On the strength of no normal precipitous filter

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Abstract

We consider a question of T. Jech and K. Prikry that asks if the existence of a precipitous filter implies the existence of a normal precipitous filter. The aim of this paper is to improve a result of [3] and to show that measurable cardinals of a higher order rather than just measurable cardinals are necessary in order to have a model with a precipitous filter but without a normal one.

1 Introduction

The notion of a precipitous filter was first introduced by T. Jech and K. Prikry in [4]:

Definition 1.1. *A filter F is* precipitous *if for every generic $G \subseteq F^+$, the ultrapower* $\mathrm{Ult}(V, G)$ *is well-founded.*

They asked whether the existence of a precipitous filter over κ implies the existence of a normal precipitous filter over κ.
H-D.Donder and J-P.Levinski [1] introduced the following notion:

Definition 1.2. *A cardinal κ is called* ∞-semi-precipitous *iff there exists a forcing notion P such that the following is forced by the weakest condition: there exists an elementary embedding $j : V \to M$ with critical point κ and M transitive.*

Clearly, if there is a precipitous filter over κ, then κ is ∞-semi-precipitous - just take P to be the forcing with the positive sets.
E. Schimmerling and B. Velickovic [8] proved that there is no precipitous ideals on $\aleph_1$ in $L[E]$ models up to at least a Woodin limit of Woodins. On the other hand $\aleph_1$ is always ∞-semi-precipitous in presence of a Woodin cardinal. So ∞-semi-precipitousness need not imply precipitousness at least in presence of large enough cardinals.
In the opposite direction the following was shown in [2](Thm. 3.11):

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Theorem 1.3. *Assume that:*

(1) $\aleph_1$ *is* ∞*-semi-precipitous.*

(2) $2^{\aleph_1} = \aleph_2$.

(3) There is no inner model satisfying $(\exists\alpha\ o(\alpha) = \alpha^{++})$.

(4) $\aleph_3$ *is not a limit of measurable cardinals in the core model.*

Then there exists a normal precipitous filter on $\aleph_1$.

There is a huge gap between a Woodin cardinal and infinitely many measurable cardinals. The purpose of this paper is to improve Theorem 1.3 and to narrow the gap. Some methods developed here likely to be useful for other purposes as well.

2 Preliminaries

We denote $\kappa := \aleph_1^V$. We shall always denote $\kappa^+ := \aleph_2^V$, even if we refer to this ordinal in the context of other models. The same for $\kappa^{++} := \aleph_3^V$. Our aim is to prove the following theorem:

Theorem 2.1. *Assume that:*

(1) κ *is* ∞*-semi-precipitous.*

(2) $2^{\aleph_0} = \kappa$ *and* $2^\kappa = \kappa^+$.

(3) There is no inner model with a strong cardinal.

(4) In the core model, the set $\{\alpha < \kappa^{++} \mid o(\alpha) \geq \alpha^+\}$ *is bounded in* κ^{++}.

Then there exists a normal precipitous filter.

The rest of this paper is dedicated to the proof theorem 2.1. From now on, we shall assume that (1)-(4) in theorem 2.1 indeed hold.
Under assumption (3) of theorem 2.1, the core model $\mathcal{K}$ exists and is of the form $L[\mathcal{U}]$ where $\mathcal{U} = \{\mathcal{U}_\gamma \mid \gamma \in On\}$ is a coherent sequence of extenders (see [5] and [6] for a definition and thorough discussion of the core model). Under assumption (4), there exists an ordinal λ such that:

(1) $\kappa^+ < \lambda < \kappa^{++}$

(2) $\lambda > \sup\{\alpha < \kappa^{++} \mid o(\alpha) \geq \alpha^+\}$

(3) λ is not a measurable cardinal in $\mathcal{K}$

We fix this λ for the remainder of this paper. For $\gamma \in [\lambda, \kappa^{++})$, $\mathcal{U}_\gamma$ is just a measure, not an extender. For such γ's, we denote $\alpha_\gamma := \text{crit}(\mathcal{U}_\gamma)$ if $\mathcal{U}_\gamma$ is not trivial, and $\alpha_\gamma := 0$ otherwise. We also denote $\beta_\gamma := o(\mathcal{U}_\gamma)$. In the other direction, for $\alpha \in [\lambda, \kappa^{++})$ and $\beta < o(\alpha)$, we denote $\mathcal{U}(\alpha, \beta) := \mathcal{U}_\gamma$ for the unique γ such that $\alpha = \alpha_\gamma$ and $\beta = \beta_\gamma$.

3 Precipitousness up to λ

The following result of [3] will be our starting point:

Theorem 3.1. *Assume that:*

(1) U *is a precipitous filter over* κ*.*

(2) $2^\kappa = \kappa^+$*.*

(3) $\kappa \Vdash_{U^+} i(\kappa) > \kappa^+$*, where* i *is defined as follows: Let* $G \subseteq U^+$ *be generic, let* f *be such that* $\kappa = [f]_G$*, and let* $G_{normal} := \{f[X] \mid X \in G\}$*. Then* $i : V \to \mathrm{Ult}(V, G_{normal})$ *is the ultrapower embedding.*

Then for every $\tau < \kappa^{++}$*, there exists a normal filter that is precipitous up to the image of* τ*.*

It was pointed out in [2] that actually the proof of 3.1 still works fine once the first item of 3.1 is replaced by κ being ∞-semi-precipitous.
In this section, we will use theorem 3.1 to show that there exists a filter F over κ which is precipitous up to the image of λ. The next sections will be dedicated to showing that F is precipitous up to κ^{++}, and is therefore fully precipitous. The proof of this section closely follows the proof of theorem 1.3 in [3] - the main difference is that in theorem 1.3, it was assumed that there is no inner model satisfying $(\exists\alpha\ o(\alpha) = \alpha^{++})$, and here we only have the weaker assumption that there is no inner model with a strong cardinal.
To prove the existence of F, we wish to use theorem 3.1, but we first need to prove that the assumptions in that theorem hold. Assumptions (1) and (2) in that theorem are already included in the assumptions of theorem 2.1, so we only need to show assumption (3).
The following result seems to be well known. A proof below was suggested by the referee and it is much shorter than those used originally.

Theorem 3.2. *Suppose that* κ *is* ∞*-semi-precipitous cardinal. Then* $\kappa^+ = (\kappa^+)^{\mathcal{K}}$*.*

Proof. By the assumption, in a generic extension $V[G]$ of V there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$. Then $j \restriction \mathcal{K}$ is an iterated ultrapower of $\mathcal{K}$ by its extenders. Denote that iteration by $\mathcal{I} = \langle \mathcal{K}_i \mid i \leq \theta \rangle$, i.e, $\mathcal{K}_0 = \mathcal{K}$, $\mathcal{K}_\theta = \mathcal{K}^M$, and for every $i < \theta$, $\mathcal{K}_{i+1} = \mathrm{Ult}(\mathcal{K}_i, E_i)$ for some $\mathcal{K}_i$-extender E_i. W.l.o.g, we can assume that this iteration is normal, i.e, extenders with lower Mitchell order are used before extenders with higher Mitchell order. For $\alpha \leq \beta \leq \theta$, we denote $j_{\alpha\beta} : \mathcal{K}_\alpha \to \mathcal{K}_\beta$ as the iteration embedding. In particular, $j \restriction \mathcal{K} = j_{0\theta}$.

We show that $E_0 \in \mathcal{K}^M$, which is impossible, since E_0 is the first extender used in the iteration from $\mathcal{K}$ to $\mathcal{K}_\theta = \mathcal{K}^M$. As in Zeman [9], view E_0 as a function on $P(\kappa)^{\mathcal{K}}$. An old argument of Kunen shows that $E_0 \in M$; this is because if $F := \langle x_\xi \mid \xi < \kappa \rangle \in V$ enumerates $P(\kappa) \cap \mathcal{K}$, then E_0 is definable from $F' := j(F) \restriction \kappa$ by: $E_0(x_\xi) = \{\eta \mid \eta \in F'(\xi)\}$. To see that E_0 is on the $\mathcal{K}^M$ sequence , by Lemma 8.3.4 of [9] it suffices to show that

$$M \models \text{“}ult(\mathcal{K}^M, E_0) \text{ is wellfounded ”}.$$

M itself is wellfounded, hence it is enough to check that

$$V[G] \models \text{“}ult(\mathcal{K}^M, E_0) \text{ is wellfounded ”}.$$

Now $\mathcal{K} := \mathcal{K}^V = \mathcal{K}^{V[G]}$. Work inside $V[G]$. We have

(1) $\mathcal{K}$ is a universal weasel,

(2) E_0 is on the $\mathcal{K}$ sequence and is thus $\mathcal{K}$–correct,

(3) the iteration from $\mathcal{K}$ to $\mathcal{K}^M = \mathcal{K}_\theta$ uses only extenders with indexes $\geq$ the index of E_0.

Thus the hypotheses of Lemma 7.3.1 of [9] hold, and $ult(\mathcal{K}_\theta, E_0)$ is wellfounded (page 274 of [9] points out that Lemma 7.3.1 goes through for premice in the absence of 0-pistol).

