Reflection and not SCH with overlapping extenders.

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Abstract

We use the forcing with overlapping extenders [4] to give a direct construction of a model of $\neg$SCH+Reflection.

1 Introduction.

In 2005 Assaf Sharon [9] constructed a model with a singular strong limit cardinal κ of cofinality ω such that $2^\kappa > \kappa^+$ and every stationary subset of κ^+ reflects. He used infinitely many supercompact cardinals for this.

Recently, A. Poveda, A. Rinot, D. Sinapova [8] and O. Ben Neria, Y. Hayut, S. Unger [2] addressed this problem again. In [8] a general schema of iteration is given. The paper [2] uses the iterated ultrapowers approach of Y. Hayut and S. Unger [7] and the overlapping extenders forcing of [4]. It extends Sharon's result to uncountable cofinality (using supercompacts) and for countable cofinality replaces supercompacts by much weaker assumptions.

The purpose of the present note is to give a strait proof of Sharon's result using the forcing with overlapping extenders but without appeal to iterated ultrapowers (still using supercompacts).

2 A model in which SCH fail at a singular cardinal and Reflection holds at its successor.

Recall that Sharon used long extenders forcing and as a result a rather complicated iteration was needed in order to destroy non-reflecting stationary sets that appear there. By using

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the forcing with overlapping extenders instead, there is no need for further iteration. This was pointed out by O. Ben Neria, Y. Hayut and S. Unger [2], however their argument was based on a delicate analyzes of iterated ultrapowers. We will use here the forcing of [4] and ideas from A. Sharon [9] instead.

Fix a regular cardinal η. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals and let $\langle E_\alpha \mid \alpha < \eta \rangle$ be a sequence of extenders such that for every $\alpha < \eta$

1. $\eta < \kappa_0$,
2. $E(\alpha)$ is a $(\kappa_\alpha, \bar{\kappa}_\eta^{++})-$extender, where $\bar{\kappa}_\eta = \bigcup_{\alpha<\eta} \kappa_\alpha$,
3. $E(\alpha) \lhd E(\alpha + 1)$,
4. there is a supercompact cardinal between $\sup_{\beta<\alpha} \kappa_\beta$ and κ_α.

Let $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq, \leq^* \rangle$ be the forcing of Section 2 of [4].
For every limit $\alpha \leq \eta$ denote $\bar{\kappa}_\alpha = \bigcup_{\alpha'<\alpha} \kappa_{\alpha'}$.
By [4], Section 2, it has the following properties:

1. $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq, \leq^* \rangle$ is a Prikry type forcing,
2. the forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$:
   (a) blows up the power of $\bar{\kappa}_\eta$ to $\bar{\kappa}_\eta^{++}$,
   (b) blows up the power of $\bar{\kappa}_\alpha$ above $\bar{\kappa}_\alpha^{+}$, for every limit $\alpha < \eta$,
   (c) preserves cardinals and cofinalities,
   (d) preserves strong limitness of each of κ_α's, for every $\alpha \leq \eta$, and $\bar{\kappa}_\alpha$'s, for every limit $\alpha \leq \eta$
   (e) does not add new subsets to κ_0.
3. For every $p \in \mathcal{P}$ and every $\mathcal{P}-$name $\underset{\sim}{\zeta}$ of an ordinal, there is $p^* \geq^* p$ such that the number of possible decisions of $\underset{\sim}{\zeta}$ above p^* is at most $\bar{\kappa}_\eta$.
   I.e. $|\{\xi \mid \exists q \geq p^*(q \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{\zeta} = \xi)\}| \leq \bar{\kappa}_\eta$. [1]

[1]This condition basically says that one entree given dense open set by taking a direct extension and then specifying finitely many coordinates. Usually, this property has the same proof, as the Prikry condition and is used to show that $\bar{\kappa}_\eta^{+}$ is preserved in $V^{\langle \mathcal{P}, \leq \rangle}$.

