mathcal{P}(E)$ (see Lemmas 3.12, 3.14 of [2] for a proof):

**Proposition 2.2** *Let $\mu < \kappa$, $q \in \mathcal{P}(E)$ and $\underset{\sim}{h}$ is a name of a function from $\mu$ to ordinals. Then there is a direct extension $p = \{\langle \gamma, p^\gamma\rangle \mid \gamma \in \mathrm{supp}(p) \setminus \{mc(p)\}\} \cup \{\langle mc(p), p^{mc(p)}, T^p\rangle\}$ of $q$ so that for every $\alpha < \mu$ there is a level $k(\alpha)$ of $T^p$ such that for any $t \in T^p$ from the level $k(\alpha)$, $(p)_t || \underset{\sim}{h}(\alpha)$.*

Let $G$ be a generic subset. For each $\alpha, \kappa \leq \alpha < \kappa^{++}$ we denote the set $\bigcup\{p^\alpha \mid p \in G\}$ by $G^\alpha$.

Then $V[G]$ satisfies the following:

- GCH holds below $\kappa$
- all the cardinals of $V$ are preserved
- $\kappa$ changes its cofinality to $\omega$ (and it is the only one that changes cofinality)
- $2^\kappa = \kappa^{++}$
- for every $\alpha, \kappa \leq \alpha < \kappa^{++}$ the sequence $G^\alpha$ is a Prikry sequence for $U_\alpha$
- for every $\alpha, \kappa \leq \alpha < \kappa^{++}$ for all but finitely many $n < \omega$ the $n$-th element $G^\alpha(n)$ of $G^\alpha$ is below $(G^\kappa(n))^{++}$
- for every $\alpha, \beta, \kappa \leq \alpha < \beta < \kappa^{++}$ for all but finitely many $n < \omega$ we have $G^\alpha(n) < G^\beta(n)$. Further we denote this simply by $G^\alpha <^* G^\beta$.

Let us denote $G^\kappa(n)$ by $\eta_n$. We may assume that $G^\alpha \in \prod_{n<\omega} \eta_n^{++}$, for every $\alpha, \kappa \leq \alpha < \kappa^{++}$. Just change the finitely many values that fall outside to zero.
We move the indexes of $G^\alpha$'s in order to start from 0 rather than from $\kappa$. Thus set $f_0 = G^\kappa, f_1 = G^{\kappa+1}$, etc.

Let us prove the following additional property:

**Proposition 2.3** *$\langle f_\alpha \mid \alpha < \kappa^{++}\rangle$ is a continuous scale in $\prod_{n<\omega} \eta_n^{++}$ mod finite.*

*Proof.* Set $f_{\kappa^{++}}(n) = \eta_n^{++}$. Let $\delta \leq \kappa^{++}$ be a limit ordinal of uncountable cofinality. We need to show that $f_\delta$ is an exact upper bound of $\langle f_\alpha \mid \alpha < \delta\rangle$. Suppose $g \in \prod_{n<\omega} \eta_n^{++}$ and $g <^* f_\delta$. Turn to $V$. Let $\underset{\sim}{g}$ be a name of $g$ and suppose that the weakest condition forces

$$“\underset{\sim}{g} <^* \underset{\sim}{f}_\delta”.$$

Assume for simplicity that it forces

$$“\forall n < \omega \quad \underset{\sim}{g}(n) < \underset{\sim}{f}_\delta(n)”.$$

Otherwise just work above an $m$ from which on we have this inequality.

By 2.2 there is a condition of the form $p \cup \{\langle \beta, \emptyset, S\rangle\}$ so that:

for each $n<\omega$ there will be a level $k(n)$ of $S$ such that for any $t$ from this level of $S$

$$(p\cup\{\langle\beta,\emptyset,S\rangle\})_t || \underset{\sim}{g}(n).$$

By shrinking $S$ if necessary, we can assume that there is no $m<k(n)$ and a subtree $S'$ of $S$ such that for some $t\in S'$ from the level $m$

$$(p\cup\{\langle\beta,\emptyset,S'\rangle\})_t || \underset{\sim}{g}(n).$$

Without loss of generality we may assume that $\delta\in \mathrm{supp}(p)$ and in particular $\delta <_E \beta$. Suppose for simplicity that $p^\delta$ is the empty sequence.
Also, shrinking $S$ if necessary, assume that every element of $S$ is permitted to $p^\delta$. Then for every $t\in S$ we will have

$$(p\cup\{\langle\beta,\emptyset,S\rangle\})_t || \underset{\sim}{f}_\delta \restriction |t| = \pi_{\beta,\delta}``t.$$