□

Theorem 3.2 implies that assumption (3) in theorem 3.1 holds - i is an iterated ultrapower of $\mathcal{K}$, so $i(\kappa) > (\kappa^+)^{\mathcal{K}} = \kappa^+$. Now, as in the proof of theorem 1.3 (the proof is detailed in [3]), we can apply theorem 3.1 and construct some filter F which is precipitous up to the image of λ. We have the following result from that proof:

Lemma 3.3. [1] *Let $G \subseteq F^+$ be generic, and let $V \supseteq \langle f_n \mid n < \omega \rangle \in V[G]$, such that for every $n < \omega$ there is some $h_n \in \mathcal{K}$ and some $g_n \in {}^\kappa\lambda \cap V$ such that $f_n = h_n \circ g_n$. Then there are some $n < m < \omega$ such that $[f_n] \leq [f_m]$.* □

In [3], this result was enough to imply the full precipitousness of F, since the assumption that there are no measurables in $[\lambda, \kappa^{++})$ implies a strong covering property between V and $\mathcal{K}$, and that strong covering property was used to prove full precipitousness. In this paper, we only assume that there

[1] In [3] it was proved under the assumption that there is no inner model of $o(\kappa) = \kappa^{++}$, but the proof still goes through when there is no inner model with a strong cardinal.

are no measurables $\alpha \in [\lambda, \kappa^{++})$ such that $o(\alpha) \geq \alpha^{+}$. Therefore, the result above does not imply precipitousness as directly as it did in [3].
We continue the proof of theorem 2.1.
Let $G \subseteq F^{+}$ be generic. Throughout this paper, we shall always use the same G, so we shall just write $[f]$ instead of $[f]_G$. Also, whenever we write "for almost all ν" or $\forall^{*}\nu$, we will mean that there exists some set $A \in G$ such that the statement holds for all $\nu \in A$.
We wish to prove that F is precipitous. It is enough to prove that for every sequence $V \supseteq \langle f_n \mid n < \omega \rangle \in V[G]$ of ordinal functions on κ, there are some $n < m < \omega$ such that $[f_n] \leq [f_m]$. We fix, for the remainder of this paper, a sequence $V \supseteq \langle f_n \mid n < \omega \rangle \in V[G]$ of ordinal functions on κ. Our aim is to show that there are some $n < m < \omega$ such that $[f_n] \leq [f_m]$.
Note that we may assume that for every $n < \omega$, $f_n : \kappa \to [\lambda, \kappa^{++})$. The reason for that is simple: if F is precipitous up to an image of κ^{++}, then it is in fact precipitous. Therefore, we can assume that $f_n : \kappa \to \kappa^{++}$. Now let $S := \{n < \omega \mid \forall^{*}\nu \ f_n(\nu) < \lambda\}$. If S is infinite, then since F is precipitous up to the image of λ, there must be some $n < m$ in S such that $[f_n] \leq [f_m]$ and we are done. So, we can assume that S is finite, and by ignoring some finite prefix we can assume that for all $n < \omega$ and for almost all ν, $f_n(\nu) > \lambda$. In other words, we can assume that $f_n : \kappa \to [\lambda, \kappa^{++})$ for every $n < \omega$.

4 Coverings

We wish to apply the covering lemma, in V, in order to cover $\mathrm{ran}(f_n)$. We state some the basic properties of the covering lemma (for a full statement and proof, see [6] and [7]).
Let $x \in V$, $|x| = \kappa$, $x \subseteq [\lambda, \kappa^{++})$. The covering lemma provides us with a function h, an ordinal ρ, and a system of indiscernibles $\mathcal{C}$ with the following properties:

(1) $h \in \mathcal{K}$ and $h : \rho \times [\kappa^{++}]^{<\omega} \to \mathcal{K}$. For a set $y \subseteq \kappa^{++}$, we denote:

$$h[y] := \{h(\delta, z) \mid \delta < \rho \text{ and } z \in [y]^{<\omega}\}$$

(2) There is some model $N \prec H(\chi)$ (for some χ large enough) such that $x \subseteq N$, ${}^{\omega}N \subseteq N$, $|N| \leq \kappa$, and $N \cap \mathcal{K} \subseteq h[\mathcal{C}]$. It is here that we apply our assumption that $2^{\aleph_0} = \kappa$ (otherwise we couldn't have ${}^{\omega}N \subseteq N$).

(3) ρ is a cardinal in $\mathcal{K}$, and $\rho < \sup(N \cap On)$.

(4) $\mathcal{C} = \bigcup\{\mathcal{C}_\gamma \mid \gamma \in \mathrm{dom}(\mathcal{C})\}$, $\mathcal{C} \subseteq [\lambda, \kappa^{++})$, and $|\mathcal{C}| \leq \kappa$. For every $\gamma \in \mathrm{dom}(\mathcal{C})$, $\mathcal{C}_\gamma \subseteq \alpha_\gamma$. We shall often use the equivalent notation $\mathcal{C}(\alpha, \beta)$ instead of $\mathcal{C}_\gamma$ (where $\alpha = \alpha_\gamma$ and $\beta = \beta_\gamma$).

(5) For every ordinal $\xi \in N$, $\xi \in h[\mathcal{C} \cap \xi]$ iff $\xi \notin \mathcal{C}$.

(6) If $c \in \mathcal{C}_\gamma$, $y \subseteq \alpha_\gamma$, and $y \in h[\mathcal{C} \cap c]$, then $c \in y$ iff $y \in \mathcal{U}_\gamma$.

(7) $\mathcal{C}$ is *h-coherent*, i.e:

  (a) For every $c \in \mathcal{C}$ there is a unique $\gamma \in h[\mathcal{C} \cap c]$ such that $c \in \mathcal{C}_\gamma$. We denote this γ by $\gamma^{\mathcal{C}}(c)$. When $\mathcal{C}$ is clear from the context, we shall just write $\gamma(c)$. We shall also use the notation $\alpha(c)$ and $\beta(c)$ for $\alpha_{\gamma(c)}$ and $\beta_{\gamma(c)}$.

  (b) Let $c \in \mathcal{C}$, $\gamma := \gamma(c)$, $\alpha := \alpha(c)$, and assume there is some $\gamma' \neq \gamma$ such that also $c \in \mathcal{C}_{\gamma'}$. Let $\alpha' := \alpha_{\gamma'}$. Then $\alpha' < \alpha$, and there is some $\gamma'' < \gamma$ such that $\alpha_{\gamma''} = \alpha$ and $\alpha' \in \mathcal{C}_{\gamma''}$.

  (c) We denote by $\mathrm{Coh}_{\gamma',\gamma}$ the least function t (in the ordering of $\mathcal{K}$) such that $\gamma' = [t]_{\mathcal{U}_\gamma}$. Let $\gamma' < \gamma$ such that $\alpha_{\gamma'} = \alpha_\gamma$, and assume there is some $c \in \mathcal{C}_\gamma$ such that $\gamma' \in h[\mathcal{C} \cap c]$. Let $\gamma'' := \mathrm{Coh}_{\gamma',\gamma}(c)$, and $c' := \min\{\xi \mid \gamma' \in h[\mathcal{C} \cap \xi]\}$. Then $\mathcal{C}_{\gamma''} = \mathcal{C}_{\gamma'} \cap (c - c')$.

(8) Some additional properties of h and $\mathcal{C}$ are:

  (a) If $\vec{c} \subseteq \mathcal{C}$, $c \in \mathcal{C}_\gamma$, $y \in \mathcal{U}_\gamma$, $y \in h[\vec{c}]$, and $c \notin y$, then $\vec{c} \cap [c, \alpha_\gamma) \neq 0$.

  (b) $\alpha(c) = \min(h[\mathcal{C} \cap c] \setminus c)$ (we slightly abuse notation here by taking the minimum over all *ordinals* in $h[\mathcal{C} \cap c]$ above c).

We shall call a triplet $h, \mathcal{C}, \rho$ as above a *covering*.

5 Choosing a Good Covering

In this subsection, we will use the notation common in [7]: given an elementary sub-model $N \prec H(\chi)$, there is a function $h^N \in \mathcal{K}$, an h^N-coherent system of indiscernibles $\mathcal{C}^N$, and an ordinal $\rho^N < \sup(N \cap On)$, such that $N \cap \mathcal{K} = h^N[\mathcal{C}^N]$. Mitchell [6], Remark 4.22, notes that if every measurable limit point of N is a member of N, then ρ^N can be picked to be $\min(On \setminus N)$. In our situation it does not seem to be a case. Still the game below allows at least keep such ρ^N constant for many different models N.
Given a sequence $V \supseteq \langle g_n \mid n < \omega\rangle \in V[G]$ of ordinal functions on κ, we wish to apply the covering lemma (in V) to cover $\mathrm{ran}(g_n)$. We wish our coverings to have several convenient properties.
Let $\mathcal{G}$ be a game (in V) with the following rules:

(R1) In step $2n$, player I chooses some function $g_n : \kappa \to [\lambda, \kappa^{++})$.

(R2) In step $2n+1$, Player II chooses a model X_n and an h^{X_n}-coherent system of indiscernibles $\mathcal{C}_n \subseteq \mathcal{C}^{X_n}$, such that $\mathrm{ran}(g_n) \subseteq h^{X_n}[\mathcal{C}_n]$.

(R3) For every $n < m < \omega$, $\mathcal{C}_n \subseteq \mathcal{C}_m$, and for every $c \in \mathcal{C}_n$, $\alpha^{X_n}(c) = \alpha^{X_m}(c)$.

(R4) In step 1, player II also chooses some ordinal $\eta < \kappa^{++}$. For every $n < \omega$, $\rho^{X_n} \leq \eta$.

Player II wins if the game continues infinitely many steps. Otherwise (i.e, a step was reached in which player II cannot make a step such that rules (R2)-(R4) hold) player I wins.

Lemma 5.1. *Player II has a winning strategy.*

Proof. Note that the game is open, and so determined. Suppose that player I has a winning strategy. Let σ be some strategy for player I. We will show that σ is not a winning strategy. Let $M' \prec H(\kappa^{+3})$ be such that $|M'| = \kappa$ and $\sigma \in M'$. Denote $M := M' \cap H(\kappa^{++})$. Let $\eta := \min(M \setminus \rho^M)$.
We shall prove the following claim.

Claim: There are $\{g_n \mid n < \omega\}$, $\{X_n, \mathcal{C}_n \mid n < \omega\}$, such that if player I plays g_n at step $2n$ and player II plays $X_n, \mathcal{C}_n$ at step $2n + 1$ (and player II chooses η at step 1), then:

(1) All plays are legal in game $\mathcal{G}$, and player I plays according to strategy σ. In particular, the game continues for ω steps, and player II wins.
(2) For every $n < \omega$:
    (a) g_n, X_n, and $\mathcal{C}_n$ are in M'.
    (b) $\rho^{X_n} \leq \eta$.
    (c) $\mathcal{C}_n \subseteq \mathcal{C}^M$, and for every $c \in \mathcal{C}_n$, $\alpha^{X_n}(c) = \alpha^M(c)$.

