Short Extenders Forcings I

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Contents

Let $\kappa = \bigcup_{n<\omega} \kappa_n$, for an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega \rangle$. Each of κ_n's is strong to a some degree below κ_{n+1}. The purpose of the present paper is to present new methods of blowing up the power of κ in situations of this type.
In Chapter 1, the simplest case the gap 3 (i.e. $2^\kappa = \kappa^{+3}$) is considered. The basic apparatus which consists of the preparation forcing which produces a structure with pistes, the main forcing with suitable structures and equivalence which allows cardinals preservation is introduced.
Chapter 2 deals with generalizations to gap 4 (again between κ and its power) and higher gaps. The main issue here is the preparation forcing that is supposed to preserve GCH.
Chapter 3 presents a small modification of the preparation forcing of the previous chapter that allows to preserve strong cardinals.
In Chapter 4, we deal with a certain PCF-configuration called *dropping cofinalities*.
In Chapter 5 techniques of dropping cofinalities are applied to specific problems like constructing models with arbitrary gaps between κ and its power from optimal large cardinal assumptions, the first fixed point of the $\aleph$–function.

The techniques developed here will be used in a subsequent paper [8] to construct a model with a countable set which pcf has cardinality $\aleph_1$.

Chapter 1

Gap 3

We introduce here a special structure that will be used in order to blow up the power of a singular cardinal κ of cofinality ω to κ^{+3}.

A knowledge of the method used for dealing with Gap 2 is assumed. We refer for this to [1] or to [10] with a smoother presentation.

1.1 The Preparation Forcing

We assume GCH.
A condition in the preparation forcing $\mathcal{P}'$, which we define below, will consists basically of an elementary chain of models of cardinality κ^{++} and a directed system elementary submodels of cardinality κ^{+}. Inside this directed system a crucial role will be played by a certain elementary chain which will be called *central line* . Let us give first a definition of both elementary chains.

Definition 1.1.1 The set $\mathcal{P}''$ consists of elements of the form

$$\langle B^{1\kappa^+}, A^{1\kappa^{++}} \rangle$$

so that the following hold:

1. $A^{1\kappa^{++}}$ is a continuous closed chain of length less than κ^{+3} of elementary submodels of $\langle H(\kappa^{+3}), \in, <, \subseteq, \kappa \rangle$ each of cardinality κ^{++}.

2. For each $X \in A^{1\kappa^{++}}$, we have $X \cap \kappa^{+3} \in On$. So, $X \supseteq \kappa^{++}$. Further we shall frequently identify such model X with the ordinal $X \cap \kappa^{+3}$ and also view $A^{1\kappa^{++}}$ as a closed set of ordinals.

3. If X is a non-limit element of the chain $A^{1\kappa^{++}}$ then

   (a) $A^{1\kappa^{++}} \restriction X := \{Y \mid Y \subset X, Y \in A^{1\kappa^{++}}\} \in X$,

   (b) ${}^{\kappa^+}X \subseteq X$.

4. $B^{1\kappa^+}$ is a continuous closed chain of length less than κ^{++} of elementary submodels of $\langle H(\kappa^{+3}), \in, <, \subseteq, \kappa\rangle$, each of cardinality κ^+. $B^{1\kappa^+}$ has a last element which we denote by $\max(B^{1\kappa^+})$.

5. For each $X \in B^{1\kappa^+}$, we have $X \cap \kappa^{++} \in On$. Hence $X \supseteq \kappa^+$.

6. If X is a non-limit element of the chain $B^{1\kappa^+}$ then

   (a) $B^{1\kappa^+} \restriction X := \langle Y \mid Y \subset X, Y \in B^{1\kappa^+}\rangle \in X$,

   (b) ${}^{\kappa}X \subseteq X$,

   (c) If $\delta < \sup(X)$ for some $\delta \in A^{1\kappa^{++}}$ (we identify here an element of $A^{1\kappa^{++}}$ with an ordinal), then $\min((X \cap On) \setminus \delta) \in A^{1\kappa^{++}}$.

The following technical notion will be needed in order to define $\mathcal{P}'$ (and will be used further as well).

Definition 1.1.2 Suppose that $\langle B^{1\kappa^+}, A^{1\kappa^{++}}\rangle \in \mathcal{P}''$, $F \in B^{1\kappa^+}$ and $F_0, F_1 \in F$. We say that the triple F_0, F_1, F is of Δ-*system type* iff

1. F_0 is the immediate predecessor of F in the chain $B^{1\kappa^+}$,

2. $F_1 \prec F$,

3. If $\delta < \sup(F_1 \cap On)$ for some $\delta \in A^{1\kappa^{++}}$, then $\min((F_1 \cap On) \setminus \delta) \in A^{1\kappa^{++}}$.

4. There are $\alpha_0, \alpha_1 \in A^{1\kappa^{++}}$ such that

   (a) $\mathrm{cof}(\alpha_0) = \mathrm{cof}(\alpha_1) = \kappa^{++}$,

   (b) $\alpha_0 \in F_0$ and $\alpha_1 \in F_1$,

   (c) $F_0 \cap F_1 \cap On = F_0 \cap \alpha_0 = F_1 \cap \alpha_1$,

   (d) either $\alpha_0 > \sup(F_1 \cap On)$ or $\alpha_1 > \sup(F_0 \cap On)$.

Intuitively, this means that F_0, F_1 behave as in a Δ-system with the common part below $\min \alpha_0, \alpha_1$.

Further let us call α_0, α_1 the witnessing ordinals for F_0, F_1, F.

The next condition will require more similarity:

5. (isomorphism condition)
   the structures

$$\langle F_0, \in, <, \subseteq, \kappa, A^{1\kappa^{++}} \cap F_0, f_{F_0} \rangle$$

   and

$$\langle F_1, \in, <, \subseteq, \kappa, A^{1\kappa^{++}} \cap F_1,\ f_{F_1} \rangle$$

   are isomorphic over $F_0 \cap F_1$, i.e. the isomorphism $\pi_{F_0 F_1}$ between them is the identity on $F_0 \cap F_1$, where $f_{F_0} : \kappa^+ \longleftrightarrow F_0$, $f_{F_1} : \kappa^+ \longleftrightarrow F_1$ are some fixed in advance bijections. It is possible for gap 3 to do without f_{F_i}.

   Note that, in particular, we will have that $\mathrm{otp}(F_0) = \mathrm{otp}(F_1)$[1] and $F_0 \cap \kappa^{++} = F_1 \cap \kappa^{++}$.

Definition 1.1.3 The set $\mathcal{P}'(\kappa)$ or, for simplicity, just $\mathcal{P}'$ consists of elements of the form

$$\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle$$

so that the following hold:

1. $A^{0\kappa^+} \in A^{1\kappa^+}$,
2. every $X \in A^{1\kappa^+}$ is either equal to $A^{0\kappa^+}$ or belongs to it,
3. $C^{\kappa^+} : A^{1\kappa^+} \to P(A^{1\kappa^+})$,
4. for every $X \in A^{1\kappa^+}$, $\langle C^{\kappa^+}(X), A^{1\kappa^{++}} \rangle \in \mathcal{P}''$ and X is the maximal model of $C^{\kappa^+}(X)$. In particular, each $C^{\kappa^+}(X)$ is an increasing continuous chain of models of cardinality κ^+.
5. (Coherence) If $X, Y \in A^{1\kappa^+}$ and $X \in C^{\kappa^+}(Y)$, then $C^{\kappa^+}(X)$ is an initial segment of $C^{\kappa^+}(Y)$ with X being the largest element of it.

   We call $C^{\kappa^+}(A^{0\kappa^+})$ *central line* of $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle$. The following conditions describe a special way in which $A^{1\kappa^+}$ is generated from the central line.

[1]Here and further by $\mathrm{otp}(X)$ we mean $\mathrm{otp}(X \cap On)$.

6. Let $B \in A^{1\kappa^+}$. Then $B \in C^{\kappa^+}(A^{0\kappa^+})$ (i.e., it is on the central line) or there are $n < \omega$ and sequences $\langle A_1, ..., A_n \rangle$, $\langle B_1, ..., B_n \rangle$ of elements of $A^{1\kappa^+}$ such that

   (a) $A_1 \in C^{\kappa^+}(A^{0\kappa^+})$ is the least model of the central line $C^{\kappa^+}(A^{0\kappa^+})$ that contains B.

   (b) A_1 is a successor model in $C^{\kappa^+}(A^{0\kappa^+})$. Let A_1^- denotes its immediate predecessor in $C^{\kappa^+}(A^{0\kappa^+})$.

   (c) The triple A_1^-, B_1, A_1 is of a Δ-system type with respect to $\langle C^{\kappa^+}(A^{0\kappa^+}), A^{1\kappa^{++}} \rangle$.

   (d) For each $m, 1 < m \leq n$,

      i. $A_m \in C^{\kappa^+}(B_{m-1})$ (i.e. it is on the central line of B_{m-1}) is the least model in $C^{\kappa^+}(B_{m-1})$ that contains B.

      ii. A_m is a successor model in $C^{\kappa^+}(B_{m-1})$. Let A_m^- denote its immediate predecessor in $C^{\kappa^+}(B_{m-1})$.

      iii. The triple A_m^-, B_m, A_m is of a Δ-system type with respect to $\langle C^{\kappa^+}(B_{m-1}), A^{1\kappa^{++}} \rangle$.

   (e) $B \in C^{\kappa^+}(B_n)$.

   We refer to the sequence $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n \rangle$ as the *piste* from $A^{0\kappa^+}$ (or from the central line) to B. Denote it by $pst(A^{0\kappa^+}, B)$.

   Let us call *n the distance of B from the central line*, denote it by $dcl(B)$. If it is on the central line, then set $dcl(B) = 0$.

   The next condition strengthens a bit the isomorphism condition (5) of Definition 1.1.2.

7. (isomorphism condition) Let $F_0, F_1, F \in A^{1\kappa^+}$ be of a Δ-system type and $X \in A^{1\kappa^+}$. Then $X \in F_0$ iff $\pi_{F_0 F_1}[X] \in F_1 \cap A^{1\kappa^+}$. This means that the structures of 1.1.2(5) remain isomorphic even if we add $F_0 \cap A^{1\kappa^+}$ to the first and $F_1 \cap A^{1\kappa^+}$ to the second.

8. (uniqueness) Let $F_0, F_1, F_1', F \in A^{1\kappa^+}$. If both triples F_0, F_1, F and F_0, F_1', F are of a Δ-system type, then $F_1 = F_1'$.

   Note that both conditions 7, 8 can be stated equivalently only in the case when F is on the central line.

Let us define also a piste to an ordinal.

Definition 1.1.4 Let $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in \mathcal{P}'$ and $\alpha \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. The sequence $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, A_{n1} \rangle$ of elements of $A^{1\kappa^+}$ is called a *piste* from $A^{0\kappa^+}$ to α iff

1. $A_1 \in C^{\kappa^+}(A^{0\kappa^+})$ is the least model of $C^{\kappa^+}(A^{0\kappa^+})$ with $\alpha \in A_1$,

2. either

   - A_1 is the least model of $C^{\kappa^+}(A^{0\kappa^+})$ and then $A_n^- = A_1$, i.e. the piste consists of A_1 alone,

     or

   - A_1^- exists, it is the immediate predecessor of A_1 on $C^{\kappa^+}(A^{0\kappa^+})$. If A_1^- is the unique immediate predecessor of A_1, or there is another one but α does belong to it, then the piste consists of $\langle A_1, A_1^- \rangle$. Otherwise, A_1^-, B_1, A_1 are of Δ-system type, $\alpha \in B_1$ and the piste continues.

3. For each $m, 1 < m \leq n$,

   (a) $A_m \in C^{\kappa^+}(B_{m-1})$ (i.e. it is on the central line of B_{m-1}) is the least model in $C^{\kappa^+}(B_{m-1})$ with $\alpha \in A_m$, either

   - A_m is the least model of $C^{\kappa^+}(B_{m-1})$ and then $B_n^- = A_m$,

     or

   - A_m^- exists, it is the immediate predecessor of A_m on $C^{\kappa^+}(B_{m-1})$. If A_m^- is the unique immediate predecessor of A_m, or there is another one but α does belong to it, then $A_n^- = A_m^-$. Otherwise, A_m^-, B_m, A_m are of Δ-system type, $\alpha \in B_m$ and the piste continues.

4. $\alpha \in A_n$ and either

   - A_n is the least model of $C^{\kappa^+}(B_{n-1})$ and then $A_n^- = A_{n1} = A_n$, i.e. the piste terminates at A_n;

     or

   - there exists the immediate predecessor of A_n in $C^{\kappa^+}(B_{m-1})$. Then A_n^- is this immediate predecessor of A_n and there is no $Z \in A^{1\kappa^+}$ such that A_n^-, Z, A_n is of a Δ-system type. In this case $A_{n1} = A_n^-$ and the piste terminates at A_n^-;

     or

   - there exists the immediate predecessor of A_n in $C^{\kappa^+}(B_{m-1})$. Then A_n^- is this immediate predecessor of A_n and there is $Z \in A^{1\kappa^+}$ such that A_n^-, Z, A_n is of a Δ-system type, witnessed by $\xi_0 \in A_n^- \cap A^{1\kappa^{++}}, \xi_1 \in Z \cap A^{1\kappa^{++}}$. Then $\alpha \notin Z$. If $\alpha \notin [\xi_1, \sup(Z)]$, then $A_{n1} = A_n^-$ and the piste to α terminates at A_n^-. If $\alpha \in [\xi_1, \sup(Z)]$, then $A_{n1} = Z$.

Note that pistes to ordinals terminate by the last model A_n to which the ordinal belongs followed by its immediate predecessor in $C^{\kappa^+}(A_n)$, whenever such predecessor exists.

Define now a well-founded relation called *the complexity of pistes*. We will use it further in inductive arguments.

Definition 1.1.5 (Complexity of pistes)
Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$.

- Suppose that $A, B \in A^{1\kappa^+}$. We say that the piste from $A^{0\kappa^+}$ to A is *simpler* than the piste from $A^{0\kappa^+}$ to B iff

  1. $A \subset B$, or
  2. $A \not\subset B, B \not\subset A, A \neq B$ and if $F \in A^{1\kappa^+}$ is the last common point of both pistes, then $A \subseteq F_0$, where F_0 is the immediate predecessor of F in $C^{\kappa^+}(F)$. Note that necessarily, there is $F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F is a triple of a Δ-system type and $B \subseteq F_1$.

- Suppose that $A \in A^{1\kappa^+}$ and $\alpha \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. We say that the piste from $A^{0\kappa^+}$ to A is *simpler* than the piste from $A^{0\kappa^+}$ to α iff

  1. A is one of the models of the piste to α,
     or
  2. if F is the last common model of the pistes, then $A \in C^{\kappa^+}(F)$, or $A \notin C^{\kappa^+}(F)$ and $A \subseteq F_0$, where F_0 is the immediate predecessor of F in $C^{\kappa^+}(F)$. Note, if the second possibility occurs, then, necessarily, there is $F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F is a triple of a Δ-system type and $\alpha \in F_1$.

- Suppose that $\alpha, \beta \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. We say that the piste from $A^{0\kappa^+}$ to α is *simpler* than the piste from $A^{0\kappa^+}$ to β iff $\alpha \neq \beta$, there is $F \in A^{1\kappa^+}$ which is the last common point of both pistes and

  1. there are $D, E \in C^{\kappa^+}(F)$ such that $\alpha \in D \in E$ and $\beta \in E \setminus D$,
     or
  2. there are $F_0, F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F are of a Δ-system type, $F_0 \in C^{\kappa^+}(F)$, $\alpha \in F_0$ and $\beta \in F_1$,

3. there are $F_0, F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F are of a Δ-system type, $F_0 \in C^{\kappa^+}(F)$,ξ_0, ξ_1 the witnessing ordinals, and $\beta \in F \setminus (F_0 \cup F_1)$, $\xi_1 \leq \beta \leq \sup(F_1)$ and $\alpha \in F_1$, or

4. there are $F_0, F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F are of a Δ-system type, $F_0 \in C^{\kappa^+}(F)$, and ξ_0, ξ_1 are the witnessing ordinals, and $\alpha \in F \setminus (F_0 \cup F_1)$, $\beta \in F_1$ and $\alpha < \xi_1$ or $\alpha > \sup(F_1)$.

Lemma 1.1.6 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ *and* $B \in A^{1\kappa^+}$. *Then*

1. $\langle\langle B, A^{1\kappa^+} \cap (B \cup \{B\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (B \cup \{B\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$.

2. *If* $B' \in A^{1\kappa^+}$ *and* $B' \subsetneq B$, *then* $B' \in B$.

Proof. We prove both statements simultaneously by an induction on $dcl(B)$ -the distance from the central line. If B is on the central line, then it is clear. Suppose that B is not on the central line. Consider the piste $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n\rangle$ from $A^{0\kappa^+}$ to B. We have

$$\langle\langle A_1^-, A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'.$$

Recall that A_1^-, B_1, A_1 are of the Δ-system type. Hence we have the isomorphism $\pi_{A_1^-,B_1}$ between A_1^- and B_1 which preserves all the relevant structure. In particular, it will move the piste from A_1^- to a model in $A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\})$ to the piste from B_1 to the corresponding under $\pi_{A_1^-,B_1}$ model of $A^{1\kappa^+} \cap (B_1 \cup \{B_1\})$. This easily implies that

$$\langle\langle B_1, A^{1\kappa^+} \cap (B_1 \cup \{B_1\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (B_1 \cup \{B_1\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'.$$

Suppose now that we have some $B' \in A^{1\kappa^+}, B' \subsetneq B_1$. If $B' \not\subseteq A_1^-$, then the piste from $A^{0\kappa^+}$ to B' goes via B_1, and hence $B' \in B_1$. Suppose that $B' \subseteq A_1^-$. It is impossible to have $B' = A_1^-$, since then

$$A_1^- \cap B_1 \supseteq B' = A_1^-,$$

which is clearly not the case. So, $B' \subsetneq A_1^-$. Then the piste from $A^{0\kappa^+}$ to B' goes via A_1^-, and hence $B' \in A_1^-$. Then $\pi_{A_1^-,B_1}(B') \in B_1$, but

$$\pi_{A_1^-,B_1}(B') = \pi_{A_1^-,B_1}``B' = B'.$$

So we are done.
Hence, $A^{1\kappa^+} \cap (B_1 \cup \{B_1\}) = A^{1\kappa^+} \cap \mathcal{P}(B_1)$.

