On $\sigma-$complete uniform ultrafilters.

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Abstract

$\sigma-$complete uniform ultrafilters where extensively studied in early seventies. Many nice results were obtained, specially by J. Ketonen (see [11]).
Recently, G. Goldberg returned to the subject and proved a remarkable result that under The Ultrapower Axiom the first strongly compact cardinal is a supercompact.
The purpose of the present paper is to provide some concrete examples of $\sigma-$complete uniform ultrafilters.

1 Some basic definitions and facts.

Definition 1.1 Let $U \subseteq \mathcal{P}(\lambda)$ be an ultrafilter on λ.

1. U is called *uniform* iff for every A of cardinality $< \lambda$, $\lambda \setminus A \in U$.
2. U is called $\kappa-$*complete* iff the intersection of any less than κ members of U is in U.
3. U is called $\kappa-$*complete exactly* iff U is $\kappa-$complete, but not κ^+-complete.

If κ is a strongly compact cardinal, then for every $\lambda \geq \kappa$ there is a uniform $\kappa-$complete ultrafilter over λ. By J. Ketonen [11] the opposite is true as well.
Here we would like to examine the existence of $\sigma-$complete uniform ultrafilters over a cardinal λ which are exactly $\kappa-$complete for some $\kappa < \lambda$ under much weaker assumptions.
Note that if U is a $\sigma-$complete uniform ultrafilter over a cardinal λ and
$j_U : V \to M_U \simeq \mathrm{Ult}(V, U)$ is the corresponding elementary embedding, then

$$\sup(j_U{}''\lambda) \leq [id]_U < j_U(\lambda).$$

Also, if U is $\kappa-$complete for some $\kappa < \lambda$, then $\mathrm{cof}(\lambda) \geq \kappa$.

The following notion replaces the normality in the present context.

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Definition 1.2 A uniform ultrafilter U over λ is called *weakly normal* iff for every $A \in U$ and every regressive $f : A \to \lambda$ there is $\alpha < \lambda$ such that the set $\{\nu \in A \mid f(\nu) < \alpha\}$ is in U.

Note that (for $\sigma-$complete ultrafilters) this is equivalent to the statement that $[id]_U$ is the least function which is not bounded by $[c_\gamma]_U$, for $\gamma < \lambda$, i.e. $[id]_U = \sup(j_U{}''\lambda)$.

Clearly, if U is exactly $\kappa-$complete ultrafilter over λ, then κ must be a measurable and the critical point of the corresponding elementary embedding $j_U : V \to M_U \simeq {}^\lambda V/U$. Further by $\kappa-$completeness we will mean the exact $\kappa-$completeness.

Let us address the strength of the existence of such ultrafilters.

Proposition 1.3 *Assume that* $\neg 0^\P$.[1]
Suppose that there exists a $\kappa-$*complete uniform ultrafilter over* $\lambda > \kappa$. *Then the following hold:*

1. κ *and* $\mathrm{cof}(\lambda)$ *are measurable cardinals in the core model,*

2. *if* $\mathrm{cof}(\lambda) < \lambda$, *then either*

   (a) λ *is a measurable cardinal in the core model,*
   or

   (b) λ *is singular limit of measurable cardinals in the core model.*

Proof. Let $\mathcal{U}$ be $\kappa-$complete uniform ultrafilter over $\lambda > \kappa$. Let $\mathcal{K}$ denotes the core model.

Consider $j_{\mathcal{U}} : V \to M_{\mathcal{U}} \simeq \mathrm{Ult}(V, \mathcal{U})$ the corresponding elementary embedding. By W. Mitchell [14], $j := j_{\mathcal{U}} \restriction \mathcal{K}$ is an iterated ultrapower of $\mathcal{K}$ by its measures or extenders. Note that κ is a critical point of j, and so, it is a measurable cardinal in $\mathcal{K}$.

If $\mathrm{cof}(\lambda)$ is not a measurable in $\mathcal{K}$, then $\sup(j''cof(\lambda)) = j(\mathrm{cof}(\lambda))$, and so, $\sup(j''\lambda) = j(\lambda)$, which is impossible due to the uniformity of $\mathcal{U}$.

Suppose now that $\mathrm{cof}(\lambda) < \lambda$ and λ is not a measurable cardinal in $\mathcal{K}$.
Assume that in $\mathcal{K}$, λ is not a limit of measurable cardinals. Let μ be a supremum of measurable cardinals of $\mathcal{K}$ below λ.
Consider $[id]_{\mathcal{U}}$. Then there are $n < \omega$ and $f : [\mu]^n \to On$ such that $j(f)(a) = [id]_{\mathcal{U}}$, for some $a \in [j(\mu)]^n = j(\mathrm{dom}(f))$. So, $[id]_{\mathcal{U}} \in \mathrm{rng}(j(f))$. Let $Z = \mathrm{rng}(f)$, then $|Z| \leq \mu$. However, $[id]_{\mathcal{U}} \in j(Z) = \mathrm{rng}(j(f))$ implies that $Z \in \mathcal{U}$ and this contradicts the uniformity of $\mathcal{U}$.
$\square$

[1] This means that there is no inner model with a sharp for a strong cardinal.

2 Basic constructions.

Given two measurable cardinals $\kappa < \lambda$, it is easy to define a uniform ultrafilter U over λ which is exactly $\kappa-$complete.
Thus, fix some normal ultrafilters $\mathcal{V}$ on κ and $\mathcal{U}$ on λ. Let $j_{\mathcal{U}} : V \to M_{\mathcal{U}}$ be the ultrapower embedding by $\mathcal{U}$. Clearly, $\mathcal{V}$ remains a normal ultrafilter in $M_{\mathcal{U}}$. Let

$$j_{\mathcal{V}}^{M_{\mathcal{U}}} : M_{\mathcal{U}} \to N := M_{\mathcal{V}}^{M_{\mathcal{U}}}$$

be its ultrapower embedding there.
Set

$$i = j_{\mathcal{V}}^{M_{\mathcal{U}}} \circ j_{\mathcal{U}}.$$

Define a desired uniform $\kappa-$complete (exactly) ultrafilter $\mathcal{W}$ on λ as follows:

$$X \in \mathcal{W} \Leftrightarrow \lambda + \kappa \in i(X).$$

Such $\mathcal{W}$ is not weakly normal, since $i''\lambda = \lambda < [id]_W = \lambda + \kappa$.
It follows also that the weakly normal ultrafilter Rudin-Keisler below $\mathcal{W}$ is actually $\mathcal{U}$.

It is not hard to construct a model with a uniform weakly normal, $\kappa-$complete ultrafilter which is not normal.

Start with a GCH model with two measurable cardinals $\kappa < \lambda$.

Add a Cohen function $f_\alpha : \alpha \to \alpha$ for every inaccessible $\alpha, \kappa < \alpha \leq \lambda$ (with the Easton support iteration).

Extend the embedding i (of the previous construction) and change the value of $f_{i(\lambda)}(\lambda)$ to κ.
The resulting extension of $\mathcal{U}$ will be as desired, and, moreover it will be Rudin-Keisler equivalent to $\mathcal{W}$ defined as above using $\lambda + \kappa$.

3 Suslin trees.

It is possible (again from two measurables) to construct a model with
a uniform $\kappa-$complete ultrafilter U over λ in which λ is not a measurable.
The basic idea goes back to K. Kunen's [13] construction of a model with a $\lambda-$saturated ideal over λ. Here we would like to add a $\lambda-$Suslin tree to λ and then to argue that in the ultrapower by a normal ultrafilter over κ $\lambda-$branches are added to its image. This will allow us to extend embeddings by measures over λ.

Theorem 3.1 *Assume GCH. Let $\kappa < \lambda$ be two measurable cardinals. Then there is a cofinality preserving generic extension which satisfies the following:*

1. *GCH,*
2. *λ is not measurable,*
3. *there is a uniform $\kappa-$complete ultrafilter over λ.*
   Moreover, every $\mathcal{U}$ which is a normal ultrafilter over λ in V extends to a uniform $\kappa-$complete ultrafilter over λ.

Proof. Fix some normal ultrafilters $\mathcal{V}$ on κ and $\mathcal{U}$ on λ. Let $j_{\mathcal{U}} : V \to M_{\mathcal{U}}$ be the ultrapower embedding by $\mathcal{U}$. Clearly, $\mathcal{V}$ remains a normal ultrafilter in $M_{\mathcal{U}}$. Let

$$i := j_{\mathcal{V}}^{M_{\mathcal{U}}} : M_{\mathcal{U}} \to N := M_{\mathcal{V}}^{M_{\mathcal{U}}}$$

be its ultrapower embedding there.

Our first attempt will be to add a $\lambda-$Suslin tree T such that its image under $j_{\mathcal{V}}$ has a branch in V.

Let $\eta, \kappa < \eta \leq \lambda$ be a 2-Mahlo cardinal. Define a forcing notion Q_η as follows:
$\langle T, \vec{f}\rangle \in Q_\eta$ iff

1. $T \subseteq {}^{\eta>}2$ is a normal tree of a successor height $ht(T)$ below η,
2. $\vec{f} = \langle f_\alpha \mid \alpha < ht(T)\rangle$, where
   (a) for every $\alpha < ht(T)$, $f_\alpha : \kappa \to Lev_\alpha(T)$,
   (b) for every $\alpha, \beta < ht(T)$, if $\alpha < \beta$, then the set $\{\nu < \kappa \mid f_\alpha(\nu) <_T f_\beta(\nu)\}$ is co-bounded in κ.
   The idea behind is that $[f_\alpha]_{\mathcal{V}}$ would eventually generate an $\eta-$branch in the ultrapower.

The order on Q_η is defined in the natural fashion by taking end extensions[2].

The following lemma is easy:

Lemma 3.2 *The forcing Q_η is $< \eta-$strategically closed.*

[2]We denote the fact that a condition p is stronger than a condition q by $p \geq q$.

Proof. Let $\tau < \eta$. Define a winning strategy for Player II (the one that plays even stages including limit ones)by induction.
Suppose that $\langle\langle T^\zeta, \vec{f^\zeta}\rangle \mid \zeta < \xi < \tau\rangle$, ξ even is a play according to this strategy. Define the $\xi-$th move of Player II. Let $T = \bigcup_{\zeta<\xi} T^\zeta$ and T^ξ will be the tree obtained from T by adding the last level. We just continue all maximal branches of T to this level.
Note that there are maximal branches, since trees in Q_η are normal, have successor heights and at each limit stage of the construction (below ξ) we do the same, i.e. all maximal branches of the union are continued.
Now, set $\vec{f}$ to be the union of all $\vec{f^\zeta}, \zeta < \xi$. Let ξ^* be the last level of T^ξ. Define f^{ξ^*} and then add it to $\vec{f}$.
Suppose first that ξ is a limit stage. Let $f^{\xi^*}(\nu), \nu < \kappa$, be the point at the level ξ^* which is a continuation of the branch $\{t \in T \mid \exists\zeta < \xi(\zeta \text{ is even and } (\vec{f^\zeta}(ht(T^\zeta))(\nu) \geq_T t)\}$. We assume by induction that all $\vec{f^\zeta}(ht(T^\zeta))(\nu)$, for even ζ are on the same branch.
In particular, for every two even ordinals $\zeta < \xi < \tau$, we will have
$\vec{f^\zeta}(ht(T^\zeta))(\nu) \leq_T \vec{f^\xi}(ht(T^\xi))(\nu)$, for every $\nu < \kappa$.
Suppose now that ξ is a successor stage.
Set

$$X = \{\nu < \kappa \mid \vec{f}^{\xi-1}(ht(T^{\xi-1}))(\nu) \geq_T \vec{f}^{\xi-2}(ht(T^{\xi-2}))(\nu)\}.$$

For every $\nu \in X$, let $f^{\xi^*}(\nu)$ be a point at the level ξ^* which is above $\vec{f}^{\xi-1}(ht(T^{\xi-1}))(\nu)$.
For every $\nu \in \kappa \setminus X$, let $f^{\xi^*}(\nu)$ be a point at the level ξ^* which is above $\vec{f}^{\xi-2}(ht(T^{\xi-2}))(\nu)$.
Clearly such defined f^{ξ^*} is as desired.
$\square$

Let now $G \subseteq Q_\eta$ be generic and $T(G)$ be the $\eta-$tree added by G.

Lemma 3.3 *$T(G)$ is an $\eta-$Suslin tree.*

Proof. Work in V. Let $\underset{\sim}{A}$ be a name of a maximal antichain in $T(G)$. Using the strategy of the previous lemma define $\langle T, \vec{f}\rangle$ of a limit height $\delta < \eta$ of cofinality κ and an increasing continuous sequence $\langle \delta_\gamma \mid \gamma < \kappa\rangle$ with limit δ such that for every $\gamma \leq \gamma' < \kappa$,

1. $\langle T \restriction \delta_\gamma + 1, \vec{f} \restriction \delta_\gamma + 1\rangle \in Q_\eta$,
2. for every $t \in T \restriction \delta_\gamma + 1$ there is $t^* \in T \restriction \delta_{\gamma+1} + 1$, $t^* \geq_T t$ or $t \geq_T t^*$,
   $\langle T \restriction \delta_{\gamma+1} + 1, \vec{f} \restriction \delta_{\gamma+1} + 1\rangle \Vdash t^* \in \underset{\sim}{A}$.