Proof: Let $m < \omega$, and assume $\{g_n \mid n < m\}$ and $\{X_n, \mathcal{C}_n \mid n < m\}$ are already defined such that (1), (2) and (3) hold.
Let g_m be the next move of player I using strategy σ, i.e:

$$g_m := \sigma(g_0, \langle \eta, X_0, \mathcal{C}_0 \rangle, g_1, \langle X_1, \mathcal{C}_1 \rangle, \ldots, g_{m-1}, \langle X_{m-1}, \mathcal{C}_{m-1} \rangle)$$

By (2), $g_m \in M'$. Let N be a covering model such that:

$$\operatorname{ran}(g_m) \cup \left(\bigcup_{n<m} X_n \right) \subseteq N \text{ and } |N| = \kappa$$

By elementarity, we can assume that $N \in M'$. Since $|M'| = \kappa$, $\kappa \in M'$, and it follows that $N \subseteq M'$, and in fact $N \subseteq M$. Then $\mathcal{C}^N \setminus \mathcal{C}^M$ is finite (see lemma 1.2 in [7]). Let $\vec{c} := \mathcal{C}^N \setminus \mathcal{C}^M$. Similarly, for every $n < m$, $\mathcal{C}_n \setminus \mathcal{C}^N$ is finite. Let $\vec{d} \subseteq \mathcal{C}^M$ be finite such that $\vec{c} \in h[\vec{d}]$, $h^N \in h[\vec{d}]$, and $\left(\bigcup_{n<m} \mathcal{C}_n \setminus \mathcal{C}^N\right) \subseteq \vec{d}$. We may assume w.l.o.g that $\vec{d}$ is a support, i.e, for every $d \in \vec{d}$, $\alpha^M(d) \in h^M[\vec{d} \cap d]$

(this can be achieved by adding finitely many indiscernibles from $\mathcal{C}^M$ to $\vec{d}$).
Define:

$$\begin{aligned}\vec{\alpha} \;:=\; & \{\alpha^N(c) \mid \exists c \in \mathcal{C}^N \cap \mathcal{C}^M \; \alpha^N(c) \neq \alpha^M(c)\} \cup \\ & \{\alpha^M(c) \mid \exists c \in \mathcal{C}^N \cap \mathcal{C}^M \; \alpha^N(c) \neq \alpha^M(c)\} \cup \\ & \vec{d} \cup \{\alpha^M(d) \mid d \in \vec{d}\}\end{aligned}$$

By theorem 1.2 in [7], this set is finite. Let $\{\alpha_0, \ldots, \alpha_{r-1}\}$ be an increasing enumeration of $\vec{\alpha}$. Let $\vec{s}_i \subseteq \mathcal{C}^M$ be minimal (in reverse lexicographical order) such that $\alpha_i \in h^M[\vec{s}_i]$. Note that we can assume that $\vec{s}_i \subseteq \vec{d}$ for every $i < r$. Let $\theta(X)$ be the conjunction of the following statements:

(a) $N \in X \prec H(\chi)$
(b) $\vec{d} \subseteq \mathcal{C}^X$, and $\vec{d}$ is a support
(c) $\mathcal{C}^N \setminus \mathcal{C}^X = \vec{c} \in h^X[\vec{d}]$
(d) $h^N \in h^X[\vec{d}]$
(e) $\forall d \in \vec{d} \; \alpha^X(d) = \alpha^M(d)$
(f) $\forall i < r \; \vec{s}_i$ is minimal in $\mathcal{C}^X$ such that $\alpha_i \in h^X[\vec{s}_i]$
(g) $\forall c \in \mathcal{C}^X \cap \mathcal{C}^N \; \alpha^X(c) \neq \alpha^N(c) \longrightarrow \exists i < r \; \alpha^X(c) = \alpha_i$
(h) $\rho^X \leq \eta$

Note that in every conjunct, we only use finitely many parameters from M. Clearly, $H(\kappa^{+3}) \models \theta(M)$. Then by elementarity there is some $X \in M'$ such that $M' \models \theta(X)$. Define:

$$\mathcal{C} := \left(\mathcal{C}^N \setminus \vec{c}\right) \cup \vec{d}$$

We wish to prove that (1) and (2) from the statement of the lemma hold, if we set $X_m := X$ and $\mathcal{C}_m := \mathcal{C}$.
(2a) is immediate, and (2b) follows from conjunct (h).
We show (2c). Clearly, $\mathcal{C} \subseteq \mathcal{C}^M$. Let $c \in \mathcal{C}$. We need to show that $\alpha^X(c) = \alpha^M(c)$. Assume otherwise. If $c \in \vec{d}$, then by conjunct (e), $\alpha^X(c) = \alpha^M(c)$ and we are done. Then we can assume that $c \in \mathcal{C}^N \setminus \vec{c}$. There are several cases.
Case 1: $\alpha^X(c) \in \vec{\alpha}$ and $\alpha^M(c) \in \vec{\alpha}$
Then there is some $i < r$ such that $\alpha^X(c) = \alpha_i$ and some $j < r$ such that $\alpha^M(c) = \alpha_j$. Conjunct (f) implies that $\vec{s}_i \subseteq c$. Now, $\alpha_j = \min(h^M[\mathcal{C}^M \cap c] \setminus c)$, and $c < \alpha_i \in h^M[\vec{s}_i]$. This implies that $\alpha_j \leq \alpha_i$. In the other direction, $\alpha_i = \min(h^X[\mathcal{C}^X \cap c] \setminus c)$, and $\alpha_j \in h^X[\vec{s}_j]$ by conjunct (f). Then also $\alpha_i \leq \alpha_j$, and $\alpha_i = \alpha_j$ - contradiction.
Case 2: $\alpha^X(c) \in \vec{\alpha}$ and $\alpha^M(c) \notin \vec{\alpha}$
Let $i < r$ be such that $\alpha^X(c) = \alpha_i$. Since $\alpha^M(c) \notin \vec{\alpha}$, it follows that $\alpha^M(c) =$

$\alpha^N(c)$. Denote $\alpha := \alpha^M(c) = \alpha^N(c)$. Note that $\alpha_i = \min(h^X[\mathcal{C}^X \cap c] \setminus c)$. By conjunct (f), $\vec{s}_i \subseteq c$, and:

$$\alpha = \min(h^M[\mathcal{C}^M \cap c] \setminus c) \leq \min(h^M[\vec{s}_i] \setminus c) \leq \alpha_i$$

The last inequality follows the fact that $\alpha_i \in h^M[\vec{s}_i]$ (by definition of $\vec{s}_i$). Then $\alpha \leq \alpha_i$, and since $\alpha = \alpha^M(c) \neq \alpha^X(c) = \alpha_i$, $\alpha < \alpha_i$.
Let $\vec{a} \subseteq \mathcal{C}^N \cap c$ be finite such that $\alpha \in h^N[\vec{a}]$. Let $\vec{b} \subseteq \mathcal{C}^X \cap c$ be finite such that $\vec{a} \in h^X[\vec{b}]$. So, by conjunct (d), $\alpha \in h^X[\vec{b} \cup \vec{d}]$, and in fact $\alpha \in h^X[(\vec{b} \cup \vec{d}) \cap (\alpha + 1)]$. But $c < \alpha < \alpha_i = \alpha^X(c)$. This implies that $(\vec{b} \cup \vec{d}) \cap (\alpha + 1) \cap [c, \alpha_i) \neq 0$. Since $\vec{b}$ is below c, it follows that $\vec{d} \cap (\alpha + 1) \cap [c, \alpha_i) \neq 0$. Define:

$$d_0 := \min(\vec{d} \cap [c, \alpha])$$

If $d_0 = c$, then $\alpha^M(c) = \alpha^M(d_0) = \alpha^X(d_0) = \alpha^X(c)$ (by conjunct (e)), which contradicts our assumption on c. Then $d_0 > c$. If $d_0 = \alpha$, then $\alpha \in \vec{\alpha}$, which is again impossible by our assumption on α. Then $d_0 < \alpha$. Now, $\alpha \in h^M[\mathcal{C}^M \cap d_0] \setminus d_0$, which implies that $\alpha^M(d_0) \leq \alpha$. Again, $\alpha^M(d_0) = \alpha$ implies that $\alpha \in \vec{\alpha}$ which is impossible, so $\alpha^M(d_0) < \alpha$.
Recall that $\alpha^M(d_0) \in h^M[\vec{d} \cap d_0]$. As before, $c < \alpha^M(d_0) < \alpha$ implies that $(\vec{d} \cap d_0) \cap [c, \alpha] \neq 0$. Let $d_1 := \max(\vec{d} \cap d_0)$. Then $d_1 \in \vec{d} \cap [c, \alpha]$ and $d_1 < d_0$, which contradicts the minimality of d_0.
Case 3: $\alpha^X(c) \notin \vec{\alpha}$ and $\alpha^M(c) \in \vec{\alpha}$
This case is symmetric to case 2, and its proof is the same.
Case 4: $\alpha^X(c) \notin \vec{\alpha}$ and $\alpha^M(c) \notin \vec{\alpha}$
Then $\alpha^X(c) = \alpha^N(c) = \alpha^M(c)$ - contradiction.
This completes the proof of (2c).
We now show (1). We need to verify that the 4 rules of game $\mathcal{G}$ are kept.
Rule (R1) is trivial, and rule (R4) follows directly from (2b).
We prove that rule (R2) is kept. Clearly, $\mathcal{C} \subseteq \mathcal{C}^X$, and:

$$\operatorname{ran}(g_m) \subseteq N \cap \mathcal{K} = h^N[\mathcal{C}^N] \subseteq h^X[(\mathcal{C}^N \setminus \vec{c}) \cup \vec{d}] = h^X[\mathcal{C}]$$

It remains to show that $\mathcal{C}$ is h^X-coherent. Let $c \in \mathcal{C}$. We need to prove that $\alpha^X(c) \in h^X[\mathcal{C} \cap c]$. If $c \in \vec{d}$, then by conjunct (b), $\alpha^X(c) \in h^X[\vec{d} \cap c]$. If $c \notin \vec{d}$, then $c \in \mathcal{C}^N$. If $\alpha^X(c) \in \vec{\alpha}$, then there is some $i < r$ such that $\alpha^X(c) \in h^X[\vec{s}_i]$, and $\vec{s}_i \subseteq \vec{d} \cap c$. Therefore, we can assume that $\alpha^X(c) \notin \vec{\alpha}$, which implies that $\alpha^X(c) = \alpha^N(c)$. Let $\vec{e} \subseteq \mathcal{C}^N \cap c$ such that $\alpha^N(c) \in h^N[\vec{e}]$. Then:

$$\alpha^X(c) = \alpha^N(c) \in h^N[\vec{e}] \subseteq h^X[((\vec{e} \setminus \vec{c}) \cup \vec{d}) \cap \alpha^X(c)]$$

By the same argument as in case 2 above, we can show that all the indiscernibles in $((\vec{e} \setminus \vec{c}) \cup \vec{d}) \cap \alpha^X(c)$ are below c.