Now we deal with B and $\langle\langle B_1, A^{1\kappa^+}\cap(B_1\cup\{B_1\}), C^{\kappa^+}\restriction A^{1\kappa^+}\cap(B_1\cup\{B_1\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$. The piste distance from B_1 to B is shorter than those from $A^{0\kappa^+}$ to B. So the induction hypothesis applies.

□

The next lemma is trivial.

Lemma 1.1.7 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ *and* $Z \in A^{1\kappa^{++}}$ *is so that* $Z \cap \kappa^{+3} \geq \sup(A^{0\kappa^+})$*. Then* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, \{Y \in A^{1\kappa^{++}} \mid Y \subseteq Z\}\rangle \in \mathcal{P}'$*.*

Let us introduce the following notation:

Definition 1.1.8 Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Then set

$$p \restriction B := \langle\langle B, A^{1\kappa^+} \cap (B \cup \{B\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (B \cup \{B\})\rangle, A^{1\kappa^{++}}\rangle.$$

We call $p \restriction B$ the restriction of p to B.
Similarly, if $Z \in A^{1\kappa^{++}}$, then set

$$p \restriction Z := \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, \{Y \in A^{1\kappa^{++}} \mid Y \subseteq Z\}\rangle.$$

Also, let $p \restriction (B, Z) := (p \restriction B) \restriction Z$, if $Z \cap \kappa^{+3} \geq \sup(B)$.

By the previous lemmas, $p \restriction (B, Z) \in \mathcal{P}'$.

The next lemma follows easily from the definitions.

Lemma 1.1.9 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$*,* $A \in A^{1\kappa^+}$ *and* $\delta \in A^{1\kappa^{++}}$*. If* $\delta < \sup(A)$*, then* $\min(A \setminus \delta) \in A^{1\kappa^{++}}$*.*

Proof. By 1.1.3(4), $\langle C^{\kappa^+}(A), A^{1\kappa^{++}}\rangle \in \mathcal{P}''$. So, it satisfies 1.1.1(6(c)) and we are done, if A is a successor model of $C^{\kappa^+}(A)$. Suppose A is a limit model of $C^{\kappa^+}(A)$. Let $\langle A_i \mid i < \eta\rangle$ be an increasing sequence of successor models of $C^{\kappa^+}(A)$ with $\bigcup_{i<\eta} A_i = A$. Now, $\delta < \sup(A)$, so starting with some $i^* < \eta$, we have $\delta < \sup(A_i)$. Just note that $i < j$ implies $A_i \in A_j$, hence $\langle \sup(A_i) \mid i < \eta\rangle$ is an increasing sequence of ordinals with limit $\sup(A)$. Set $\alpha_i = \min(A_i \setminus \delta)$, for each $i, i^* \leq i < \eta$. By 1.1.1(6(c)), $\alpha_i \in A^{1\kappa^{++}}$. Clearly, $i \geq j$ implies $\alpha_i \leq \alpha_j$. Hence, the sequence $\langle \alpha_i \mid i^* \leq i < \eta\rangle$ is eventually constant. Let α^* be this constant value. Then $\min(A \setminus \delta) = \alpha^*$ and we are done.

□

Definition 1.1.10 Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $A, B \in A^{1\kappa^+}$. We say that A *satisfies the intersection property with respect to* B or shortly $ip(A, B)$ iff either

1. $A \supseteq B$, or

2. $B \supseteq A$, or

3. $A \not\supseteq B, B \not\supseteq A$, and then there are $A' \in A^{1\kappa^+} \cap (A \cup \{A\})$ and $\eta \in A^{1\kappa^{++}} \cap A'$ such that

$$A \cap B = A' \cap \eta,$$

or just

$$A \cap B = A'.$$

Let $ipb(A, B)$ denotes that both $ip(A, B)$ and $ip(B, A)$ hold.

Lemma 1.1.11 *(The intersection lemma) Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $X, Y \in A^{1\kappa^+}$. Then $ipb(X, Y)$.*

Proof. Assume that $X \not\supseteq Y, Y \not\supseteq X$.
Consider the pistes $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n\rangle$ from $A^{0\kappa^+}$ to X and $\langle D_1, D_1^-, E_1, ..., D_{m-1}, D_{m-1}^-, E_{m-1}, D_m, D_m^-, E_m\rangle$ from $A^{0\kappa^+}$ to Y.
Let $B_k = E_k$ be the last place up to which the pistes coincide. Then we have both A_{k+1}, D_{k+1} in $C^{\kappa^+}(B_k)$ but at different places.
Suppose first that A_{k+1} is above D_{k+1}. Then $A_{k+1}^- = D_{k+1}$ or $A_{k+1}^- \supset D_{k+1}$, and then $D_{k+1} \in A_{k+1}^-$. Now, $A_{k+1}^-, B_{k+1}, A_{k+1}$ are of a Δ-system type. Hence by Definition 1.1.2(4), there are ordinals $\alpha_0, \alpha_1 \in A^{1\kappa^{++}} \cap A_{k+1}, \alpha_0 \in A_{k+1}^-$ and $\alpha_1 \in B_{k+1}$ such that

$$A_{k+1}^- \cap B_{k+1} = A_{k+1}^- \cap \alpha_0 = B_{k+1} \cap \alpha_1.$$

Recall that $X \subseteq B_{k+1}$ and $Y \subseteq A_{k+1}^-$. Hence,

$$X \cap Y = (X \cap B_{k+1}) \cap (Y \cap A_{k+1}^-) = (X \cap \alpha_1) \cap (Y \cap \alpha_0).$$

Let us use (7) of 1.1.3. Then

$$X' = \pi_{B_{k+1}, A_{k+1}^-}[X] \in A_{k+1}^- \cap A^{1\kappa^+}.$$

Also,

$$X \cap \alpha_1 = X' \cap \alpha_0,$$

since the isomorphism π_{B_{k+1}, A_{k+1}^-} is the identity over $B_{k+1} \cap A_{k+1}^-$. Hence,

$$X \cap Y = X \cap \alpha_1 \cap Y = X' \cap \alpha_0 \cap Y.$$

Consider

$$p := \langle\langle A^-_{k+1}, A^{1\kappa^+} \cap (A^-_{k+1} \cup \{A^-_{k+1}\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (A^-_{k+1} \cup \{A^-_{k+1}\})\rangle, A^{1\kappa^{++}}\rangle.$$

By Lemma 1.1.6, it is in $\mathcal{P}'$. We can apply the inductive hypothesis to p,X' and Y, since the piste from A^-_{k+1} to X' is shorter than those from $A^{0\kappa^+}$ to X (it is just a copy under $\pi_{B_{k+1},A^-_{k+1}}$ of the final segment $\langle B_{k+1}, ..., A_n, A^-_n, B_n\rangle$ of the original piste to X from $A^{0\kappa^+}$). Hence there are $Y' \in A^{1\kappa^+} \cap (Y \cup \{Y\})$ and $\eta \in A^{1\kappa^{++}} \cap Y'$ such that

$$X' \cap Y = Y' \cap \eta.$$

Then

$$X \cap Y = X' \cap \alpha_0 \cap Y = Y' \cap \eta \cap \alpha_0.$$

If $\alpha_0 \in Y'$, then we are done. Suppose otherwise. If $\alpha_0 \geq \sup(Y')$, then we can just remove it from the intersection above. If $\alpha_0 < \sup(Y')$, then replace it by $\min(Y' \setminus \alpha_0)$, which is in $A^{1\kappa^{++}}$ by Lemma 1.1.9.
This shows $ip(Y, X)$. Finally, using $\pi_{A^-_{k+1},B_{k+1}}$ and moving Y to B_{k+1}, the same argument shows $ip(X, Y)$.
□

It is easy to deduce the following generalization using an induction:

Lemma 1.1.12 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ *and* $A_1, ..., A_n \in A^{1\kappa^+}$*, for some* $n < \omega$ *. Then there are* $A' \in A^{1\kappa^+} \cap (A_1 \cup \{A_1\})$ *and* $\eta \in A^{1\kappa^{++}} \cap A'$ *such that* $A_1 \cap ... \cap A_n = A' \cap \eta$ *or just* $A_1 \cap ... \cap A_n = A'$.

We need to allow a possibility to change the component C^{κ^+} in elements of $\mathcal{P}'$ and replace one central line by another. It is essential for the definition of an order on $\mathcal{P}'$ given below.

Definition 1.1.13 Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Define $swt(p, B)$ (here swt stands for switch) to be

$$\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, D^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle\ ,$$

where D^{κ^+} is obtained from C^{κ^+} as follows:
$D^{\kappa^+} = C^{\kappa^+}$ unless B has exactly two immediate predecessors in $A^{1\kappa^+}$. If $B_0 \neq B_1$ are such predecessors of B and, say $B_0 \in C^{\kappa^+}(B)$, then we set $D^{\kappa^+}(B) = C^{\kappa^+}(B_1)^\frown B$. Extend D^{κ^+} on the rest in the obvious fashion just replacing $C^{\kappa^+}(B_0)$ by $C^{\kappa^+}(B_1)$ for models including B and then moving over isomorphic models.
Intuitively, we switched here from B_0 to B_1.

Note that $swt(swt(p, B), B) = p$.

Let us further denote $swt(p, B)$ also by $swt(p, B_0, B_1)$.

Define $q = swt(p, B_1, \dots, B_n)$ by applying the operation swt n-times:

$p_{i+1} = swt(p_i, B_i)$, for each $1 \le i \le n$, where $p_1 = p$ and $q = p_{n+1}$.

The following simple observation will be useful further.

Lemma 1.1.14 *Let $p = \langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), C^{\kappa^+}(p)\rangle, A^{1\kappa^{++}}(p)\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}(p)$. Then there are $E_1, ..., E_m \in A^{1\kappa^+}(p)$ such that $B \in C^{\kappa^+}(q)(A^{0\kappa^+}(p))$, where*

$$q = \langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), C^{\kappa^+}(q)\rangle, A^{1\kappa^{++}}(p)\rangle = swt(p, E_1, ..., E_m).$$

Proof. If $B \in C^{\kappa^+}(p)(A^{0\kappa^+}(p))$, then let $q = p$. Suppose otherwise. Consider the piste $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n\rangle$ from $A^{0\kappa^+}$ to B. Then

$$q = \langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), C^{\kappa^+}(q)\rangle, A^{1\kappa^{++}}(p)\rangle = swt(p, A_1^-, B_1, A_2^-, B_2, ..., A_n^-, B_n)$$

will be as desired.

□

Definition 1.1.15 Let $r, q \in \mathcal{P}'$. Then $r \ge q$ (r is stronger than q) iff there is $p = swt(r, B_1, \dots, B_n)$ for some $B_1, \dots, B_n$ appearing in r so that the following hold, where

$$p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$$
$$q = \langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle$$

(1.1) $\qquad A^{1\kappa^{++}} \cap (\max(B^{1\kappa^{++}}) + 1) = B^{1\kappa^{++}}$

(1.2) $\qquad B^{0\kappa^+} \in C^{\kappa^+}(A^{0\kappa^+})$ and $D^{\kappa^+}(B^{0\kappa^+})$ is an initial segment of $C^{\kappa^+}(A^{0\kappa^+})$

(1.3) $\qquad q = p \restriction (B^{0\kappa^+}, \max(B^{1\kappa^{++}}))$(as it was defined in 1.1.8).

Remarks (1) Note that if $t = swt(p, B_0, \dots, B_n)$ is defined, then $t \ge p$ and

$$p = swt(swt(p, B_0, \dots, B_n), B_n, B_{n-1}, \dots, B_0) = swt(t, B_n, \dots, B_0) \ge t$$

. Hence the switching produces equivalent conditions.

(2) We need to allow $swt(p, B)$ for the Δ-system argument. Since in this argument two conditions are combined into one and so C^{κ^+} should pick one of them only. Also it is needed for proving a strategic closure of the forcing.

(3) The use of finite sequences $B_0, \dots, B_n$ is needed in order to insure transitivity of the order $\leq$ on $\mathcal{P}'$.

Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle,\ A^{1\kappa^{++}}\rangle \in \mathcal{P}'$. Set $p\backslash\kappa^{++} = A^{1\kappa^{++}}$. Define $\mathcal{P}'_{\geq\kappa^{++}}$ to be the set of all $p\backslash\kappa^{++}$ for $p \in \mathcal{P}'$.

The next lemma is obvious.

Lemma 1.1.16 *$\langle \mathcal{P}'_{\geq\kappa^{++}}, \leq \rangle$ is κ^{+3}-closed.*

Set $p \restriction \kappa^{++} = \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle$ where $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle,\ A^{1\kappa^{++}}\rangle \in \mathcal{P}'$.

Let $G(\mathcal{P}'_{\geq\kappa^{++}})$ be a generic subset of $\mathcal{P}'_{\geq\kappa^{++}}$. Define $\mathcal{P}'_{<\kappa^{++}}$ to be the set of all $p \restriction \kappa^{++}$ for $p \in \mathcal{P}'$ with $p\backslash\kappa^{++} \in G(\mathcal{P}'_{\geq\kappa^{++}})$.

Let $p \in \mathcal{P}'$ and $q \in \mathcal{P}'_{\geq\kappa^{++}}$. Then $q^\frown p$ denotes the set obtained from p by adding q to the last component of p, i.e. to $A^{1\kappa^{++}}$.

The following lemma is trivial.

Lemma 1.1.17 *Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$, $\max(A^{1\kappa^{++}}) \geq \sup(A^{0\kappa^+})$, $q \in \mathcal{P}'_{\geq\kappa^{++}}$ and $q \geq_{\mathcal{P}'_{\geq\kappa^{++}}} p\backslash\kappa^{++}$. Then $q^\frown p \in \mathcal{P}'$ and $q^\frown p \geq p$.*

It follows now that $\mathcal{P}'$ projects to $\mathcal{P}'_{\geq\kappa^{++}}$.

Let us turn to the chain condition.

Lemma 1.1.18 *The forcing $\mathcal{P}'_{<\kappa^{++}}$ satisfies κ^{+3}-c.c. in $V^{\mathcal{P}'_{\geq\kappa^{++}}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \Vdash_{\mathcal{P}'_{\geq\kappa^{++}}} (\langle \underset{\sim}{p}_\alpha = \langle \underset{\sim}{A}^{0\kappa^+}_\alpha, \underset{\sim}{A}^{1\kappa^+}_\alpha, \underset{\sim}{C}^{\kappa^+}_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle \text{ is an antichain in } \underset{\sim}{\mathcal{P}'}_{<\kappa^{++}})$$

Without loss of generality we can assume that each $A^{0\kappa^+}_\alpha$ is forced to be a successor model, otherwise just extend conditions by adding one additional model on the top. Define by induction, using Lemma 1.1.16, an increasing sequence $\langle q_\alpha \mid \alpha < \kappa^{+3}\rangle$ of elements of $\mathcal{P}'_{\geq\kappa^{++}}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{+3}\rangle$, $p_\alpha = \langle A^{0\kappa^+}_\alpha, A^{1\kappa^+}_\alpha, C^{\kappa^+}_\alpha\rangle$ so that for every $\alpha < \kappa^{+3}$

$$q_\alpha \Vdash_{\mathcal{P}'_{\geq\kappa^{++}}} \langle \underset{\sim}{A}^{0\kappa^+}_\alpha, \underset{\sim}{A}^{1\kappa^+}_\alpha, \underset{\sim}{C}^{\kappa^+}_\alpha\rangle = \check{p}_\alpha \ .$$

For a limit $\alpha < \kappa^{+3}$ let

$$\overline{q}_\alpha = \bigcup_{\beta<\alpha} q_\beta \cup \{\sup \bigcup_{\beta<\alpha} q_\beta\}$$

and q_α be its extension deciding $\underset{\sim}{p}_\alpha$. Also assume that $\max q_\alpha \geq \sup(A^{0\kappa^+}_\alpha \cap \kappa^{+3})$.

We form a Δ-system. By shrinking if necessary assume that for some stationary $S \subseteq \kappa^{+3}$ and $\delta < \kappa^{+3}$ we have the following for every $\alpha < \beta$ in S:

(a) $A^{0\kappa^+}_\alpha \cap \alpha = A^{0\kappa^+}_\beta \cap \beta \subseteq \delta$

(b) $(A^{0\kappa^+}_\alpha \cap \kappa^{+3})\backslash\alpha \neq \emptyset$

(c) $\sup(A^{0\kappa^+}_\alpha \cap \kappa^{+3}) < \beta$

(d) $\sup \overline{q}_\alpha = \alpha + 1$

(e)

$$\langle A^{0\kappa^+}_\alpha, \in, \leq, \subseteq, \kappa, C^{\kappa^+}_\alpha, f_{A^{0\kappa^+}_\alpha},\ A^{1\kappa^+}_\alpha, q_\alpha \cap A^{0\kappa^+}_\alpha\rangle$$
$$\langle A^{0\kappa^+}_\beta, \in, \leq, \subseteq, \kappa, C^{\kappa^+}_\beta, f_{A^{0\kappa^+}_\beta},\ A^{1\kappa^+}_\beta, q_\beta \cap A^{0\kappa^+}_\beta\rangle$$

are isomorphic over δ, i.e. by isomorphism fixing every ordinal below δ, where

$$f_{A^{0\kappa^+}_\alpha} : \kappa^+ \longleftrightarrow A^{0\kappa^+}_\alpha$$

and

$$f_{A^{0\kappa^+}_\beta} : \kappa^+ \longleftrightarrow A^{0\kappa^+}_\beta$$

are the fixed enumerations.