3. For every $\nu < \kappa$, $f_{\delta_\gamma}(\nu) \leq_T f_{\delta_{\gamma'}}(\nu)$.

Denote by A^* the set of all such t^*'s.

We would like now to freeze $\underset{\sim}{A}$. As usual, to do so we will continue to the level δ only branches that pass via an element of A^*. An additional thing to take care here is an extension of $\vec{f}$. Namely we need to define a function f_δ from κ to the level δ of the tree which respects the item 2b of the definition of Q_η.

Let $f_{\delta_\nu} = \vec{f}(\delta_\nu)$, for every $\nu < \kappa$.

Consider first $f_{\delta_0}(0)$. Pick a $\delta-$branch through T which passes via $f_{\delta_0}(0)$ and an element of A^*. Note that there is such a branch, since, following the strategy, we continue at limit stages of the construction all maximal branches. Continue it to the level δ and set $f_\delta(0)$ to be this continuation.

Proceed by induction on $\mu < \kappa$ and define $f_\delta(\mu)$ be above $f_{\delta_\mu}(\mu)$ and an element of A^*.

Note that $f_{\delta_\mu}(\mu) \geq_T f_{\delta_{\mu'}}(\mu)$, for every $\mu' \leq \mu$.

Then, for every $\mu < \kappa$, we will have that for every $\nu, \mu \leq \nu < \kappa$, $f_\delta(\nu) \geq_T f_{\delta_\mu}(\nu)$. Clearly, the last set is co-bounded, and so, we are done.

$\square$

In particular the natural forcing for adding a branch into $T(G)$ satisfies $\eta-$c.c.. Consider the other forcing F associated with G.

Let f be in F iff there is $\langle T, \vec{f} \rangle \in G$ such that for some $\alpha < ht(T)$, $f = f_\alpha$. Set $f^1 \geq_F f^2$ iff for every $\nu < \kappa$, $f^1(\nu) \geq_{T(G)} f^2(\nu)$.

The intuition behind is that we would like to extend $G(T)$ to the level η. A generic for F allows to define $\eta-$branches and f_η, i.e. the function to the level η of such tree.

Let us argue that the forcing $\langle F, \leq_F \rangle$ satisfies $\eta-$c.c..

Lemma 3.4 *$\langle F, \leq_F \rangle$ satisfies $\eta-$c.c..*

Proof. Work in V. Let $\underset{\sim}{A}$ be a name of a maximal antichain in $\langle F, \leq_F \rangle$.

η is a 2-Mahlo cardinal[3], so there an elementary submodel N of a large enough fragment of V such that $\underset{\sim}{A} \in N$ and $N \cap \eta = \eta'$, ${}^{\eta'>}N \subseteq N$, for some Mahlo cardinal $\eta' < \eta$.

We will construct by induction a pair $\langle T, \vec{f} \rangle$ with T of height η', using the strategy defined in Lemma 3.2. Player II will follow the strategy at even stages and at odd stages we will deal with $\underset{\sim}{A}$. Note that, according to this strategy, all maximal branches are continued at

[3]It is possible to use weaker assumptions, but it does matter much since λ is a measurable, and so, there are many such cardinals below it.

limit steps, so we need not worry what to do there. Let us concentrate on the construction at odd stages.

For every $s \in T$ we will define by induction an $\eta'-$branch b_s such that:

for every $\eta'' < \eta'$, for every $\langle s_\rho \mid \rho < \kappa \rangle \in [T \restriction \eta'' + 1]^\kappa$ with the set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ co-bounded, there is $\eta''', \eta'' \leq \eta''' < \eta'$ such that $\{(\rho, b_{s_\rho}(\eta''')) \mid \rho < \kappa\}$ is forced (by an initial segment of $\langle T, \vec{f} \rangle$) to be above an element of A.

Proceed as follows.

Start with a condition in $Q_\eta \cap N$. Extend it inside N to a condition $\langle T^0, \vec{f^0} \rangle$ which decides an element of A. Let g be this element and $\alpha_g < ht(T^0)$ its level.

Then the set $\{\nu < \kappa \mid g(\nu) <_{T^0} f^0_{ht(T^0)}(\nu)\}$ is co-bounded, since $g = f^0_{\alpha_g}$.

Branches $\langle b_{g(\nu)} \mid \nu < \kappa \rangle$ will be used to form f_α, for every even level $\alpha \leq \eta'$, i.e. at stages where Player II plays.

We continue by induction using the fact that $\eta'^\kappa = \eta' = 2^{<\eta'}$.

Thus, fix an enumeration $\pi : \eta' \to [2^{<\eta'}]^\kappa$ such that the pre-image of each element of $[2^{<\eta'}]^\kappa$ is stationary.

Suppose that for some $\eta'' < \eta'$, $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1 \rangle$ is defined.

Consider $\pi(\eta'')$. If $\pi(\eta'') \notin [T \restriction \eta''+1]^\kappa$, then nothing special is done. If $\pi(\eta'') \in [T \restriction \eta''+1]^\kappa$, then let $\pi(\eta'') = \langle s_\rho \mid \rho < \kappa \rangle$.

If the set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ is not co-bounded, then again, do nothing special.

If the set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ is co-bounded, but $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is not forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1 \rangle$ to be a condition, then do nothing special.

Suppose now that the set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ is co-bounded and $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1 \rangle$ to be a condition.

If $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1 \rangle$ to be above an element of A, then do nothing special.

If $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is not forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1 \rangle$ to be above an element of A, then we extend $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1 \rangle$ to a condition $\langle T', \vec{f'} \rangle$ which decides an element of A above $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$. Let $\eta''', \eta'' < \eta''' < ht(T')$ be the level of such element of A. Denote this element by $\langle a_\rho \mid \rho < \kappa \rangle$. Then for every $\rho < \kappa$, $b_{s_\rho}(\eta'') <_{T'} a_\rho$. Pick now $a'_\rho >_{T'} a_\rho$ at the level $ht(T') - 1$ such that $a'_\rho = f'_{ht(T')-1}(\rho)$, if $f'_{ht(T')-1}(\rho) >_{T'} a_\rho$, and arbitrary otherwise[4].

Set $b_{s_\rho} \restriction ht(T') = \{x \in T' \mid x \leq a'_\rho\}$.

Define $b_s \restriction ht(T')$ to be an arbitrary extension of $b_s \restriction \eta'' + 1$, if $s \neq s_\rho$, for some $\rho < \kappa$.

[4]Note that for co-boundedly many ρ's, we will have $f'_{ht(T')-1}(\rho) >_{T'} a_\rho$.

Now we add to T the last level (the level η') by continuing branches $b_s, s \in T$ only. Set $f_{\eta'}(\nu)$ to be the continuation of $b_{g(\nu)}$ to the level η', for every $\nu < \kappa$.
Denote by $\langle T^*, \vec{f}\rangle$ such extension.
Let us show that the condition $\langle T^*, \vec{f}\rangle$ forces that $A \subseteq [T]^\kappa$. Suppose otherwise. Then there is $\langle S, \vec{g}\rangle \geq_{Q_\eta} \langle T^*, \vec{f}\rangle$ and $\xi, \eta' < \xi < ht(S)$ such that

$$\langle S, \vec{g}\rangle \Vdash g_\xi \in \underset{\sim}{A}.$$

For every $\rho < \kappa$, let a_ρ the predecessor of $g_\xi(\rho)$ at level η'. Then, for every $\rho < \kappa$, there is $s_\rho \in T$ such that $a_\rho = b_{s_\rho}$. Find $\eta'' < \eta'$ such that $\pi(\eta'') = \langle s_\rho \mid \rho < \kappa\rangle$ and $\langle s_\rho \mid \rho < \kappa\rangle \in [T \restriction \eta'' + 1]^\kappa$.
The set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ co-bounded, since $b_{s_\rho}(\eta'')$ is the predecessor of $g_\xi(\rho)$ at the level η'' and the set $\{\rho < \kappa \mid g_\xi(\rho) >_S f_{\eta''}(\rho)\}$ co-bounded, by the definition of Q_η.
Then, by the construction, there is $\eta''', \eta'' \leq \eta''' < \eta'$ such that
$\{(\rho, b_{s_\rho}(\eta''')) \mid \rho < \kappa\}$ is forced (by an initial segment of $\langle T, \vec{f}\rangle$) to be above an element of A.
Which is impossible since A is an anti-chain.
Contradiction.
□

We force in the interval (κ, λ) a $\eta-$Suslin tree $T(\eta)$ with the forcing Q_η, and then, we force with the forcing $\langle F_\eta, \leq_{F_\eta}\rangle$, defined above, branches trough $T(\eta)$, for every 2-Mahlo cardinal η.
More precisely, Easton support iteration is used between κ and λ. At all but 2-Mahlo cardinals the forcing is trivial and if $\eta, \kappa < \eta < \lambda$ is a 2-Mahlo cardinal, then we force with $Q_\eta * F_\eta$. Denote this iteration by $\mathbb{P}$.

Now, over λ itself, force only a $\lambda-$Suslin tree $T(\lambda)$ with Q_λ.
Note that $j_{\mathcal{V}}(\lambda) = \lambda$.
Let $G \subseteq \mathbb{P} * Q_\lambda$ be a generic.
We extend $j_{\mathcal{V}}$ in a natural fashion to $j^*_{\mathcal{V}} : V[G] \to M_{\mathcal{V}}[G^*]$. Namely, $j_{\mathcal{V}}{''}G$ will generate $G^* \subset j_{\mathcal{V}}(\mathbb{P} * Q_\lambda)$ generic over $M_{\mathcal{V}}$, since the forcing $\mathbb{P} * Q_\lambda$ is more than $\kappa-$strategically closed.
Then λ does not move and, by elementarity, $j^*_{\mathcal{V}}(T(\lambda))$ is a $\lambda-$Suslin tree in $M_{\mathcal{V}}[G^*]$.
Let $\langle f^G_\alpha \mid \alpha < \lambda\rangle$ be the part of $G \restriction Q_\lambda$ which consists of functions.
For every $\alpha < \beta < \lambda$, the set $\{\nu < \kappa \mid f^G_\alpha(\nu) \not<_{T(\lambda)} f^G_\beta(\nu)\}$ is bounded in κ by some $\delta(\alpha, \beta)$.
By elementarity, the set

$$\{\nu < j_{\mathcal{V}}(\kappa) \mid j^*_{\mathcal{V}}(f^G_\alpha)(\nu) <_{j^*_{\mathcal{V}}(T(\lambda))} j^*_{\mathcal{V}}(f^G_\beta)(\nu)\} \supseteq j_{\mathcal{V}}(\kappa) \setminus \delta(\alpha, \beta).$$

Note that there are unboundedly many $\alpha < \lambda$ such that $j_{\mathcal{V}}(\alpha) = \alpha$, for example, every inaccessible $\alpha, \kappa < \alpha < \lambda$ is a fixed point of $j_{\mathcal{V}}$.
Set

$$b_\nu := \{t \in j^*_{\mathcal{V}}(T(\lambda)) \mid \exists \alpha < \lambda (t <_{j^*_{\mathcal{V}}(T(\lambda))} j^*_{\mathcal{V}}(f^G_\alpha)(\nu))\},$$

for every $\nu, \kappa \leq \nu < j_{\mathcal{V}}(\kappa)$. Then the sequence $\langle b_\nu \mid \kappa \leq \nu < j_{\mathcal{V}}(\kappa)\rangle$ is in $V[G]$, and it is a sequence of $\lambda-$branches through $j^*_{\mathcal{V}}(T(\lambda))$.
Consider

$$H = \{f^{G^*}_\gamma \mid \gamma < \lambda,\ \text{exists } \alpha, \gamma < \alpha < \lambda \text{ for every } \nu < j_{\mathcal{V}}(\kappa)(f^{G^*}_\gamma(\nu) <_{j^*_{\mathcal{V}}(T(\lambda))} j^*_{\mathcal{V}}(f^G_\alpha)(\nu))\}.$$

Then H is close to produce a $M_{\mathcal{V}}[G^*]-$generic set for the forcing $\langle F_\lambda, \leq_{F_\lambda} \rangle$ with the tree $j^*_{\mathcal{V}}(T(\lambda))$ in $M_{\mathcal{V}}[G^*]$. It follows from the fact that in $M_{\mathcal{V}}[G^*]$ the forcing satisfies $\lambda-$c.c.. Thus, suppose that $A \subseteq F_\lambda$ is a maximal anti-chain. Then there is $\delta < \lambda$ such that each function in A acts at a level of $j^*_{\mathcal{V}}(T(\lambda))$ below δ. Take an inaccessible $\alpha, \delta < \alpha < \lambda$. Consider $j^*_{\mathcal{V}}(f^G_\alpha)$. It is compatible with an element of A, but since $j^*_{\mathcal{V}}(f^G_\alpha)$ acts at the level $\alpha > \delta$, it should be above this element. The only problem is that for $\beta, \alpha < \beta < \lambda$, $j^*_{\mathcal{V}}(f^G_\alpha)(\nu)$ need not be below $j^*_{\mathcal{V}}(f^G_\beta)(\nu)$ for every ν, but rather for $\nu \geq \kappa$. It is not hard to fix this and we do it below.
We would like to use $\mathcal{U}$ to extend further the elementary embedding and eventually to produce an extension of $\mathcal{U}$ which is as desired.
The problematic point is that $j^*_{\mathcal{V}}(T(\lambda))$ extended to the level λ by extending the branches $\langle b_\nu \mid \kappa \leq \nu < j_{\mathcal{V}}(\kappa)\rangle$ to the level λ is not normal. Namely there will be many $t \in j^*_{\mathcal{V}}(T(\lambda))$ without a $\lambda-$branch through t.