Let us argue that for every $n<\omega$, $k(n)\leq n$. Fix $n<\omega$. Let $t\in S$ and $t=\langle t(1),...,t(n)\rangle$. Then

$$(p\cup\{\langle\beta,\emptyset,S\rangle\})_t \Vdash \underset{\sim}{\eta}_n = (t(n))^0.$$

Hence,

$$(p\cup\{\langle\beta,\emptyset,S\rangle\})_t \Vdash \underset{\sim}{g}(n) < (t(n)^0)^{++}.$$

Now, once the number of possible values of $\underset{\sim}{g}(n)$ is bounded below $\kappa$ it is easy to find a subtree $S'$ of $S$ such that

$$(p\cup\{\langle\beta,\emptyset,S'\rangle\})_t || \underset{\sim}{g}(n).$$

But by the assumptions we made above, this is possible only when $k(n)\leq n$.

Now $k(n)\leq n$, for each $n<\omega$, implies that for any $t\in S$

$$(p\cup\{\langle\beta,\emptyset,S\rangle\})_t || \underset{\sim}{g} \restriction |t|.$$

The weakest condition forces "$\underset{\sim}{g} <^* \underset{\sim}{f}_\delta$", hence

$$(p\cup\{\langle\beta,\emptyset,S\rangle\})_t \Vdash \forall n<|t| \quad \underset{\sim}{g}(n) < \pi_{\beta\delta}(t(n)),$$

for any $t\in S$. Define functions $g'$ and $f'$ on $S'$ as follows:

$$g'(t)=\langle \nu_0,...,\nu_{|t|-1}\rangle \text{ iff } \forall n<|t| \quad (p\cup\{\langle\beta,\emptyset,S\rangle\})_t \Vdash \underset{\sim}{g}(n)=\nu_n\rangle$$

and

$$f'(t) = \langle \pi_{\beta\delta}(t(0)), ..., \pi_{\beta\delta}(t(|t|-1))\rangle.$$

By shrinking $S$ to some $S'$ if necessary, we can assume that $\mathrm{cof}(f'(t)(k)) = \mathrm{cof}(\delta)$, if $\mathrm{cof}(\delta) < \kappa$ and $\mathrm{cof}(f'(t)(k)) = ((t(k))^0)^+$, if $\mathrm{cof}(\delta) = \kappa^+$, for each $k < |t|$.

Let us split the proof into two cases according to the cofinality of $\delta$.

**Case 1.** $\mathrm{cof}(\delta) = \kappa^+$.
For each $t \in S'$ the set $\mathrm{Suc}_{S'}(t) \in U_\beta$. Define functions $g'_t, f'_t$ on this set as follows: $g'_t(\nu) = g'(t^\frown\nu)$ and $f'_t(\nu) = f'(t^\frown\nu)$. Then in $M$ we will have

$$j(g'_t)(\beta) < j(f'_t)(\beta) = j(\pi_{\beta\delta})(\beta) = \delta.$$

Note that the total number of t's in $S'$ is $\kappa$. We assumed that $\mathrm{cof}(\delta) = \kappa^+$. Hence there will be $\alpha < \delta$ such that

$$\alpha > j(g'_t)(\beta),$$

for every $t \in S'$. Extend now the condition $p \cup \{\langle \beta, \emptyset, S'\rangle\}$ by adding $\alpha$ to its support. Then the resulting condition will force "$\underset{\sim}{g} <^* \underset{\sim}{f_\alpha}$".