4. The forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq^* \rangle$ is equivalent to the product of Cohen forcings $Cohen(\kappa_\alpha^+, \bar{\kappa}_\eta^{++})$.[2]

   Namely, we just remove or ignore sets of measure one A^p_α in each coordinate $p(\alpha) = \langle f^p_\alpha, A^p_\alpha \rangle$ of a condition $p = \langle p(\alpha) \mid \alpha < \eta \rangle \in \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$. More precisely, if $p = \langle p(\alpha) \mid \alpha < \eta \rangle$ and $q = \langle q(\alpha) \mid \alpha < \eta \rangle$ are in $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$, then set $p \sim q$ iff for every $\alpha < \eta$

   (a) $p(\alpha)$ is non-pure iff $q(\alpha)$ is non-pure. Require then that $p(\alpha) = q(\alpha)$.

   (b) If $p(\alpha) = \langle f^p_\alpha, A^p_\alpha \rangle$, i.e. is pure, then $q(\alpha) = \langle g^p_\alpha, B^p_\alpha \rangle$ is pure as well, and require that $f^p_\alpha = g^p_\alpha$.

   Then $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}/\sim, \leq^* \rangle$ is the product of Cohen forcings.

Let us assume (or make) that all relevant supercompact cardinals were made indestructible under directed closed forcings using the Laver forcing.
Then force with $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$. Denote further $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$ by $\mathcal{P}$. We claim that the resulting generic extension is as desired, i.e. it satisfies $2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$ and every stationary subset of $\bar{\kappa}_\eta^+$ reflects.
$2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$ follows by (2(a)) above. Let deal with the reflection. Denote $\bar{\kappa}_\eta$ by λ.

Theorem 2.1 *In $V^{\langle \mathcal{P}, \leq \rangle}$, every stationary subset of $\bar{\kappa}_\eta^+$ reflects.*

Proof. Assume for simplicity that $\eta = \omega$. The argument follows closely Section III of [9], only the long extenders forcing is replaced by $\mathcal{P}$.

Let $\underset{\sim}{S}$ be a canonical name of a stationary subset of $\bar{\kappa}_\omega^+$, i.e.,

$$\underset{\sim}{S} = \{\langle \alpha, p \rangle \mid p \in \mathcal{P} \text{ and } p \Vdash_{\mathcal{P}} \alpha \in \underset{\sim}{S}\}.$$

Suppose for simplicity that $\underset{\sim}{S}$ concentrates on a fixed cofinality below the least supercompact.

For every $n < \omega$, set

$$\underset{\sim}{S}_n = \{\langle \alpha, p \rangle \mid \ell(p) = n\}, \mathcal{P}_n = \{p \in \mathcal{P} \mid \ell(p) = n\} \text{ and } \leq_n^* = \leq \restriction \mathcal{P}_n.$$

The following was proved in [9] (Claim 1.1.1):

[2] This is the crucial difference from the long extenders Prikry forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$ of Sec. 2 of [3]. The conditions in $\mathcal{P}$ consist basically of two parts one of cardinality $< \kappa_n, (n < \omega)$ (assignment functions) and another of cardinality κ_ω (Cohen functions). As a result, $\langle \mathcal{P}, \leq^* \rangle$ collapses κ_ω^+ and this allowed Asaf Sharon [9] to build a non-reflecting stationary set.
In the present setting both parts are put into one of cardinality κ_n.

Lemma 2.2 *Suppose that for some $m < \omega$, we have $p \in \mathcal{P}_m$ such that $p \Vdash_{\mathcal{P}_m} (\underset{\sim}{S}_m$ is stationary $)$. Then there is $q \geq^* p$, $q \Vdash_{\mathcal{P}} (\underset{\sim}{S}$ reflects $)$.*

So, it is enough to show that for every $p \in \mathcal{P}$ there is $q \geq p$ such that

$$q \Vdash_{\mathcal{P}_{\ell(q)}} \underset{\sim}{S}_{\ell(q)} \text{ is stationary.}$$

Suppose otherwise. Then, as in [9], there are $p \in \mathcal{P}$ and $\mathcal{P}_n-$names $\underset{\sim}{C}_n, n < \omega$ such that for every $q \geq p$,

$$q \Vdash_{\mathcal{P}_{\ell(q)}} \underset{\sim}{C}_{\ell(q)} \text{ is a club in } \bar{\kappa}^+_\omega \text{ and } \underset{\sim}{C}_{\ell(q)} \cap \underset{\sim}{S}_{\ell(q)} = \emptyset.$$