In order to prevent this, let us revise the construction and use homogeneous trees instead of arbitrary ones.

Define a forcing notion Q^{hom}_η as follows:
$\langle T, \vec{f}\rangle \in Q^{hom}_\eta$ iff

1. $T \subseteq {}^{\eta>}2$ is a normal tree of a successor height $ht(T)$ below η,
2. T is homogeneous in the following sense:
   if $s_0, s_1 \in T$ are on a same level, then $T_{s_0} = T_{s_1}$[5].
3. $\vec{f} = \langle f_\alpha \mid \alpha < ht(T)\rangle$, where
   (a) for every $\alpha < ht(T)$, $f_\alpha : \kappa \to Lev_\alpha(T)$,

[5]If $s \in T$, then T_s denotes $\{t \restriction \eta \setminus |s| \mid t \in T, t \geq_T s\}$.

(b) for every $\alpha, \beta < ht(T)$, if $\alpha < \beta$, then the set $\{\nu < \kappa \mid f_\alpha(\nu) <_T f_\beta(\nu)\}$ is co-bounded in κ.

The order on Q_η^{hom} is defined in the natural fashion by taking end extensions.

The next lemma is as in the previous setting:

Lemma 3.5 *The forcing Q_η^{hom} is $< \eta-$strategically closed.*

Let now $G \subseteq Q_\eta^{hom}$ be generic and $T(G)$ be the $\eta-$tree added by G.

Lemma 3.6 *$T(G)$ is an $\eta-$Suslin tree.*

Proof. Work in V. Let $\underset{\sim}{A}$ be a name of a maximal antichain in $T(G)$.
Fix an increasing continuous elementary chain $\langle N_\gamma \mid \gamma < \kappa\rangle$ of elementary submodels of H_χ for χ large enough such that

1. $Q_\eta^{hom}, \underset{\sim}{A} \in N_0$,

2. for every $\gamma < \kappa$, $N_\gamma \cap \lambda = \delta_\gamma$ for some cardinal δ_γ,

3. for every successor $\gamma < \kappa$, δ_γ is a regular cardinal and ${}^{\delta_\gamma >}N_\gamma \subseteq N_\gamma$.

For every successor $\gamma < \kappa$, we work inside N_γ and define by induction a homogeneous normal tree T^γ of height δ_γ, functions $\vec{f}^\gamma$ for T^γ and $\langle b_t^\gamma \mid t \in T^\gamma\rangle$ such that

1. T^γ end-extends $T^{\gamma-1}$,

2. $\vec{f}^\gamma$ end-extends $\vec{f}^{\gamma-1}$,

3. for every $t \in T^\gamma$, b_t^γ is a $\delta_\gamma-$branch through T^γ,

4. if $t \in T^{\gamma-1}$, then $b_t^\gamma \restriction \delta_{\gamma-1} = b_t^{\gamma-1}$,

5. $\langle T^\gamma, \vec{f}^\gamma\rangle$ decides $\underset{\sim}{A} \cap T^\gamma$,

6. for every $t \in T^\gamma$, $\langle T^\gamma, \vec{f}^\gamma\rangle$ forces that an element of $\underset{\sim}{A}$ belongs to b_t^γ,

7. for every $t, s \in T_\gamma$ which are on a same level $\alpha < \delta_\gamma$, $b_t^\gamma \setminus \alpha + 1 = b_s^\gamma \setminus \alpha + 1$.

Now, we extend *all* $\delta_\gamma-$branches of T^γ to the level δ_γ, and not only the chosen b_t^γ's. In addition, for every $\alpha \leq \delta_\gamma$, we will have a function $f_\alpha = f_\alpha^\gamma$ from κ to the $\alpha-$th level. They are defined together with trees as in Lemma 3.3.

Set $\delta_\kappa = \bigcup_{\gamma<\kappa} \delta_\gamma$, $T^\kappa = \bigcup_{\gamma<\kappa} T^\gamma$ and for every $t \in T^\kappa$, $b_t = \bigcup_{\gamma<\kappa} b_t^\gamma$.
Extend T^κ to the level δ_κ, but now continue only the branches b_t, $t \in T^\kappa$.
Let us define f_{δ_κ}, i.e. the function for the level δ_κ.
Consider first $f_{\delta_0}(0)$. Set $f_{\delta_\kappa}(0)$ be the continuation of $b_{f_{\delta_0}(0)}$ to the level δ_κ.
Given $\gamma < \kappa$, set $f_{\delta_\kappa}(\gamma)$ be the continuation of $b_{f_{\delta_\gamma}(\gamma)}$ to the level δ_κ.
Then for every $\gamma < \kappa$,

$$\{\nu < \kappa \mid f_{\delta_\kappa}(\nu) >_{T^\kappa} f_{\delta_\gamma}(\nu)\} \supseteq \kappa \setminus \gamma.$$

□

In particular the natural forcing for adding a branch into $T(G)$ satisfies $\eta-$c.c.. Consider the other forcing F associated with G.
Let f be in F iff there are $\xi_f < \kappa$ and $\langle T, \vec{f}\rangle \in G$ such that for some $\alpha < ht(T)$, $f = f_\alpha \setminus \xi_f$.
Set $f^1 \geq_F f^2$ iff $\xi_{f^1} = \xi_{f^2}$ and for every $\nu, \xi_{f^1} \leq \nu < \kappa$, $f^1(\nu) \geq_{T(G)} f^2(\nu)$.
F can be viewed as a set of pairs $\langle f, \xi_f\rangle$.
The intuition behind this is that we would like to extend $G(T)$ to the level η. An $\eta-$branch (together with homogeneity) allows to extend the tree $T(G)$ and a generic for F allows to define f_η, i.e. the function to the level η of such tree.

The next lemma is a "homogeneous" analog of Lemma 3.4

Lemma 3.7 *$\langle F, \leq_F \rangle$ satisfies $\eta-$c.c..*

Proof. Let $A \subseteq F$ be an anti-chain. Our aim is to show that $|A| < \eta$. We can split A into at most $\kappa-$many anti-chains according to the values of ξ_f's of its members. It is enough to show that each part of such splitting has cardinality less than η. So deal with them separately. Assume for simplicity that for every $f \in A$, $\xi_f = 0$.

Work in V. Let $\underset{\sim}{A}$ be a name of A.
Proceed as in 3.4.
η is a 2-Mahlo cardinal, so there an elementary submodel N of a large enough fragment of V such that $\underset{\sim}{A} \in N$ and $N \cap \eta = \eta'$, ${}^{\eta'>}N \subseteq N$, for some Mahlo cardinal $\eta' < \eta$.

We will construct by induction a pair $\langle T, \vec{f}\rangle$ with T of height η', using the strategy defined in Lemma 3.2. Player II will follow the strategy at even stages and at odd stages we will deal with $\underset{\sim}{A}$. Note that, according to this strategy, all maximal branches are continued at limit steps, so we need not worry what to do there. The main attention will be to the construction at odd stages.

We will associate a maximal branch b_s through s, for every $s \in T$. Only this branches will be continued to the final level η', as it was done in 3.4. This will insure the following:

for every $\eta'' < \eta'$, *for every* $\langle s_\rho \mid \rho < \kappa \rangle \in [T \restriction \eta'' + 1]^\kappa$ *with the set* $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ *co-bounded,*
if there is an extension of $\langle T, \vec{f} \rangle$ *which decides an element of* A *above* $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$, *then there is* $\eta''', \eta'' \leq \eta''' < \eta'$ *such that*
$\{(\rho, b_{s_\rho}(\eta''')) \mid \rho < \kappa\}$ *is forced (by an initial segment of* $\langle T, \vec{f} \rangle$*) to be above an element of* A.

There is an additional requirement here - a homogeneity of the final tree. In order to insure this, the inductive construction of T will provide the following:

$(*)_T$ *for every* $s, t \in T$ *which are on a same level* $\alpha < \eta'$, *for every* b_x *with* $s \in b_x$ *there is* $y \in T$ *such that* $t \in b_y$ *and* $b_x \setminus \alpha = b_y \setminus \alpha$.

Proceed as follows.

Start with a condition $\langle T^0, \vec{f}^0 \rangle \in Q_\eta \cap N$.

Let us define a sequence of maximal branches $\langle b_s^{T_0} \mid s \in T_0 \rangle$ through T^0 which has the property $(*)_{T^0}$. Such $b_s^{T_0}$'s will be initial segments of the final branches b_s's through T. Let $\eta_0 + 1$ be the height of T^0. Proceed by induction on levels $\alpha \leq \eta_0$.

Consider the first level.

Suppose that $\langle 0 \rangle$ is there. Pick some b at the last level of T^0 (i.e. a $0-1$ sequence of the length η_0) which extends $\langle 0 \rangle$, or simply, its first element is 0. Set $b_{\langle 0 \rangle}^{T_0} = b$. Also, set $b_s^{T_0} = b$ for every $s \in T^0$ which is on b, i.e. for every initial segment of b.

Now deal with the next element (if there is such) of the first level of T^0.

So consider $\langle 1 \rangle$. By homogeneity of T^0, we have $T^0_{\langle 0 \rangle} = T^0_{\langle 1 \rangle}$. In particular, $b_1 := \langle 1 \rangle^\frown b \setminus 1 \in T^0$. Set $b_{\langle 1 \rangle}^{T_0} = b_1$ and let $b_s^{T_0} = b_1$, for every $s \in T^0$ which is on b_1, i.e. for every initial segment of b_1.

Now, suppose we got to a level α and there is s at this level with $b_s^{T_0}$ undefined. Set $b_s^{T_0} = s^\frown b_{\langle 0 \rangle}^{T^0} \setminus \alpha$. Also, set $b_t^{T_0} = b_s^{T_0}$ for every $t \in T^0$ which is on $b_s^{T_0}$ and is above the level α.

Fix an enumeration $\pi : \eta' \to [2^{<\eta'}]^\kappa$ such that the pre-image of each element of $[2^{<\eta'}]^\kappa$ is stationary.

Suppose that for some $\eta'' < \eta'$, $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1 \rangle$ is defined.

Consider $\pi(\eta'')$. If $\pi(\eta'') \notin [T \restriction \eta'' + 1]^\kappa$, then nothing special is done. If $\pi(\eta'') \in [T \restriction \eta'' + 1]^\kappa$, then let $\pi(\eta'') = \langle s_\rho \mid \rho < \kappa \rangle$.

If the set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ is not co-bounded, then again, do nothing special.
If the set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ is co-bounded, but $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is not forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1\rangle$ to be a condition, then do nothing special.
Suppose now that the set $\{\rho < \kappa \mid b_{s_\rho}(\eta'') = f_{\eta''}(\rho)\}$ is co-bounded and $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1\rangle$ to be a condition.
If $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1\rangle$ to be above an element of A, then do nothing special.
If $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is not forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1\rangle$ to be above an element of A, and no extension $\langle T', \vec{f'}\rangle$ of $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1\rangle$ decides an element of A above $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$, then do nothing special. Note that now this may happens, since we deal with an antichain which is not maximal.
If $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$ is not forced by $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1\rangle$ to be above an element of A and there is an extension $\langle T', \vec{f'}\rangle$ of $\langle T \restriction \eta'' + 1, \vec{f} \restriction \eta'' + 1\rangle$ which decides an element of A above $\{(\rho, b_{s_\rho}(\eta'')) \mid \rho < \kappa\}$.
By extending if necessary, we can assume that $ht(T') - 1$ is an inaccessible cardinal.
Let $\eta''', \eta'' < \eta''' < ht(T')$ be the level of such element of A. Denote this element by $\langle a_\rho \mid \rho < \kappa\rangle$. Then for every $\rho < \kappa$, $b_{s_\rho}(\eta'') <_{T'} a_\rho$. Pick now $a'_\rho >_{T'} a_\rho$ at the level $ht(T') - 1$ such that $a'_\rho = f'_{ht(T')}(\rho)$, if $f'_{ht(T')-1}(\rho) >_{T'} a_\rho$, and arbitrary otherwise[6].
Set $b^{T'}_{s_\rho} = b_{s_\rho} \restriction ht(T') = \{x \in T' \mid x \leq_{T'} a'_\rho\}$.
Set $b^{T'}_t = b_t \restriction ht(T') = \{x \in T' \mid x \leq_{T'} a'_\rho\}$, for every $t \in T', t \leq_{T'} a'_\rho$ which sits on a level $> \eta''$.