We claim that for $\alpha < \beta$ in S it is possible to extend q_β to a condition forcing compatibility of p_α and p_β. Proceed as follows. Pick A to be an elementary submodel of cardinality κ^+ so that

(i) $A^{1\kappa^+}_\alpha, A^{1\kappa^+}_\beta \in A$

(ii) $C^{\kappa^+}_\alpha, C^{\kappa^+}_\beta \in A$

(iii) $q_\beta \in A$.

Extend q_β to $q = q_\beta \cup \sup(A \cap \kappa^{+3})$. Set $p = \langle A, A^{1\kappa^+}, C^{\kappa^+}\rangle$, where $A^{1\kappa^+} := A^{1\kappa^+}_\alpha \cup A^{1\kappa^+}_\beta \cup \{A\}$, $C^{\kappa^+} := C^{\kappa+}_\alpha \cup C^{\kappa^+}_\beta \cup \langle A, C^{\kappa^+}_\beta(A^{0\kappa^+}_\beta)^\frown A\rangle\rangle$.
Clearly, $\langle C^{\kappa^+}(A), q\rangle \in \mathcal{P}''$.
The triple $A^{0\kappa^+}_\beta, A^{0\kappa^+}_\alpha, A$ is of a Δ-system type relatively to q, by (e) above. It follows that $\langle p, q\rangle \in \mathcal{P}'$. Thus the condition (6) of Definition 1.1.3 holds since each of $\langle p_\alpha, q\rangle, \langle p_\beta, q\rangle$ satisfies it. The condition (7) of Definition 1.1.3 follows from (e) above and since both $\langle p_\alpha, q\rangle, \langle p_\beta, q\rangle$ satisfy it.

□

Lemma 1.1.19 *$\mathcal{P}'$ is κ^{++}-strategically closed.*

Proof. We define a winning strategy for the player playing at even stages. Thus suppose $\langle p_j \mid j < i \rangle$, $p_j = \langle \langle A_j^{0\kappa^+}, A_j^{1\kappa^+}, C_j^{\kappa^+} \rangle, A_j^{1\kappa^{++}} \rangle$ is a play according to this strategy up to an even stage $i < \kappa^{++}$. Set first

$$B_i^{0\kappa^+} = \bigcup_{j<i} A_j^{0\kappa^+}, B_i^{1\kappa^+} = \bigcup_{j<i} A_j^{1\kappa^+} \cup \{B_i^{0\kappa^+}\},$$

$$D_i^{\kappa^+} = \bigcup_{j<i} C_j^{\kappa^+} \cup \{\langle B_i^{0\kappa^+}, \{B_i^{0\kappa^+}\} \cup \{C_j^{\kappa^+}(A_j^{0\kappa^+}) \mid \ j \text{ is even}\} \rangle\}$$

and

$$B_i^{1\kappa^{++}} = \bigcup_{j<i} A_j^{1\kappa^{++}} \cup \{\sup \bigcup_{j<i} A_j^{1\kappa^{++}}\}.$$

Then pick $A_i^{0\kappa^+}$ to be a model of cardinality κ^+ such that

(a) ${}^{\kappa}A_i^{0\kappa^+} \subseteq A_i^{0\kappa^+}$

(b) $B_i^{0\kappa^+}, B_i^{1\kappa^+}, D_i^{\kappa^+}, B_i^{1\kappa^{++}} \in A_i^{0\kappa^+}$.

Set $A_i^{1\kappa^+} = B_i^{1\kappa^+} \cup \{A_i^{0\kappa^+}\}$, $C_i^{\kappa^+} = D_i^{\kappa^+} \cup \{\langle A_i^{0\kappa^+}, D_i^{\kappa^+}(B_i^{0\kappa^+}) \cup \{A_i^{0\kappa^+}\} \rangle\}$ and $A_i^{1\kappa^{++}} = B_i^{1\kappa^{++}} \cup \{\sup(A_i^{0\kappa^+} \cap \kappa^{+3})\}$. As an inductive assumption we assume that at each even stage $j < i$, p_j was defined in the same fashion. Then $p_i = \langle \langle A_i^{0\kappa^+}, A_i^{1\kappa^+}, C_i^{\kappa^+} \rangle, A_i^{1\kappa^{++}} \rangle$ will be a condition in $\mathcal{P}'$ stronger than each p_j for $j < i$. The switching may be required here once moving from an odd stage to its immediate successor even stage.

□

Let us point out in addition the following:

Lemma 1.1.20 *Let G be a generic subset of $\mathcal{P}'$. Then the set*

$$S = \{A \mid \exists \langle \langle A^{0\kappa^+}, A^{1\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in G \quad A = A^{0\kappa^+}\}$$

is stationary subset of $[H(\kappa^{+3})]^{\leq \kappa^+}$ in $V[G]$.

Proof. Suppose otherwise. Then there are $p \in G$ and $\underset{\sim}{C}$ such that

$$p \Vdash \underset{\sim}{C} \text{ is a closed unbounded subset of } [H(\kappa^{+3})]^{\leq \kappa^+} \text{ disjoint to } \underset{\sim}{S}.$$

Work in V. Pick an elementary submodel M of $H(\chi)$, for large enough χ, such that $|M| = \kappa^+$, ${}^{\kappa}M \subseteq M$ and $\mathcal{P}', p, \underset{\sim}{C}, \underset{\sim}{S} \in M$. Let $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ be a list of all dense open subsets of $\mathcal{P}'$

which are M.
We use Lemma 1.1.19 and ${}^\kappa M \subseteq M$ in order to construct an increasing sequence $\langle\langle\langle A_\beta^{0\kappa^+}, A_\beta^{1\kappa^+}\rangle, A_\beta^{1\kappa^{++}}\rangle \mid \beta \leq \kappa^+\rangle$ of elements of $\mathcal{P}'$ above p such that for every $\alpha < \kappa^+$

1. $\langle\langle\langle A_\beta^{0\kappa^+}, A_\beta^{1\kappa^+}\rangle, A_\beta^{1\kappa^{++}}\rangle \mid \beta < \alpha\rangle \in M$,

2. $\langle\langle A_{\alpha+1}^{0\kappa^+}, A_{\alpha+1}^{1\kappa^+}\rangle, A_{\alpha+1}^{1\kappa^{++}}\rangle \in D_\alpha$,

3. $A_\alpha^{0\kappa^+} = \bigcup_{\beta<\alpha} A_\beta^{0\kappa^+}$, if α is a limit ordinal.

Then, clearly, $A_{\kappa^+}^{0\kappa^+} = M \cap H(\kappa^{+3})$ and $\langle\langle A_{\kappa^+}^{0\kappa^+}, A_{\kappa^+}^{1\kappa^+}\rangle, A_{\kappa^+}^{1\kappa^{++}}\rangle \Vdash A_{\kappa^+}^{0\kappa^+} \in \underset{\sim}{C}$. But also $\langle\langle A_{\kappa^+}^{0\kappa^+}, A_{\kappa^+}^{1\kappa^+}\rangle, A_{\kappa^+}^{1\kappa^{++}}\rangle \Vdash A_{\kappa^+}^{0\kappa^+} \in \underset{\sim}{S}$. Which is impossible. Contradiction.
□

1.2 Suitable structures and assignment functions

In the gap 2 case (see for example [1]) assignment functions a_n (those connecting the level κ with level $\kappa_n, n < \omega$) were order preserving. In other words a_n is an isomorphism between structures in the language containing only the predicate for the order relation. Here, in the gap 3 case (and beyond), a_n's will be isomorphisms between structures in more complicated languages.

Let us start with two definitions which will specify relevant structures.

Definition 1.2.1 A three sorted structure $\langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ is called *suitable structure (over κ)* iff

1. X has a maximal under inclusion element. Denote it by $\max(X)$.

2. $Y \subseteq \max(X)$,

3. C is a binary relation X,

4. $\langle\langle \max(X), X, C\rangle, Y\rangle \in \mathcal{P}'$, where for every $A \in X$ we identify $C(A)$ with the set $\{B \in X \mid \langle A, B\rangle \in C\}$.

5. $Z = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X \cup Y\}$.

Note that by Lemma 1.1.11, an intersection $t_1 \cap ... \cap t_n$ above is really simple, thus it is equal to an element of X or of Y or to $s \cap \alpha$, where $s \in X$ and $\alpha \in Y$.
Since $\langle\langle \max(X), X, C\rangle, Y\rangle \in \mathcal{P}'$ (item 4), it is possible to talk about pistes in a suitable structure $\langle\langle X, Y, Z\rangle, C, \in, \subseteq \rangle$.
Further we will consider suitable structures also over cardinals μ different from κ. The definition of such structures is the same only $\mathcal{P}' = \mathcal{P}'(\kappa)$ in the item 4 above should be replaced by $\mathcal{P}'(\mu)$.

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.

Definition 1.2.2 A suitable structure $\langle\langle X, Y, Z\rangle, C, \in, \subseteq \rangle$ is called *suitable generic structure* iff there is $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ such that

1. $\langle\langle X, Y, Z\rangle, C, \in, \subseteq \rangle$ is a substructure (not necessarily elementary) of $\langle\langle A^{1\kappa^+}, A^{1\kappa^{++}}, \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in A^{1\kappa^+} \cup A^{1\kappa^{++}}\}\rangle$,
   $C^{\kappa^+}, \in, \subseteq \rangle$,

2. $\max(X) \in C^{\kappa^+}(A^{0\kappa^+})$,

3. $\langle\langle \max(X), X, C\rangle, Y\rangle$ and $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ agree about the pistes to members of X and to ordinals in $\max(X) \cap Y$. In other words we require that all the elements of pistes in $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ to elements of X and to ordinals in $\max(X) \cap Y$ are in X.

Note that, as a condition in $\mathcal{P}'$, $\langle\langle \max(X), X, C\rangle, Y\rangle$ need not be weaker than $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$, and hence it need not be in $G(\mathcal{P}')$. Thus, for example, $A^{1\kappa^{++}}$ need not be an end extension of Y.
Note also, that any stronger condition $\langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ with $C^{\kappa^+}(A^{0\kappa^+})$ being an initial segment of $D^{\kappa^+}(B^{0\kappa^+})$ will witness that $\langle\langle X, Y, Z\rangle, C, \in, \subseteq \rangle$ is a suitable generic structure.

Lemma 1.2.3 *Let $\langle\langle X, Y, Z\rangle, C, \in, \subseteq \rangle$ be a suitable generic structure as witnessed by $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$. Suppose that $F_0, F_1, F \in A^{1\kappa^+}$, $F_0, F \in C^{\kappa^+}(A^{0\kappa^+})$ is a triple of a Δ-system type with α_0, α_1 as in Definition 1.1.2, and $\alpha_1 \in Y$. Then $F_0, F_1 \in X \cap \max(X), F \in X, \alpha_0 \in \max(X) \cap Y$.*

Proof. The piste to α_1 from $\max(X)$ (or the same from $A^{0\kappa^+}$) passes through F and turns to F_1. Hence, by 1.2.2(4), $F_0, F_1, F \in X$. Recall that by 1.2.1(4) we have $\langle\langle \max(X), X, C\rangle, Y\rangle \in$

$\mathcal{P}'$. Hence F_0, F_1, F are of a Δ-system type in $\langle\langle \max(X), X, C\rangle, Y\rangle$. Then there are $\alpha_0' \in F_0 \cap Y, \alpha_1' \in F_1 \cap Y$ such that

$$F_0 \cap F_1 = F_0 \cap \alpha_0' = F_1 \cap \alpha_1'.$$

But, also

$$F_0 \cap F_1 = F_1 \cap \alpha_1$$

and $\alpha_1, \alpha_1' \in Y \subseteq A^{1\kappa^{++}}$.
Hence, $\alpha_1 = \alpha_1'$. Finally, $\alpha_0' = \pi_{F_1,F_0}(\alpha_1) = \alpha_0$. Hence, $\alpha_0 \in \max(X) \cap Y$.
□

Lemma 1.2.4 *Let $p = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ and $p' = \langle\langle X', Y', Z'\rangle, C', \in, \subseteq\ \rangle$ be isomorphic suitable structures (even over different cardinals) and a an isomorphism between them. Suppose that F_0, F_1, F is a triple in X of a Δ-system type and $\alpha_0 \in F_0 \cap Y, \alpha_1 \in F_1 \cap Y$ are witnessing this ordinals. Then $a(F_0), a(F_1), a(F)$ is a triple in X' of a Δ-system type witnessed by $a(\alpha_0)$ and $a(\alpha_1)$.*

Proof. Obviously, α_0 and α_1 are uniquely determined by F_0 and F_1.
Denote $a(F_0)$ by F_0',$a(F_1)$ by F_1',$a(F)$ by F', $a(\alpha_0)$ by α_0' and $a(\alpha_1)$ by α_1'. Now, $F_0', F_1' \in F'$, moreover F_0' is the immediate predecessor of F' in $C(F')$ and F_1' is an additional predecessor of F' under the inclusion relation, since a is an isomorphism between p and p'. Note that by 1.2.1(4) this implies that F_0', F_1', F' is a Δ-system type triple in p'.
Let $\alpha_0'' \in F_0' \cap a(Y)$ and $\alpha_1'' \in F_1' \cap a(Y)$ be such that

$$F_0' \cap F_1' = F_0' \cap \alpha_0'' = F_1' \cap \alpha_1''.$$

Also $\alpha_0' \in F_0' \cap a(Y)$ and $\alpha_1' \in F_1' \cap a(Y)$, since a respects $\in$-relation. But then, necessarily, $\alpha_0' = \alpha_0'', \alpha_1' = \alpha_1''$.
□

Lemma 1.2.5 *Let $p = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ and $p' = \langle\langle X', Y', Z'\rangle, C', \in, \subseteq\ \rangle$ be isomorphic suitable structures (even over different cardinals) and a an isomorphism between them. Then a respects pistes, i.e. for every $A \in X$ and $B \in (X \cup Y) \cap A$, a maps the piste between A and B in p onto the piste between $a(A)$ and $a(B)$.*

Proof. Induction on pistes length. Thus, if B in $C(A)$ or if $B \in Y$ and the piste to it from A involves only $C(A)$, then the isomorphism a guaranties the same for the images. Suppose

that the piste proceeds with splitting. Let F_0, F_1, F be the first split on the way to B, i.e. $F \in C(A)$, the triple F_0, F_1, F is of a Δ-system type, $B \nsubseteq F_0$ (or, if $B \in Y$, $B \notin F_0$) and $B \subseteq F_1$ (or $B \in F_1 \cup \{F_1\}$). By the previous lemma (Lemma 1.2.4), $a(F_0), a(F_1), a(F)$ is a triple in X' of a Δ-system type. a is isomorphism, hence $a(F) \in C'(a(A)), a(F_0) \in C'(a(F_0))$, $a(B) \nsubseteq a(F_0)$ (or, if $B \in Y$, $a(B) \notin a(F_0)$) and $a(B) \subseteq a(F_1)$ (or $a(B) \in a(F_1) \cup \{a(F_1)\}$). But this means that the piste from $a(A)$ to $a(B)$ goes via $a(F_1)$. Now we can apply induction to the piste from F_1 to B, since it is shorter than the original one from A to B.
□

Lemma 1.2.6 *Let $p = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ and $p' = \langle\langle X', Y', Z'\rangle, C', \in, \subseteq\ \rangle$ be isomorphic suitable structures (even over different cardinals), a an isomorphism between them and $F_0, F_1, F \in X$ a triple of a Δ-system type. Then a respects π_{F_0,F_1}, i.e. for every $A \in F_0 \cap (X \cup Y)$ we have $a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A))$.*

Proof. Let $F_0, F_1, F \in X$ be a triple of a Δ-system type and $A \in F_0 \cap (X \cup Y)$. We prove the lemma by induction on the length of the piste from F_0 to A.
Suppose first that $A \in C(F_0)$ (or in case $A \in Y$ the piste to A involves only $C(F_0)$). The isomorphism a moves $C(F_0)$ to $C'(a(F_0))$ and $C(F_1)$ to $C'(a(F_1))$. By Lemma 1.2.4, the triple $a(F_0), a(F_1), a(F)$ is of a Δ-system type. So, $\pi_{a(F_0),a(F_1)}$ moves $C'(a(F_0))$ onto $C'(a(F_1))$ respecting the inclusion relation. Then $\pi_{a(F_0),a(F_1)}(a(A))$ should be an element of $C(a(F_1))$ at the same place as $a(A)$ in $C(a(F_0))$, which, in turn is at the same place as A in $C(F_0)$ and $\pi_{F_0,F_1}(A)$ in $C(F_1)$. Hence

$$a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A)).$$

Suppose now that $A \notin C(F_0)$. Let H_0, H_1, H be the first splitting on the way to A from F_0. The induction applies to H_1, A . Hence

$$a(\pi_{H_1,H_0}(A)) = \pi_{a(H_1),a(H_0)}(a(A)).$$

Let $A' = \pi_{H_1,H_0}(A)$. Apply the induction to F_0, A'. Then

$$a(\pi_{F_0,F_1}(A')) = \pi_{a(F_0),a(F_1)}(a(A')).$$

Again, apply induction to F_0, H_0 and F_0, H_1. So,

$$a(\pi_{F_0,F_1}(H_0)) = \pi_{a(F_0),a(F_1)}(a(H_0))$$

and

$$a(\pi_{F_0,F_1}(H_1)) = \pi_{a(F_0),a(F_1)}(a(H_1)).$$

Finally,

$$\pi_{F_0,F_1}(A) = \pi_{\pi_{F_0,F_1}(H_0),\pi_{F_0,F_1}(H_1)}(\pi_{F_0,F_1}(A')).$$

Applying a, we obtain

$$a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A)).$$

□

Note that the proofs of Lemmas 1.2.5, 1.2.6 rely only on Lemmas 1.2.3 and 1.2.4 do not use the component of suitable structures consisting of intersections. Let us isolate a weaker notion that still will capture all the essential parts.