Finally, we show rule (R3). If $m = 0$, there is nothing to prove. Assume that $m > 0$, and let $n < m$. By induction, $\mathcal{C}_n \subseteq \mathcal{C}^M$, so:

$$\mathcal{C}_n \subseteq (\mathcal{C}^N \cap \mathcal{C}^M) \cup (\mathcal{C}_n \setminus \mathcal{C}^N) \subseteq (\mathcal{C}^N \setminus \vec{c}) \cup \vec{d} = \mathcal{C}$$

Let $c \in \mathcal{C}_n \cap \mathcal{C}$. Then by (3):

$$\alpha^{X_n}(c) = \alpha^M(c) = \alpha^X(c)$$

And we are done. $\square$***Claim***

The claim proves that σ is not a winning strategy for player I. Since σ was arbitrary, there is no winning strategy for player I. But the game $\mathcal{G}$ is an open game for player I, so one of the players has a winning strategy. Then there must be a winning strategy for player II. $\square$

We now go back to our fixed set of functions $\langle f_n \mid n < \omega \rangle$. This lemma provides us with a very convenient covering of the ranges of these functions:

Corollary 5.2. *For every $n < \omega$ there is a model X_n, and an h^{X_n}-coherent system of indiscernibles $\mathcal{C}_n$ such that:*

(1) $\operatorname{ran}(f_n) \subseteq h^{X_n}[\mathcal{C}_n]$
(2) For every $n < m$, $\mathcal{C}_n \subseteq \mathcal{C}_m$.
(3) For every $n < m$ and every $c \in \mathcal{C}_n$, $\alpha^{X_n}(c) = \alpha^{X_m}(c)$ and $\beta^{X_n}(c) = \beta^{X_m}(c)$ (or, equivalently, $\gamma^{X_n}(c) = \gamma^{X_m}(c)$).
(4) For every $n < m$, $\rho^{X_n} \leq \lambda$.

Proof. Let σ be a winning strategy of player II in the game $\mathcal{G}$. For the first move, let player I play the constant function 0, and let $\eta, X_{-1}, \mathcal{C}_{-1}$ be the response of player II using strategy σ. Next, let player I play f_0, and let $X_0, \mathcal{C}_0$ be the response of player II by strategy σ. Continue in a similar fashion. In general, for $n < \omega$:

$$\langle X_n, \mathcal{C}_m \rangle := \sigma(0, \langle \eta, X_{-1}, \mathcal{C}_{-1} \rangle, f_0, \langle X_0, \mathcal{C}_0 \rangle, \ldots, f_{n-1}, \langle X_{n-1}, \mathcal{C}_{m-1} \rangle)$$

Thus we have defined X_n and $\mathcal{C}_n$ for every $n < \omega$. Note that (1), (2), and the first part of (3) follow directly from the rules of the game $\mathcal{G}$. The second part of (3) is proved in the following claim:

Claim: For all $c \in \mathcal{C}_n$, $\beta^{X_n}(c) = \beta^{X_m}(c)$.

Proof: Let $\alpha \in [\lambda, \kappa^{++})$ be measurable in $\mathcal{K}$. By assumption (4) in theorem 2.1, $o(\alpha) < \alpha^+$. Then there is a sequence of disjoint sets $\langle Z(\alpha, \beta) \mid \beta < o(\alpha)\rangle \in \mathcal{K}$, such that for every $\beta < o(\alpha)$, $Z(\alpha, \beta) \in \mathcal{U}(\alpha, \beta)$. Consider the sequence $\vec{Z} := \langle Z(\alpha, \beta) \mid \alpha < \kappa^{++}$ is measurable and $\beta < o(\alpha)\rangle$. By elementarity, $\vec{Z} \in X_n$. Since $\vec{Z}$ is definable in $\mathcal{K}$ without parameters, $\vec{Z} \in h^{X_n}[0]$.
Let $c \in \mathcal{C}_n$. Let $\beta_n := \beta^{X_n}(c)$ and $\beta_m := \beta^{X_m}(c)$, and assume that $\beta_n \neq \beta_m$. By the rules of the game $\mathcal{G}$, there is some α such that $\alpha = \alpha^{X_n}(c) = \alpha^{X_m}(c)$. Since $(\alpha, \beta_n) \in h^{X_n}[\mathcal{C}_n \cap c]$, also $Z(\alpha, \beta_n) \in h^{X_n}[\mathcal{C}_n \cap c]$. This implies that $c \in Z(\alpha, \beta_n)$. By the same argument, also $c \in Z(\alpha, \beta_m)$. Then:

$$c \in Z(\alpha, \beta_n) \cap Z(\alpha, \beta_m) = 0$$

Contradiction. $\square$***Claim***

What remains is to prove (4). Note that the choice of η did not depend on the filter F or on the specific choice of the sequence $\langle f_n \mid n < \omega\rangle$. So, w.l.o.g, we can assume that F is precipitous up to the image of $\max\{\lambda, \eta\}$. To simplify the proof, we can assume that $\lambda \geq \eta$ (recall that at the end of section 2, λ was just some ordinal picked arbitrarily below κ^{++} and above some bounded subset of κ^{++}, so requiring that $\lambda \geq \eta$ is justified).
This completes the proof of the corollary. $\square$

For the remainder of this paper, we fix X_n and $\mathcal{C}_n$ as in corollary 5.2. We denote $h_n := h^{X_n}$.

6 The Ultrafilter $W_{\vec{h},\vec{\delta}}$

Throughout this section, we work in $\mathcal{K}$.
For every function $h \in \mathcal{K}$, we define a function $\bar{h}$ as follows. If $h(x) \in \kappa^{++}$, then $\bar{h}(x)$ is the minimal $\gamma \geq h(x)$ such that $\mathcal{U}_\gamma$ is a full measure over a measurable cardinal $\alpha_\gamma \geq \lambda$. Otherwise, we just set $\bar{h}(x)$ to be the minimal γ such that $\alpha_\gamma \geq \lambda$.
Note that $\mathcal{U}_{\bar{h}(x)}$ is always a full measure over a measurable $\alpha_{\bar{h}(x)} \in [\lambda, \kappa^{++})$, and $h(x) = \bar{h}(x)$ iff $\mathcal{U}_{h(x)}$ is a full measure and $\alpha_{h(x)} \in [\lambda, \kappa^{++})$ (we are using our assumption that κ^{++} is a limit of measurable cardinals in $\mathcal{K}$, otherwise the result follows by [3]).
Denote $S_r := {}^r\kappa^{++}$. For a set $X \subseteq S_r$ and a sequence $\vec{\xi} \in S_{r-1}$, we define:

$$p(X, \vec{\xi}) := \{\mu \mid \vec{\xi}^\frown\langle\mu\rangle \in X\}$$

Let $\vec{\delta} \in [\lambda]^{<\omega}$, and $\vec{h} = \langle h_0, \ldots, h_{r-1} \rangle$ be a sequence of functions in $\mathcal{K}$. We define the ultrafilter $W_{\vec{h},\vec{\delta}}$ over S_r by induction on r.
If $r = 0$, then $S_r = \{0\}$, and $W_{\vec{h},\vec{\delta}} := \{\{0\}\}$.
Otherwise, let $\vec{h}' := \langle h_0, \ldots, h_{r-2} \rangle$. $W_{\vec{h},\vec{\delta}}$ is defined to be the set of $X \subseteq S_r$ for which there is a set $Y \in W_{\vec{h}',\vec{\delta}}$ such that the following holds: for every $\vec{\xi} \in Y$, if $\gamma := \bar{h}_{r-1}(\vec{\delta}, \vec{\xi})$, then $p(X, \vec{\xi}) \cap \alpha_\gamma \in \mathcal{U}_\gamma$. Recall that by the definition of $\bar{h}_{r-1}$, $\mathcal{U}_\gamma$ is a full measure over α_γ, which is a measurable cardinal $\geq \lambda$. We denote that set Y by $Y(X)$.