**Case 2.** $\mathrm{cof}(\delta) < \kappa$.
Let $\mu = \mathrm{cof}(\delta)$. Then, by the assumptions made, $\omega < \mu < \kappa$. Fix a cofinal sequence $\langle \delta_\xi \mid \xi < \mu\rangle$ in $\delta$. Extend the condition $p \cup \{\langle \beta, \emptyset, S'\rangle\}$ if necessary, to insure that the set $\{\delta_\xi \mid \xi < \mu\}$ is contained in the support. Assume that it is already the case. For each $\xi < \mu$ and $t \in S'$ set $f'_\xi(t) = \langle \pi_{\beta\delta_\xi}(t(0)), ..., \pi_{\beta\delta_\xi}(t(|t|-1))\rangle$.
We can now easily shrink $S'$ level by level to $S''$ and insure the following:

- for each level $n$ there is $\xi_n < \mu$ such that for every $t$ from this level we have $g'(t) < f'_{\xi_n}(t)$.

Pick some $\xi^* < \mu$ such that for every $n < \omega \quad \xi_n < \xi^*$. Then the following will easily hold:

$$p \cup \{\langle \beta, \emptyset, S''\rangle\} \Vdash \underset{\sim}{g} <^* \underset{\sim}{f_{\xi^*}}.$$

$\square$

**Proposition 2.4** *In $V[G]$ there is no continuous tree-like scale of the length $\kappa^{++}$ in any $\prod A$ modulo an ideal $I$.*

*Proof.* Suppose otherwise. The only product of cardinals below $\kappa$ which produces scales of the length $\kappa^{++}$ is $\prod_{n<\omega}\eta_n^{++}$. So we can assume without loss of generality that there is a tree-like continuous scale $\langle t_\alpha \mid \alpha < \kappa^{++}\rangle$ in $\prod_{n<\omega}\eta_n^{++}$. Suppose for simplicity that it is a scale mod finite.

Now using the uniqueness of an exact upper bound and 2.3, we obtain a club $C \subseteq \kappa^{++}$ such that for each $\alpha \in C$ of uncountable cofinality $t_\alpha(n) = f_\alpha(n)$, for all but finitely many $n$'s. The forcing $\mathcal{P}$ satisfies $\kappa^{++}$-c.c. So, we can assume that $C \in V$ and the weakest condition forces the above. Work now in $V$
Set $C' = \{\alpha \in C \mid \mathrm{cof}(\alpha) \notin \{\omega, \kappa\}\}$.
For each $\alpha \in C'$ pick a condition $p_\alpha$ and a number $n_\alpha$ such that

$$p_\alpha \Vdash \forall m > n_\alpha \quad \underset{\sim}{t}_\alpha(m) = \underset{\sim}{f}_\alpha(m).$$

Without loss of generality we can assume that $\alpha \in \mathrm{supp}(p_\alpha)$.
Consider the set $\{\mathrm{supp}(p_\alpha) \mid \alpha \in C'\}$. This a set of $\kappa^{++}$ subsets of $\kappa^{++}$ each of cardinality at most $\kappa$. Form a $\Delta$-system. Let $S \subseteq C'$ be stationary and such that

- $\{\mathrm{supp}(p_\alpha) \mid \alpha \in S\}$ is a $\Delta$-system
- all $n_\alpha$'s with $\alpha \in S$ are the same.

Let $n^*$ denotes the common value of $n_\alpha$'s for $\alpha \in S$. Denote the kernel of the $\Delta$-system by $\widetilde{S}$. We can assume that $\mathrm{supp}(p_\alpha) \cap \alpha = \widetilde{S}$.
Pick some $\xi < \kappa^{++}$ such that $\xi >_E \alpha$ for every $\alpha \in \widetilde{S}$. Note that it is possible since the set $\widetilde{S}$ has cardinality at most $\kappa$. Now, for each $\alpha \in S$ we pick $\alpha^* >_E \xi, mc(p_\alpha)$. Extend every $p_\alpha(\alpha \in S)$ to a condition $p^*_\alpha$ by adding $\alpha^*$ as a maximal coordinate (i.e. $\mathrm{supp}(p^*_\alpha) = \mathrm{supp}(p_\alpha) \cup \{\alpha^*\}$)and by shrinking $\pi_{\alpha^*\alpha}$"$T^{p^*_\alpha} \subseteq T^{p_\alpha}$ in order to insure the following:

**(*)** if for some $\gamma \in \widetilde{S}$ $\quad\nu$ is addable to $p^\gamma_\alpha$, then $\pi_{\alpha^*\gamma}(\nu) = \pi_{\xi\gamma}(\pi_{\alpha^*\xi}(\nu))$.