Definition 1.2.7 A two sorted structure $\langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ is called *weak suitable structure* iff

1. X has a maximal under inclusion element. Denote it $\max(X)$,
2. $Y \subseteq \max(X)$,
3. C is a binary relation X,
4. $\langle\langle \max(X), X, C\rangle, Y\rangle \in \mathcal{P}'$, where for every $A \in X$ we identify $C(A)$ with the set $\{B \in X \mid \langle A, B\rangle \in C\}$.

The following analogs of Lemmas 1.2.5, 1.2.6 were actually proved above:

Lemma 1.2.8 *Let $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ and $p' = \langle\langle X', Y'\rangle, C', \in, \subseteq\ \rangle$ be isomorphic weak suitable structures (even over different cardinals) and a an isomorphism between them. Then a respects pistes, i.e. for every $A \in X$ and $B \in (X \cup Y) \cap A$, a maps the piste between A and B in p onto the piste between $a(A)$ and $a(B)$.*

Lemma 1.2.9 *Let $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ and $p' = \langle\langle X', Y'\rangle, C', \in, \subseteq, \rangle$ be isomorphic weak suitable structures (even over different cardinals), a an isomorphism between them and $F_0, F_1, F \in X$ a triple of a Δ-system type. Then a respects π_{F_0,F_1}, i.e. for every $A \in F_0 \cap (X \cup Y)$ we have $a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A))$.*

Let $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ be a weak suitable structure. Consider $Z = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X \cup Y\}$. Then $\langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ is a suitable structure. Let us call it *expansion of* p to a suitable structure.

Lemma 1.2.10 *Suppose that* $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ *and* $p' = \langle\langle X', Y'\rangle, C', \in, \subseteq\ \rangle$ *are isomorphic weak suitable structures (even over different cardinals). Then their expansions are isomorphic as well.*

Proof. Let a be the isomorphism between p and p'. We show that it extends to an isomorphism between the expansions. Let $Z = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X \cup Y\}$ and $Z' = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X' \cup Y'\}$. Extend a to a function b in the obvious fashion: $b \restriction \mathrm{dom}(a) = a$ and $b(t_1 \cap ... \cap t_n) = a(t_1) \cap ... \cap a(t_n)$, for any $t_1, ..., t_n \in X \cup Y$. We need to check that such defined b is a function and an isomorphism.

Note first that for every $A, B \in X$, $A' \in (A \cup \{A\}) \cap X$ and $\alpha \in Y \cap A'$ such that $A \cap B = A' \cap \alpha$ we have $a(A) \cap a(B) = a(A') \cap a(\alpha)$. Use induction on the pistes complexity from $\max(X)$ to A, B as in Lemma 1.1.11. The inductive step follows since a preserves Δ-system triples. Also, by Lemmas 1.2.8,1.2.9, a respects pistes and images under Δ-system triples isomorphisms.

Similarly, if instead of two sets we have finitely many $A_1, ..., A_n \in X$, $A' \in (A_1 \cup \{A_1\}) \cap X$ and $\alpha \in Y \cap A'$ such that $A_1 \cap ... \cap A_n = A' \cap \alpha$, then $a(A_1) \cap ... \cap a(A_n) = a(A') \cap a(\alpha)$. Also, the same holds if some (or actually one) of A_i's is in Y, i.e. is an ordinal.

Now, by Lemma 1.1.12, for every $A_1, ..., A_n \in X$ there are $A' \in (A_1 \cup \{A_1\}) \cap X$ and $\eta \in Y \cap A'$ such that $A_1 \cap ... \cap A_n = A' \cap \eta$, or just $A_1 \cap ... \cap A_n = A'$.

An alternative proof that works for higher gaps as well proceeds as follows. Suppose that

$$A_1 \cap ... \cap A_n = B_1 \cap ... \cap B_n,$$

for some $A_1, ..., A_n, B_1, ..., B_n \in X \cup Y$. We need to show that then

$$a(A_1) \cap ... \cap a(A_n) = a(B_1) \cap ... \cap a(B_n).$$

The proof is by induction on complexity of the pistes to components of the intersections. Thus, suppose that A_1 has a maximal piste complexity among the components of the intersection. Consider the pistes from $\max(X)$ to A_1 and to A_2. Go to the last point until which the pistes coincide. Then, as in the proof of Lemma 1.1.11, we replace A_1 by $A'_1 \in X$ and $\alpha_1 \in Y$ which are simpler than A_1 in the piste sense and such that

$$A_1 \cap A_2 = A'_1 \cap \alpha_1 \cap A_2.$$

Now the induction applies.

□

Our setting for the gap 3, which is almost identical to those for the gap 2 in [1], [3], is as follows:

κ is a limit of an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ such that for every $n < \omega$, κ_n caries a $(\kappa_n, \kappa_n^{+n+3})$–extender E_n (in the gap 2 case $(\kappa_n, \kappa_n^{+n+2})$–extender was used). The order $\leq_{E_n}$ of the extender E_n and the corresponding to it projection maps $\pi^{E_n}_{\alpha\beta}, \alpha \geq_{E_n} \beta$ are defined as in [2].

Fix $n < \omega$. We define an analog $\mathcal{P}'_n$ of $\mathcal{P}'$ on the level n just replacing κ by κ_n^{+n}. An assignment function a_n will be an isomorphism between a suitable generic structure of cardinality less than κ_n over κ and a suitable structure over κ_n^{+n}.

Define Q_{n0}.

Definition 1.2.11 Let Q_{n0} be the set of the triples $\langle a, A, f \rangle$ so that:

1. f is partial function from κ^{+3} to κ_n of cardinality at most κ
2. a is an isomorphism between a suitable generic structure $\langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle$ of cardinality less than κ_n and a suitable structure $\langle \langle X', Y', Z' \rangle, C', \in, \subseteq \rangle$ in $\mathcal{P}'_n$ so that
   (a) $\max(X')$ is above every $t \in X' \cup Y'$ in the order $\leq_{E_n}$ of the extender E_n, (or actually, the ordinal which codes $\max(X')$ in the fixed in advance nice codding of $[\kappa_n^{+n+3}]^{<\kappa_n}$. We need that each element of $[\kappa_n^{+n+3}]^{<\kappa_n}$ is coded by a stationary many ordinals below κ_n^{+n+3}).
   Further let as denote $\max(X')$ by $\max(\mathrm{rng}(a))$.
   (b) if $t \in X' \cup Y'$ then for some $k, 2 < k < \omega$, $t \prec H(\chi^{+k})$, with χ big enough fixed in advance. (Alternatively, it is possible to work with a subset of κ_n^{+n+3} only and further require it is a restriction of such model to κ_n^{+n+3}.) We deal with elementary submodels of $H(\chi^{+k})$, instead of those of $H(\kappa_n^{+n+3})$.
   Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of a model $t \in X \cup Y$ will be an elementary submodel of $H(\chi^{+k})$.
   The way to compare such models $t_1 \prec H(\chi^{+k_1}), t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
   move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{n,\max(X')}$,

4. for every ordinals α, β, γ which are elements of Y' or the ordinals coding models in X' we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \quad \text{implies}$$
$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho))$$

for every $\rho \in \pi``_{\max(X'),\alpha}(A)$,

5. $\pi_{\max(X'),\alpha}(\nu) > \pi_{\max(X'),\beta}(\nu)$, for every $\alpha > \beta$ in Y', $\nu \in A$

Define a partial order on Q_{n0} as follows.

Definition 1.2.12 Let $\langle a, A, f\rangle$ and $\langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $a \supseteq b$,

2. $f \supseteq g$,

3. $\pi_{\max(\text{rng}(a)),\max(\text{rng}(b))}``A \subseteq B$,

4. $\text{dom}(f) \cap Y^b = \text{dom}(g) \cap Y^b$, where Y^b is the second component (i.e. the set of ordinals) of the suitable structure on which b is defined.
   Note that here we do not require disjointness of the domain of g and of Y^b, but as it will follow from the further definition of non-direct extension, the values given by g will be those that eventually counts.

Definition 1.2.13 Q_{n1} consists of all partial functions $f : \kappa^{+3} \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

Definition 1.2.14 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq^*_n = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f\rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown\nu = f \cup \{\langle \alpha, \pi_{\max(\text{rng}(a)),a(\alpha)}(\nu) \mid \alpha \in \text{dom}(a) \setminus \text{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\text{dom}(a) \setminus \text{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $\text{dom}(a)$ are used to produce non direct extensions, models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq^*_n q$ or $p \in Q_{n1}, q = \langle b, B, g\rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown\nu$.

Definition 1.2.15 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) for every $n < \omega$, $\quad p_n \in Q_n$,

(2) there is $\ell(p) < \omega$ such that

  (i) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,

  (ii) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$,

  (iii) there is $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in G(\mathcal{P}')$ which witnesses that $\mathrm{dom}(a_n)$ is a suitable generic structure (i.e. $\mathrm{dom}(a_n)$ and $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle$ satisfy 1.2.2), simultaneously for every $n, l(p) \leq n < \omega$.

(3) for every $n \geq m \geq \ell(p)$, $\quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,

(4) for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \kappa_m^{+m+3}$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \kappa_m^{+m+3}$. It is possible to define being k-good this way as well).

(5) For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.

The orders $\leq_{\mathcal{P}}, \leq^*_{\mathcal{P}}$ are defined as in the gap 2 case in [1].

Next lemma deals with extensions of elements of $\mathcal{P}$. The analogs for the gap 2 are trivial.

Lemma 1.2.16 *Let $p \in \mathcal{P}$ and $\langle \langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+} \rangle, B^{1\kappa^{++}} \rangle \in G(\mathcal{P}')$. Then*

1. *for every $\alpha \in B^{1\kappa^{++}}$ there is $q \geq^* p$ such that $\alpha \in \mathrm{dom}(a_n(q))$ for all but finitely many n's;*

2. *for every $A \in B^{1\kappa^+}$ there is $q \geq^* p$ such that $A \in \mathrm{dom}(a_n(q))$ for all but finitely many n's. Moreover, if $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \geq \langle \langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+} \rangle, B^{1\kappa^{++}} \rangle$ witnesses a generic suitability of p and $A \in C^{\kappa^+}(A^{0\kappa^+})$, then the addition of A does not require adding of ordinals and the only models that probably will be added together with A are its images under Δ-system type isomorphisms for triples in p.*

Proof. Pick some $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in G(\mathcal{P}')$ stronger than $\langle \langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+} \rangle, B^{1\kappa^{++}} \rangle$ such that

1. $\alpha \in A^{1\kappa^{++}}$,

2. $A \in A^{1\kappa^{+}}$,

3. $\langle\langle A^{0\kappa^{+}}, A^{1\kappa^{+}}, C^{\kappa^{+}}\rangle, A^{1\kappa^{++}}\rangle$ witnesses that $\mathrm{dom}(a_n(p))$ is a suitable generic structure (i.e. $\mathrm{dom}(a_n(p))$ and $\langle\langle A^{0\kappa^{+}}, A^{1\kappa^{+}}, C^{\kappa^{+}}\rangle, A^{1\kappa^{++}}\rangle$ satisfy 1.2.2), for every $n, l(p) \leq n < \omega$.

Note first that it is easy to add to p any $A \in C^{\kappa^{+}}(A^{0\kappa^{+}})$ such that the maximal models of p_n's belong to A. Just at each level $n \geq l(p)$ pick an elementary submodel of $H(\chi^{+\omega})$ of cardinality κ_n^{+n+1} which includes $\mathrm{rng}(a_n)$ as an element. Map A to such a model.

Hence it is enough to deal with α, A which are the members of the maximal model of p, just otherwise, we can add first $A^{0\kappa^{+}}$.

We proof the lemma simultaneously for α and A by induction on the piste distance or complexity.

Fix $n \geq l(p)$. Let $\mathrm{dom}(a_n(p)) = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\rangle$.

Suppose that the piste to α involves only the central line. The general case is treated similarly.

Let $A_1 \in C^{\kappa^{+}}(\max(X))$ be the least model of $C^{\kappa^{+}}(\max(X))$ with $\alpha \in A_1$. We assume that $A_1 \in X$. Just otherwise use the induction to add it. This is possible, since the piste to A_1 is simpler than those to α.

Case 1. A_1 is the least model of $C^{\kappa^{+}}(\max(X))$.
The piste to α from $\max(X)$ (or from $A^{0\kappa^{+}}$) consists of A_1 alone. So, in order to add α we do not have to add models or other ordinals first.
Consider $\beta_1 = \min((A_1 \cap Y) \setminus \alpha)$ and $\gamma_1 = \max(A_1 \cap Y \cap \alpha)$ whenever defined. Suppose that both β_1 and γ_1 are defined. If one of them or both are undefined then the argument below will be only simpler.
Let us denote $a_n(\beta_1)$ by β_1^*, $a_n(\gamma_1)$ by γ_1^*, $a_n(X)$ by X^* and $a_n(A_1)$ by A_1^*. Let C^* be the function that corresponds to C in $\mathrm{rng}(a_n)$. Then $A_1^* \in C^*(\max(X^*))$. Also, $\beta_1^*, \gamma_1^* \in A_1^* \cap a_n{}''Y$ and $\gamma_1^* < \beta_1^*$.
Assume that A_1^* and β_1^* are k-good, for some $k >> 2$ [2]. Pick now $M \in A_1^*$ such that

1. $M \in \beta_1^*$,

2. $|M| = \kappa_n^{+n+2}$,

[2]We use the definition of k-goodness as defined in [1].

3. M is $k-1$-good,

4. $\gamma_1^* \in M$.

Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.

Case 2. A_1 is not the least element of $C^{\kappa^+}(\max(X))$.
Then we will need to add also the immediate predecessor A_1^- of A_1 in $C^{\kappa^+}(\max(X))$. Do this using the induction.
Split the argument into three cases.
Case 2.1. $\alpha > \sup(A_1^-)$.
Then we proceed exactly as in Case 1 above only require in addition that $a_n(A_1^-) \in M$.
Case 2.2. $\alpha = \sup(A_1^-)$.
Set $B = a_n(A_1)$. Then, its immediate predecessor $B^- = a_n(A_1^-)$. Pick $k < \omega$ such that $B^- \prec H(\chi^{+k+1})$ and $B \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$. Then $H(\chi^{+k}) \in B^-$. Hence

$$B^- \models \forall \nu < \kappa_n^{+n+3} \forall t \in [H(\chi^{+k})]^{<\kappa^{+n+3}} \exists M \prec H(\chi^{+k}) \quad (M \supseteq \nu \cup t \text{ and } |M| < \kappa_n^{+n+3}).$$

Let $\delta = \sup(B^- \cap \kappa_n^{+n+3})$. Set M to be the Skolem hull of $\delta \cup (B^- \cap H(\chi^{+k}))$ in $H(\chi^{+k})$. Then $M \cap \kappa_n^{+n+3} = \delta$. Also, $M \in B$.
Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.
Case 2.3. $\alpha < \sup(A_1^-)$.
Consider $\alpha_1 = \min(A_1^- \setminus \alpha)$. We need to add α_1 before α and this can be done using the induction, since the piste to α_1 is simpler than those to α. So assume that α_1 is already in Y. Note that $\mathrm{cof}(\alpha_1) = \kappa^{++}$, since $A_1^- \supseteq \kappa^+$ and it is an elementary submodel of $H(\kappa^{+3})$.
We split the proof now into two cases.
Case 2.3.1. $\alpha = \sup(\alpha_1 \cap A_1^-)$.
This case is similar to Case 2.2 above. Set $B = a_n(A_1)$. Then, its immediate predecessor $B^- = a_n(A_1^-)$. Let $E = a_n(\alpha_1)$.
Pick $k < \omega$ such that $E \prec H(\chi^{+k+1})$, $B^- \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$ and $B \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$. Then $H(\chi^{+k}) \in E \cap B^-$.

$$E \cap B^- \models \forall \nu < \kappa_n^{+n+3} \forall t \in [H(\chi^{+k})]^{<\kappa^{+n+3}}$$

$$\exists M \prec H(\chi^{+k}) \quad (M \supseteq \nu \cup t \text{ and } |M| < \kappa_n^{+n+3}).$$

Let $\delta = \sup(E \cap B^- \cap \kappa_n^{+n+3})$. Set M to be the Skolem hull of $\delta \cup (E \cap B^- \cap H(\chi^{+k}))$ in $H(\chi^{+k})$. Then $M \cap \kappa_n^{+n+3} = \delta$. Also, $M \in B$.

Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.

Case 2.3.2. $\alpha > \sup(\alpha_1 \cap A_1^-)$.

Consider $\beta_1 = \min((A_1 \cap Y) \setminus \alpha)$ and $\gamma_1 = \max(A_1 \cap Y \cap \alpha)$ whenever defined. Suppose that both β_1 and γ_1 are defined. If one of them or both are undefined then the argument below will be only simpler.