Suppose for simplicity that $p = 0_{\mathcal{P}}$.
Fix $n < \omega$. Consider the forcing $\mathcal{P}_n$ and $\underset{\sim}{C}_n$.
Here $\mathcal{P}_n$ is just a full support product of Cohen forcings $\langle Q_k \mid k < \omega \rangle$, where for every $k < n$, Q_k is a Cohen forcing which adds less than κ_{k+1} new subsets to a cardinal $< \kappa_k$, and so its cardinality $< \kappa_k$.
Q_n is a Cohen forcing which adds $\bar{\kappa}^{++}_\omega-$many subsets to a cardinal $< \kappa_n$, and, for every $k, n < k < \omega$, Q_k is a Cohen forcing which adds $\bar{\kappa}^{++}_\omega-$many subsets to κ^+_k.
In particular, for every $k < \omega$, Q_k satisfies κ^{++}_k-c.c. and $\kappa^{++}_k < \bar{\kappa}^+_\omega$.
Now, using the chain condition, for every $m < \omega$, we can find a $\prod_{m<k<\omega} Q_k-$name $\underset{\sim}{C}^m_n$ which is forced to be a club subset of $\underset{\sim}{C}_n$.
It is possible to make $\underset{\sim}{C}^m_n$'s decreasing.

If $\vec{f} \in \prod_{k<\omega} Q_k$ and $m' \leq m < \omega$, then let us view $\vec{f} \restriction [m, \omega) \in \prod_{m \leq k<\omega} Q_k$ also as a condition in $\prod_{m' \leq k<\omega} Q_k$, just put the empty function at each coordinate in the interval $[m', m)$. Clearly, then $\vec{f} \restriction [m', \omega)$ will be a stronger condition than $\vec{f} \restriction [m, \omega)$ in the forcing $\prod_{m' \leq k<\omega} Q_k$.
So, if, for some α, $\vec{f} \restriction [m, \omega) \Vdash_{\prod_{m \leq k<\omega} Q_k} \alpha \in \underset{\sim}{C}^m_n$, then $\vec{f} \restriction [m, \omega) \Vdash_{\prod_{r \leq k<\omega} Q_k} \alpha \in \underset{\sim}{C}^r_n$, for every $r \leq m$, since $\underset{\sim}{C}^i_n$'s are decreasing.
Hence, if for every large enough $m < \omega$, $\vec{f} \restriction [m, \omega) \Vdash_{\prod_{m \leq k<\omega} Q_k} \alpha \in \underset{\sim}{C}^m_n$, then for every $m < \omega$, $\vec{f} \restriction [m, \omega) \Vdash_{\prod_{m \leq k<\omega} Q_k} \alpha \in \underset{\sim}{C}^m_n$.

Now we use an idea from [2] and consider the forcing $\prod_{k<\omega} Q_k/finite$.

Lemma 2.3 *There is a $\prod_{k<\omega} Q_k/finite-$name $\underset{\sim}{C}^\omega_n$ of a club in $\bar{\kappa}^+_\omega$ such that for every $\vec{f} \in \prod_{k<\omega} Q_k$, if $\vec{f}/finite \Vdash_{\prod_{k<\omega} Q_k/finite} \alpha \in \underset{\sim}{C}^\omega_n$, then for every $m < \omega$, $\vec{f} \restriction [m, \omega) \Vdash_{\prod_{m \leq k<\omega} Q_k} \alpha \in \underset{\sim}{C}^m_n$.*

Proof. Let H be a generic subset of $\prod_{k<\omega} Q_k/finite$.

Work in $V[H]$ and define

$$C_n^\omega = \{\alpha < \bar{\kappa}_\omega^+ \mid \exists \vec{f} \in H \forall m < \omega (\vec{f} \upharpoonright [m,\omega) \Vdash_{\prod_{m \le k < \omega} Q_k} \alpha \in \underset{\sim}{C}_n^m)\}.$$

Claim 1 C_n^ω is unbounded in $\bar{\kappa}_\omega^+$.