We can assume the following, by shrinking $S$ more, if necessary, and using that $2^\kappa = \kappa^+$ in $V$:

1. $\min(S) > \sup(\widetilde{S})$
2. for each $\alpha, \beta \in S$ and $\gamma \in \widetilde{S}$ we have $p^\gamma_\alpha = p^\gamma_\beta$
3. the trees of $p^*_\alpha$'s for $\alpha \in S$ are the same (but not the maximal coordinates)
4. for every $\alpha, \beta \in S$ we have $\pi_{\alpha^*\alpha} = \pi_{\beta^*\beta}$

5. for every $\alpha, \beta \in S$ and $\gamma \in \widetilde{S} \cup \{\xi\}$ the projections $\pi_{\alpha^*\gamma}$ and $\pi_{\beta^*\gamma}$ are the same.

Let $T$ be this common tree. Denote by $A$ the first splitting level of $T$, i.e.

$$A = Suc_T(p_\alpha^{*mc}) = Suc_T(p_\beta^{*mc}).$$

Again, by shrinking if necessary, let us assume that the finite sequences for the normal measure as well as those for the maximal coordinates are the same. Extending conditions if necessary (but without changing their supports) we can assume the lengths of these sequences are at least $n^*$. Denote their length by $n'$.
Now fix two different members $\alpha, \beta$ of $S$ which are generators of the extender $E$. Recall that an ordinal $\tau$ is called a generator of $E$, if for each $k < \omega, \mu_1, ..., \mu_k < \tau$ and $f : [\kappa]^k \to \kappa$ $j(f)(\mu_1, ..., \mu_k) \neq \tau$. Clearly, the set of generators of $E$ contains a club in $\kappa^{++}$. So, there are such $\alpha, \beta$ in $S$.

**Claim 1** There are two different elements $\nu, \nu'$ of $A$ such that

1. $(\nu)^0 = (\nu')^0$

2. $\pi_{\alpha^*\xi}(\nu) = \pi_{\alpha^*\xi}(\nu')$ (the same with $\alpha^*$ replaced by $\beta^*$, since $\pi_{\alpha^*\xi} = \pi_{\beta^*\xi}$)

3. $\pi_{\alpha^*\alpha}(\nu) \neq \pi_{\beta^*\beta}(\nu')$.

*Proof.* Suppose otherwise. Then for each $\nu \neq \nu'$ in $A$ we have $\pi_{\alpha^*\xi}(\nu) = \pi_{\alpha^*\xi}(\nu')$ implies $\pi_{\alpha^*\alpha}(\nu) = \pi_{\beta^*\beta}(\nu')$. Then also $\pi_{\alpha^*\alpha}(\nu) = \pi_{\alpha^*\alpha}(\nu')$, since $\pi_{\alpha^*\alpha} = \pi_{\beta^*\beta}$. This allows us to define a function $h$ on $\pi_{\alpha^*\xi}$"$A$ as follows:

$$h(\tau) = \chi \text{ if for some } \nu \in A, \quad \pi_{\alpha^*\xi}(\nu) = \tau \text{ and } \pi_{\alpha^*\alpha}(\nu) = \chi.$$

Now, in M (the ultrapower by $E$) we will have $j(h)(\xi) = \alpha$, since $j(\pi_{\alpha^*\xi})(\alpha^*) = \xi$ and $j(\pi_{\alpha^*\alpha})(\alpha^*) = \alpha$. This is impossible since $\alpha$ is a generator and $\xi < \alpha$. Contradiction.
$\square$ of the claim.

Pick now two different elements $\nu, \nu'$ of $A$ which satisfy the statement of the claim. Denote $\pi_{\alpha^*\alpha}(\nu)$ by $\tau$ and $\pi_{\beta^*\beta}(\nu') = \pi_{\alpha^*\alpha}(\nu')$ by $\tau'$.
Extend $p_\alpha^*$ to $q_\alpha$ and $p_\beta^*$ to $q_\beta$ by adding $\nu$ and $\nu'$ respectively (we add them to the maximal coordinates of the conditions and then project to the permitted coordinates). Note that for each $\gamma \in \widetilde{S}$ we will have $q_\alpha^\gamma = q_\beta^\gamma$. It follows from the condition 2 of Claim 1 and the condition