Let us denote $a_n(\beta_1)$ by β_1^*, $a_n(\gamma_1)$ by γ_1^*, $a_n(X)$ by X^* and $a_n(A_1)$ by A_1^*. Let C^* be the function that corresponds to C in $\mathrm{rng}(a_n)$. Then $A_1^* \in C^*(\max(X^*))$ and $a_n(A_1^-)$ is the immediate predecessor of A_1^* in $C^*(A_1^*)$. Also, $\beta^*, \gamma^* \in A_1^* \cap a_n{}''Y$ and $\gamma^* < \beta^*$.

Assume that A_1^* and β_1^* are k-good, for some $k >> 2$. Pick now $M \in A_1^*$ such that

1. $M \in \beta_1^*$,
2. $|M| = \kappa_n^{+n+2}$,
3. M is $k-1$-good,
4. $\gamma_1^*, a_n(A_1^-) \cap a_n(\alpha_1) \in M$.

Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.

Set

$$Y_1 = Y \cup \{\alpha' \mid \alpha' \text{ is the image of}$$

$$\alpha \text{ under } \Delta- \text{ system types triples (of } X) \text{ isomorphisms } \}.$$

Claim 1.2.16.1 *Y_1 is a closed set.*

Proof. We just prove that every limit point of Y_1 is a limit point of Y, and hence, is in Y. It is enough to deal limits of ω-sequences, since if every limit of an ω-sequence from Y_1 is in Y, then any limit will be in Y, because Y is closed.

Such images are generated as follows. Pick the smallest triple $F_0^1, F_1^1, F^1 \in X$ of a Δ-system type with $F_0^1, F^1 \in C(\max(X))$ and $F_0^1 \subseteq A$. We add $\alpha^1 = \pi_{F_0^1, F_1^1}(\alpha)$ to Y. Note that it is possible to have $\alpha = \alpha^1$. Let $\xi_0^1 \in F_0^1 \cap Y, \xi_1^1 \in F_1^1 \cap Y$ be as in Definition 1.1.2(4d). Then $\alpha^1 > \alpha$ implies $\xi_0^1 \leq \alpha < \xi_1^1 \leq \alpha^1$.

Then pick the smallest triple $F_0^2, F_1^2, F^2 \in X$ of a Δ-system type with $F_0^2, F^2 \in C(\max(X))$ and $F_0^2 \subseteq F^1$. We add $\alpha^{20} = \pi_{F_0^2, F_1^2}(\alpha)$ and $\alpha^{21} = \pi_{F_0^2, F_1^2}(\alpha^1)$ to Y. Again it is possible to have $\alpha^{2i} \in \{\alpha, \alpha^1\}$, where $i < 2$. Let $\xi_0^2 \in F_0^2 \cap Y, \xi_1^2 \in F_1^2 \cap Y$ be as in Definition 1.1.2(4d). Again, if one of the new α^{2i}'s is above its pre-image, then the corresponding ξ_i^2 will be above

$\sup(F_0^2)$, and so, above both α, α^1.
Continue further all the way up to $\max(X)$. This way all the images of α are generated. Note that we move up over the central line of X.
At each stage j in the process the same effect observed above will take place- if one of α^{ji}'s is above its pre-image, then the corresponding ξ_i^j will be above $\sup(F_0^j)$, and so, above all the images $\alpha^{j'i'}$ of α generated at stages $j' < j$. But all such ξ_i^j are in Y. Hence, their limit, which is the same as those of increasing sequence of α^{ji}'s, is in Y as well.
$\square$ of the claim.

Turn now to the adding of a model.
Assume first that a model A is on the central line. Let us observe that no collision with ordinals in Y can occur. Thus if some $\alpha \in Y, \alpha \notin A$ and $\sup(A) > \alpha$ (if $\alpha = \sup(A)$, then by the piste closure we must have $A \in X$), then the same should hold with images, i.e. the image A^* of A must have supremum above $\alpha^* := a_n(\alpha)$ and $\alpha^* \notin A^*$. There may be infinitely many such α's and then, in general, it will be impossible to find A^*. In present situation, we have the advantage - X is closed under pistes to ordinals of Y. This means, in particular, that there is $B_\alpha \in C(\max(X))$ such that $\alpha \in B_\alpha$ and B_α is the least model of $C(\max(X))$, or B_α has the immediate predecessor B_α^- in $C(\max(X))$ and $\alpha \notin B_\alpha^-$. In our case the first possibility is just impossible. Thus, we assumed that $A \in C^{\kappa^+}(A^{0\kappa^+})$, $\alpha \in B_\alpha \setminus A$. So, B_α is not the least element of $C^{\kappa^+}(A^{0\kappa^+})$, which by 1.2.2(3) implies that B_α is not the least element of $C(\max(X))$ as well.
Hence, B_α^- exists and $A \subseteq B_\alpha^-$.
Consider now a set

$$T = \{B_\alpha^- \mid \alpha \in Y, \alpha \notin A, \sup(A) > \alpha\}.$$

T is a subset of the closed chain $C(\max(X))$. Let E be the least element of T under the inclusion. Then $A \subset E$, since $T \subseteq C^{\kappa^+}(A^{0\kappa^+})$ and so, both E and A are inside the chain $C^{\kappa^+}(A^{0\kappa^+})$, but E is of the form B_α^-, for some $\alpha \in B_\alpha \setminus A$, and $B_\alpha^- \in X, A \notin X$.
Now it is easy to add A in a fashion similar to adding an ordinal above.
First we pick the least $D \in C(E)$ which contains A. Let F be the last model of $C(E)$ inside D. Note that D can be a limit model of $C^{\kappa^+}(A^{0\kappa^+})$ and so D^- may not exist. Even if D^- exists, still it cannot be in X, since otherwise $A = D^-$ will be in X.
Set $\beta = \min((D \cap Y) \setminus \sup(A))$ whenever defined. Suppose that β is defined. If it is undefined then the argument below will be only simpler. Note that necessarily $\beta > \sup(A)$. Otherwise, $\sup(A) = \beta$ and it is in Y. Then the largest model W of $C^{\kappa^+}(A^{0\kappa^+})$ with $\sup(A) \notin W$ must be in X (pistes closure to ordinals). But then $W = A$, since $W \neq A$ will imply $W \in A$ or

$A \in W$, both possibilities are clearly impossible.
Note that every $\gamma \in D \cap Y \cap \beta$ is in F. Otherwise, let some $\gamma \in D \cap Y \cap \beta$ be not in F. The piste to γ goes via D but does continue further on $C^{\kappa^+}(D)$. Hence, D must be a successor model of $C^{\kappa^+}(A^{0\kappa^+})$ and D^- must be in X, which is impossible, as was observed above.

Let us denote $a_n(\beta)$ by β^*, $a_n(D)$ by D^*, $a_n(X)$ by X^* and $a_n(F)$ by F^*. Let C^* be the function that corresponds to C in $\mathrm{rng}(a_n)$. Then $D^*, F^* \in C^*(X^*)$ and $\beta^* \in D^* \cap a_n{}''Y$. Assume that D^* and β^* are k-good, for some $k >> 2$. Pick now $M \in D^*$ such that

1. $M \in \beta^*$,
2. $|M| = \kappa_n^{+n+1}$,
3. M is $k-1$-good,
4. $F^* \in M$.

Now, extend a_n by mapping A to M and all the images of it under Δ-system types triples isomorphisms to those of M.
Note that no new ordinals were added in the process and only models that are images of A under Δ-system types isomorphisms for triples in X were added.

Suppose that A is not on the central line. In this case we are supposed to add to p the whole piste from $A^{0\kappa^+}$ to A. We can concentrate, using the induction, only on the case of a Δ-system triple. Namely given $F_0, F_1, F \in A^{1\kappa^{++}}$ of a Δ-system type with F_0 being the immediate predecessor of F in $C^{\kappa^+}(A^{0\kappa^+})$. We need to add F_1 (and probably also F_0, F if they are not inside) to p. F_0, F are on the central line, hence we may assume that they are in p. Let $\alpha_0, \alpha_1 \in F \cap A^{1\kappa^{++}}$ be so that $\alpha_0 \in F_0, \alpha_1 \in F_1$, $F_0 \cap F_1 = \alpha_0 \cap F_0 = \alpha_1 \cap F_1$ and either $\alpha_0 > \sup(F_1)$ or $\alpha_1 > \sup(F_0)$. By the argument above, we can assume that α_0 is already in p.
Note that $F_1 \notin p$ implies that $\alpha_1 \notin p$, since otherwise the piste to α_1 must be in p, by the definition of a suitable structure, but F_1 which is a part of this piste (actually the final model of it) is not in p. This provides a freedom to define the image of α_1 which will be crucial further in choosing the image of F_1.

Fix $n \geq l(p)$. We need to add F_1 to $\mathrm{dom}(a_n(p))$. Let $\mathrm{dom}(a_n(p)) = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\rangle$. We assume that $F_0, F \in C(\max(X))$ and $\alpha_0 \in Y$.
Note that $Y \cap [\alpha_1, \sup(F_1)] = \emptyset$, since if some $\xi \in Y \cap [\alpha_1, \sup(F_1)]$, then all models of the piste to ξ are in X, but F_1 is one of them.
Split into two cases.

Case 1. $\alpha_0 > \alpha_1$.

Then $\sup(F_1) < \alpha_0$. Consider the images F_0^*, F^* and M_0 of F_0, F and α_0 under a_n.

Let us deal first with a little bit simplified situation, but which still contains the main elements of the construction.

Subcase 1.A. No elements of $Y \cap (\sup(F_0 \cap \alpha_0), \alpha_0)$ are in $\mathrm{dom}(a_n) \cap F$.

By Definition 1.1.2, we have $\mathrm{cof}(\alpha_0) = \kappa^{++}$. Hence $\mathrm{cof}(M_0 \cap \kappa_n^{+n+3}) = \kappa_n^{+n+2}$. So, ${}^{\kappa_n^{+n+1}>}M_0 \subseteq M_0$. In particular, $M_0 \cap F_0^* \in M_0$. Clearly, $M_0 \cap F_0^* \in F^*$, as well. We assume that M_0 is k-good for k big enough. Hence there is a $k-1$-good $M_1 \in M_0$ realizing the same $k-1$ type over $M_0 \cap F_0^*$ as M_0 does. By elementarity, we can find such M_1 inside F^*. Finally, pick F_1^* to be an element of $F^* \cap M_0$ which realizes over $\langle M_0 \cap F_0^*, M_1\rangle$ the same $k-1$ type as F_0^* realizes over $\langle M_0 \cap F_0^*, M_0\rangle$.
Extend a_n by mapping F_1 to F_1^* and all the images of it under Δ-system types triples isomorphisms. In particular, M_1 is added as the image of M_0 under $\pi_{F_0^*, F_1^*}$.

Turn now to a general case.

Subcase 1.B. There are elements of $Y \cap (\sup(F_0 \cap \alpha_0), \alpha_0)$ in $\mathrm{dom}(a_n) \cap F$.

Let γ denotes the last such element below α_1 and β the first such element above α_1. If one of them does not exists, then the argument below applies with obvious simplifications. Note that, as was observed above, there is no elements of Y in the interval $[\alpha_1, \sup(F_1)]$.

Denote $a_n(\beta)$ by N and $a_n(\gamma)$ by γ^*. We assume that M_0 and N are k-good for k big enough. $\sup(F_0^* \cap M_0) \cap \kappa_n^{+n+3} < N \cap \kappa_n^{+n+3}$, hence $F_0^* \cap M_0 \cap \kappa_n^{+n+3} \in N$ (as a set of ordinals of small cardinality). There is a $k-1$-good $M_1 \in N$ realizing the same $k-1$ type over $F_0^* \cap M_0 \cap \kappa_n^{+n+3}$ as M_0 does and with $\gamma^* \in M_1$. By elementarity, we can find such M_1 inside F^*. Finally, pick F_1^* to be an element of $F^* \cap N$ which realizes over $\langle F_0^* \cap M_0 \cap \kappa_n^{+n+3}, M_1\rangle$ the same $k-1$ type as F_0^* realizes over $\langle M_0 \cap F_0^*, M_0\rangle$.
Extend a_n by mapping F_1 to F_1^* and all the images of it under Δ-system types triples isomorphisms. In particular, M_1 is added as the image of M_0 under $\pi_{F_0^*, F_1^*}$.

Case 2. $\alpha_0 < \alpha_1$.

The construction is similar. The only change is that we pick M_1 above M_0.

This completes the inductive construction, and hence the proof of the lemma.

□

The ordering $\leq^*$ on $\mathcal{P}$ and $\leq_n$ on Q_{n0} seems to be not closed in the present situation. Thus it is possible to find an increasing sequence of $\aleph_0$ conditions $\langle\langle a_{ni}, A_{ni}, f_{ni}\rangle \mid i < \omega\rangle$ in Q_{n0} with no simple upper bound. The reason is that the union of maximal models of these conditions, i.e. $\bigcup_{i<\omega} \max(\mathrm{dom}\, a_{ni})$ need not be in $A^{1\kappa^+}$ for any $A^{1\kappa^+}$ in $G(\mathcal{P}')$. The next lemma shows that still $\leq_n$ and so also $\leq^*$ share a kind of strategic closure.

Lemma 1.2.17 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_n \rangle$ does not add new sequences of ordinals of the length $< \kappa_n$, i.e. it is (κ_n, ∞) – distributive.*

Proof. Let $\delta < \kappa_n$ and $\underset{\sim}{h}$ be a Q_{n0}-name of a function from δ to ordinals. Without loss of generality assume that δ is a regular cardinal.

Using genericity of $G(\mathcal{P}')$ (or stationarity of the set $\{A^{0\kappa^+} | A^{0\kappa^+}$ appears in an element of $G(\mathcal{P}')\}$, see 1.1.20) it is not hard to find elementary submodel M of some $H(\nu)$ for ν big enough so that

(a) Q_{n0}, $\underset{\sim}{h}$, $\mathcal{P}' \in M$,

(b) $|M| = \kappa^+$, $M \supseteq \kappa^+$,

(c) there is $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ such that $A^{0\kappa^+} = M \cap H(\kappa^{+3})$ and $\max(A^{1\kappa^{++}} \cap \kappa^{+3}) = \sup(M \cap \kappa^{+3})$.

(d) $cf(M \cap \kappa^{++}) = \delta$,

(e) ${}^{\delta>}M \subseteq M$.

Note that for such M, $M^* = M \cap H(\kappa^{+3})$ must be a limit model, since by Definition 1.1.1(6) successor models are closed under κ sequences, but M^* is not by (d) above.

We have $C^{\kappa^+}(M^*) \setminus \{M^*\} \subseteq M^*$. Let $B \in C^{\kappa^+}(M^*) \setminus \{M^*\}$. We claim that then $C^{\kappa^+}(B) \in M$. Thus, by elementarity there are $B^{1\kappa^+}, D^{\kappa^+}, B^{1\kappa^{++}} \in M$ such that

$$\langle\langle B, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle \in G(\mathcal{P}') \cap M.$$

Note that $C^{\kappa^+} \restriction B^{1\kappa^+}$ may be different from D^{κ^+}, but by the definition of order on $\mathcal{P}'$ (1.1.15) and since $B \in C^{\kappa^+}(M^*)$, there are $E_1, ..., E_n \in B^{1\kappa^+}$ such that the switch with $E_1, ..., E_n$ turns D^{κ^+} into $C^{\kappa^+} \restriction B^{1\kappa^+}$. But $B^{1\kappa^+} \in M$ and $|B^{1\kappa^+}| \leq \kappa^+$. Hence $B^{1\kappa^+} \subseteq M$, since $M \supseteq \kappa^+$. So $E_1, ..., E_n \in M$, and then the corresponding switch is in M as well. This implies that its result $C^{\kappa^+} \restriction B^{1\kappa^+}$ is in M.

The cofinality of $C^{\kappa^+}(M^*) \setminus \{M^*\}$ under the inclusion must be δ, since it is an $\in$-increasing continuous sequence of elements of M^* with limit M^* and by (d) above $\mathrm{cof}(M^* \cap \kappa^{++}) = \delta$. Fix an increasing continuous sequence $\langle A_i \mid i < \delta\rangle$ of elements of $C^{\kappa^+}(M^*) \setminus \{M^*\}$ such that $\bigcup_{i<\delta} A_i = M^*$, A_0 is a successor model and for each limit model A_i in the sequence A_{i+1} is its immediate successor in $C^{\kappa^+}(M^*)$. By (e), each initial segment of it will be in M. Now we decide inside M one by one values of $\underset{\sim}{h}$ and put models from $\langle A_i \mid i < \delta\rangle$ to be maximal models of conditions used. This way we insure that unions of such conditions is a condition.

We define by induction an increasing sequence of conditions

$$\langle\langle a(i), A(i), f(i)\rangle | i \leq \delta\rangle.$$

and an increasing continuous subsequence

$$\langle A_{k_i} | i < \delta\rangle \text{ of } \langle A_i | i < \delta\rangle$$

such that for each $i < \delta$

(1) $\langle a(i), A(i), f(i)\rangle \in M$,

(2) $\langle a(i+1), A(i+1), f(i+1)\rangle$ decides $\underset{\sim}{h}(i)$,

(3) $A_{k_i}, A_{k_{i+1}} \in \mathrm{dom}(a(i))$, $A_{k_{i+1}}$ is the maximal model of $\mathrm{dom}(a(i))$ and $\langle\langle A_{k_{i+1}}, T, C^{\kappa^+} \restriction T\rangle, R\rangle \in G(\mathcal{P}') \cap M$ witnesses a generic suitability of $\mathrm{dom}(a(i))$, for some T, R, with $R \subseteq A_{k_{i+1}} \cup \sup(A_{k_{i+1}})$.