Apply the homogeneity of T' now.
For every $\rho, 0 < \rho < \kappa$ there is $a'_{\rho,0}$ at the last level of T' which is above $b_{s_0}(\eta'')$ and $a'_\rho \setminus \eta'' = a'_{\rho,0} \setminus \eta''$.
Proceed by induction on levels of T' above η'' and define $b^{T'}_t = b_t \restriction ht(T')$ for t's in $T'_{b_{s_0}(\eta'')}$ which are $\leq_{T'} a'_{\rho,0}, \rho < \kappa$.
Thus, suppose $t \in T'_{b_{s_0}(\eta'')}$ is like this and $b_t \restriction ht(T')$ is still undefined. Pick $\rho < \kappa$ to be the least such that $t \leq_{T'} a'_{\rho,0}$ and set $b^{T'}_t = b_t \restriction ht(T') = a'_{\rho,0}$[7].

Repeat the same process above every $b_{s_\mu}(\eta'')$, $0 < \mu < \kappa$, i.e., with μ replacing 0.

There may be still s's in T' with $b^{T'}_s$ undefined. Proceed by induction on levels of T' starting from η''.
Thus, let, for example, $s \in Lev_{\eta''}(T) \setminus \{b_{s_\rho}(\eta'') \mid \rho < \kappa\}$.

[6]Note that for co-boundedly many ρ's, we will have $f'_{ht(T')-1}(\rho) >_{T'} a_\rho$.
[7]Recall that nodes of T' are just sequences and we identify between them.

Use the homogeneity of T'. For every $\rho < \kappa$ there is $a'_{s,\rho} >_{T'} s$ at the last level $ht(T') - 1$ of T' such that $a'_{s,\rho} \setminus \eta'' = a'_\rho \setminus \eta''$. We set $b_s^{T'} = b_s \restriction ht(T') = a'_{s,0}$ and repeat the settings made above $b_{s_0}(\eta'')$ to those above s.

Let us argue that the choice of the branches $b_s^{T'}$ above gives the property $(*)_{T'}$.

Claim. *$(*)_{T'}$ holds.*

Proof. First note that by the construction, for every $t \in T'$, there is $\rho < \kappa$ such that $b_t^{T'} \setminus ht(t) = a'_\rho \setminus ht(t)$.

The second point is that the branches which lead to a'_ρ's, $\rho < \kappa$, split completely before the final level $ht(T') - 1$, i.e., there is $\tilde{\eta}, \eta'' < \tilde{\eta} < ht(T') - 1$ such that $\langle a'_\rho(\tilde{\eta}) \mid \rho < \kappa \rangle$ are different points of the level $\tilde{\eta}$ of T'. This follows from an inaccessibility of $ht(T') - 1$.

So, by the construction, for every $t \in T', \rho < \kappa$, there will be $x \geq_{T'} t$ such that $b_x^{T'} \setminus ht(t) = a'_\rho \setminus ht(t)$.

Now $(*)_{T'}$ follows.

$\square$ of the claim.

This completes the construction of T and $\langle b_s \mid s \in T \rangle$.

We continue only the branches $b_s, s \in T$ to the level η'.

The rest of the proof is as in 3.4.

$\square$

Now we are ready to complete the proof of 3.1.

Proceed as it was done after Lemma 3.4.

We force in the interval (κ, λ) a $\eta-$Suslin tree $T(\eta)$ with the forcing Q_η, and then, we force with the forcing $\langle F_\eta, \leq_{F_\eta} \rangle$, defined above, branches trough $T(\eta)$, for every 2-Mahlo cardinal η.

More precisely, Easton support iteration is used between κ and λ. At all but 2-Mahlo cardinals the forcing is trivial and if $\eta, \kappa < \eta < \lambda$ is a 2-Mahlo cardinal, then we force with $Q_\eta * F_\eta$. Denote this iteration by $\mathbb{P}$.

Now, over λ itself, force only a $\lambda-$Suslin tree $T(\lambda)$ with Q_λ.

Note that $j_{\mathcal{V}}(\lambda) = \lambda$.

Let $G \subseteq \mathbb{P} * Q_\lambda$ be a generic.

We extend $j_{\mathcal{V}}$ in a natural fashion to $j^*_{\mathcal{V}} : V[G] \to M_{\mathcal{V}}[G^*]$. Namely, $j_{\mathcal{V}}{}''G$ will generate $G^* \subset j_{\mathcal{V}}(\mathbb{P} * Q_\lambda)$ generic over $M_{\mathcal{V}}$, since the forcing $\mathbb{P} * Q_\lambda$ is more than $\kappa-$strategically closed.

Then λ does not move and, by elementarity, $j^*_{\mathcal{V}}(T(\lambda))$ is a $\lambda-$Suslin tree in $M_{\mathcal{V}}[G^*]$.

Let $\langle f^G_\alpha \mid \alpha < \lambda \rangle$ be the part of $G \restriction Q_\lambda$ which consists of functions.

For every $\alpha<\beta<\lambda$, the set $\{\nu<\kappa \mid f^G_\alpha(\nu) \not<_{T(\lambda)} f^G_\beta(\nu)\}$ is bounded in κ by some $\delta(\alpha,\beta)$. By elementarity, the set

$$\{\nu<j_{\mathcal{V}}(\kappa) \mid j^*_{\mathcal{V}}(f^G_\alpha)(\nu) <_{j^*_{\mathcal{V}}(T(\lambda))} j^*_{\mathcal{V}}(f^G_\beta)(\nu)\} \supseteq j_{\mathcal{V}}(\kappa)\setminus\delta(\alpha,\beta).$$

Note that there are unboundedly many $\alpha<\lambda$ such that $j_{\mathcal{V}}(\alpha)=\alpha$, for example, every inaccessible $\alpha, \kappa<\alpha<\lambda$ is a fixed point of $j_{\mathcal{V}}$.
Set

$$b_\nu := \{t \in j^*_{\mathcal{V}}(T(\lambda)) \mid \exists\alpha<\lambda(t <_{j^*_{\mathcal{V}}(T(\lambda))} j^*_{\mathcal{V}}(f^G_\alpha)(\nu))\},$$

for every $\nu, \kappa\leq\nu<j_{\mathcal{V}}(\kappa)$. Then the sequence $\langle b_\nu \mid \kappa\leq\nu<j_{\mathcal{V}}(\kappa)\rangle$ is in $V[G]$, and it is a sequence of $\lambda-$branches through $j^*_{\mathcal{V}}(T(\lambda))$.
Consider

$$H=\{\langle f^{G^*}_\gamma,\kappa\rangle \mid \gamma<\lambda, \text{ exists } \alpha,\gamma<\alpha<\lambda, \text{ for every } \nu,\kappa\leq\nu<j_{\mathcal{V}}(\kappa)$$

$$(f^{G^*}_\gamma(\nu) <_{j^*_{\mathcal{V}}(T(\lambda))} j^*_{\mathcal{V}}(f^G_\alpha)(\nu))\}.$$

Let us argue now that such defined H is a $M_{\mathcal{V}}[G^*]-$generic set for the forcing $\langle F_\lambda, \leq_{F_\lambda}\rangle$ with the tree $j^*_{\mathcal{V}}(T(\lambda))$ in $M_{\mathcal{V}}[G^*]$, with $\xi_f=\kappa$.
First, we have that for every $\beta,\alpha<\beta<\lambda$, $j^*_{\mathcal{V}}(f^G_\alpha)(\nu)$ is below $j^*_{\mathcal{V}}(f^G_\beta)(\nu)$ for every ν, $\kappa\leq\nu<j_{\mathcal{V}}(\kappa)$.
Second, let us argue that H meets maximal antichains. It follows from the fact that in $M_{\mathcal{V}}[G^*]$ the forcing satisfies $\lambda-$c.c.. Thus, suppose that $A\subseteq F_\lambda$ is a maximal anti-chain. Then there is $\delta<\lambda$ such that each function in A acts at a level of $j^*_{\mathcal{V}}(T(\lambda))$ below δ. Take an inaccessible $\alpha,\delta<\alpha<\lambda$.
Consider $\langle j^*_{\mathcal{V}}(f^G_\alpha),\kappa\rangle$. It is compatible with an element of A. Let $\langle g,\kappa\rangle$ be such an element. Then the level on which g acts should be below δ. However, $j^*_{\mathcal{V}}(f^G_\alpha)$ acts at the level $\alpha>\delta$. Hence, for every $\nu,\kappa\leq\nu<j_{\mathcal{V}}(\kappa)$, $j^*_{\mathcal{V}}(f^G_\alpha) >_{j^*_{\mathcal{V}}(T(\lambda))} g(\nu)$.
So, $g\in H\cap A$.

Let $\mathcal{U}$ be a normal ultrafilter over λ in V. Consider its elementary embedding $j_{\mathcal{U}}: V\to M_U$. The critical point of $j_{\mathcal{U}}$ is λ, and so, $\mathcal{V}$ does not move.
Let $i:V\to M$ be the composition of $j_{\mathcal{U}}$ with $j^{M_{\mathcal{U}}}_{\mathcal{V}}: M_{\mathcal{U}}\to M\simeq \mathrm{Ult}(M_{\mathcal{U}},\mathcal{V})$.
It is well-known that M is just the ultrapower by $\mathcal{U}\times\mathcal{V}$ (or equivaletly by $\mathcal{V}\times\mathcal{U}$) and i is just the corresponding elementary embedding.

Consider the forcing $j_{\mathcal{U}}(\mathbb{P}*Q_\lambda)$ in $M_{\mathcal{U}}$. It splits into the part $\mathbb{P}*Q_\lambda*F_\lambda$ and the part $\mathbb{P}_{(\kappa,j_{\mathcal{U}}(\lambda))}*Q_{j_{\mathcal{U}}(\lambda)}$ above λ.

The part above λ is more than $\lambda-$strategically closed. So, we can construct a master condition sequence $\langle \underset{\sim}{q}_\zeta \mid \zeta < \lambda^+\rangle$.
Now use $j_{\mathcal{V}}^{M_{\mathcal{U}}}$ and move to M. Then $\langle j_{\mathcal{V}}^{M_{\mathcal{U}}}(\underset{\sim}{q}_\zeta) \mid \zeta < \lambda^+\rangle$ will be a master condition sequence there. We refer to J. Cummings handbook article [5] for this type of arguments.
Finally, plugging in G^* and H, the embedding i extends to

$$i^* : V[G] \to M[G^*, H, R],$$

where R is generated by $\langle j_{\mathcal{V}}^{M_{\mathcal{U}}}(\underset{\sim}{q}_\zeta)_{G^* * H} \mid \zeta < \lambda^+\rangle$.
Define $\mathcal{U}^*$ in $V[G]$ as follows:

$$X \in \mathcal{U}^* \text{ iff } \lambda \in i^*(X).$$

Clearly, $\mathcal{U}^* \supseteq \mathcal{U}$, and so $\mathcal{U}^*$ is a uniform ultrafilter over λ.
It is not hard to see that the embedding i^* is actually the ultrapower embedding by $\mathcal{U}^*$, since the only generators of the original i are κ and λ, but we can obtain κ from H. So, only λ remains.
Clearly, $\sup(i''\lambda) = \lambda$. So, $i_{\mathcal{U}^*} = i^* \supseteq i$ implies that $\mathcal{U}^*$ is a weakly normal ultrafilter over λ.
Finally, λ is not a measurable cardinal in $V[G]$, since there exists a $\lambda-$Suslin tree.
$\square$

3.1 Extensions with different degrees of completeness.

The construction above provides an extension of normal ultrafilter $\mathcal{U}$ over λ to a $\kappa-$complete uniform weakly normal ultrafilter.
Let us argue that from a bit stronger assumptions it is possible to obtain extensions with different degrees of completeness.

Theorem 3.8 *Assume GCH. Let $\kappa < \lambda$ be two measurable cardinals.*
Assume that κ is a limit of measurable cardinals. Then there is a cofinality preserving generic extension which satisfies the following:

1. *GCH,*
2. *λ is not measurable,*
3. *there is a uniform $\kappa-$complete ultrafilter over λ.*
   Moreover, every $\mathcal{U}$ which is a normal ultrafilter over λ in V extends to a uniform $\kappa-$complete ultrafilter over λ.

4. *For every $\delta < \kappa$ there is $\delta', \delta \leq \delta' < \kappa$ and a uniform $\delta'-$complete ultrafilter over λ. Moreover, every $\mathcal{U}$ which is a normal ultrafilter over λ in V extends to a uniform $\delta'-$complete ultrafilter over λ.*

Proof. We will combine the construction of the previous section with those of Theorem 3.1. Fix some normal ultrafilters $\mathcal{V}$ on κ and $\mathcal{U}$ on λ. Fix in addition normal ultrafilters $\mathcal{V}_\delta$ over δ, for every $\delta < \kappa$
Iterate first the Cohen forcing and add function $f_\alpha : \alpha \to \alpha$ for every inaccessible $\alpha, \alpha \leq \kappa$.
Each of the ultrafilters above extends to a normal ultrafilter.
Let $\mathcal{V}'_\delta$ and $\mathcal{U}'$ be an extensions of $\mathcal{V}_\delta$ and $\mathcal{U}$, for every measurable $\delta < \kappa$.
Further we shall abuse the notation and refer to them as $\mathcal{V}_\delta$ and $\mathcal{U}$.
Over κ let us define extensions $\mathcal{V}(\delta)$ of $\mathcal{V}$, for every measurable $\delta \leq \kappa$. Pick $\mathcal{V}(\kappa)$ to be a normal ultrafilter which extends $\mathcal{V}$.
For every measurable $\delta < \kappa$, consider in V,

$$i_\delta := j_{\mathcal{V}} \circ j_{\mathcal{V}_\delta} : V \to M(\delta).$$

Extend the embedding i_δ to i^*_δ and change
the value of $f_{i_\delta(\kappa)}(\kappa)$ to δ. Let $\mathcal{V}(\delta)$ be the corresponding extension of $\mathcal{V}$.
Then it is a uniform weakly normal $\delta-$complete ultrafilter over κ.