(*) above.
Then

$$q_\alpha \Vdash \underset{\sim}{f}_\alpha(n') = \tau$$

and

$$q_\beta \Vdash \underset{\sim}{f}_\beta(n') = \tau'.$$

Pick now any $\rho \in A$ above both $\nu$ and $\nu'$. Set $\zeta = \pi_{\alpha^*\alpha}(\rho) = \pi_{\beta^*\beta}(\rho)$. Extend $q_\alpha$ to $r_\alpha$ and $q_\beta$ to $r_\beta$ by adding $\rho$. Then

$$r_\alpha \Vdash \underset{\sim}{f}_\alpha(n') = \tau, \underset{\sim}{f}_\alpha(n'+1) = \zeta$$

and

$$r_\beta \Vdash \underset{\sim}{f}_\beta(n') = \tau', \underset{\sim}{f}_\alpha(n'+1) = \zeta.$$

Let us find a common extension $s$ of $r_\alpha$ and $r_\beta$.
Pick some $\mu$ above both $mc(r_\alpha) = \alpha^*$ and $mc(r_\beta) = \beta^*$ in the order of the extender $E$. Let $B \in U_\mu$ be such that $\pi_{\mu,mc(r_\alpha)}$"$B \subseteq A \setminus \rho + 1$ and $\pi_{\mu,mc(r_\beta)}$"$B \subseteq A \setminus \rho + 1$. Combine $r_\alpha$ and $r_\beta$ together into a condition $s$ with the maximal coordinate $\mu$ and the set of measure one $B$.
Now,

$$s \Vdash \underset{\sim}{t}_\alpha(n'+1) = \underset{\sim}{t}_\beta(n'+1), \underset{\sim}{t}_\alpha(n') \neq \underset{\sim}{t}_\beta(n').$$

Contradiction.
□

# 3 Some generalizations.

**1.** It is possible to replace $\kappa^{++}$ by any regular $\lambda > \kappa^+$. The only little change will be needed in the proof of 2.4 once dealing with the kernel of the $\Delta$-system. The kernel has cardinality at most $\kappa$. So the number of generators that it can cover is again at most $\kappa$. Hence we can shrink the set $S$ in order to avoid them all.
Another way is to take increasing subsequences of the length $\kappa^{++}$ of both $t_\alpha$'s and $f_\alpha$'s and to deal with them only.
**2.** It is possible to move the construction to $\aleph_\omega$. Thus, combine the present arguments with those of [2], Section 4.
**3.** Luis Pereira introduced the following weakening of the notion of a tree-like scale.
Let $\bar{t} = \langle t_\alpha \mid \alpha < \lambda \rangle$ be a scale in a product $\prod_{n<\omega} \kappa_n$ of regular cardinals below $\kappa$. For each $n < m < \omega$ and $\alpha < \kappa$ set

$$\bar{t}_{n,m}(\alpha) = \{t_\delta(n) \mid t_\delta(m) = \alpha\}.$$

A scale $\bar{t}$ is called *essentially tree-like* iff for every $n < m < \omega$ and $\alpha < \kappa$ the set $\bar{t}_{n,m}(\alpha)$ is a nonstationary subset of $\kappa_n$.
Pereira asked whether a continuous essentially tree-like scale always exists, provided that $\kappa$ is a strong limit of cofinality $\omega$ and $2^\kappa = \lambda > \kappa^+$.
Let us show that the arguments of the previous section can be extended to provide a negative answer.

**Proposition 3.1** *In $V[G]$ there is no continuous essentially tree-like scale of the length $\kappa^{++}$ in any $\prod A$ modulo an ideal $I$.*

*Proof.* The proof is similar to those of 2.4. Here instead of putting two condition together we will need to combine infinitely many. The main point will be a more careful choice of generators.
Let us prove first two lemmas.