Lemma 6.1. *$W_{\vec{h},\vec{\delta}}$ is an ultrafilter over S_r, which is $|\lambda|^+$-complete.*

Proof. The proof is by induction on r.
If $r = 0$, there is nothing to prove.
Otherwise, let $\vec{h}'$ be as before. Let $X \subseteq S_r$. Assume that $X \notin W_{\vec{h},\vec{\delta}}$, and denote $\overline{X} := S_r \setminus X$. We need to prove that $\overline{X} \in W_{\vec{h},\vec{\delta}}$.
Let $Y := \{\vec{\xi} \restriction (r-1) \mid \vec{\xi} \in X\}$, and $\overline{Y} := S_{r-1} \setminus Y$.
If $Y \notin W_{\vec{h}',\vec{\delta}}$, then, by the induction hypothesis, $\overline{Y} \in W_{\vec{h}',\vec{\delta}}$, and:

$$W_{\vec{h},\vec{\delta}} \ni \{\vec{\xi}^\frown\langle\mu\rangle \mid \vec{\xi} \in \overline{Y} \wedge \mu < \kappa^{++}\} \subseteq \overline{X}$$

Which implies that $\overline{X} \in W_{\vec{h},\vec{\delta}}$.
Assume that $Y \in W_{\vec{h}',\vec{\delta}}$. For every $\vec{\xi} \in S_{r-1}$, denote $\gamma(\vec{\xi}) := \bar{h}_{r-1}(\vec{\delta}, \vec{\xi})$, $\alpha(\vec{\xi}) := \alpha_{\gamma(\vec{\xi})}$, and $U(\vec{\xi}) := \mathcal{U}_{\gamma(\vec{\xi})}$. Recall that $\alpha(\vec{\xi}) \geq \lambda$. Let:

$$Y' := \{\vec{\xi} \in Y \mid p(X, \vec{\xi}) \cap \alpha(\vec{\xi}) \notin U(\vec{\xi})\}$$

If $Y' \notin W_{\vec{h}',\vec{\delta}}$, then $Y \setminus Y' \in W_{\vec{h}',\vec{\delta}}$, and by definition $X \in W_{\vec{h},\vec{\delta}}$ which contradicts our initial assumption. Then $Y' \in W_{\vec{h}',\vec{\delta}}$. Let $\vec{\xi} \in Y'$. Since $U(\vec{\xi})$ is an ultrafilter, $p(X, \vec{\xi}) \cap \alpha(\vec{\xi}) \notin U(\vec{\xi})$ implies that:

$$\alpha(\vec{\xi}) \setminus p(X, \vec{\xi}) \in U(\vec{\xi})$$

In other words, $p(\overline{X}, \vec{\xi}) \cap \alpha(\vec{\xi}) \in U(\vec{\xi})$. This holds for every $\vec{\xi} \in Y'$, and $Y' \in W_{\vec{h}',\vec{\delta}}$. Then by definition, $\overline{X} \in W_{\vec{h},\vec{\delta}}$.
We turn to the proof of the $|\lambda|^+$-completeness. Let $\{X_\eta \mid \eta < \lambda\} \subseteq W_{\vec{h},\vec{\delta}}$. Let $X := \bigcap_{\eta<\lambda} X_\eta$, $Y_\eta := Y(X_\eta) \in W_{\vec{h}',\vec{\delta}}$, and $Y := \bigcap_{\eta<\lambda} Y_\eta$. By the induction hypothesis, $Y \in W_{\vec{h}',\vec{\delta}}$. Let $\vec{\xi} \in Y$. Then for every $\eta < \lambda$, $\vec{\xi} \in Y_\eta$ and $p(X_\eta, \vec{\xi}) \cap \alpha(\vec{\xi}) \in U(\vec{\xi})$. Since $\alpha(\vec{\xi}) > \lambda$, $U(\vec{\xi})$ is $|\lambda|^+$-complete, and:

$$p(X, \vec{\xi}) \cap \alpha(\vec{\xi}) = \bigcap_{\eta<\lambda} p(X_\eta, \vec{\xi}) \cap \alpha(\vec{\xi}) \in U(\vec{\xi})$$

And $X \in W_{\vec{h},\vec{\delta}}$. □

We shall prove an important characterization of the ultrapower of $\mathcal{K}$ by $W_{\vec{h},\vec{\delta}}$ as an iterated ultrapower of $\mathcal{K}$ by its measures.
Let $\mathcal{K}_0 := \mathcal{K}$, and $\psi_0 : \mathcal{K} \to \mathcal{K}_0$ be the identity.
Let $i < r$, and assume we already defined $\gamma_0, \dots, \gamma_{i-1}$, $\alpha_0, \dots, \alpha_{i-1}$, $\mathcal{K}_i$ and ψ_i. We define γ_i, α_i, $\mathcal{K}_{i+1}$ and ψ_{i+1}. Define $\gamma_i := \psi_i(\bar{h}_i)(\vec{\delta}, \langle \alpha_0, \dots, \alpha_{i-1} \rangle)$. Let α_i and U_i be such that:

$$\mathcal{K}_i \models U_i = \mathcal{U}_{\gamma_i} \text{ and } \alpha_i = \alpha_{\gamma_i}$$

And let $j : \mathcal{K}_i \to \mathcal{K}_{i+1} \cong Ult(\mathcal{K}_i, U_i)$ be the ultrapower embedding. Define $\psi_{i+1} := j \circ \psi_i : \mathcal{K} \to \mathcal{K}_{i+1}$.
Thus, we described a process that terminates after r steps. The process defines a finite iterated ultrapower of $\mathcal{K}$, ending with the model $\mathcal{K}_r$, where $\psi_r : \mathcal{K} \to \mathcal{K}_r$ is the iterated ultrapower embedding, $U_0, \dots, U_{r-1}$ are the measures, and $\alpha_0, \dots, \alpha_{r-1}$ are the critical points.

Lemma 6.2. *Let $\psi : \mathcal{K} \to M \cong Ult(\mathcal{K}, W_{\vec{h},\vec{\delta}})$ be the ultrapower embedding. Then $M = \mathcal{K}_r$, and $\psi = \psi_r$*

Proof. Denote $W := W_{\vec{h},\vec{\delta}}$. The proof is by induction on r.
If $r = 0$, there is nothing to prove, since in that case $M = \mathcal{K} = \mathcal{K}_r$.
Assume that $r > 0$. Define $\tau : M \to \mathcal{K}_r$ by $\tau([t]_W) := \psi_r(t)(\alpha_0, \dots, \alpha_{r-1})$. We want to show that τ is well-defined, i.e, it does not depend on the choice of representative from the equivalence class $[t]_W$.
Let $t_1, t_2 : S_r \to \mathcal{K}$, such that $[t_1]_W = [t_2]_W$. Then there is a set $X \in W$ such that:

$$\forall \vec{\xi} \in X \quad t_1(\vec{\xi}) = t_2(\vec{\xi})$$

Let $Y := Y(X)$. For every $\vec{\xi} \in S_{r-1}$, denote $\gamma(\vec{\xi}) := \bar{h}_{r-1}(\vec{\delta}, \vec{\xi})$ and $U(\vec{\xi}) := \mathcal{U}_{\gamma(\vec{\xi})}$. Then:

$$\forall \vec{\xi} \in Y \exists Z \in U(\vec{\xi}) \forall \mu \in Z \quad t_1(\vec{\xi}^\frown \mu) = t_2(\vec{\xi}^\frown \mu)$$

Let $t'_1 : S_{r-1} \to \mathcal{K}$ be such that $t'_1(\vec{\xi})(\mu) := t_1(\vec{\xi}^\frown \mu)$. Define t'_2 similarly. Then:

$$\forall \vec{\xi} \in Y \quad [t'_1(\vec{\xi})]_{U(\vec{\xi})} = [t'_2(\vec{\xi})]_{U(\vec{\xi})}$$

Define $t^*_1(\vec{\xi}) := [t'_1(\vec{\xi})]_{U(\vec{\xi})}$ and $t^*_2(\vec{\xi}) := [t'_2(\vec{\xi})]_{U(\vec{\xi})}$. Then for every $\vec{\xi} \in Y$, $t^*_1(\vec{\xi}) = t^*_2(\vec{\xi})$. Using the induction hypothesis, we have:

$$\begin{array}{ll} \psi_{r-1}(t^*_1)(\alpha_0, \dots, \alpha_{r-2}) = \psi_{r-1}(t^*_2)(\alpha_0, \dots, \alpha_{r-2}) & \Longrightarrow \\ [\psi_{r-1}(t'_1)(\alpha_0, \dots, \alpha_{r-2})]_{U_{r-1}} = [\psi_{r-1}(t'_2)(\alpha_0, \dots, \alpha_{r-2})]_{U_{r-1}} & \Longrightarrow \\ \psi_r(t'_1)(\alpha_0, \dots, \alpha_{r-2})(\alpha_{r-1}) = \psi_r(t'_2)(\alpha_0, \dots, \alpha_{r-2})(\alpha_{r-1}) & \Longrightarrow \\ \psi_r(t_1)(\alpha_0, \dots, \alpha_{r-1}) = \psi_r(t_2)(\alpha_0, \dots, \alpha_{r-1}) & \end{array}$$

This completes the proof that τ is well defined.
Clearly, $\mathrm{ran}(\tau) = \mathcal{K}_r$. But τ is an elementary embedding - the proof for atomic formulas is identical to the proof that τ is well-defined (just replace $=$ by $\in$), and the induction step for non-atomic formulas is trivial. Then τ must be the identity, and $M = \mathcal{K}_r$. In addition, for every $x \in \mathcal{K}$:

$$\psi_r(x) = \psi_r(\mathrm{Const}_x)(\alpha_0, \dots, \alpha_{r-1}) = \tau([\mathrm{Const}_x]_W) = \tau(\psi(x)) = \psi(x)$$

Where Const_x is the constant function with value x. Then $\psi_r = \psi$ and we are done. $\square$

7 Indiscernibles and Large Sets

Pick some $n < \omega$. We shall prove some results regarding f_n and its covering $h_n, \mathcal{C}_n, \rho_n$, in relation to the ultrafilters we defined in the previous section. Since in this section, we shall always deal with the same n, we shall remove the subscript n and simply have $f := f_n$, $h := h_n$, $\mathcal{C} := \mathcal{C}_n$ etc.

Lemma 7.1. *There exists a set $A \in G$, functions $g : \kappa \to [\lambda]^{<\omega}$ and $\vec{c} : \kappa \to [\mathcal{C}]^{<\omega}$ in V, a sequence of functions $\vec{h} = \langle h_0, \dots, h_{r-1}\rangle \in \mathcal{K}$ and a function $h' \in \mathcal{K}$, such that:*

(1) For every $\nu \in A$, $\vec{c}(\nu)$ is an increasing sequence of indiscernibles of length r, and we denote $\vec{c}(\nu) = \langle c_0(\nu), \dots, c_{r-1}(\nu)\rangle \subseteq \mathcal{C}$. For every $i < r$, we denote $\gamma_i(\nu) := h_i(g(\nu), \vec{c}(\nu) \cap c_i(\nu))$, and $c_i(\nu) \in \mathcal{C}_{\gamma_i(\nu)}$.