Proof. Work in V and then use the density argument. Let $\rho < \bar{\kappa}_\omega^+$. Find $\alpha_0, \rho < \alpha_0 < \bar{\kappa}_\omega^+$ and $f_0 \in \prod_{k<\omega} Q_k$ such that

$$f_0 \Vdash_{\prod_{k<\omega} Q_k} \alpha_0 \in \underset{\sim}{C}_n^0.$$

Next, we find $\alpha_1, \alpha_0 < \alpha_1 < \bar{\kappa}_\omega^+$ and $f_1 \in \prod_{k<\omega} Q_k, f_1 \ge f_0$ such that

$$f_1 \upharpoonright [1,\omega) \Vdash_{\prod_{1 \le k < \omega} Q_k} \alpha_1 \in \underset{\sim}{C}_n^1.$$

Note that since this clubs are decreasing, we will have

$$f_1 \upharpoonright [1,\omega) \Vdash_{\prod_{k<\omega} Q_k} \alpha_1 \in \underset{\sim}{C}_n^0.$$

Continue in the similar fashion by induction and define two increasing sequences $\langle \alpha_m \mid m < \omega \rangle$ and $\langle f_m \mid m < \omega \rangle$ such that

$$f_m \upharpoonright [m,\omega) \Vdash_{\prod_{m \le k < \omega} Q_k} \alpha_m \in \underset{\sim}{C}_n^m,$$

for every $m < \omega$. Also, since the clubs are decreasing, we will have

$$f_m \upharpoonright [m,\omega) \Vdash_{\prod_{r \le k < \omega} Q_k} \alpha_m \in \underset{\sim}{C}_n^r,$$

for every $r \le m < \omega$.
Finally, let $\alpha = \bigcup_{m<\omega} \alpha_m$ and $f = \bigcup_{m<\omega} f_m$.
Then, for every $m < \omega$, $f \upharpoonright [m,\omega) \Vdash_{\prod_{m \le k < \omega} Q_k} \alpha \in \underset{\sim}{C}_n^m$.
So, $f/finite \Vdash_{\prod_{k<\omega} Q_k/finite} \alpha \in \underset{\sim}{C}_n^\omega$, and we are done.
$\square$ of the claim.

Claim 2 C_n^ω is a closed subset of $\bar{\kappa}_\omega^+$.

Proof. Note that $\bar{\kappa}_\omega^+$ is a successor of singular cardinal, so we need to deal only with sequences of a length below $\bar{\kappa}_\omega$.
Work in V and then use the density argument.
So let $\zeta < \bar{\kappa}_\omega$ and $\langle \underset{\sim}{\alpha}_\xi \mid \xi < \zeta \rangle$ be an increasing sequence of elements of $\underset{\sim}{C}_n^\omega$.
Pick $m_0 < \omega$ to be large enough such that $\kappa_{m_0} > \zeta$.

Similar to the previous claim, we define an increasing sequence $\langle f_\xi \mid \xi < \zeta\rangle$ of conditions in $\prod_{m_0<k<\omega} Q_k$ which decide $\underset{\sim}{\alpha}_\xi$'s. Set $\alpha_\zeta = \bigcup_{\xi<\zeta}\alpha_\xi$ and $f_\zeta = \bigcup_{\xi<\zeta} f_\xi$.
Then

$$f_\zeta/finite \Vdash_{\prod_{k<\omega} Q_k/finite} \alpha_\zeta \in \underset{\sim}{C}^\omega_n \wedge \alpha_\zeta = \bigcup_{\xi<\zeta} \underset{\sim}{\alpha}_\xi.$$

□ of the claim.

□

Let us argue now that in $V[G(\mathcal{P})]$ we can find H which is a generic subset H of $\prod_{k<\omega} Q_k/finite$.

Set

$$H = \{\langle f_m \mid m < \omega\rangle/finite \mid \exists p = \langle p_k \mid k < \omega\rangle \in G(\mathcal{P})\exists m_0 < \omega\forall m > m_0(p_m = \langle f_m, A_m\rangle)\}.$$

Let us show a genericity of H. So, let $D \in V$ be a dense open subset of $\prod_{k<\omega} Q_k/finite$. Define $D' \subseteq \mathcal{P}$ as follows.

$$D' = \{p = \langle p_k \mid k < \omega\rangle \in \mathcal{P} \mid \exists m_0 < \omega(\langle f^p_m \mid m_0 < m < \omega\rangle \in D\},$$

where for $m \geq \ell(p), p_m = \langle f^p_m, A^p_m\rangle$.