We construct a Suslin tree as in 3.1. Now, any of $\mathcal{V}(\delta)$, $\delta \leq \kappa$, can be used to move κ, and so, to produce $\lambda-$branches in the tree.
This in turn will produce extensions of $\mathcal{U}$ with various degrees of completeness.
□

J. Bagaria and M. Magidor [2] considered the following variation of strong compactness:
Every $\lambda-$complete filter can be extended to a $\kappa-$complete ultrafilter.

This clearly holds if λ is a strongly compact, but in general λ need not be even a measurable.
Thus J. Bagaria and M. Magidor [2] showed that it is possible to have such situation even with a singular λ.

Let us point out that our constructions can be used to provide additional examples.

Theorem 3.9 *Assume GCH. Let λ be a supercompact cardinal and $\kappa < \lambda$ be a measurable. Then there is a cofinality preserving generic extension which satisfies the following:*

1. *GCH,*

2. *λ is not measurable,*

3. *every $\lambda-$complete filter can be extended to a $\kappa-$complete ultrafilter.*

Proof. We use the construction of 3.1, only instead of using a normal ultrafilter $\mathcal{U}$ over λ and extending its embedding $j_{\mathcal{U}}$, supercompact ultrafilters over $\mathcal{P}_\lambda(\mu)$ for various μ's are used.

Suppose that in the extension F is a $\lambda-$complete filter on an ordinal and let μ be a regular large enough such that $|F| \leq \mu$. Let $F = \{Z_\alpha \mid \alpha < \mu\}$. Pick names $\underset{\sim}{F}$ and $\underset{\sim}{Z}_\alpha$'s for F and Z_α's. Assume that the weakest condition already forces all of this.
Work in V. Let $\mathcal{V}$ be a normal ultrafilter over κ, $\mathcal{U}$ a normal ultrafilter over $\mathcal{P}_\lambda(\mu)$ and $i = j^{M_{\mathcal{V}}}_{j_{\mathcal{V}}(\mathcal{U})}$.
Then $j_{\mathcal{V}}(\underset{\sim}{F})$ is forced to consist of $j_{\mathcal{V}}(\{\underset{\sim}{Z}_\alpha \mid \alpha < \mu\})$.
Set $j_{\mathcal{V}}(\{\underset{\sim}{Z}_\alpha \mid \alpha < \mu\}) = \{\underset{\sim}{Z}'_\alpha \mid \alpha < j_{\mathcal{V}}(\mu)\}$.
The ultrapower $M := M^{M_{\mathcal{V}}}_{j_{\mathcal{V}}(\mathcal{U})}$ of $M_{\mathcal{V}}$ by $j_{\mathcal{V}}(\mathcal{U})$ is closed, inside $M_{\mathcal{V}}$, under $j_{\mathcal{V}}(\mu)-$sequences of its elements. In particular, $\{i(\underset{\sim}{Z}'_\alpha) \mid \alpha < j_{\mathcal{V}}(\mu)\} \in M$.
We have $\sigma = i \circ j_{\mathcal{V}} : V \to M$.
It extends in the generic extension to $\sigma^* : V[G] \to M[G^*]$, as in 3.1.
By elementarity, $\sigma^*(F)$ is a $\sigma(\lambda) > j_{\mathcal{V}}(\mu)$ complete filter. Also, $Z''_\alpha := (i(\underset{\sim}{Z}'_\alpha))_{G^*} \in \sigma^*(F)$, for every $\alpha < j_{\mathcal{V}}(\mu)$.
Hence, there is some $\eta \in \bigcap_{\alpha < j_{\mathcal{V}}(\mu)} Z''_\alpha$.
Now we can define a desired extension F^* of F be setting

$$Z \in F^* \Leftrightarrow \eta \in \sigma^*(Z).$$

□

It is possible to modify the construction a bit in order to insure that λ is the least cardinal which satisfies (3), i.e. every $\lambda-$complete filter can be extended to a $\sigma-$complete ultrafilter. We just follow Y. Kimchi and M. Magidor [12] and add non-reflecting stationary subsets of

$$\{\alpha \in (\gamma, \gamma^+) \mid \mathrm{cof}(\alpha) = \omega\},$$

for cofinally many $\gamma < \lambda$ preserving supercompactness of λ.

The next result is similar, only 3.8 is used instead of 3.1.

Theorem 3.10 *Assume GCH. Let λ be a supercompact cardinal and $\kappa < \lambda$ be a measurable. Assume that κ is a limit of measurable cardinals. Then there is a cofinality preserving generic extension which satisfies the following:*

1. GCH,

2. λ is not measurable,

3. every $\lambda-$complete filter can be extended to a $\kappa-$complete ultrafilter,

4. for every $\delta < \kappa$ there is $\delta', \delta \leq \delta' < \kappa$ such that every $\lambda-$complete filter can be extended to a $\delta'-$complete ultrafilter.

Again, using Y. Kimchi and M. Magidor [12] it is possible to insure that λ is the least such that every $\lambda-$complete filter can be extended to a $\sigma-$complete ultrafilter.

4 $\sigma-$complete ultrafilters over singular cardinals.

Let κ be a measurable cardinal.

Note that if $\lambda > \kappa$ is a singular cardinal of cofinality κ, then it is easy to define a $\kappa-$complete ultrafilter over λ.
Namely, let $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ be a cofinal in λ sequence and let $\mathcal{V}$ be a $\kappa-$complete ultrafilter over κ.
Set $X \subseteq \lambda$ to be in $\mathcal{U}$ iff $\{\alpha < \kappa \mid \lambda_\alpha \in X\} \in \mathcal{V}$.
Clearly, such $\mathcal{U}$ is not uniform on λ, but rather concentrates on a set of cardinality κ and it is is a copy of $\mathcal{V}$ to this set.

It is not hard to get a uniform ultrafilter.

Proposition 4.1 *Let κ be a measurable cardinal and $\lambda > \kappa$ a singular cardinal of cofinality κ. Suppose that λ is a limit of measurable cardinals. Then there exists a uniform $\kappa-$complete ultrafilter over λ.*

Proof. Fix a cofinal in λ sequence $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ whose members are measurable cardinals.
Let $\mathcal{V}$ be a $\kappa-$complete ultrafilter over κ and $\mathcal{V}_\alpha$ be a $\lambda_\alpha-$complete ultrafilter over λ_α, for every $\alpha < \kappa$. Define $\mathcal{U}$ over λ as follows:

$$X \in \mathcal{U} \text{ iff } \{\alpha < \kappa \mid X \cap \lambda_\alpha \in \mathcal{V}_\alpha\} \in \mathcal{V}.$$

It is easy to see that $\mathcal{U}$ is as desired.

□

The following is easy:

Proposition 4.2 *Suppose that U is a $\kappa-$complete weakly normal ultrafilter over a cardinal λ of cofinality κ.*
Then U concentrates on a set of cardinality κ. In particular, U is not uniform unless $\kappa = \lambda$.

Proof. By weak normality $[id]_U = \sup(j_U{}''\lambda)$.
Let $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ be an increasing continuous cofinal in λ sequence. Then in M_U,
$\sup(\{j_U(\lambda_\alpha) \mid \alpha < \kappa\}) = \sup(j_U{}''\lambda) = [id]_U$.
So, $[id]_U \in j_U(\{\lambda_\alpha \mid \alpha < \kappa\})$, since the image of a closed set is a closed set.
$\square$

Proposition 4.3 *Assume that κ is a measurable cardinal and $\lambda > \kappa$ with $o(\lambda) = \kappa$. Then there is a cardinal preserving forcing extension such that $\mathrm{cof}(\lambda) = \kappa$ and there is a $\kappa-$complete uniform ultrafilter over λ which extends*
a normal ultrafilter $\mathcal{U}$ over λ in V. In addition, $\mathcal{U}$ extends to a $\kappa-$complete ultrafilter over λ which concentrates on a set of cardinality κ.

Proof. We will use the Magidor forcing and the idea similar to Bagaria-Magidor [2].
Assume GCH for simplicity. Let $\mathcal{V}$ be a normal ultrafilter over κ,
$\vec{U} = \langle \mathcal{U}(\lambda, \alpha) \mid \alpha < \kappa \rangle$ be a coherent sequence of normal measures over λ. We refer, for example, to [6] for the relevant definitions.
We would like to change the cofinality of λ to κ using the Magidor forcing $\mathbb{M}_{\vec{U}}$.

Apply first $\mathcal{V}$. Let

$$i := j_{\mathcal{V}} : V \to M_{\mathcal{V}} = \mathrm{Ult}(V, \mathcal{V}).$$

Let $i(\vec{U}) = \langle \mathcal{U}'(\lambda, \alpha) \mid \alpha < i(\kappa) \rangle$. Then, $\mathcal{U}'(\lambda, \alpha) = \mathcal{U}(\lambda, \alpha) \cap M_{\mathcal{V}}$, for every $\alpha < \kappa$.
Now, in $M_{\mathcal{V}}$, each $\mathcal{U}'(\lambda, \alpha)$, $\alpha < i(\kappa)$, is a normal ultrafilter. In particular, $\mathcal{U}'(\lambda, \kappa)$ is such. So, if C is a closed unbounded subset of λ in V, then $i(C)$ is such in $M_{\mathcal{V}}$, and so, $i(C) \in \mathcal{U}'(\lambda, \kappa)$.
Still in $M_{\mathcal{V}}$, we apply $\mathcal{U}'(\lambda, \kappa)$:

$$j^{M_{\mathcal{V}}}_{\mathcal{U}'(\lambda,\kappa)} : M_{\mathcal{V}} \to M^{M_{\mathcal{V}}}_{\mathcal{U}'(\lambda,\kappa)}.$$

Denote $\mathcal{U}'(\lambda, \kappa)$ by $\mathcal{U}_1$, $j^{M_{\mathcal{V}}}_{\mathcal{U}'(\lambda,\kappa)}$ by j_1, $M^{M_{\mathcal{V}}}_{\mathcal{U}'(\lambda,\kappa)}$ by M_1 and $j_1(\lambda)$ by λ_1. Clearly, $crit(j_1) = \lambda$, and so, $\lambda < \lambda_1$. Also, $\lambda \in j_1(i(C)) \in j_1(\mathcal{U}_1)$.
Consider $j_2 := j^{M_1}_{j_1(\mathcal{U}'(\lambda,0))} : M_1 \to M_2 := M^{M_1}_{j_1(\mathcal{U}'(\lambda,0))}$.
Denote $j_2(\lambda_1)$ by λ_2.

Then, for every $X \subseteq \lambda$,

$$X \in \mathcal{U}(\lambda,0) \Leftrightarrow i(X) \in \mathcal{U}'(\lambda,0) \Leftrightarrow j_1(i(X)) \in j_1(\mathcal{U}'(\lambda,0)) \Leftrightarrow \lambda_1 \in j_2(j_1(i(X))).$$

Denote $j_2 \circ j_1 \circ i$ by σ. Then $\sigma(\lambda) = \lambda_2$ and $o(\lambda_2) = i(\kappa)$ in M_2.

Force now with $\mathbb{M}_{\vec{U}}$. Let G be a generic.
Note that (identifing G with the Magidor sequence which it generates)
it will be $\mathbb{M}_{\vec{\mathcal{U}}\restriction(\lambda,\kappa)}-$generic over $M_{\mathcal{V}}$.
Define U over λ in $V[G]$ as follows:

$X \in U$ iff, in M_2, there is $p \in \sigma(\mathbb{M}_{\vec{\mathcal{U}}\restriction(\lambda,\kappa)}) = \mathbb{M}_{\sigma(\vec{\mathcal{U}})\restriction(j_{\mathcal{U}}(\lambda),i(\kappa))}$ such that

$$p \restriction \lambda+1 \in G, p \setminus \lambda+1 \geq^* \langle \lambda_2, \lambda_2 \setminus \lambda_1 + 1\rangle \text{ and } p \Vdash (\lambda_1 \in \sigma(\utilde{X})).$$

The requirement $p \setminus \lambda+1 \geq^* \langle \lambda_2, \lambda_2 \setminus \lambda_1 + 1\rangle$ insures that the first element of the Magidor sequence in which is above λ will be above λ_1, as well. This will be used for a uniformity argument.
Then, U is a $\kappa-$complete ultrafilter which extends the normal ultrafilter $\mathcal{U}(\lambda,0)$
over λ in V.
It is possible to argue that $j_U \restriction V$ is further iteration of σ starting with λ_2 and using the measures of the coherent sequence on it. The critical points of these embeddings will generate the Magidor sequence over the image of M_2.
Let us argue now that U is uniform.