(2) For every $\nu \in A$, $f(\nu) = h'(g(\nu), \vec{c}(\nu))$.

(3) For every distinct $\nu_1, \nu_2 \in A$, $\vec{c}(\nu_1) \cap \vec{c}(\nu_2) = 0$.

(4) For every $i < r$, $\sup_{\nu\in A} c_i(\nu) = \min_{B\in G} \sup_{\nu\in B} c_i(\nu)$.

(5) For every $i < r$, the function $\nu \mapsto \alpha(c(\nu))$ is either constant or strictly increasing on A, and:

- *(a) If it is strictly increasing on A, then $\gamma_i(\nu) = \gamma(c_i(\nu))$.*
- *(b) If it is constant on A, with constant value α, let $c_* := \sup_{\nu\in A} c_i(\nu)$.*
  - *(i) If $c_* = \alpha$, then $\gamma_i(\nu) = \gamma(c_i(\nu))$.*
  - *(ii) If $c_* < \alpha$, then $c_* \in \mathcal{C}_{\gamma_*}$ for some γ_* such that $\alpha_{\gamma_*} = \alpha$. In this case, $\gamma_i(\nu) = \mathrm{Coh}_{\gamma_*, \gamma(c(\nu))}(c_*)$.*

  In particular, $\sup_{\nu\in A} c_i(\nu) = \alpha_{\gamma_i(\nu)}$

Proof. For every $\nu < \kappa$, let $\vec{d}(\nu)$ be some support for $f(\nu)$ - i.e, for every $d \in \vec{d}(\nu)$, $\gamma(d) \in h[\vec{d}(\nu) \cap d]$, and $f(\nu) \in h[\vec{d}(\nu)]$. For every $d \in \vec{d}(\nu)$, there is

some $\delta < \rho$ and some $\vec{d'} \subseteq \vec{d}(\nu) \cap d$ such that $\gamma(d) = h(\delta, \vec{d'})$. By corollary 5.2, $\rho \leq \lambda$, so $\delta < \lambda$. Similarly, there is some $\delta < \lambda$ and some $\vec{d'} \subseteq \vec{d}(\nu)$ such that $f(\nu) = h(\delta, \vec{d'})$. Let $g(\nu) \in [\kappa]^{<\omega}$ be the collection of all those δ's.
Since G is normal, there is some $A \in G$ such that $\{\vec{d}(\nu) \mid \nu \in A\}$ is a Δ-system with kernel $\vec{e}$. We may shrink A and assume that for every $\nu \in A$, $|\vec{d}(\nu)| = r'$ and $|g(\nu)| = k$.
Define $\vec{c}(\nu) := \vec{d}(\nu) \setminus \vec{e}$. Clearly, (3) holds.
Again, we can shrink A such that $|\vec{c}(\nu)| = r$ for every $\nu \in A$. Denote $\vec{c}(\nu) = \langle c_0(\nu), \ldots, c_{r-1}(\nu)\rangle$ and $g(\nu) = \langle g_0(\nu), \ldots, g_{k-1}(\nu)\rangle$.
Let $i < r$. We wish to define the function h_i. First, we define a function $h'_i \in \mathcal{K}$, such that $\gamma(c_i(\nu)) = h'_i(g(\nu), \vec{c}(\nu) \cap c_i(\nu))$ for every $\nu \in A$. We can assume, by shrinking A, that there is some $\vec{e'} \subseteq \vec{e}$, some $\ell_0 < \cdots < \ell_{p-1} < i$, and some $j < k$ such that for every $\nu \in A$, $\gamma(c_i(\nu)) = h(g_j(\nu), c_{\ell_0}(\nu), \ldots, c_{\ell_{p-1}}(\nu), \vec{e'})$.
Define:

$$h'_i(\langle\delta_0, \ldots, \delta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{i-1}\rangle) := h(\delta_j, \xi_{\ell_0}, \ldots, \xi_{\ell_{p-1}}, \vec{e'})$$

And indeed $\gamma(c_i(\nu)) = h'_i(g(\nu), \vec{c}(\nu) \cap c_i(\nu))$.
We now define the function h_i. We first shrink A such that both sequences $\langle c_i(\nu) \mid \nu \in A\rangle$ and $\langle \alpha(c_i(\nu)) \mid \nu \in A\rangle$ are either constant or strictly increasing, and such that (4) holds, i.e:

$$\sup_{\nu \in A} c_i(\nu) = \min_{B \in G} \sup_{\nu \in B} c_i(\nu)$$

Note that $\langle c_i(\nu) \mid \nu \in A\rangle$ must be strictly increasing, since $\vec{c}(\nu_1) \cap \vec{c}(\nu_2) = 0$ for every $\nu_1, \nu_2 \in A$. If $\langle \alpha(c_i(\nu)) \mid \nu \in A\rangle$ is strictly increasing, define $h_i := h'_i$. Otherwise, there is some α such that $\alpha(c_i(\nu)) = \alpha$ for every $\nu \in A$. Let $c_* := \sup_{\nu \in A} c_i(\nu)$. If $c_* = \alpha$, define $h_i := h'_i$ as before. Otherwise, $c_* \in \mathcal{C}_{\gamma_*}$ for some γ_* such that $\alpha = \alpha_{\gamma_*}$. In this case, we define:

$$h_i(x) := \mathrm{Coh}_{\gamma_*, h'_i(x)}(c_*)$$

Thus, we have defined h_i such that (5) holds.
Let $\gamma_i(\nu) := h_i(g(\nu), \vec{c}(\nu) \cap c_i(\nu))$. To prove (1), we need to show that $c_i(\nu) \in \mathcal{C}_{\gamma_i(\nu)}$. If $h_i = h'_i$, then $\gamma_i(\nu) = \gamma(c_i(\nu))$ and we are done. Otherwise, $\gamma_i(\nu) = \mathrm{Coh}_{\gamma_*, \gamma(c_i(\nu))}(c_*)$ - and by the properties of the coherence function, $c_i(\nu) \in \mathcal{C}_{\gamma_i(\nu)}$, and $c_* = \alpha_{\gamma_i(\nu)}$.
The function h' is defined similarly to h'_i, such that $f(\nu) = h'(g(\nu), \vec{c}(\nu))$, and (2) holds. □

We now prove a very useful property of the covering, that given a sequence of large sets, the appropriate indiscernibles are almost always contained in these sets.

Lemma 7.2. *Let $\vec{X} = \langle X_{\vec{\delta}} \mid \vec{\delta} \in [\lambda]^{<\omega}\rangle \in \mathcal{K}$ be a sequence of sets such that for every $\vec{\delta} \in [\lambda]^{<\omega}$, $X_{\vec{\delta}} \in W_{\vec{h},\vec{\delta}}$ (where $\vec{h}, A, \vec{c}, g$ are the objects from conclusion of Lemma 7.1). Then for almost all $\nu \in A$, $\vec{c}(\nu) \in X_{g(\nu)}$.*

Proof. The proof is by induction on r (the length of $\vec{c}(\nu)$).
If $r = 0$, then $\vec{c}(\nu) = 0$, $X_{g(\nu)} = \{0\}$ for every $\nu \in A$, and the result trivially holds.
Assume that $r > 0$. Let $c(\nu) := \max(\vec{c}(\nu))$ and $\vec{d}(\nu) := \vec{c}(\nu) \setminus \{c(\nu)\}$. By induction, we can assume that for every set $\vec{X}$ as in the lemma, $\vec{d}(\nu) \in Y(X_{g(\nu)})$ for almost all $\nu \in A$. Recall from the previous lemma that the sequence $\langle c(\nu) \mid \nu \in A\rangle$ is strictly increasing. Assume towards a contradiction that there is some $B \in G \cap \mathcal{P}(A)$ and some $\vec{X}$ as in the lemma such that for every $\nu \in B$, $\vec{c}(\nu) \notin X_{g(\nu)}$. By elementarity, we can assume that $\vec{X} \in h[\mathcal{C}]$. Let $\vec{e} \subseteq \mathcal{C}$ be finite such that $\vec{X} \in h[\vec{e}]$. We can assume w.l.o.g that the kernel of the Δ-system from lemma 7.1 (used in the definition of $\vec{h}$) is contained in $\vec{e}$.
For every $\nu \in B$, let $\gamma(\nu) := h_{r-1}(g(\nu), \vec{d}(\nu))$, $\alpha(\nu) := \alpha_{\gamma(\nu)}$, and $U(\nu) := \mathcal{U}_{\gamma(\nu)}$. By definition of the ultrafilter $W_{\vec{h},g(\nu_n)}$:

$$Z(\nu) := p(X_{g(\nu)}, \vec{d}(\nu)) \cap \alpha(\nu) \in U(\nu)$$

Note that $Z(\nu) \in h[\vec{e} \cup \vec{d}(\nu)]$, $Z(\nu) \in U(\nu)$, but $c(\nu) \notin Z(\nu)$. This implies that there is some $e(\nu) \in (\vec{e} \cup \vec{d}(\nu)) \cap [c(\nu), \alpha(\nu))$. If there is some $\nu \in B$ for which $e(\nu) \in d(\nu)$, then $c(\nu) > \max(d(\nu)) \geq e(\nu)$, contradictory to the choice of $e(\nu)$. Then for every $\nu \in B$, $e(\nu) \in \vec{e}$. Since $\vec{e}$ is finite, we can shrink B such that there is some $e \in \vec{e}$ such that $e(\nu) = e$ for every $\nu \in B$.
By lemma 7.1, the function $\alpha(\nu)$ is either strictly increasing or constant on A (and on B).
Case 1: $\alpha(\nu)$ is strictly increasing on B
Recall that in this case, $\alpha(\nu) = \alpha(c(\nu)) = \min(h[c(\nu)] \setminus c(\nu))$. Let $\nu_1 < \nu_2$ be in B. Then $c(\nu_1) < c(\nu_2) < e < \alpha(\nu_1) < \alpha(\nu_2)$. This is a contradiction, since:

$$\alpha(\nu_1) \geq \min(h[c(\nu_2)] \setminus c(\nu_2)) = \alpha(\nu_2)$$

Case 2: $\alpha(\nu)$ is constant on B
Let α be that constant value. By lemma 7.1, $\sup_{\nu \in B} c(\nu) = \alpha$. But:

$$\sup\{c(\nu) \mid \nu \in B\} \leq e < \alpha$$

Contradiction.