Claim 3 D' is dense in $\langle \mathcal{P}, \leq\rangle$ and even in $\langle \mathcal{P}, \leq^*\rangle$.

Proof. Let $q = \langle q_k \mid k < \omega\rangle \in \mathcal{P}$. For every $m \geq \ell(q)$, q_m is of the form $\langle f^q_m, A^q_m\rangle$. Consider $\vec{f} = \langle f^q_m \mid \ell(q) \leq m < \omega\rangle$. There is $\vec{g} = \langle g_m \mid m < \omega\rangle \in D$ such that $\vec{g} \geq_{\prod_{k<\omega} Q_k/finite} \vec{f}$. Now define $\vec{h} = \langle h_m \mid m < \omega\rangle$ as follows:
$h_m = g_m$, for every $m < \ell(q)$; for every $m \geq \ell(q)$, if g_m does not extend f_m, then let $h_m = f_m$ (note that there are only finitely many m's like this); if g_m extends f_m, then $h_m = g_m$.
Now we pick sets of measure one B_m which project to subsets of A^q_m such that $p = q \restriction \ell(q)^\frown\langle\langle h_m, B_m\rangle \ell(q) \leq m < \omega$ is a condition in $\mathcal{P}$. Then $p \geq^* q$, by its definition, and also, $p \in D'$.
□ of the claim.

Pick now some $p \in D' \cap G(\mathcal{P})$, then $\langle f^p_m \mid \ell(p) \leq m < \omega\rangle \in H$.

Given such generic H inside $V[G(\mathcal{P})]$, we will there all the corresponding clubs C^ω_n, for every $n < \omega$.
Set $C = \bigcap_{n<\omega} C^\omega_n$. Then $C \subseteq \bar{\kappa}^+_\omega$ is a club as well.
Pick some $\alpha \in C \cap S$. Then there is some $p \in G(\mathcal{P})$ which forces all this. Take $n = \ell(p)$.

Then $\langle \alpha, p\rangle \in \underset{\sim}{S}_n$ and also, $p \Vdash_{\mathcal{P}_n} \alpha \in C_n$, which is impossible.
Contradiction.
□

3 ¬SCH and the reflection for a club.

We generalize the result to club many cardinals:

Theorem 3.1 *Suppose that θ is the least inaccessible cardinal which is a limit of supercompact cardinals.*
Then there is cofinality preserving extension so that

- *θ remaining inaccessible,*
- *there is a club in θ consisting of singular strong limit cardinals ν such that*
    1. $2^\nu > \nu^+$,
    2. *every stationary subset of ν^+ reflects.*

Proof. The construction of the previous section can be applied here, only replace η by an inaccessible cardinal θ.
Let $\langle \delta_\alpha \mid \alpha < \theta\rangle$ be an increasing sequence of supercompact cardinals. Set $\kappa_\alpha = \delta_{\alpha+1}$, for every $\alpha < \theta$. Clearly, each κ_α is strong. Repeat the previous construction using the sequence $\langle \kappa_\alpha \mid \alpha < \theta\rangle$.
Note that given a limit $\alpha < \theta$, we do not know in advance (i.e. without forcing with $E(\alpha)$) what will be $2^{\bar{\kappa}_\alpha}$, where, as before, $\bar{\kappa}_\alpha = \bigcup_{\beta<\alpha} \kappa_\beta$. So, if we have only boundedly many supercompacts below κ_α, then it is possible that there will be no supercompact in the interval $(2^{\bar{\kappa}_\alpha}, \kappa_\alpha)$. However, having a supercompact inside $(\kappa_\alpha, \kappa_{\alpha+1})$, we can repeat the argument of the previous section just using $\kappa_{\alpha+1}$ as the first strong in this argument.
□

Finally note that it is possible to combine the previous results on AP [5] and Tree Property [6] with the present one, since the same forcing is used in all of them. So we obtain the following:

Theorem 3.2 *Suppose that θ is the least inaccessible cardinal which is a limit of supercompact cardinals.*
Then there is cofinality preserving extension so that

- *θ remaining inaccessible,*
- *there is a club in θ consisting of singular strong limit cardinals ν such that*
  1. $2^\nu > \nu^+$,
  2. $\neg AP_{\nu^+}$,
  3. *the tree property holds at ν^+,*
  4. *every stationary subset of ν^+ reflects.*

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