There is no problem with $A(i)$'s and $f(i)$'s in this construction. Thus we have enough completeness to take intersections of $A(i)$'s and unions of $f(i)$'s. The only problematic part is $a(i)$. So let us concentrate only on building of $a(i)$'s.

i=0

Then let us pick some $Z_0 \prec Z_1 \prec H(\chi^{+\omega}) \cap M$ of cardinality κ_n^{+n+1}, closed under κ_n^{+n} - sequences of its elements and $Z_0 \in Z_1$. Set $a(0) = \langle\langle A_0, Z_0\rangle, \langle A_1, Z_1\rangle\rangle$.

i+1

Then we first extend $\langle a(i), A(i), f(i)\rangle$ to a condition $\langle a(i)', A(i)', f(i)'\rangle \in M$ which decides $\underset{\sim}{h}(i)$. Then perform *swt* (see 1.1.13) to turn $\langle a(i)', A(i)', f(i)'\rangle$ into an equivalent condition $\langle a(i)'', A(i)', f(i)'\rangle$ with $A_{k_i} \in C^{\kappa^+}(\max(\mathrm{dom}(a(i)'')))$. Pick a successor model A_j (from the cofinal sequence $\langle A_i \mid i < \delta\rangle$) including $\max(\mathrm{dom}(a(i)''))$. Set $k_{i+2} = j$ and add it to $\mathrm{dom}(a(i)'')$, using *swt* inside A_j if necessary. Finally we add A_{j+1}.

i is a limit ordinal

Then we need to turn $a = \bigcup_{j<i} a(j)$ into condition. For this we will need to add to $\mathrm{dom}(a)$ models and ordinals which are limits of elements of $\mathrm{dom}(a)$. First we extend a by adding to it $\langle A_{k_i}, \bigcup_{j<i} a(A_{k_j})\rangle$, where $k_i = \cup_{j<i} k_j$. Then for each non decreasing sequence $\langle \alpha_j | j < i\rangle$ of ordinals in $\mathrm{dom}(a)$ we add the pair $\langle \cup_{j<i} \alpha_j, \cup_{j<i}(a(\alpha_j) \cap H(\chi^{+\ell}))\rangle$, if it is

not already in the dom(a), where $\ell \leq \omega$ the maximal such that for unboundedly many j's in i $a(\alpha_j) \prec H(\chi^{+\ell})$, if the maximum exists or $\ell >> n$ otherwise. Finally, for each model $B \in \text{dom}(a)$ if there is a nondecreasing sequence $\langle B_j | j < i \rangle$ of elements of $C^{\kappa^+}(B)$ in dom(a) and B is the least possible (under inclusion or with least sup) including the sequence, then we add the pair $\langle \cup_{j<i} B_j, \cup_{j<i}(a(B_j) \cap H(\chi^{+\ell})) \rangle$, if it is not already in the dom(a), where $\ell \leq \omega$ is the minimum between the least k such that $a(B) \subseteq H(\chi^{+k})$ and the maximal ℓ' such that for unboundedly many j's in i $a(B_j) \prec H(\chi^{+\ell'})$, if the maximum exists

or

it is k, if the maximum does not exist and $k < \omega$,

or

$\ell >> n$, if the maximum does not exist and $k = \omega$.

We will need to extend a bit more if the following hold:

1. $B \in \text{dom}(a)$,
2. $\langle B_j \mid j < i \rangle$ is a nondecreasing sequence of elements of $C^{\kappa^+}(B)$ in dom(a),
3. B is the least element of dom(a) such that $\bigcup_{j<i} B_j \in B$,
4. $\langle \alpha_j \mid j < i \rangle$ is a sequence of ordinals such that
   (a) $\alpha_j \in B_j$,
   (b) $\alpha_j \in \text{dom}(a)$,
   (c) $\bigcup_{j<i} \alpha_j \notin \text{dom}(a)$.

Set $\alpha = \bigcup_{j<i} \alpha_j$. Then $\alpha \in B$.
Let us consider two cases.
Case 1. $\alpha \notin \bigcup_{j<i} B_j$.
If B is the real immediate successor of $\bigcup_{j<i} B_j$, i.e. the one in $C^{\kappa^+}(A^{0\kappa^+})$ of $G(\mathcal{P}')$, then the extension made above suffices. Otherwise, we need to add the real successor of $\bigcup_{j<i} B_j$ in order to insure pistes to ordinals closure. Denote such successor by E. We map it to a model E^* such that $\bigcup_{j<i}(a(B_j) \cap H(\chi^{+\ell})) \prec E^* \prec a(B) \cap H(\chi^{+\ell}))$ and E^* is good enough, where ℓ is as above. Note that each $\gamma \in B \cap \text{dom}(a)$ is already in B_j, for some $j < i$, by pistes to ordinals closure of dom(a). Finally we map α to $\bigcup_{j<i}(a(B_j) \cap a(\alpha_j))$.
Case 2. $\alpha \in \bigcup_{j<i} B_j$.
Let E be the smallest model in $C^{\kappa^+}(B)$ with $\alpha \in E$.
Subcase 2.1. E is the least (under the inclusion) element of $C^{\kappa^+}(B)$.

If for some $j < i$ we have $\alpha_j \in E$, then by the piste closure of $\mathrm{dom}(a)$, the model E is in $\mathrm{dom}(a)$. It is easy now to extend a by adding only α which is mapped to an appropriate element of $a(E)$.
Suppose that for each $j < i$, $\alpha_j \notin E$. Consider α_0. Let D_0 be the largest model in $C^{\kappa^+}(B)$ with $\alpha_0 \notin D_0$. By the piste closure of $\mathrm{dom}(a)$, we have $D_0 \in \mathrm{dom}(a)$. Assume that $D_0 \neq E$, otherwise proceed as above. Clearly $D_0 \supset E$, and hence

$\alpha_0 < \alpha < \sup(D_0)$. Then $\alpha_{01} := \min(D_0 \setminus \alpha_0) \in \mathrm{dom}(a)$. So, $\alpha_0 < \alpha_{01} < \alpha$. Let D_{01} be the largest model in $C^{\kappa^+}(B)$ with $\alpha_{01} \notin D_0$. By the piste closure of $\mathrm{dom}(a)$, we have $D_{01} \in \mathrm{dom}(a)$. Again, we assume that $D_{01} \neq E$. Clearly $D_0 \supset D_{01} \supset E$, and hence $\alpha_{01} < \alpha < \sup(D_{01})$. Then $\alpha_{02} := \min(D_{01} \cap \alpha_{01}) \in \mathrm{dom}(a)$. So, $\alpha_0 < \alpha_{01} < \alpha_{02} < \alpha$. We continue and define D_{02} etc. The sequence of such D_{0k} will be $\in$-decreasing, and hence at certain stage $D_{0k} = E$.
Subcase 2.2. E is not the least (under the inclusion) element of $C^{\kappa^+}(B)$.
Then E has the immediate predecessor E^- in $C^{\kappa^+}(B)$. Suppose first that α is a limit point of E^-. Note that then necessarily E^- is a limit model, as successor ones are closed under $< \kappa^+$-sequences.
Claim 1.2.17.1 *There is an increasing sequence $\langle \alpha'_j \mid j < i\rangle$ in $E^- \cap \mathrm{dom}(a)$ with limit α.*

Proof. Let $j < i$. If $\alpha_j \in E^-$, then we take it. Suppose that $\alpha_j \notin E^-$. Pick D_j to be the largest model in $C^{\kappa^+}(B)$ with $\alpha_j \notin D_j$. Then, $D_j \in \mathrm{dom}(a)$, and clearly, $D_j \supseteq E^-$. Also, $\alpha_j < \alpha$ and α is a limit point of E^-. Hence $\alpha_j < \sup(D_j)$. Then $\alpha_{j1} := \min(D_j \setminus \alpha_j) \in D_j \cap \mathrm{dom}(a)$. If $\alpha_{j1} \in E^-$, then we pick it. Otherwise, continue and consider D_{j1} the largest model in $C^{\kappa^+}(B)$ with $\alpha_{j1} \notin D_{j1}$. Then, $D_{j1} \in \mathrm{dom}(a)$, and clearly, $D_{j1} \supseteq E^-$. Also, $\alpha_{j1} < \alpha$ and α is a limit point of E^-. Hence $\alpha_{j1} < \sup(D_{j1})$. Then $\alpha_{j2} := \min(D_{j1} \setminus \alpha_{j1}) \in D_{j1} \cap \mathrm{dom}(a)$. If $\alpha_{j2} \in E^-$, then we pick it. Otherwise, continue. After finitely many steps we will reach some such $\alpha_{jk} \in E^-$.
□ of the claim.
Let $\langle \alpha'_j \mid j < i\rangle$ be given by the claim. For each $j < i$ let K_j be the least model of $C^{\kappa^+}(B)$ with $\alpha'_j \in K_j$. Then $E^- = \bigcup_{j<i} K_j$, since, clearly $E^- \supseteq \bigcup_{j<i} K_j$ and if $E^- \not\supseteq \bigcup_{j<i} K_j$, then α will be in the immediate successor $K \in C^{\kappa^+}(B)$ of $\bigcup_{j<i} K_j$, but $K \subseteq E^-$ and $\alpha \notin E^-$. Now we are in situation of Case 1 with $\langle \alpha_j \mid i < j\rangle$ replaced by $\langle \alpha'_j \mid i < j\rangle$ and $\langle B_j \mid i < j\rangle$ by $\langle K_j \mid j < i\rangle$.

Suppose now that α is not a limit point of E^-. Pick $j^* < i$ such that for every $j, j^* \leq j < i$, $\sup(E^- \cap \alpha) < \alpha_j$. If for some $j, j^* \leq j < i$, $\alpha_j \in E$, then E will be the least model of $C^{\kappa^+}(B)$ with α_j inside, and hence $E, E^- \in \mathrm{dom}(a)$, due to the piste closure of $\mathrm{dom}(a)$.

Suppose that for each $j, j^* \leq j < i$, $\alpha_j \notin E$. Fix such j. Pick D_j to be the largest model in $C^{\kappa^+}(B)$ with $\alpha_j \notin D_j$. Then, $D_j \in \mathrm{dom}(a)$, and clearly, $D_j \supseteq E$. If $D_j = E$, then $E \in \mathrm{dom}(a)$. Then, also $E^- \in \mathrm{dom}(a)$, since $\alpha_{j1} := \min(E \setminus \alpha_j) \in E \cap \mathrm{dom}(a)$, but E^- is the largest model in $C^{\kappa^+}(B)$ with α_{j1} not inside, and hence it must be in $\mathrm{dom}(a)$ by the piste closure.
Suppose that $D_j \neq E$. Consider $\alpha_{j1} := \min(D_j \setminus \alpha_j) \in D_j \cap \mathrm{dom}(a)$. Clearly, $\alpha_{j1} < \alpha$, since $E \subseteq D_j$ and $\alpha \in E$. If $\alpha_{j1} \in E$, then E will be the least model of $C^{\kappa^+}(B)$ with α_{j1} inside, since $\alpha_{j1} \notin E^-$. Then $E, E^- \in \mathrm{dom}(a)$.
If $\alpha_{j1} \notin E$, then we continue and pick D_{j1} to be the largest model in $C^{\kappa^+}(B)$ with $\alpha_{j1} \notin D_{j1}$. Then, $D_{j1} \in \mathrm{dom}(a)$, and clearly, $D_{j1} \supseteq E$. If $D_{j1} = E$, then $E \in \mathrm{dom}(a)$. Then, also $E^- \in \mathrm{dom}(a)$, since $\alpha_{j2} := \min(E \setminus \alpha_{j1}) \in E \cap \mathrm{dom}(a)$, but E^- is the largest model in $C^{\kappa^+}(B)$ with α_{j2} not inside, and hence it must be in $\mathrm{dom}(a)$ by the piste closure.
If $D_{j1} \neq E$, then we continue in the same fashion to define α_{j2}, D_{j2} etc. After finitely many steps we will have $E = D_{jk}$ or $\alpha_{jk} \in E$. Both imply $E, E^- \in \mathrm{dom}(a)$.

Finally denote the resulting extension of a by b.

Claim 1.2.17.2 $\mathrm{dom}(b)$ *is a suitable generic structure.*
Proof. Let as check the condition (6c) of Definition 1.1.1. Thus let $A, \alpha \in \mathrm{dom}(b)$, $A \in C(\max(\mathrm{dom}(b)))$ a non-limit model and $\sup(A) > \alpha$. We need to show that $\min(A \setminus \alpha) \in \mathrm{dom}(b)$.
Case 1. $A \in \mathrm{dom}(a(l))$ *for some* $l < i$.
If $\alpha \in \mathrm{dom}(a)$, then for some $j < i$ big enough we will have $A, \alpha \in \mathrm{dom}(a_j)$, and then $\min(A \setminus \alpha) \in \mathrm{dom}(a_j)$. Note that if α is a non-limit element of $\mathrm{dom}(b)$, then $\alpha \in \mathrm{dom}(a)$.
Suppose that α is a limit point of $\mathrm{dom}(b)$ and $\alpha \notin \mathrm{dom}(a)$. Let $\langle \alpha_j | j < i \rangle$ be a nondecreasing sequence from $\mathrm{dom}(a)$ converging to α. By (6c) of Definition 1.1.1, $\gamma_j = \min(A \setminus \alpha_j) \in \mathrm{dom}(a)$. If $\langle \gamma_j | j < i \rangle$ is eventually constant, then the constant value will be as desired. Suppose otherwise. Then $\langle \gamma_j | j < i \rangle$ will be also a converging to α sequence. But remember that A is non-limit, hence ${}^{\kappa}A \subseteq A$, and so $\alpha \in A$. Then $\min(A \setminus \alpha) = \alpha \in \mathrm{dom}(b)$ and we are done.
Case 2. $A \notin \mathrm{dom}(a)$.
Assume that $\alpha \notin A$, just otherwise $\min(A \setminus \alpha) = \alpha$ and we are done. Denote $\min(A \setminus \alpha)$ by α^*
Subcase 2.1. $\alpha \in \mathrm{dom}(a)$.
Consider then the smallest model E_α in $C(\max(\mathrm{dom}(b)))$ with α inside. Let E_α^- be its immediate predecessor in $C(\max(\mathrm{dom}(b)))$. Then $A \subseteq E_\alpha^-$, since $\alpha \notin A$, and $A \neq E_\alpha^-$, since

$E_\alpha^- \in \mathrm{dom}(a)$ and $A \notin \mathrm{dom}(a)$. Then $\sup(E_\alpha^-) > \alpha$, hence $\alpha_1 := \min(E_\alpha^- \setminus \alpha) > \alpha$ and $\alpha_1 \in \mathrm{dom}(a)$. $E_\alpha^- \supseteq A$ implies that $\alpha_1 \leq \alpha^*$. If $\alpha_1 = \alpha^*$, then $\alpha^* \in \mathrm{dom}(a)$ and we are done. Suppose otherwise. Then $\alpha_1 < \alpha^*$. Consider then the smallest model E_{α_1} in $C(\max(\mathrm{dom}(b)))$ with α_1 inside. Let $E_{\alpha_1}^-$ be its immediate predecessor in $C(\max(\mathrm{dom}(b)))$. Then $A \subseteq E_{\alpha_1}^-$, since $\alpha_1 \notin A$, and $A \neq E_{\alpha_1}^-$, since $E_{\alpha_1}^- \in \mathrm{dom}(a)$ and $A \notin \mathrm{dom}(a)$. Then $\sup(E_{\alpha_1}^-) > \alpha_1$, since $\alpha^* \in E_{\alpha_1}^-$ and $\alpha^* > \alpha_1$. Hence $\alpha_2 := \min(E_{\alpha_1}^- \setminus \alpha_1) > \alpha_1$ and $\alpha_2 \in \mathrm{dom}(a)$. If $\alpha_2 = \alpha^*$, then $\alpha^* \in \mathrm{dom}(a)$ and we are done. Otherwise, $\alpha_2 < \alpha^*$. We continue and consider $E_{\alpha_2}, E_{\alpha_2^-}$ etc. Note that the sequence of models E_{α_m} constructed this way is decreasing. So the process stops after finitely many steps. Which means that $\alpha^* \in \mathrm{dom}(a)$.

Subcase 2.2. $\alpha \notin \mathrm{dom}(a)$.

Then α is a limit of an increasing sequence $\langle \alpha_j \mid j < i \rangle$ of elements of $\mathrm{dom}(a)$.

If an unbounded subsequence of the sequence $\langle \alpha_j \mid j < i \rangle$ is in A, then α will be in A as well, since A is a non-limit model and so is closed under δ sequences of its elements. Hence there is $j^* < i$ such that for every $j, j^* \leq j < i, \alpha_j \notin A$. Let $j^* \leq j < i$. We have $\sup(A) > \alpha > \alpha_j$. Set $\alpha_j^* = \min(A \setminus \alpha_j)$. By Subcase 2.1, $\alpha_j^* \in \mathrm{dom}(a)$. If $\alpha_j^* > \alpha$, then $\alpha_j^* = \alpha^*$ and we are done. Assume, hence that $\alpha_j^* < \alpha$, for every $j < i$. But the sequence $\langle \alpha_j^* \mid j < i \rangle$ is a sequence of elements of A which converges to α. So, $\alpha \in A$. Contradiction.

$\square$ of the claim.

The next claim is similar.

Claim 1.2.17.3 $\mathrm{rng}(b)$ *is a suitable structure over* κ_n.

We need to check that b is an isomorphism between the suitable structures $\mathrm{dom}(b)$ and $\mathrm{rng}(b)$. By Lemma 1.2.10, it is enough to show that the restriction of b is an isomorphism between the corresponding weak suitable structures. But this is obvious, since no Δ-system type triples are added at limit stages.