**Lemma 3.2** *The following set contains a club:*

$$\{\alpha < \kappa^{++} \mid \forall X \in U_\alpha \text{ the set } \{\beta < \kappa^{++} \mid X \in U_\beta\} \text{ is stationary }\}.$$

*Proof.* Suppose otherwise. Let $S \subseteq \kappa^{++}$ be a stationary set so that for every $\alpha \in S$ there are $X_\alpha \in U_\alpha$ and a club $C_\alpha \subset \kappa^{++} \backslash \alpha + 1$ such that $X_\alpha \notin U_\beta$, for each $\beta \in C_\alpha$. Set

$$C = \Delta_{\alpha \in S} C_\alpha.$$

Recall that $2^\kappa = \kappa^+$. Hence there are $X^* \subseteq \kappa$ and a stationary $S^* \subseteq S \cap C$ such that for each $\alpha \in S^*$ we have $X_\alpha = X^*$. Pick two elements $\alpha < \beta$ of $S^*$. Then $\beta \in C$ implies $\beta \in C_\alpha$. So, $X_\alpha \notin U_\beta$. But $X_\alpha = X^* = X_\beta$ and $X_\beta \in U_\beta$. Contradiction.
$\square$ of the lemma.

Set $C = \{\alpha < \kappa^{++} \mid \forall X \in U_\alpha$ the set $\{\beta < \kappa^{++} \mid X \in U_\beta\}$ is stationary $\}$.

**Lemma 3.3** *Let $\alpha \in C$. Then for every set $X \in U_\alpha$ there is a set $Y \in U_\alpha, Y \subseteq X$ such that for every $\nu \in Y$ the set*

$$\{\nu' < ((\nu)^0)^{++} \mid \nu' \in X\}$$

*is a stationary subset of $((\nu)^0)^{++}$.*

*Proof.* Let $X \in U_\alpha$. The set $Z := \{\beta < \kappa^{++} \mid X \in U_\beta\}$ is stationary, by the choice of $C$. Remember that $X \in U_\beta$ implies $\beta \in j(X)$. So, $j(X) \cap \kappa^{++} \supseteq Z$. In particular, in $M$, we can conclude that $j(X) \cap \kappa^{++}$ is stationary. Reflecting this down we obtain that the set

$$\{\nu < \kappa \mid X \cap (\nu^0)^{++} \text{ is a stationary subset of } (\nu^0)^{++}\}$$

is in $U_\alpha$.
$\square$ of the lemma.

Now we are ready to fill in the missing point.
Preserve the notation of 2.4.
Without loss of generality we may assume that $S \subseteq C$, otherwise just replace it by $S \cap C$.
Let $\alpha \in S$. By Lemma 3.3, there is $\eta < \kappa$ such that the following set

$$D = \{\pi_{\alpha^*\alpha}(\nu) \mid \nu \in A, (\nu)^0 = \eta\}$$

is a stationary subset of $\eta^{++}$.
We have $\min(S) > \xi$, so by shrinking $A$ if necessary, it is possible to assume that $\nu \in A$ implies $\pi_{\alpha^*\alpha}(\nu) > \pi_{\alpha^*\xi}(\nu)$.
For each $\tau \in D$ pick the least $\nu_\tau \in A$ with $\pi_{\alpha^*\alpha}(\nu_\tau) = \tau$. Define a regressive function $h$ on $D$ by setting $h(\tau) = \pi_{\alpha^*\xi}(\nu_\tau)$. There is a stationary $D' \subseteq D$ on which $h$ is constant.
Now continue as in 2.4 only instead of two different points $\alpha, \beta$ in $S$ let us use $\eta^{++}$ many. Thus, let $\langle \alpha_\tau \mid \tau \in D' \rangle$ be an increasing sequence of elements of $S$. For each $\tau \in D'$ we extend $p^*_{\alpha_\tau}$ to $q_{\alpha_\tau}$ by adding $\nu_\tau$ to the maximal coordinate. As in 2.4, the following will hold:

1. for each $\gamma \in \widetilde{S}, \tau, \tau' \in D' \quad q^\gamma_{\alpha_\tau} = q^\gamma_{\alpha_{\tau'}}$
2. for each $\tau \in D' \quad q_{\alpha_\tau} \Vdash \underset{\sim}{f}_{\alpha_\tau}(n') = \tau$

Pick now any $\rho \in A$ above all $\nu_\tau$, $\tau \in D'$. Set $\zeta = \pi_{\alpha^*\alpha_\tau}(\rho)$, for some (any) $\tau \in D'$. Extend each $q_{\alpha_\tau}$ $(\tau \in D')$ to $r_{\alpha_\tau}$ by adding $\rho$ to the maximal coordinate. Then, for every $\tau \in D'$ we will have the following:

$$r_{\alpha_\tau} \Vdash \underset{\sim}{f}_{\alpha_\tau}(n') = \tau, \underset{\sim}{f}_{\alpha_\tau}(n'+1) = \zeta.$$