□

8 Embeddings

Denote $\psi_{\vec{h},\vec{\delta}} : \mathcal{K} \to \mathcal{K}_{\vec{h},\vec{\delta}} = \mathrm{Ult}(\mathcal{K}, W_{\vec{h},\vec{\delta}})$ to be the ultrapower embedding.
Let $\varphi : \mathcal{K} \to \mathcal{K}'$ be the complete iteration of $\mathcal{K}$, using all the measures and extenders (and their images) in $\mathcal{K}$ with critical points below κ^{++}. By lemma 6.2, $\mathcal{K}_{\vec{h},\vec{\delta}}$ is just a finite iteration of $\mathcal{K}$ by its measures, so there is some elementary embedding $\varphi_{\vec{h},\vec{\delta}} : \mathcal{K}_{\vec{h},\vec{\delta}} \to \mathcal{K}'$ such that $\varphi = \varphi_{\vec{h},\vec{\delta}} \circ \psi_{\vec{h},\vec{\delta}}$.
For every $n < \omega$, we consider the covering $h_n, \mathcal{C}_n, \rho_n$ of $\mathrm{ran}(f_n)$. Let A_n, $\vec{h}_n$, g_n, $\vec{c}_n$ be defined for this covering as in lemma 7.1. We write $\vec{h}_n = \langle h_{n,0}, \ldots, h_{n,r_n-1} \rangle$

Lemma 8.1. *We can assume that for every $n < m$ and for almost all ν, $\vec{c}_n(\nu) \subseteq \vec{c}_m(\nu)$.*

Proof. Let $m < \omega$. Recall the proof of lemma 7.1 (by which $\vec{c}_m(\nu)$ was defined) - we started from some support $\vec{d}_m(\nu)$ for $f_m(\nu)$, and then applied a Δ-system argument. Since we can choose any support we want, and since $\mathcal{C}_n \subseteq \mathcal{C}_m$ for every $n < m$ (by lemma 5.2), we can choose $\vec{d}_m(\nu)$ to be some support for $f_m(\nu)$ such that $\bigcup_{n<m} \vec{c}_n(\nu) \subseteq \vec{d}_m(\nu)$. We can now apply the Δ-system argument from lemma 7.1 to define $c_m(\nu)$. Note that still $c_n(\nu) \subseteq c_m(\nu)$, since by induction, the elements of $c_n(\nu)$ are not part of the Δ-system. $\square$

The next lemma proves a crucial property, related to the comparison of two different coverings.

Lemma 8.2. *Fix some $n < m < \omega$. Let $A \in G \cap \mathcal{P}(A_n \cap A_m)$ and $I \subseteq \omega$ be finite such that for all $\nu \in A$, $\vec{c}_m(\nu) \restriction I = \vec{c}_n(\nu)$. Let $\nu \in A$. Denote $W_i := W_{\vec{h}_i, g_i(\nu)}$, $\mathcal{K}_i := \mathrm{Ult}(\mathcal{K}, W_i)$, $\psi_i := \psi_{\vec{h}_i, g_i(\nu)}$, and $\varphi_i := \varphi_{\vec{h}_i, g_i(\nu)}$ for $i \in \{n, m\}$. Then we can assume (by shrinking A if necessary), that there is an elementary embedding Γ such that the following diagram commutes:*

ψ_n ψ_m φ_n

$\mathcal{K}$

Proof. To prove the lemma, it is enough to find some iterated ultrapower embedding (i.u.e) $\Gamma : \mathcal{K}_n \to \mathcal{K}_m$. Then the commutativity of the diagram above follows from uniqueness of iterations (the Dodd-Jensen Lemma, see Corollary 4.3.11 of [9]): for example, both φ_n and $\varphi_m \circ \Gamma$ are i.u.e's of $\mathcal{K}_n$ to $\mathcal{K}'$, so $\varphi_n = \varphi_m \circ \Gamma$.
We shall prove the existence of such an i.u.e by induction on $r_n := |\vec{c}_n(\nu)|$ and $r_m := |\vec{c}_m(\nu)|$. If $r_m = 0$ or $r_n = 0$, then there is nothing to prove. Therefore, we can assume that $0 < r_n \leq r_m$. For $i \in \{n, m\}$, denote $\vec{c}_i^*(\nu) := \vec{c}_i(\nu) \restriction (r_i - 1)$, and let $\mathcal{K}_i^*$, ψ_i^* be the corresponding model and embedding. Let $\vec{\alpha}_i^*$ be the critical points of the iteration of $\mathcal{K}$ to $\mathcal{K}_i^*$, and W_i^* be the corresponding ultrafilter. Let $j_i : \mathcal{K}_i^* \to \mathcal{K}_i$ be the ultrapower embedding, using the ultrafilter U_i with critical point α_i. Let γ_i be such that $U_i = (\mathcal{U}_{\gamma_i})^{\mathcal{K}_i^*}$.
We also denote $c_i(\nu) := \max(\vec{c}_i(\nu))$ and $t_i(x) := h_{i,r_i-1}(g_i(\nu), x)$.
If $r_m - 1 \notin I$, then we can apply the induction hypothesis to $\vec{c}_n$ and $\vec{c}_m^*$, and get some i.u.e $\Gamma^* : \mathcal{K}_n \to \mathcal{K}_m^*$. Then we can simply define $\Gamma := j_m \circ \Gamma^*$.
We can therefore assume that $r_m - 1 \in I$, i.e, $c_n(\nu) = c_m(\nu)$.

Claim: $t_n(c_n^*(\nu)) = t_m(c_m^*(\nu))$

Proof: For $i \in \{n, m\}$, set $\gamma_i(\nu) := t_i(c_i^*(\nu))$. Denote $\alpha(\nu) := \alpha^{\mathcal{C}_n}(c_n(\nu)) = \alpha^{\mathcal{C}_m}(c_m(\nu))$. By lemma 7.1, there are 2 cases. If $\alpha(\nu)$ is strictly increasing on A, then $\gamma_n(\nu) = \gamma^{\mathcal{C}_n}(c_n(\nu)) = \gamma^{\mathcal{C}_m}(c_m(\nu)) = \gamma_m(\nu)$. Otherwise, let α be the constant value of $\alpha(\nu)$, and set $c_* := \sup_{\nu \in A} c_n(\nu) = \sup_{\nu \in A} c_m(\nu)$. If $c_* = \alpha$ then it is the same case. Otherwise, let γ_* be such that $c_* \in \mathcal{C}_{\gamma_*}$ and $\alpha = \alpha_{\gamma_*}$. Then:

$$\gamma_n(\nu) = \mathrm{Coh}_{\gamma_*, \gamma(c_n(\nu))}(c_*) = \mathrm{Coh}_{\gamma_*, \gamma(c_m(\nu))}(c_*) = \gamma_m(\nu)$$

In any case, $\gamma_n(\nu) = \gamma_m(\nu)$. $\square$***Claim***

Let $I^* := I \setminus \{r_m - 1\}$. We apply the induction hypothesis to $\vec{c}_n^*$ and $\vec{c}_m^*$, and get an i.u.e $\Gamma^* : \mathcal{K}_n^* \to \mathcal{K}_m^*$. By the claim, we have:

$$t_m(\vec{c}_m^*(\nu)) = t_n(\vec{c}_n^*(\nu)) = t_n(\vec{c}_m^*(\nu) \restriction I^*)$$

Define $X := \{\vec{\xi} \in S_{r_m - 1} \mid t_m(\vec{\xi}) = t_n(\vec{\xi} \restriction I^*)\}$. By lemma 7.2, we can shrink A and have that $\vec{c}_m^*(\nu) \in X \in W_m^*$. Then:

$$\psi_m^*(t_m)(\vec{\alpha}_m^*) = \psi_m^*(t_n)(\vec{\alpha}_m^* \restriction I^*) = \Gamma^*(\psi_n^*(t_n)(\alpha_n^*))$$

The equality above can be written as $\gamma_m = \Gamma^*(\gamma_n)$, which implies that $U_m = \Gamma^*(U_n)$ and $\alpha_m = \Gamma^*(\alpha_n)$.
We can now define the embedding $\Gamma : \mathcal{K}_n \to \mathcal{K}_m$:

$$\Gamma([t]_{U_n}) := [\Gamma^*(t)]_{U_m}$$

First we show that Γ is well defined. Assume that $[t]_{U_n} = [u]_{U_n}$. Then:

$$\mathcal{K}_n^* \models \{\xi < \alpha_n \mid t(\xi) = u(\xi)\} \in U_n$$

Applying Γ^* to the above gives us:

$$\mathcal{K}_m^* \models \{\xi < \alpha_m \mid \Gamma^*(t)(\xi) = \Gamma^*(u)(\xi)\} \in U_m$$

Which implies that $[\Gamma^*(t)]_{U_m} = [\Gamma^*(u)]_{U_m}$, and Γ is well defined. The proof that Γ is an elementary embedding is similar (just replace $=$ by $\in$). It remains to show that Γ is an i.u.e. By definition, $\Gamma \circ j_n = j_m \circ \Gamma^*$. Then, by the minimality of an i.u.e, and since $j_m \circ \Gamma^*$ is an i.u.e, also $\Gamma \circ j_n$ is an i.u.e. In particular, Γ is an i.u.e. □

9 The Main Proof

In this section, we shall prove theorem 2.1. We wish to prove that there are some $n < m$ such that $[f_n] \leq [f_m]$.