$\square$

It is possible to work in V rather than in $V[G(\mathcal{P}')]$ or M. Combining arguments of 1.1.19 and the previous lemma it is not hard to show the following:

Lemma 1.2.18 $\mathcal{P}' * Q_{n0}$ *is* $< \kappa_n$*-strategically closed.*

Lemma 1.2.19 $\langle \mathcal{P}, \leq^* \rangle$ *does not add new sequences of ordinals of the length* $< \kappa_0$.

Proof. Repeat the argument of Lemma 1.2.17 with $\mathcal{P}$ replacing Q_{n0}.

$\square$

The argument of Lemma 1.2.17 can be used in a standard fashion to show the Prikry condition (i.e. the standard argument runs inside elementary submodel M with δ replaced by κ^+).

Lemma 1.2.20 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Finally we define $\to$ on $\mathcal{P}$ similar to those of [1] or [3].

Lemma 1.2.21 *$\langle \mathcal{P}, \to \rangle$ satisfies κ^{++}-c.c.*

Proof. Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++}\rangle$ be a name of an antichain of the length κ^{++}. Using 1.1.19 we find an increasing sequence $\langle\langle\langle A_\alpha^{0\kappa^+}, A_\alpha^{1\kappa^+}, C_\alpha^{\kappa^+}\rangle, A_\alpha^{1\kappa^{++}}\rangle \mid \alpha < \kappa^{++}\rangle$ of elements of $\mathcal{P}'$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ so that for every $\alpha < \kappa^{++}$ the following hold:

(a) $\langle\langle A_{\alpha+1}^{0\kappa^+}, A_{\alpha+1}^{1\kappa^+}, C_{\alpha+1}^{\kappa^+}\rangle, A_{\alpha+1}^{1\kappa^{++}}\rangle \Vdash \underset{\sim}{p}_\alpha \leq \check{p}_\alpha$,

(b) $\bigcup_{\beta<\alpha} A_\beta^{0\kappa^+} = A_\alpha^{0\kappa^+}$, if α is a limit ordinal,

(c) ${}^\kappa A_{\alpha+1}^{0\kappa^+} \subseteq A_{\alpha+1}^{0\kappa^+}$,

(d) $A_{\alpha+1}^{0\kappa^+}$ is a successor model,

(e) $\langle A_\beta^{1\kappa^+} \mid \beta < \alpha\rangle \in A_{\alpha+1}^{0\kappa^+}$,

(f) for every $\alpha \leq \beta < \kappa^{++}$ we have

$$C_\alpha^{\kappa^+}(A_\alpha^{0\kappa^+}) \text{ is an initial segment of } C_\beta^{\kappa^+}(A_\beta^{0\kappa^+}),$$

(g) $p_\alpha = \langle p_{\alpha n} \mid n < \omega\rangle$,

(h) for every $n \geq l(p_\alpha)$, $A_{\alpha+1}^{0\kappa^+}$ is the maximal model of $\mathrm{dom}(a_{\alpha n})$ and $A_\alpha^{0\kappa^+} \in \mathrm{dom}(a_{\alpha n})$, where $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle$.
Actually this condition is the reason for not requiring the equality in (a) above.

Let $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle$ for every $\alpha < \kappa^{++}$ and $n \geq \ l(p_\alpha)$.
Let $\alpha < \kappa^{++}$. Fix some

$$\langle\langle B_{\alpha+1}^{0\kappa^+}, B_{\alpha+1}^{1\kappa^+}, D_{\alpha+1}^{\kappa^+}\rangle, B_{\alpha+1}^{1\kappa^{++}}\rangle \leq_{\mathcal{P}'} \langle\langle A_{\alpha+1}^{0\kappa^+}, A_{\alpha+1}^{1\kappa^+}, C_{\alpha+1}^{\kappa^+}\rangle, A_{\alpha+1}^{1\kappa^{++}}\rangle$$

which witnesses a generic suitability of structure $\mathrm{dom}(a_{\alpha n})$ for each $n, l(p_\alpha) \leq n < \omega$, as in Definition 1.2.2. Note that $B^{0\kappa^+}_{\alpha+1}$ need not be in $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$ and even if it does, then $D^{\kappa^+}_{\alpha+1}(B^{0\kappa^+}_{\alpha+1})$ need not be an initial segment of $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$. By the definition of the order $\leq_{\mathcal{P}'}$ (Definition 1.1.15) there are $m < \omega$ and $E_1, ..., E_m \in A^{1\kappa^+}_{\alpha+1}$ such that

$$swt(\langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle, E_1, ..., E_m) \text{ and } \langle\langle B^{0\kappa^+}_{\alpha+1}, B^{1\kappa^+}_{\alpha+1}, D^{\kappa^+}_{\alpha+1}\rangle, B^{1\kappa^{++}}_{\alpha+1}\rangle$$

satisfy (1)-(3) of Definition 1.1.15.
By Lemma 1.2.16 it is possible to add all $E_i (i = 1, ..., m)$ to $\mathrm{dom}(a_{\alpha n})$, for a final segment of n's. By adding and taking non-direct extension if necessary, we can assume that E_i's are already in $\mathrm{dom}(a_{\alpha n})$, for every $n \geq l(p_\alpha)$.
Now we can apply the opposite switch (i.e. the one starting with E_m, then E_{m-1}, ...,and finally E_1) to $\mathrm{dom}(a_{\alpha n})$ (and the corresponding to it under $a_{\alpha n}$ to $\mathrm{rng}(a_{\alpha n})$). Denote the result still by $a_{\alpha n}$.
Finally, $\langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle$ will witness a generic suitability of structure $\mathrm{dom}(a_{\alpha n})$ for each $n, l(p_\alpha) \leq n < \omega$.
In particular, we have now that the central line of $\mathrm{dom}(a_{\alpha n})$ is a part of $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$ and $A^{0\kappa^+}_{\alpha}$ is on it, for every $n, l(p_\alpha) \leq n < \omega$.

Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^{++}$ the following holds:

(1) $\ell = \ell(p_\alpha) = \ell(p_\beta)$,

(2) for every $n < \ell$ $\quad p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1} i.e. $p_{\alpha n} \cup p_{\beta n}$ is a function,

(3) for every n, $\ell \leq n < \omega$, $\quad \langle \mathrm{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$,

(4) for every n, $\omega > n \geq \ell$, $\quad \mathrm{rng}(a_{\alpha n}) = \mathrm{rng}(a_{\beta n})$.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let α be in S. For each $n, \ell \leq n < \omega$, there will be $\beta(\alpha, n) < \alpha$ such that

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} \subseteq A^{0\kappa^+}_{\beta(\alpha,n)}.$$

Just recall that $|a_{\alpha n}| < \kappa_n$. Shrink S to a stationary subset S^* so that for some $\alpha^* < \min S^*$ of cofinality κ^+ we will have $\beta(\alpha, n) < \alpha^*$, whenever $\alpha \in S^*, \ell \leq n < \omega$. Now, the cardinality of $A^{0\kappa^+}_{\alpha^*}$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\alpha, \beta \in S^*, \ell \leq n < \omega$

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} = \mathrm{dom}(a_{\beta n}) \cap A^{0\kappa^+}_{\beta}.$$

Let us add $A^{0\kappa^+}_{\alpha^*}$ to each p_α with $\alpha \in S^*$.

By 1.2.16(2), we can add it without adding ordinals and the only other models that probably were added are the images of $A^{0\kappa^+}_{\alpha^*}$ under Δ-system type isomorphisms. Denote the result for simplicity by p_α as well.

Let now $\beta < \alpha$ be ordinals in S^*. We claim that p_β and p_α are compatible in $\langle \mathcal{P}, \to \rangle$. First extend p_α by adding $A^{0\kappa^+}_{\beta+2}$. This will not add other additional models or ordinals except the images of $A^{0\kappa^+}_{\beta+2}$ under isomorphisms to p_α, as was remarked above.

Let p be the resulting extension. Denote p_β by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$ and $p_n = \langle a_n, A_n, f_n \rangle$. Let $q_n = \langle b_n, B_n, g_n \rangle$. Without loss of generality we may assume that $a_n(A^{0\kappa^+}_{\beta+2})$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $\mathrm{rng}(b_n)$ inside $a_n(A^{0\kappa^+}_{\beta+2})$ over the common parts $\mathrm{dom}(b_n)$ and $\mathrm{dom}(a_n)$. This will produce $q'_n = \langle b'_n, B_n, g_n \rangle$ which is $k_n - 1$-equivalent to q_n and with $\mathrm{rng}(b'_n) \subseteq a_n(A^{0\kappa^+}_{\beta+2})$. Doing the above for all $n \geq \ell(p)$ we will obtain $q' = \langle q'_n \mid n < \omega \rangle$ equivalent to q (i.e. $q' \longleftrightarrow q$).

Extend q' to q'' by adding to it $\langle A^{0\kappa^+}_{\beta+2}, a_n(A^{0\kappa^+}_{\beta+2}) \rangle$ as the maximal set for every $n \geq \ell(p)$. Recall that $A^{0\kappa^+}_{\beta+1}$ was its maximal model. So we add a top model. Hence no additional models or ordinals are added at all. Let $q''_n = \langle b''_n, B_n, g_n \rangle$, for every $n \geq \ell(p)$.

Combine now p and q'' together. Thus for each $n \geq \ell(p)$ we add b''_n to a_n as well as all of its isomorphic images under Δ-system type isomorphisms of triples in a_n. The rest of the parts are combined in the obvious fashion (we put together the functions and intersect sets of measure one moving first to the same measure). Note that this is possible due to the intersection properties, since the relevant models that witness problematic intersections are in $A^{0\kappa^+}_{\alpha}$ and in $A^{0\kappa^+}_{\beta}$ respectively, and so are in the kernel $A^{0\kappa^+}_{\alpha^*}$.
For example let us show that ordinals (i.e. the images of members of $A^{1\kappa^{++}}_{\alpha+1}$ and of $A^{1\kappa^{++}}_{\beta+1}$) are in the right order.
Let $\eta_\alpha \in A^{1\kappa^{++}}_{\alpha+1}$ and let $\eta_\beta \in A^{1\kappa^{++}}_{\beta+1}$ be the corresponding to it ordinal in q''. We need to argue that $\eta_\alpha \geq \eta_\beta$. The only problematic case is once there is some ζ in the kernel above η_α, η_β. Consider then $\zeta_\alpha \in A^{0\kappa^+}_{\alpha} \cap A^{1\kappa^{++}}_{\alpha+1}$ such that $A^{0\kappa^+}_{\alpha} \cap \eta_\alpha = A^{0\kappa^+}_{\alpha} \cap \zeta_\alpha$, i.e. ζ_α is the first element of $A^{0\kappa^+}_{\alpha}$ above η_α. Then, necessarily, ζ_α is in the kernel. Hence $\zeta_\alpha \in A^{0\kappa^+}_{\beta} \cap A^{1\kappa^{++}}_{\beta+1}$ and $A^{0\kappa^+}_{\beta} \cap \eta_\beta = A^{0\kappa^+}_{\beta} \cap \zeta_\alpha$, i.e. ζ_α is the first element of $A^{0\kappa^+}_{\beta}$ above η_β. But $\alpha > \beta$ and α is a limit ordinal, hence $A^{0\kappa^+}_{\alpha} \supseteq A^{0\kappa^+}_{\beta+2}$. In particular, $\eta_\beta \in A^{0\kappa^+}_{\alpha}$. This implies $\eta_\beta < \eta_\alpha$.

Add if necessary $A^{0\kappa^+}_{\alpha+3}$ as a new top model in order to insure 1.2.11(2a). Let $r = \langle r_n | n < \omega \rangle$ be the result, where $r_n = \langle c_n, C_n, h_n \rangle$, for $n \geq \ell(p)$.

Claim 1.2.21.1 For each $\gamma, \alpha+3<\gamma<\kappa^{++}$,

$$\langle\langle A^{0\kappa^+}_\gamma, A^{1\kappa^+}_\gamma, C^{\kappa^+}_\gamma\rangle, A^{1\kappa^{++}}_\gamma\rangle \Vdash r \in \underset{\sim}{\mathcal{P}}.$$

Proof. Let $\gamma \in (\alpha+3, \kappa^{++})$ and $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$ with $\langle\langle A^{0\kappa^+}_\gamma, A^{1\kappa^+}_\gamma, C^{\kappa^+}_\gamma\rangle, A^{1\kappa^{++}}_\gamma\rangle \in G(\mathcal{P}')$.

Fix $n \geq \ell(p)$. The main points here are that b''_n and a_n agree on the common part and adding of b''_n to a_n does not require other additions of models or of ordinals except the images of b''_n under Δ -system type isomorphisms for triples in a_n.

We need to check that $\mathrm{dom}(c_n)$ is a suitable generic structure and $\mathrm{rng}(c_n)$ is a suitable structure. Let us deal with $\mathrm{dom}(c_n)$. The range is similar. By Lemma 1.2.10 it is enough to deal with a weak suitable structures. Let $\langle\langle X, Y\rangle, C, \in, \subseteq\rangle$ be the corresponding redact of $\mathrm{dom}(c_n)$.

Clearly, $\langle\langle X, Y\rangle, C, \in, \subseteq\rangle$ is a submodel of $\langle\langle A^{1\kappa^+}_\gamma, A^{1\kappa^{++}}_\gamma\rangle, C^{\kappa^+}_\gamma, \in, \subseteq\rangle$.

Let us check that the structures $\langle\langle X, Y\rangle, C, \in, \subseteq\rangle$ and $\langle\langle A^{1\kappa^+}_\gamma, A^{1\kappa^{++}}_\gamma\rangle, C^{\kappa^+}_\gamma, \in, \subseteq\rangle$ agree about pistes to members of X and to ordinals in Y. This will show, in particular that $\langle\langle X, Y\rangle, C, \in, \subseteq\rangle$ is pistes closed and, hence $\langle\langle \max(X), X, C\rangle, Y\rangle \in \mathcal{P}'$.

Fix $t \in X \cup Y$ (a model or an ordinal). Note that, by the choice of the top model $\max(X)$ of X we have $\max(X) \in C^{\kappa^+}_\gamma(A^{0\kappa^+}_\gamma)$. Hence, the piste from $A^{0\kappa^+}_\gamma$ to t will go via $\max(X)$. If t appears in $\mathrm{dom}(a_n)$, then the continuation of the piste will be inside $\mathrm{dom}(a_n)$, since $\max(a_n) = A^{0\kappa^+}_{\alpha+1} \in C(\max(X))$. It will co-inside with the piste from $A^{0\kappa^+}_{\alpha+1}$ to t, since $\mathrm{dom}(a_n)$ is a suitable structure. Hence all the members of the piste are in $X \cup Y$.

Note that if t is in the common part, i.e. if t appears in both $\mathrm{dom}(a_n)$ and $\mathrm{dom}(b_n)$, then $t \in A^{0\kappa^+}_{\alpha^*}$. So the piste to t passes through $A^{0\kappa^+}_{\alpha^*}$, since $A^{0\kappa^+}_{\alpha^*} \in C^{\kappa^+}(A^{0\kappa^+}_\gamma)$.

If t appears in $\mathrm{dom}(b''_n) = \mathrm{dom}(b_n) \cup \{A^{0\kappa^+}_{\beta+2}\}$, then the piste to t will proceed via $A^{0\kappa^+}_{\beta+2}$, since $t \in A^{0\kappa^+}_{\beta+2}$ and $A^{0\kappa^+}_{\beta+2} \in C(\max(X))$. Now, it will co-inside with the piste from $A^{0\kappa^+}_{\beta+1}$ to t, since $\mathrm{dom}(b_n)$ is a suitable structure and $A^{0\kappa^+}_{\beta+1} \in C(A^{0\kappa^+}_{\beta+2})$.

The agreement between the pistes follows.

$\square$ of the claim.

Now we have $r \geq p, q''$. Hence, $p \to r$ and $q \to r$. Contradiction.

$\square$

Combining the previous lemmas together, we obtain the following:

Theorem 1.2.22 *$V^{\mathcal{P}' * \langle \mathcal{P}, \to\rangle}$ is a cardinal preserving extension of V which satisfies $2^\kappa = \kappa^{+3}$.*

Chapter 2

Gaps 4 and above-more symmetry.

It is a slightly changed version of [7] which allows more symmetry in the following sense: for any two isomorphic models $A, B \in A^{1,\eta}$ from a generic $G \subseteq \mathcal{P}'$ the structure of models from G of cardinalities smaller than η in A is the same as those in B, i.e. $\pi_{A,B}[E] \in G \cap B$ whenever $E \in G \cap A$ of cardinality $< \eta$.
This symmetry makes the forcing $\mathcal{P}'_{<\eta}$ equivalent to $N_2 \cap \mathcal{P}'_{<\eta}$, see [7],1.23,1.24.

2.1 One difference between gap 3 and higher gaps

Let $\mathcal{P}'(3)$ denote the preparation forcing for the gap 3 defined in the previous chapter. Let G be a generic subset of $\mathcal{P}'(3)$. Consider

$$S = \{A \mid \exists \langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in G \quad A = A^{0\kappa^+}\}.$$

It was shown (Lemma 1.1.20) that S is a stationary subset of $[H(\kappa^{+3})]^{\leq \kappa^+}$. Let us point out in addition the following:

Proposition 2.1.1 *If $A, B \in S$ and $otp(A \cap \kappa^{+3}) = otp(B \cap \kappa^{+3})$, then A and B are isomorphic by an isomorphism which is an identity over $A \cap B$.*

Proof. Induction on pistes complexity.
□

The purpose of this note will be to show that this proposition fails already in the gap 4 case.