Lemma 4.4 *U is a uniform ultrafilter over λ in $V[G]$.*

Proof. Suppose otherwise. Then there is a set $Z \subseteq \lambda$ of cardinality less than λ inside U. Let $|Z| = \delta < \lambda$.
It follows that in M_U,$|j_U(Z)| = j_U(\delta) < j_U(\lambda)$, but $j_U \restriction \lambda = i \restriction \lambda$, and so, in M_U,$|j_U(Z)| = j_U(\delta) = i(\delta) < \lambda$.
Now, λ_1 is a regular cardinal in M_U. Hence, $j_U(Z) \cap \lambda_1$ is bounded in λ_1. Pick some bound $\xi < \lambda_1$.
Then $\xi = j_U(h)(\kappa,\lambda)$, for some $h : \kappa \times \lambda \to \lambda, h \in V$. Without loss of generality, we can assume that $h : \lambda \to \lambda$, since otherwise ξ can be replaced by a bigger ordinal which is still below λ_1 but is represented by such a function.
Now, in $V[G]$, define a function $g : \lambda \to \lambda$ as follows:

$$g(\nu) = \min(Z \setminus h(\nu)).$$

Then, $j_U(g)(\lambda) = \lambda_1$.
Note that λ is the $\kappa-$th element of the Magidor sequence for $j_U(\lambda)$ in M_U.
Denote the Magidor sequence of G by $\langle \lambda_\nu \mid \nu < \kappa \rangle$.
We are interested in $\langle g(\lambda_\nu) \mid \nu < \kappa \rangle$.
Note that in M_U, the $\kappa + 1-$th element of the Magidor sequence for $j_U(\lambda)$ is already above λ_1. So, $g(\lambda_\nu) < \lambda_{\nu+1}$, for every $\nu < \kappa$.
Using standard arguments on Magidor forcing, it is possible to find $f : \lambda \to \lambda, f \in V$ such that $f(\lambda_\nu) \geq g(\lambda_\nu)$, for every $\nu < \kappa$.
Then, in M_U, $j_U(f)(\lambda) \geq j_U(g)(\lambda) = \lambda_1$.
But this is impossible, since $j_U \restriction V$ starts with σ and $\sigma(f)(\lambda) = j_1(i(f)(\lambda)) < \lambda_1$.
□

It is not hard to find in $V[G]$ an extension of $\mathcal{U}(\lambda, 0)$ to a $\kappa-$complete ultrafilter over λ which concentrates on a set of cardinality κ. Just note that the set $\{\lambda_{\nu+1} \mid \nu < \kappa\}$ is almost contained in every set in $\mathcal{U}(\lambda, 0)$. We can just copy $\mathcal{V}$ onto this set which will produce the desired extension.
□

It is possible to obtain a similar result with $\lambda = \kappa^{+\kappa}$ in the extension.

Proposition 4.5 *Assume that κ is a measurable cardinal and $\lambda > \kappa$ is $\mathcal{P}^2(\lambda)-$hyper-measurable. Then in a forcing extension $\lambda = \kappa^{+\kappa}$ and there is a $\kappa-$complete uniform ultrafilter over λ which extends a measure over λ in V.*

Proof. We will use the Radin forcing with guiding generic for collapses (see for example J. Cummings [4]) and the idea similar to Bagaria-Magidor [2]. Note that the $\kappa-$complete ultrafilter over λ of Bagaria-Magidor [2] (with translation to the present context) is not uniform, but rather concentrates on a set of cardinality κ, namely on the ordinals of the Radin sequence. But we can use a different one which has an additional generator, i.e. $\langle \lambda, \mu \rangle$. The argument for showing uniformity, repeats basically those of Lemma 4.4.
□

5 Blowing up the power of κ to λ.

We would like to construct a model with $2^\kappa \geq \lambda$ and with a uniform $\kappa-$complete ultrafilter U over λ.
Note that it is easy to do so starting from a supercompact cardinal κ and blowing up

its power preserving supercompactness. However, in such construction κ^+ will not be a continuity point of j_U, i.e. $j_U(\kappa^+) > \sup(j_U{}''\kappa^+)$.
We will start from a weaker assumptions and in our model κ^+ will be a continuity point of j_U.

Assume GCH. Let κ be a superstrong cardinal with a regular target, i.e. there is $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa)$ is a regular cardinal in V and $M \supseteq V_{j(\kappa)}$. Assume that j is the embedding by an extender.
Let $\lambda, \kappa < \lambda < j(\kappa)$ be a measurable cardinal.
We can assume that $\lambda = j(h_\lambda)(\kappa)$ for some function $h_\lambda : \kappa \to \kappa$, see [6] for the argument.

We would like to add $\lambda-$many Cohen functions from κ to κ. Do it as follows: define the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq j(\kappa), \beta < j(\kappa)\rangle$ where

1. $P_0 = \emptyset$;

2. if β is not inaccessible or β is an inaccessible, but β is not closed under $j(h_\lambda)$, then let Q_β be a trivial forcing;

3. if β is an inaccessible and β is closed under $j(h_\lambda)$, then let $Q_\beta = Cohen(\beta, j(h_\lambda)(\beta))$.

Back in V, let $\mathcal{U}$ be a normal ultrafilter over λ. Consider, in M, its image $j(\mathcal{U})$. Let $i : M \to N$ be its ultrapower embedding. Set $\sigma = i \circ j$.
Clearly,

$$X \in \mathcal{U} \Leftrightarrow j(X) \in j(\mathcal{U}) \Leftrightarrow j(\lambda) \in \sigma(X).$$

So,

$$\mathcal{U} = \{X \subseteq \lambda \mid j(\lambda) \in \sigma(X)\}.$$

We would like to extend the embedding σ. In order to do this, let us force Cohen subsets to $j(\kappa)$ over $V^{P_{j(\kappa)}}$. Force with $Cohen(j(\kappa), j(\lambda))$.
Note that $|j(\kappa)| = |j(\lambda)| = |j(\lambda^+)| = |\sigma(\lambda)|$, since j is the embedding by an extender. So, we can arrange $\sigma(\lambda)-$Cohen functions from $j(\kappa)$ to $j(\kappa)$.
Note that if $G \subseteq Cohen(j(\kappa), \sigma(\lambda))-$generic over $V^{P_{j(\kappa)}}$,
G is not $Cohen(j(\kappa), \sigma(\lambda))-$generic over $M^{P_{j(\kappa)}}$.
Moreover, G is not a subset of $Cohen(j(\kappa), \sigma(\lambda))^{M^{P_{j(\kappa)}}}$. Namely, $j''\kappa^+ \notin M$, but, in V,

$$\langle j(\alpha), \langle 0, 0\rangle\rangle \mid \alpha < \kappa^+\rangle$$

is a legitimate condition in $Cohen(j(\kappa), \sigma(\lambda))$.
In order to overcome this, we proceed as follows. Let $\langle f_\alpha \mid \alpha < \sigma(\lambda)\rangle$ be a sequence that

is $Cohen(j(\kappa), \sigma(\lambda))-$generic over $V^{P_{j(\kappa)}}$. Due to $j(\kappa)^+-$c.c. of the forcing, they will be $Cohen(j(\kappa), \sigma(\lambda))-$generic over $M^{P_{j(\kappa)}}$ as well.
Namely,

$$G' = \{p \in Cohen(j(\kappa), \sigma(\lambda))^{M^{P_{j(\kappa)}}} \mid \forall \alpha \in \mathrm{dom}(p)(p(\alpha) \subseteq f_\alpha)\}$$

will be $Cohen(j(\kappa), \sigma(\lambda))-$generic over $M^{P_{j(\kappa)}}$.
Thus, let in $M^{P_{j(\kappa)}}$, A be a maximal antichain in this forcing. Then $|A| \leq j(\kappa)$. So, there is $B \subseteq \sigma(\lambda)$ of size $j(\kappa)$ (in $M^{P_{j(\kappa)}}$) such that
$A \subseteq Cohen(j(\kappa), B)^{M^{P_{j(\kappa)}}} = Cohen(j(\kappa), B)^{V^{P_{j(\kappa)}}}$.
But then $\langle f_\alpha \mid \alpha \in B\rangle$ will generate $G' \restriction B$ and $G' \restriction B$ will be $V^{P_{j(\kappa)}}-$generic for $Cohen(j(\kappa), B)$ and so will intersect A.
Finally, we need to change these functions a bit in order to include $\sigma'' G(Cohen(\kappa, \lambda)) = j'' G(Cohen(\kappa, \lambda))$, where $G(Cohen(\kappa, \lambda)) \subseteq Cohen(\kappa, \lambda)$ is generic. Note that for every set of ordinals A of cardinality less than $j(\kappa)$ in M, $|A \cap j''\lambda| \leq \kappa$, since the extender producing j is on κ. So, the usual argument of Woodin allows to make the desired change, see Cummings [5]. The rest of the argument is standard.
Let $\sigma^* : V[G(P_{j(\kappa)+1})] \to N[G^*]$ be the resulting extension of σ.
Define

$$U = \{X \subseteq \lambda \mid j(\lambda) \in \sigma^*(X)\}.$$

Then $U \supseteq \mathcal{U}$ be a weakly normal uniform $\kappa-$complete ultrafilter.

Note that $j_U(\kappa) < j(\kappa)$, since $|j_U(\kappa)| \leq \lambda^\kappa = \lambda < j(\kappa)$.
So only a part of the embedding σ^* is actually used.

Note that all regular cardinals in the interval (κ, λ) are continuity points of σ^* and j_U. Menachem Magidor (unpublished) gave an other very elegant construction of a uniform ultrafilter with this property. He started with a supercompact cardinal and used intersection of dense open sets for Cohen forcing to generate the desired ultrafilter.

If we use a $(\kappa, \lambda^+)-$extender E instead of $(\kappa, j(\kappa))-$extender, then the only missing element will be a generic for $Cohen(j_E(\kappa), j_E(\lambda))$ over M_E. H. Woodin used originally the forcing $Cohen(\kappa^+, \lambda^+)$ over V in order to produce the desired generic. Recently Yoav Ben Shalom [3] showed that this additional Cohen forcing is unneeded, if we would like to blow up the power of κ below $\kappa^{+\kappa}$. It is unclear whether this is true in general, and, in particular in the case that we consider (i.e. blowing up the power of κ to a measurable λ).
However, if κ is λ^+-supercompact, then, using the standard methods, it is possible to blow up the power of κ to λ (or to λ^+). Moreover, if W is a normal measure over $\mathcal{P}_\kappa(\lambda^+)$, then the corresponding embedding $j_W : V \to M_W \simeq {}^{\mathcal{P}_\kappa(\lambda^+)}V/W$ extends in the generic extension

to $j^*_W : V[G] \to M[G^*]$ (as well as W to W^* which gives the embedding). The only forcing performed at cardinals $\geq \kappa$ is $Cohen(\kappa, \lambda)$. Now, let us derive a $(\kappa, \lambda^+)-$extender E^* (actually it is equivalent to a measure over κ) from j^*_W:

$$X \in E^*_a \Leftrightarrow a \in j^*_W(X),$$

for every finite $a \subseteq \lambda^+$ and $X \subseteq [\kappa]^{|a|}$. Then E^* is a natural extension of a $(\kappa, \lambda^+)-$extender E (now not equivalent to a measure over κ) from j_W in the same fashion.
Now the ultrapower M_{E^*} by E^* need not be in general a generic extension of the corresponding ultrapower M_E by E. It depends on particular M_W-generics for $Cohen(j_W(\kappa), j_W(\lambda))$. However, it is possible always to find E^* which is $(\kappa, \tau)-$extender for some τ of cardinality λ^+ which restriction E to V will be like this. For example pick an elementary submodel Y of large enough portion of M_W of cardinality $j_W(\lambda^+)$ such that $Y[G^*]$ is elementary in the extension $M_W[G^*]$. Then the transitive collapse of $Y \cap j_W(\kappa)$ will generate such E^* and E.

This means, in particular, that we have M_E-generic subsets of the missing Cohen part $Cohen(j_E(\kappa), j_E(\lambda))$.
We will explore this idea further in the next section.

6 On a question of J. Hamkins.

In [10], J. Hamkins introduced the following large cardinal:

Definition 6.1 A cardinal κ is called *strongly tall* iff for every $\delta > \kappa$ there is a uniform $\kappa-$complete ultrafilter U such that $j_U(\kappa) > \delta$.

Clearly, if κ is strongly compact then it is strongly tall. Hamkins [10] asked if the opposite is true as well.
In [1], a negative answer was claimed, however G. Goldberg found a gap in the argument.[8]
Goldberg proved several very nice results in the opposite direction. Thus for example he showed that under GCH strong tallness implies strong compactness.
The next proposition shows that still - the answer to the question is negative.

Proposition 6.2 *Assume GCH and let $\kappa < \lambda$ be supercompact cardinals. Then there is a cofinality preserving extension in which the following hold:*

1. *κ is strongly tall, but not strongly compact,*

[8]The mistake is due solely to the second author.