- Let X_n, h_n, $\mathcal{C}_n$ be a covering of $\text{ran}(f_n)$ as defined after corollary 5.2.
- Let $A_n, \vec{h}_n, h'_n, g_n, \vec{c}_n, r_n$ be as in lemma 7.1. All the properties from lemmas 7.1, 7.2, 8.1 and 8.2 hold.
- We define a function $h_{n,\vec{\delta}} : S_{r_n} \to \kappa^{++}$ as $h_{n,\vec{\delta}}(\vec{\xi}) := h'_n(\vec{\delta}, \vec{\xi})$. Then $h_{n,g_n(\nu)}(\vec{c}_n(\nu)) = f_n(\nu)$ for all $\nu \in A_n$.

Let $n < m < \omega$. By lemma 8.1, there is some $A_{n,m} \in G$ such that for every $\nu \in A_{n,m}$, $\vec{c}_n(\nu) \subseteq \vec{c}_m(\nu)$. We can assume that $A_{n,m} \subseteq A_n \cap A_m$.
By shrinking $A_{n,m}$ further if necessary, we can assume there is some $I(n,m) \subseteq r_m$ such that for every $\nu \in A_{n,m}$, $\vec{c}_m(\nu) \restriction I(n,m) = \vec{c}_n(\nu)$.
Define a function $h_{n,m,\vec{\delta}} : S_{r_m} \to \kappa^{++}$ as:

$$h_{n,m,\vec{\delta}}(\vec{\xi}) := h_{n,\vec{\delta}}(\vec{\xi} \restriction I(n,m))$$

Then, for every $\nu \in A_{n,m}$:

$$f_n(\nu) = h_{n,g_n(\nu)}(\vec{c}_n(\nu)) = h_{n,g_n(\nu)}(\vec{c}_m(\nu) \restriction I(n,m)) = h_{n,m,g_n(\nu)}(\vec{c}_m(\nu))$$

We are now ready to prove the main result of this paper.

Proof of theorem 2.1. We need to show that there is some $n < m$ such that $[f_n] \leq [f_m]$.
To simplify notations, denote for every $n < \omega$ and $\vec{\delta} \in [\lambda]^{<\omega}$:

$$\begin{array}{lll} W^n_{\vec{\delta}} := W_{\vec{h}_n,\vec{\delta}} & [h_{n,\vec{\delta}}] := [h_{n,\vec{\delta}}]_{W^n_{\vec{\delta}}} & \\ \mathcal{K}^n_{\vec{\delta}} := \mathcal{K}_{\vec{h}_n,\vec{\delta}} & \psi^n_{\vec{\delta}} := \psi_{\vec{h}_n,\vec{\delta}} & \varphi^n_{\vec{\delta}} := \varphi_{\vec{h}_n,\vec{\delta}} \end{array}$$

Recall that $\varphi = \varphi^n_{\vec{\delta}} \circ \psi^n_{\vec{\delta}}$, where $\varphi : \mathcal{K} \to \mathcal{K}'$ is the complete iteration of $\mathcal{K}$ using all the measures and extenders in $\mathcal{K}$ with critical points below κ^{++} $\mathcal{K}^n_{\vec{\delta}_n}$
All these embeddings are definable in $\mathcal{K}$.
For every $n < \omega$, and $\nu < \kappa$, let:

$$f'_n(\nu) := \varphi^n_{g_n(\nu)}([h_{n,g_n(\nu)}])$$

By lemma 3.3, there must be some $n < m < \omega$ such that $[f'_n] \leq [f'_m]$. We fix such $n < m$ for the remainder of the proof. We will eventually $\psi^n_{\vec{\delta}_n}$ show that also $[f_n] \leq [f_m]$. $\psi^m_{\vec{\delta}_m}$
We work in $\mathcal{K}$ and construct sets $\langle X_{\vec{\delta}_n,\vec{\delta}_m} \mid \vec{\delta}_n, \vec{\delta}_m \in [\lambda]^{<\omega} \rangle$ as follows.
Let $\vec{\delta}_n, \vec{\delta}_m \in [\lambda]^{<\omega}$. Consider the following assumptions: $\mathcal{K}^m_{\vec{\delta}_m}$

(A1) $\varphi^n_{\vec{\delta}_n}([h_{n,\vec{\delta}_n}]) \leq \varphi^m_{\vec{\delta}_m}([h_{m,\vec{\delta}_m}])$ $\varphi^n_{\vec{\delta}_n}$

(A2) There is an elementary embedding $\Gamma : \mathcal{K}^n_{\vec{\delta}_n} \to \mathcal{K}^m_{\vec{\delta}_m}$ such that the following diagram commutes:

$\mathcal{K}$

and the fourth follows from the third by assumption (A3). Then there is some set $X_{\vec{\delta}_n,\vec{\delta}_m} \in W^m_{\vec{\delta}_m}$ such that:

$$(*) \quad \forall \vec{\xi} \in X_{\vec{\delta}_n,\vec{\delta}_m} \quad h_{n,m,\vec{\delta}_n}(\vec{\xi}) \leq h_{m,\vec{\delta}_m}(\vec{\xi})$$

By lemma 6.1, $W^m_{\vec{\delta}_m}$ is complete enough so we can define:

$$X_{\vec{\delta}_m} := \bigcap_{\vec{\delta}_n \in [\lambda]^{<\omega}} X_{\vec{\delta}_n,\vec{\delta}_m} \in W^m_{\vec{\delta}_m}$$

This completes the construction. All the construction was performed in $\mathcal{K}$, so $\langle X_{\vec{\delta}} \mid \vec{\delta} \in [\lambda]^{<\omega} \rangle \in \mathcal{K}$.
Now, let $A \in G$ be such that $A \subseteq A_{n,m}$, and for every $\nu \in A$:

(1) $f'_n(\nu) \leq f'_m(\nu)$
(2) $\vec{c}_n(\nu) \subseteq \vec{c}_m(\nu)$
(3) $\vec{c}_m(\nu) \in X_{g_m(\nu)}$

(1) holds for almost all ν since $[f'_n] \leq [f'_m]$, (2) holds since $A \subseteq A_{n,m}$, and (3) holds for almost all ν by lemma 7.2. Let $\nu \in A$. It is enough to show that $f_n(\nu) \leq f_m(\nu)$.
Denote $\vec{\delta}_n := g_n(\nu)$ and $\vec{\delta}_m := g_m(\nu)$. Note that (1) implies assumption (A1), and lemma 8.2 implies assumptions (A2) and (A3). Then $(*)$ holds, and for every $\vec{\xi} \in X_{\vec{\delta}_n,\vec{\delta}_m}$:

$$h_{n,m,\vec{\delta}_n}(\vec{\xi}) \leq h_{m,\vec{\delta}_m}(\vec{\xi})$$

In particular, since $\vec{c}_m(\nu) \in X_{\vec{\delta}_m} \subseteq X_{\vec{\delta}_n,\vec{\delta}_m}$:

$$f_n(\nu) = h_{n,m,\vec{\delta}_n}(\vec{c}_m(\nu)) \leq h_{m,\vec{\delta}_m}(\vec{c}_m(\nu)) = f_m(\nu)$$

And we are done. □

This completes the proof of theorem 2.1.

10 Concluding remarks

The situation for cardinals larger than $\aleph_1$ is less clear. For successor cardinals, we can generalize theorem 2.1 if we allow some additional assumptions. In [3], 1.3 generalizes to arbitrary successors cardinals as follows:

Theorem (2.7 of [3]). *Let κ be a successor cardinal, and let κ^- be the predecessor of κ. Assume that:*

(1) F is a precipitous filter over κ, and $\{\nu < \kappa \mid \mathrm{cf}(\nu) = \kappa^-\} \in F_{normal}$ (where F_{normal} is the projection of F to a normal filter).

(2) $2^\kappa = \kappa^+$ and $(\kappa^-)^{<\kappa^-} = \kappa^-$.

(3) $\kappa \Vdash_{F^+} i(\kappa) > \kappa^+$ (where i is defined as in theorem 3.1)

Then for every $\tau < \kappa^{++}$, there exists a normal filter that is precipitous up to the image of τ.

Now, let κ be some successor cardinal for which the assumptions in the theorem above holds. We can repeat the proof in this paper for that successor cardinal κ, using the theorem above instead of theorem 3.1. Note that in this paper, we never actually used the assumption that $\kappa = \aleph_1$ (only that κ is uncountable), so the proof still applies, and we have the following generalization of theorem 2.1:

Theorem 10.1. *Let κ be a successor cardinal, and let κ^- be the predecessor of κ. Assume that:*

(1) F is a precipitous filter over κ, and $\{\nu < \kappa \mid \mathrm{cf}(\nu) = \kappa^-\} \in F_{normal}$.

(2) $\kappa^{\aleph_0} = \kappa$, $2^\kappa = \kappa^+$, and $(\kappa^-)^{<\kappa^-} = \kappa^-$.

(3) There is no inner model with a strong cardinal.

(4) In the core model, the set $\{\alpha < \kappa^{++} \mid o(\alpha) \geq \alpha^+\}$ is bounded in κ^{++}.

Then there exists a normal precipitous filter.

It is possible to replace the existence of a precipitous filter in the assumptions of this theorem by ∞-semi-precipitous.
Probably a more exciting task is to try to push the consistency strength of "no normal precipitous filter on $\aleph_1$" further up. It seems likely that the assumption $\{\alpha < \aleph_3 \mid o(\alpha) \geq \alpha^+\}$ is bounded in $\aleph_3$ can be weaken to $\{\alpha < \aleph_3 \mid \alpha \text{ is a weak repeat point }\}$ is bounded in $\aleph_3$. But going further will require to deal with $\beta^M(c)$'s which may depend on covering models M more and more. Probably at certain point this may give a hint for a forcing construction of a model with a precipitous filter but without a normal one.
An other direction is to try to remove GCH type assumptions that we made here.

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