Find now a common extension $s$ of all $r_{\alpha_\tau}$'s $(\tau \in D')$. Thus, pick some $\mu$ which is above all $\alpha_\tau, \tau \in D'$ in the order $<_E$ of the extender $E$. Let $B \in U_\mu$ be such that $\pi_{\mu, mc(r_{\alpha_\tau})}$"$B \subseteq A \backslash \rho + 1$. Combine all $r_{\alpha_\tau}$'s together into a condition $s$ with the maximal coordinate $\mu$ and the set of measure one $B$. Note that it is possible by the condition 1 above, since there is a

full agreement about the projections to the common part.
Now, for any two different elements $\tau, \tau' \in D'$ we will have

$$s \Vdash \underset{\sim}{t}_{\alpha_\tau}(n'+1) = \underset{\sim}{t}_{\alpha_{\tau'}}(n'+1), \underset{\sim}{t}_{\alpha_\tau}(n') \neq \underset{\sim}{t}_{\alpha_{\tau'}}(n').$$

Finally note that no new bounded subsets are added to $\kappa$ in the generic extension. Hence $D'$ remains stationary. So, $\langle t_\alpha \mid \alpha < \kappa^{++} \rangle$ is not essentially tree-like. Contradiction.
$\square$

**4.** The crucial ordinals $\alpha$ of 2.4, 3.1 on which tree-like property breaks down have generally cofinality $\kappa^+$. The reason is that they are elements of the set $S$ which forms a $\Delta$-system.
Pereira asked whether it is possible to break the tree-like property on ordinals of cofinality below $\kappa$. Let us show that a little modification of the constructions above allows to do this.

**Proposition 3.4** *Let $\langle t_\alpha \mid \alpha < \kappa^{++} \rangle$ be a continuous scale in $V[G]$. Then for every regular uncountable cardinal $\eta < \kappa$ there are $\alpha, \beta < \kappa^{++}$ of cofinality $\eta$ and $n < \omega$ such that $t_\alpha(n) \neq t_\beta(n)$ but $t_\alpha(n+1) = t_\beta(n+1)$.*

**Remark 3.5** A similar statement is true once dealing with essentially tree-like scales.

*Proof.* The only product of cardinals below $\kappa$ which produces scales of the length $\kappa^{++}$ is $\prod_{n<\omega} \eta_n^{++}$. So, $\langle t_\alpha \mid \alpha < \kappa^{++} \rangle$ is a scale in $\prod_{n<\omega} \eta_n^{++}$. Suppose for simplicity that it is a scale mod finite.

By uniqueness of exact upper bounds and 2.3, we obtain a club $C \subseteq \kappa^{++}$ such that for each $\alpha \in C$ of uncountable cofinality $t_\alpha(n) = f_\alpha(n)$, for all but finitely many $n$'s. The forcing $\mathcal{P}$ satisfies $\kappa^{++}$-c.c. So, we can assume that $C \in V$ and the weakest condition forces the above. Work in $V$.
For each $\alpha < \kappa^{++}$ let $\tilde{\alpha}$ be the first element of $C$ above $\alpha$ of cofinality $\eta$. For each $\alpha < \kappa^{++}$ pick a condition $p_\alpha$ and a number $n_\alpha$ such that

$$p_\alpha \Vdash \forall m > n_\alpha \quad \underset{\sim}{t}_{\tilde{\alpha}}(m) = \underset{\sim}{f}_{\tilde{\alpha}}(m).$$

Without loss of generality we can assume that $\tilde{\alpha} \in \mathrm{supp}(p_\alpha)$. Continue as in 2.4 and form a $\Delta$-system.
The rest of the argument repeats those of 2.4 only instead of dealing with $\alpha$-th coordinates (i.e. $p^\alpha$ , $\underset{\sim}{f}_\alpha$ and $\underset{\sim}{t}_\alpha$) we deal with $\tilde{\alpha}$-th (i.e. $p^{\tilde{\alpha}}$ , $\underset{\sim}{f}_{\tilde{\alpha}}$ and $\underset{\sim}{t}_{\tilde{\alpha}}$).
$\square$

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