2. $2^\kappa = \lambda$,

3. every $\lambda-$*complete filter extends to a* $\kappa-$*complete ultrafilter.*

Proof. The argument will be similar to the one of the previous section, only instead of a single normal ultrafilter over λ we will use normal ultrafilters over $\mathcal{P}_\lambda(\mu)$ for μ's above λ.

We force the iteration similar the one of the previous section:
the Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa \rangle,$$

where for every $\beta < \kappa$, $\underset{\sim}{Q}_\beta$ is trivial unless β is a strongly inaccessible in V^{P_β},
and if β is a strongly inaccessible in V^{P_β}, then we first force an ordinal $\bar{\beta} < \kappa$ with an atomic forcing and then force with the Cohen forcing $Cohen(\beta, \bar{\beta})$ to add $\bar{\beta}-$many Cohen subsets to β.
Finally, over κ, $\underset{\sim}{Q}_\kappa = Cohen(\kappa, \lambda)$.

Back in V, let $\mu \geq \lambda$ be a regular. Pick a normal ultrafilter $\mathcal{U}$ over $\mathcal{P}_\lambda(\mu)$.
Let $\mathcal{V}$ be a normal ultrafilter over $\mathcal{P}_\kappa(\mu^+)$. Proceed as in the previous section with $j = j_\mathcal{V}$ and $i = j^{M_\mathcal{V}}_{j(\mathcal{U})}$.
Now, as it was explained at the end of the previous section, by supercompactness of κ we will have an appropriate generic for $Cohen(j(\kappa), j(\mu^+))$.
This allows to define a uniform $\kappa-$complete ultrafilter $U(\mu)$ over μ, namely, it will extend the one in V generated by $\sup(j_\mathcal{U}{''}\mu)$, i.e. $\{X \subseteq \mu \mid \sup(j_\mathcal{U}{''}\mu) \in j_\mathcal{U}(X)\}$.

In order to destroy the strong compactness κ we just add in addition a non-reflecting stationary subset to κ^+ of some fixed cofinality below κ. Do this as a product and not an iteration, i.e. over V^{P_κ} we force with $Cohen(\kappa, \lambda) \times R$, where R denotes the forcing adding such a set.

Let us argue that in the final model every $\lambda-$complete filter extends to a $\kappa-$complete ultrafilter.
Suppose that F is such a filter on an ordinal and let μ be a regular large enough such that $|F| \leq \mu$. Let $F = \{Z_\alpha \mid \alpha < \mu\}$. Pick names $\underset{\sim}{F}$ and $\underset{\sim}{Z}_\alpha$'s for F and Z_α's. Assume that the weakest condition already forces all of this.
Work in V. Let $\mathcal{V}, \mathcal{U}, i$ be as above.
Then $j_\mathcal{V}(\underset{\sim}{F})$ is forced to consist of $j_\mathcal{V}(\{Z_\alpha \mid \alpha < \mu\})$.
Set $j_\mathcal{V}(\{Z_\alpha \mid \alpha < \mu\}) = \{\underset{\sim}{Z}'_\alpha \mid \alpha < j_\mathcal{V}(\mu)\}$.
The ultrapower $M := M^{M_\mathcal{V}}_{j_\mathcal{V}(\mathcal{U})}$ of $M_\mathcal{V}$ by $j_\mathcal{V}(\mathcal{U})$ is closed, inside $M_\mathcal{V}$, under $j_\mathcal{V}(\mu)-$sequences

of its elements. In particular, $\{i(\underset{\sim}{Z}'_\alpha) \mid \alpha < j_{\mathcal{V}}(\mu)\} \in M$.
We have $\sigma = i \circ j_{\mathcal{V}} : V \to M$.
It extends in the generic extension to $\sigma^* : V[G] \to M[G^*]$, where G is a generic subset of $P_\kappa * (Cohen(\kappa, \lambda) \times R)$.
By elementarity, $\sigma^*(F)$ is $\sigma(\lambda) > j_{\mathcal{V}}(\mu)$ complete filter. Also, $Z''_\alpha := (i(\underset{\sim}{Z})'_\alpha)_{G^*} \in \sigma^*(F)$, for every $\alpha < j_{\mathcal{V}}(\mu)$.
Hence, there is some $\eta \in \bigcap_{\alpha < j_{\mathcal{V}}(\mu)} Z''_\alpha$.
Now we can define a desired extension F^* of F be setting

$$Z \in F^* \Leftrightarrow \eta \in \sigma^*(Z).$$

Claim 1 κ is strongly tall in the generic extension.

Proof. Let $\mu > \lambda$ be a cardinal of countable cofinality.
Let F be a $\lambda-$complete uniform ultrafilter over μ^+ in V with $j_F(\lambda) > \mu$,
say $F = \{X \subseteq \mu^+ \mid \sup(j_{\mathcal{U}}{}''\mu^+) \in j_{\mathcal{U}}(X)\}$, where $\mathcal{U}$ is a normal ultrafilter over $\mathcal{P}_\lambda(\mu^+)$.
Then, F is still a $\lambda-$complete filter in $V[G]$, since the forcing used has $\lambda-$c.c. and so every $A \subseteq F, |A| < \lambda$ can be covered by some $B \in V, |B| < \lambda$.
Let $F^* \supseteq F$ be a $\kappa-$complete ultrafilter.
Note that $\kappa \leq \mathrm{crit}(j_{F^*}) \leq \lambda$ is a measurable cardinal and $2^\kappa = \lambda$, hence $\mathrm{crit}(j_{F^*}) = \kappa$.
In V, $j_F(\lambda) > \mu$. So there are $j_F(\lambda)-$many functions from μ^+ to λ increasing mod F. We have $F^* \supseteq F$, hence they will be such mod F^*. In particular, $j_{F^*}(\lambda) > \mu$. In $V[G]$, $2^\kappa = \lambda$, so by elementarity, in M_{F^*}, $2^{j_{F^*}(\kappa)} = j_{F^*}(\lambda) > \mu$.
Recall that we have GCH above λ in $V[G]$. Hence, $j_{F^*}(\kappa) > \mu$.
$\square$ of the claim.
$\square$

7 An example of a not $(\kappa, \kappa^+)-$regular uniform $\kappa-$complete ultrafilter over κ^+.

In [11], J. Ketonen proved many interesting results concerning regularity of $\sigma-$complete ultrafilters. The existence of a not $(\kappa, \kappa^+)-$regular[9] uniform $\kappa-$complete ultrafilter over κ^+ remained open, at least to the best of our knowledge.

The example below shows that it is possible to have such ultrafilters.

[9] An ultrafilter U is called (κ, λ) regular iff there is subset of U of cardinality λ such that any $\kappa-$members of it have empty intersection.

Proposition 7.1 *Suppose that κ is a huge cardinal. Then there is an extension with a $\kappa-$complete uniform weakly normal ultrafilter U over κ^+ such that*
$\{\nu < \kappa^+ \mid \mathrm{cof}(\nu) = \kappa\} \in U$.
In particular, by Ketonen, [11], U is not $(\kappa, \kappa^+)-$regular.

Proof. Assume GCH and let $\mathcal{V}$ be a normal ultrafilter over $\mathcal{P}^\kappa(\kappa_1)$[10] witnessing hugeness of κ, with $j_\mathcal{V}(\kappa) = \kappa_1$.

Let us use the almost huge part of the embedding. Namely, for every $\lambda < \kappa_1$, set

$$W_\lambda = \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid j_\mathcal{V}{}''\lambda \in j_\mathcal{V}(X)\}.$$

Let $j : V \to M$ be the corresponding direct limit embedding of the system $\langle W_\lambda \mid \lambda < \kappa_1\rangle$.

Note that the following hold:

1. $j(\kappa) = \kappa_1$,
2. $j \restriction \kappa_1 = j_\mathcal{V} \restriction \kappa_1$,
3. ${}^{\kappa_1>}M \subseteq M$,
4. $j''\kappa_1 \notin M$,
5. $j(\kappa_1) = \sup(j_\mathcal{V}{}''\kappa_1)$,
6. $|\kappa_2| = \kappa_1$, where $\kappa_2 = j(\kappa_1)$,
7. $j(\kappa_1^+) = \bigcup j''\kappa_1^+ = \kappa_1^+$.

By the elementarity of j, κ_1 is a huge cardinal with the target κ_2 in M, as witnessed by $j(\mathcal{V})$.
Form an ultrapower N of M using $j(\mathcal{V})$. Let $i : M \to N$ be the corresponding embedding. Then $i(\kappa_1) = \kappa_2$, ${}^{\kappa_2}N \cap M \subseteq N$, $\delta_2 := \sup(i''\kappa_2) < \kappa_3 := i(\kappa_2)$. So, ${}^{\kappa_1>}N \subseteq N$. In addition note that in M, $|\kappa_3| = \kappa_2^+$, and so, in V, $\kappa_1^+ = j(\kappa_1^+) = (\kappa_2^+)^M < \kappa_3 < \kappa_1^{++}$.

Set $\sigma = i \circ j$. Then $\sigma : V \to N$, $\sigma(\kappa) = \kappa_2, \sigma(\kappa_1) = \kappa_3$.
$N \models \mathrm{cof}(\delta_2) = \kappa_2 = \sigma(\kappa)$.
Also, if $\xi < \delta_2$, then there is $\xi' < \kappa_2$ such that $\xi < i(\xi')$. But $\kappa_2 = \sup(j''\kappa_1)$, hence there is $\zeta < \kappa_1$ such that $\xi' < j(\zeta)$, and so,

$$\xi < i(\xi') < i(j(\zeta)) = \sigma(\zeta), \text{ which means that } \sigma''\kappa_1 \text{ is unbounded in } \delta_2.$$

[10] $\mathcal{P}^\kappa(\lambda) = \{X \subseteq \lambda \mid otp(X) = \kappa\}$.

Consider now

$$\mathcal{U} = \{Z \subseteq \kappa_1 \mid \delta_2 \in \sigma(Z)\}.$$

It is a uniform, weakly normal, $\kappa-$complete ultrafilter over κ_1 and $\{\nu < \kappa_1 \mid \mathrm{cof}(\nu) = \kappa\} \in \mathcal{U}$.

We would like now to force in order to turn κ_1 into κ^+.

Do this as follows.

Define by induction the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ Suppose that $\alpha < \kappa$ and P_α is defined. Define Q_α. Let Q_α be trivial unless α is a strongly inaccessible in V^{P_α}.

Suppose that α is a strongly inaccessible in V^{P_α}. Then choose (force with an atomic forcing) an inaccessible cardinal $\bar{\alpha}, \alpha < \bar{\alpha} < \kappa$ and set $Q_\alpha = Col(\alpha, < \bar{\alpha})$.

In V^{P_κ}, we force finally with $Col(\kappa, < \kappa_1)$.

Let $G_\kappa * H$ be a generic subset of $P_\kappa * Col(\kappa, < \kappa_1)$.

Extend σ to $\sigma^* : V[G_\kappa * H] \to N^* = N[G_\kappa * H * G_{(\kappa_1,\kappa_2)} * H^*]$.

In order to get $G_{(\kappa_1,\kappa_2)} * H^*$, we will force with $(P_{(\kappa_1,\kappa_2)} * Col(\kappa_2, < \kappa_3))^N$ over $V[G_\kappa * H]$.

It is a $< \kappa_1-$closed forcing of cardinality κ_1^+, which is equivalent to $Col(\kappa_1, \kappa_1^+)$.

We have, in N^*, $\sigma^*(Col(\kappa_1, \kappa_1^+)) = Col(\kappa_2, \kappa_2^+)$.

So, using a generic for $Col(\kappa_1, \kappa_1^+)$, we can generate an N^*-generic for $Col(\kappa_2, \kappa_2^+)^{N^*}$.

Let R be $V[G_\kappa * H]-$generic subset of $Col(\kappa_1, \kappa_1^+)$ and R^* the corresponding to it N^*-generic for $Col(\kappa_2, \kappa_2^+)^{N^*}$.

Turn now to the master condition.

We need to insure that $\sigma'' G_\kappa * H * R \subseteq G_\kappa * H * G_{(\kappa_1,\kappa_2)} * H^* * R^*$. The problematic parts here are H^* and R^*. In general, it should not be the case that $\sigma'' H \subseteq H^*$ and $\sigma'' R \subseteq R^*$. However, it is possible, using the Woodin method, see [5], to change H^* and R^* a bit in order to satisfy the inclusions above.

The property used for this is that $|\sigma''\kappa_1 \cap x| < \kappa_1$, for any $x \in N$ with $|x|^N < \kappa_2$.

Let us argue that this is true.

Suppose otherwise. Let x be a set of ordinals in N, $|x|^N < \kappa_2$ and $|\sigma''\kappa_1 \cap x| = \kappa_1$.

Then there is $y \subseteq \kappa_1$ of cardinality κ_1 such that $\sigma'' y = \sigma''\kappa_1 \cap x$. So, $\sigma'' y$ is covered by the set x.

We have $\sigma''\kappa_1$ that is unbounded in δ_2 and that in M (and so in N) the cofinality of δ_2 is κ_2, as was shown above.

But then, also $\sigma'' y$ will be unbounded in δ_2, which implies that $x \cap \delta_2$ is unbounded in δ_2.

This is impossible, since $x \in N$ and $|x|^N < \kappa_2$.

After such changes, we will have an extension of σ:

$$\sigma^{**} : V[G_\kappa * H * R] \to N^* = N[G_\kappa * H * G_{(\kappa_1,\kappa_2)} * H^* * R^*].$$

Finally, in $V[G_\kappa * H * R]$, we define an extension of $\mathcal{U}$:

$$\mathcal{U}^* = \{Z \subseteq \kappa_1 \mid \delta_2 \in \sigma^{**}(Z)\}.$$

It is a uniform, weakly normal, $\kappa-$complete ultrafilter over κ_1 and
$\{\nu < \kappa_1 \mid \mathrm{cof}(\nu) = \kappa\} \in \mathcal{U}^*$. However now $\kappa_1 = \kappa^+$.
$\square$

8 A remark on the Prikry forcing.

Let $\mathcal{U}$ be a $\kappa-$complete uniform ultrafilter over a regular cardinal $\lambda > \kappa$.
In the classical paper [15], K. Prikry showed that $\mathcal{U}$ can be used in order to change the cofinality of λ to ω. Conditions are of the form $\langle t, T\rangle$, where $t \in [\lambda]^{<\omega}$ and $T \subseteq [\lambda]^{<\omega}$ is a tree with a trunk t with splittings in $\mathcal{U}$, i.e. for every $s \in T, s \geq_T t$, $Suc_T(s) \in \mathcal{U}$. Denote this forcing by $\mathcal{P}_{\mathcal{U}}$.

Let $G \subseteq \mathcal{P}_{\mathcal{U}}$ be generic.

Proposition 8.1 *The following hold in $V[G]$:*

1. *κ and λ change their cofinality to ω,*
2. *if δ is a regular cardinal in V and singular in $V[G]$, then $\mathrm{cof}(\delta) = \omega$ in $V[G]$,*
3. *a regular in V cardinal changes its cofinality in $V[G]$ iff it is a discontinuity point of $j_{\mathcal{U}}$.*
4. *Suppose that $\delta, \kappa < \delta \leq \lambda$ is a successor cardinal of V which changes its cofinality in $V[G]$. Then there is a limit cardinal $\mu, \kappa \leq \mu < \delta$, such that, in $V[G]$,*
   (a) *μ is a cardinal,*
   (b) $\mathrm{cof}(\mu) = \omega$,
   (c) *all regular cardinals of V in the interval $[\mu, \delta]$ change their cofinality to ω,*
   (d) $\mathrm{cof}^V(\mu) \geq \kappa$.

Proof. (1) is clear.

(2). Suppose that δ is a regular cardinal in V and singular in $V[G]$. Suppose for a moment that it has an uncountable cofinality $\mu < \delta$ there. Let $\langle \delta_\xi \mid \xi < \mu \rangle$ be a cofinal sequence. For every $\xi < \mu$, pick a condition $\langle t_\xi, A_\xi \rangle \in G$ which decides δ_ξ, i.e. forces that the $\xi-$th element of the cofinal sequence is δ_ξ. Then there a single t such that for $\mu-$many ξ's, we have $t_\xi = t$. Denote by Y the set of all such ξ's. Clearly, $\mathrm{otp}(Y) = \mu, \bigcup_{\xi \in Y} \delta_\xi = \delta$. But now in V, consider the set

$$Z = \{\tau \mid \exists \xi < \mu \exists A \in \mathcal{U}(\langle t, A \rangle \Vdash (\check{\tau} = \underset{\sim}{\delta}_\xi))\}.$$

Then $Z = Y$, which is impossible, since $Z \in V$ and $\delta > \mu$ is regular in V.

(3). Let us suppose now that a regular cardinal δ is a discontinuity point of $j_\mathcal{U}$, i.e. $\sup(j_\mathcal{U}{}''\delta) < j_\mathcal{U}(\delta)$. Set

$$\mathcal{U}_\delta = \{X \subseteq \delta \mid \sup(j_\mathcal{U}{}''\delta) \in j_\mathcal{U}(X)\}.$$

Then $\mathcal{U}_\delta$ is a uniform at least $\kappa-$complete weakly normal ultrafilter over δ. In addition it is Rudin-Keisler below $\mathcal{U}$, since any function which represents $\sup(j_\mathcal{U}{}''\delta)$ in $M_\mathcal{U}$ will project $\mathcal{U}$ onto $\mathcal{U}_\delta$. Let $\pi_\delta : \lambda \to \delta$ be such a function.

Now, if $\langle \lambda_n \mid n < \omega \rangle$ is a Prikry sequence for $\mathcal{U}$, then $\langle \pi_\delta(\lambda_n) \mid n < \omega \rangle$ will be a Prikry sequence for $\mathcal{U}_\delta$.

Suppose now that some regular in V cardinal δ changes its cofinality in $V[G]$. Let us argue that it must be a discontinuity point of $j_\mathcal{U}$.

Suppose otherwise, i.e. $\sup(j_\mathcal{U}{}''\delta) = j_\mathcal{U}(\delta)$.

Clearly, $\kappa < \delta < \lambda$. By the observation above, $\mathrm{cof}(\delta) = \omega$ in $V[G]$. Pick a cofinal sequence $\langle \delta_n \mid n < \omega \rangle$.

Use a standard argument to find a condition in G which non-direct extensions of a same length decide values for $\underset{\sim}{\delta}_n$'s. Just work in V and build inductively a tree with splittings in $\mathcal{U}$ which graduately decide $\underset{\sim}{\delta}_n$'s.

Namely, we construct a tree T such that for every $t \in T$, $Suc_T(t) \in \mathcal{U}$ and there is an increasing sequence $\langle m_n \mid n < \omega \rangle$ of natural numbers such that for every $n < \omega$ and $t \in T, |t| = m_n + 1$, $\langle t, T_t \rangle || \underset{\sim}{\delta}_n$, where $T_t = \{s \in T \mid s \geq_T t\}$.

Suppose for simplicity that $m_n = n$, for all $n < \omega$.

Define a function $f_0 : \lambda \to \delta$ which prescribes values of $\underset{\sim}{\delta}_0$.

Namely, we set $f_0(\nu) = \xi$, if $\langle \nu \rangle \in Lev_1(T)$ and $\langle \nu, T_{\langle \nu \rangle} \rangle \Vdash \check{\xi} = \underset{\sim}{\delta}_0$.

Now, by the elementarity, $j_\mathcal{U}(f_0)([id]_\mathcal{U}) < j_\mathcal{U}(\delta)$, but $\sup(j_\mathcal{U}{}''\delta) = j_\mathcal{U}(\delta)$, hence there will be $\alpha_0 < \delta$ such that $\mathrm{rng}(f_0) \subseteq \alpha_0 \ (\mathrm{mod}(\mathcal{U}))$.

Shrink T if necessary, and deal with the second level. Let $\nu_1 \in Lev_1(T)$. Consider $Suc_T(\nu_1)$. Define $f_{\nu_1 1} : Suc_T(\nu_1) \to \delta$ which prescribes values of $\underset{\sim}{\delta}_1$.
Namely, we set $f_{\nu_1 1}(\nu) = \xi$, if $\langle \nu \rangle \in Suc_T(\nu_1)$ and $\langle \nu_1, \nu, T_{\langle \nu_1, \nu \rangle} \rangle \Vdash \check{\xi} = \underset{\sim}{\delta}_1$.
As above, there will be $\alpha_{\nu_1 1} < \delta$ such that $\mathrm{rng}(f_{\nu_1 1}) \subseteq \alpha_{\nu_1 1}$ $(\mathrm{mod}(\mathcal{U}))$.
Consider now the function $\nu_1 \mapsto \alpha_{\nu_1 1}$. By the same argument there is $\alpha_1 < \delta$ which contains its range $(\mathrm{mod}(\mathcal{U}))$.
But this means that α_1 will bound all the decision of $\underset{\sim}{\delta}_1$ made by any choice of $t \in T, |t| = 2$.
Proceed further in the similar fashion by induction on $n < \omega$ and define such $\alpha_n < \delta$ for $\underset{\sim}{\delta}_n$.
Finally set $\alpha := \bigcup_{n<\omega} \alpha_n$. It will be below δ and will bound all $\underset{\sim}{\delta}_n$'s, which is impossible.
Contradiction. So, δ must be a discontinuity point of $j_{\mathcal{U}}$, and we are done.

Finally, let us deal with the last item (4).
So, suppose that $\delta, \kappa < \delta \leq \lambda$ is a successor cardinal of V which change a cofinality in $V[G]$.
Then the cofinality of δ must be ω in $V[G]$, and in particular it is not a cardinal. Let $\mu = |\delta|$ (in $V[G]$). Clearly, $\mu \geq \kappa$, since κ remains a cardinal.
Suppose for a moment that $\mathrm{cof}^{V[G]}(\mu) \neq \omega$.
By S. Shelah [16], (Lemma 4.9, p.304), then $(\delta^+)^V$ is collapsed as well. Now, if $\delta = \lambda$, then we are done since $\mathcal{P}_{\mathcal{U}}$ satisfies λ^+-c.c.
Suppose that $\delta < \lambda$. Consider $\eta = (\mu^+)^V$. Then $\eta \leq \delta$ and η is collapsed, and so, changes its cofinality. Then, by Item 2, $\mathrm{cof}(\eta) = \omega$ in $V[G]$ and by Item 3, η is a discontinuity point of $j_{\mathcal{U}}$.
Consider the projection $\mathcal{U}_\eta$ of $\mathcal{U}$ to η:

$$\mathcal{U}_\eta = \{X \subseteq \eta \mid \sup(j_{\mathcal{U}}{}''\eta) \in j_{\mathcal{U}}(X)\}.$$

Then $\mathcal{U}_\eta$ is a uniform at least $\kappa-$complete weakly normal ultrafilter over η.
The Prikry forcing $\mathcal{P}_{\mathcal{U}_\eta}$ with $\mathcal{U}_\eta$ is a subforcing of $\mathcal{P}_{\mathcal{U}}$. Apply the argument to $\mathcal{P}_{\mathcal{U}_\eta}$. So both $\eta = (\mu^+)^V$ and μ should have cofinality ω in $V^{\mathcal{P}_{\mathcal{U}_\eta}}$, but the last model is contained in $V[G]$ and it is assumed that $\mathrm{cof}^{V[G]}(\mu) \neq \omega$.
Contradiction.

Finally, let us argue that $\mathrm{cof}^V(\mu) \geq \kappa$. Suppose otherwise. Then $\mathrm{cof}^V(\mu) = \omega$, since no cardinal below κ changes its cofinality.
Consider, as above, $\eta = (\mu^+)^V$. Then $\eta \leq \delta$ and η is collapsed, and so, changes its cofinality.
Hence, η is a discontinuity point of $j_{\mathcal{U}}$. Then, by Ketonen [11], 1.7, a final segment of regular cardinals below μ consists of discontinuity points of $j_{\mathcal{U}}$, since $\eta = (\mu^+)^V$. So, all of them change cofinality, and hence μ is not a cardinal in $V[G]$.[11] Contradiction.

[11]Basically, in $M_{\mathcal{U}}$, $\mathrm{cof}(\sup(j''_{\mathcal{U}}\eta)) < j_{\mathcal{U}}(\mu) = \bigcup_{n<\omega} j_{\mathcal{U}}(\mu_n)$, where $\langle \mu_n \mid n < \omega \rangle \in V$ is a cofinal in μ

$\square$

9 Moving ordinals by $\sigma-$complete uniform ultrafilters.

Answering a question of D. Fremlin, it was shown in [7] that:
if U_0, U_1 are $\kappa-$complete ultrafilters over a measurable cardinal κ, then $|j_{U_0}(\tau)| = |j_{U_1}(\tau)|$, for every ordinal τ.

Let us note that this may break down for uniform ultrafilters with the same degree of completeness and over the same cardinal.

Suppose that $\mu < \lambda < \kappa$ are measurable cardinals. Pick a $\mu-$complete ultrafilter $\mathcal{W}$ over μ, a $\lambda-$complete ultrafilter $\mathcal{V}$ over λ and a $\kappa-$complete ultrafilter $\mathcal{U}$ over κ.
Consider $U_0' = \mathcal{W} \times \mathcal{V} \times \mathcal{U}$ and $U_1' = \mathcal{W} \times \mathcal{U}$.
Then U_0' is a uniform $\mu-$complete ultrafilter over $\mu \times \lambda \times \kappa$ and U_1' is a uniform $\mu-$complete ultrafilter over $\mu \times \kappa$.
Let U_0 be an equivalent to U_0' ultrafilter over κ and U_1 be an equivalent to U_1' ultrafilter over κ.
Then $j_{U_1}(\lambda) = \lambda$, but $j_{U_1}(\lambda) > \lambda^+$.

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sequence. Hence for some $n^* < \omega$ we will have in $M_{\mathcal{U}}$, $\mathrm{cof}(\sup(j''_{\mathcal{U}}\eta)) < j_{\mathcal{U}}(\mu_{n^*})$. But then, every regular cardinal in the interval $[\mu_{n^*}, \mu)$ will be a discontinuity point of $j_{\mathcal{U}}$.

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