Theorem 2.1.2 *Let $\lambda < \mu$ be cardinals such that*

1. *μ is regular,*

2. $\lambda^{++} < \mu$,

3. $2^\lambda = \lambda^+$,

4. *for every* $\delta, \lambda^+ < \delta < \mu$, $\delta^{\lambda^+} = \delta$.

Suppose that S *is an unbounded subset of* $[H(\mu)]^\lambda$.
Then there are $A, B \in S$ *with* $otp(A \cap \mu) = otp(B \cap \mu)$, *but the isomorphism between* $A \cap \mu$ *and* $B \cap \mu$ *is not the identity on* $A \cap B \cap \mu$.

Proof. Suppose otherwise. Let S be an unbounded subset of $[H(\mu)]^\lambda$ witnessing this.
Consider a sequence $\langle M_\alpha \mid \alpha < \mu \rangle$ such that for every $\alpha < \mu$

1. $\langle M_\alpha, \in, <, M_\alpha \cap S \rangle \prec \langle H(\mu), \in, <, S \rangle$,

2. $|M_\alpha| = \lambda^+$,

3. $M_\alpha \supseteq \lambda^+$,

4. ${}^\lambda M_\alpha \subseteq M_\alpha$,

5. $\beta \neq \alpha$ implies $M_\beta \neq M_\alpha$.

Form a Δ-system and shrink the sequence $\langle M_\alpha \mid \alpha < \mu \rangle$ to a sequence $\langle M_\alpha \mid \alpha \in Z \rangle$ such that for every $\alpha, \beta \in Z, \alpha < \beta$ the following hold:

1. $M_\alpha \cap \alpha = M_\beta \cap \beta$,

2. $\sup(M_\alpha \cap \mu) < \beta$,

3. $\langle M_\alpha, \in, <, M_\alpha \cap S \rangle \simeq \langle M_\beta, \in, <, M_\beta \cap S \rangle$ and the isomorphism is the identity on the common part.

Fix some $\alpha \neq \beta$ in Z. Pick an ordinal $\tau \in M_\alpha$ above $\sup(M_\alpha \cap M_\beta \cap \mu)$.
Now we use unboundedness S and find $A \in S$ with $\tau, \pi_{M_\alpha, M_\beta}(\tau) \in A$.
Consider $A \cap M_\alpha$. This set belongs to M_α, since M_α is closed under λ-sequences of its elements. By elementarity it is possible to find $A_\alpha \in M_\alpha$ such that

- $A_\alpha \supseteq M_\alpha \cap A$,

- $otp(A_\alpha \cap \mu) = otp(A \cap \mu)$,

- $A_\alpha \in S$.

Set $A_\beta = \pi_{M_\alpha, M_\beta}(A_\alpha)$. Then $otp(A_\alpha \cap \mu) = otp(A_\beta \cap \mu)$ and $A_\beta \in S$, by (3) above. Note also that the isomorphism π_{A_α, A_β} is just $\pi_{M_\alpha, M_\beta}(A_\alpha) \upharpoonright A_\alpha$. By (1) above and the choice of τ we have $A_\alpha \cap A_\beta \cap \mu \subseteq A_\alpha \cap \tau$. Hence $\tau' := \pi_{A_\alpha, A_\beta}(\tau) \neq \tau$. But $\pi_{A_\alpha, A_\beta}(\tau) = \pi_{M_\alpha, M_\beta}(\tau)$ and the last component is in A. So, $\tau' \in A \cap A_\beta$.
Now,

$$\pi_{A, A_\beta}(\tau) = \pi_{A_\alpha, A_\beta}(\pi_{A, A_\alpha}(\tau)).$$

But $\tau \in A \cap A_\alpha, A, A_\alpha \in S$, so $\pi_{A, A_\alpha}(\tau) = \tau$. Then

$$\pi_{A, A_\beta}(\tau) = \pi_{A_\alpha, A_\beta}(\tau) = \tau'.$$

Which is impossible, since $\tau' \in A \cap A_\beta, A, A_\beta \in S$ and $\tau \neq \tau'$.
□

Without GCH type assumptions the theorem above consistently fails. Thus one can use a "baby" version of the arbitrary gap preparation forcing $\mathcal{P}'$ which will be defined in the next section:

$$\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \in s \rangle,$$

with only requirement that models of the same order type are isomorphic over their intersection.

We do not know if for the gap 3 always there is S as in Proposition 2.1.1 (or even only unbounded set like this). Our conjecture is that it should not be the case. On the other hand in L-like models it may exist due to morass structures inside.
Note also that once we have such S, then it is quite hard to eliminate it. Cardinals should be collapsed or change their cofinality.

Carmi Merimovich asked the following question:

Suppose $N_1, N_2 \prec \langle H(\omega_2), \in, < \rangle$ are countable and $otp(N_1 \cap \omega_2) = otp(N_2 \cap \omega_2)$. Does it necessarily imply that $N_1 \cong N_2$?

The following provides a rather complete answer.

Proposition 2.1.3 *Suppose that $V = L$, $N_1, N_2 \prec \langle H(\omega_2), \in, < \rangle$ are countable and $otp(N_1 \cap \omega_2) = otp(N_2 \cap \omega_2)$. Then $N_1 \cong N_2$.*

Proof. By Condensation Lemma, both N_1 and N_2 are isomorphic to L_α, where $\alpha = otp(N_1 \cap \omega_2) = otp(N_2 \cap \omega_2)$.
□

Proposition 2.1.4 *Suppose that $2^{\aleph_0} > \aleph_1$, $N_1, N_2 \prec \langle H(\omega_2), \in, < \rangle$ are countable, otp$(N_1 \cap \omega_2) =$ otp$(N_2 \cap \omega_2)$, but $N_1 \cap \omega_2 \neq N_2 \cap \omega_2$. Then $N_1 \not\cong N_2$.*

Proof. We have $N_1 \cap \omega_2 \neq N_2 \cap \omega_2$ and $2^{\aleph_0} \geq \aleph_2$, hence there is a real r in $N_1 \setminus N_2$. Now if $N_1 \cong N_2$, then the isomorphism function π_{N_1,N_2} is the identity on ω. In particular $\pi_{N_1,N_2}(r) = r$. Contradiction.
□

The same conclusion can be derived even under GCH.

Proposition 2.1.5 *Assume GCH. Suppose that $N_1, N_2 \prec \langle H(\omega_2), \in, < \rangle$ be countable isomorphic structures such that $\min((N_2 \cap \omega_2) \setminus (N_1 \cap \omega_2)) > \sup(N_1 \cap \omega_2)$. Let P be Cohen forcing which adds a function from $\aleph_2$ to 2 with conditions of a size $\leq \aleph_1$. Then there is a generic $G \subseteq P$ such that $N_1[G], N_2[G] \prec \langle H(\omega_2)^{V[G]}, \in, < \rangle$, but $N_1[G] \not\cong N_2[G]$.*

Proof. Let $\eta_1 = \min((N_1 \cap \omega_2) \setminus (N_2 \cap \omega_2))$ and $\eta_2 = \min((N_2 \cap \omega_2) \setminus (N_1 \cap \omega_2))$. Then, clearly, $\pi_{N_1,N_2}(\eta_1) = \eta_2$. Now let us pick $g_1 : N_1 \cap \omega_2 \to 2$ to be P-generic function over N_1. Let $g_1(\eta_1) = 0$. Consider $g = g_1 \restriction \eta_1$. Clearly, $g \in N_1 \cap P$ and $\pi_{N_1,N_2}(g) = g$. Set $g' = g \cup \{\langle \eta_2, 1 \rangle\}$. Extend now g' to $g_2 : N_2 \cap \omega_2 \to 2$ which is P-generic function over N_2. Set $g'' = g_1 \cup g_2$. Then $g'' \in P$, since $\eta_2 = \min((N_2 \cap \omega_2) \setminus (N_1 \cap \omega_2)) > \sup(N_1 \cap \omega_2)$. Let now G be a generic subset of P with $g'' \in G$. By the construction we have $N_1[G], N_2[G] \prec \langle H(\omega_2)^{V[G]}, \in, < \rangle$, but $\bigcup G(\eta_1) \neq \bigcup G(\eta_2)$, and so the models $N_1[G], N_2[G]$ cannot be isomorphic.
□

2.2 The Preparation Forcing

We assume GCH. Fix two cardinals κ and θ such that $\kappa < \theta$ and θ is regular.

We define a set which is parallel to $\mathcal{P}''$ of Gap 3, i.e. the set of central lines.

Definition 2.2.1 The set $\mathcal{P}'''$ consists of sequences of the form $\langle C^\tau \mid \tau \in s \rangle$ such that

1. s is a closed set of cardinals from the interval $[\kappa^+, \theta]$ satisfying the following:
    (a) $|s \cap \delta| < \delta$ for each inaccessible $\delta \in [\kappa^+, \theta]$
    (b) $\kappa^+, \theta \in s$
    (c) if $\rho^+ \in s$ and $\rho \geq \kappa^+$, then $\rho \in s$
    (d) if $\rho \in s$ is singular, then s is unbounded in ρ and $\rho^+ \in s$.

If there is no inaccessible cardinals inside the interval $[\kappa^+, \theta]$, then s can be taken to be the set of all the cardinals of this interval.

2. For every $\tau \in s$, C^τ is a continuous closed chain of a length less than τ^+ of elementary submodels of $\langle H(\theta^+), \in, <, \subseteq, \kappa\rangle$ each of cardinality τ

   such that

   (a) for each element $X \in C^\tau$ we have $X \cap \tau^+ \in On$ and, hence $X \supseteq \tau$,
   Further we shall denote $\text{otp}(X \cap \theta^+)$ by simply $\text{otp}(X)$.

   (b) If $X \in C^\tau$ and there is $Y \in C^\rho, Y \supset X$, for some $\rho \in s \setminus \tau + 1$, then there is $Y \in C^{\tau^*}, Y \supset X$ such that for each $\rho \in s \setminus \tau + 1$ if $Z \in C^\rho$ and $Z \supset X$, then $Z \supseteq Y$, where $\tau^* = \min(s \setminus \tau + 1)$.

   (c) If X is a non-limit element of the chain C^τ then

      i. $C^\tau \restriction X := \{Y \mid Y \subset X, Y \in C^\tau\} \in X$,

      ii. ${}^{\text{cof}(\tau)>}X \subseteq X$,

      iii. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^\rho$ with $\sup(Y) \geq \sup(X)$, then $X \subseteq Y$,

      iv. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^\rho$ with $\sup(Y) < \sup(X)$, then there are $\rho' \in (s \setminus \rho) \cap X$ and $Y' \in C^{\rho'} \cap X$ such that $Y' \supseteq Y$ and $Y \cap X = Y' \cap X$.
      Note that $\rho' = \rho$, unless there are inaccessible cardinals.

      v. If $\xi \in (s \setminus \tau + 1) \cap X$ and $C^\xi \cap X \neq \emptyset$, then

      $$\bigcup\{Y \in C^\xi \mid Y \in X\} \in X.$$

      Denote this union by $(X)_\xi$.
      Note that if for some $\tau \in s, \xi \in s \cap \tau$ and $Z \in C^\tau$ there is no $\rho \in s \setminus \tau, A \in C^\xi$ with $(A)_\rho$ defined and so that $Z \subseteq (A)_\rho$, then $Z \supseteq B$ for each $B \in C^\xi$. Since, if for some $B \in C^\xi$ we have $\sup(Z \cap \theta^+) < \sup(B \cap \theta^+)$, then, by the condition (iv) above, there are $\rho \in s \setminus \tau, Y \in C^\rho \cap B$ such that $Z \subseteq Y$ and $Z \cap B = Y \cap B$. So, $(B)_\rho$ exists and $Z \subseteq (B)_\rho$.

      vi. $\langle C^\xi \cap (X)_\xi \mid \xi \in s \setminus \tau + 1, (X)_\xi \text{ is defined } \rangle \in X$.

3. If $\langle \xi_j \mid j < i\rangle$ is an increasing sequence of elements of s, $\xi = \bigcup_{j<i} \xi_j$ and $\langle X_j \mid j < i\rangle$ is an increasing (under the inclusion) sequence such that $X_j \in C^{\xi_j}$ for each $j < i$, then $X = \bigcup_{j<i} X_j$ is in C^ξ.

The next set will be needed here in order to define a Δ-system type triple.

Definition 2.2.2 The set $\mathcal{P}''$ consists of all sequences of triples

$$\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$$

such that for every $\tau \in s$ the following hold:

1. $|A^{1\tau}| \leq \tau$,
2. $A^{0\tau} \in A^{1\tau}$,
3. every $X \in A^{1\tau}$ has cardinality τ and is either equal to $A^{0\tau}$ or belongs to it,
4. $C^\tau : A^{1\tau} \to P(A^{1\tau})$,
5. $\langle C^\tau(A^{0\tau}) \mid \tau \in s\rangle \in \mathcal{P}'''$,
6. (Coherence)
   if $X, Y \in C^\tau(A^{0\tau})$ and $X \in C^\tau(Y)$, then $C^\tau(X)$ is an initial segment of $C^\tau(Y)$ with X being the largest element of it.
7. Let $B \in C^\tau(A^{0\tau})$ and $s' = \{\rho \in s \cap \tau \mid \exists X \in C^\rho(A^{0\rho}) \quad X \subseteq B\}$. For each $\rho \in s'$ let B_ρ be the largest element of $C^\rho(A^{0\rho})$ contained in B. Then

$$\langle C^\rho(B_\rho) \mid \rho \in s'\rangle^\frown\langle C^\tau(B)\rangle^\frown\langle C^\xi(A^{0\xi}) \mid \xi \in s \setminus \tau+1\rangle \in \mathcal{P}'''.$$

Now we define Δ-system type triples. The definition is more involved than those in the gap 3 case. The basic reason is that instead of using a single central line consisting of ordinals there, we may have here many other central lines. Over each of them Δ-system type triple may appear (thus, for example for the gap 4: there will be Δ-system type triples for κ^+ relatively to lines of models of cardinality κ^{++}, and those of cardinality κ^{++} relatively to lines of cardinality κ^{+3}, i.e. ordinals). We define simultaneously also switching using the induction on the rank of sets.

Definition 2.2.3 Suppose that $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}''$, $F \in C^\tau(A^{0\tau})$, for some $\tau \in s, \tau < \theta$ and $F_0, F_1 \in F$. We say that the triple F_0, F_1, F is of Δ-*system type* iff

1. F_0 is the immediate predecessor of F in $C^\tau(A^{0\tau})$
2. $F_1 \prec F$,

3. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^\rho(A^{0\rho})$ with $\sup(Y) \geq \sup(F_1)$, then $F_1 \subseteq Y$,

4. if for some $\rho \in s, \rho > \tau$ we have $Y \in C^\rho(A^{0\rho})$ with $\sup(Y) < \sup(F_1)$, then there are $\rho' \in (s \setminus \rho) \cap F_1$ and $Y' \in C^{\rho'}(A^{0\rho'}) \cap F_1$ such that $Y' \supseteq Y$ and $Y \cap F_1 = Y' \cap F_1$.
   Here we need to consider two possibilities: $\tau^+ \in s$ or $\tau^+ \notin s$ and then $\min(s \setminus \tau + 1)$ is an inaccessible cardinal. We will treat both possibilities similar. Denote $\min(s \setminus \tau + 1)$ by τ^*. So τ^* is either τ^+ or τ^* is an inaccessible.

5. There is $H_i \in A^{1\tau^*} \cap F_i$ which is maximal under inclusion, where $i \in \{0, 1\}$. Moreover $H_0 \in C^{\tau^*}(A^{0\tau^*})$.

6. There are $G_0, G_1 \in C^{\tau^*}(A^{0\tau^*}) \cap F$ such that

   (a) $\mathrm{cof}(G_0 \cap (\tau^*)^+) = \mathrm{cof}(G_1 \cap (\tau^*)^+) = \tau^*$,

   (b) $G_0 \in F_0$ and $G_1 \in F_1$

   (c) $F_0 \cap F_1 = F_0 \cap G_0 = F_1 \cap G_1$,

   (d) either $G_0 \in G_1$ or $G_1 \in G_0$,

   (e) there is a switch of $p \setminus \tau + 1 := \langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \setminus \tau + 1 \rangle$ which involves models only with supremums below $\max(\sup(F_0 \cap \theta^+), \sup(F_1 \cap \theta^+))$ which leaves H_0 on the central line for τ^* and moves H_1, G_0, G_1 to the central line. Moreover, all the models involved in the switch are in F.
   Here we use the induction on the ranks of sets.

   Further let us call G_0, G_1 the witnessing models for F_0, F_1, F.
   We may refer to H_0, H_1 and the models used in the switch as witnessing models as well.

   The next condition will require more similarity:

7. (isomorphism condition)
   the structures

   $$\langle F_0, \in, <, \subseteq, \kappa, \tau, C^\tau(F_0), \langle A^{1\rho} \cap F_0 \mid \rho \in (s \setminus \tau) \cap F_0 \rangle, \langle C^\rho \restriction A^{1\rho} \cap F_0 \mid \rho \in s \setminus \tau \rangle, f_{F_0} \rangle$$

   and

   $$\langle F_1, \in, <, \subseteq, \kappa, \tau, C^\tau(F_1), \langle A^{1\rho} \cap F_1 \mid \rho \in (s \setminus \tau) \cap F_1 \rangle, \langle C^\rho \restriction A^{1\rho} \cap F_1 \mid \rho \in s \setminus \tau \rangle, f_{F_1} \rangle$$

are isomorphic over $F_0 \cap F_1$, i.e. the isomorphism $\pi_{F_0 F_1}$ between them is the identity on $F_0 \cap F_1$, where $f_{F_0} : \tau \longleftrightarrow F_0$, $f_{F_1} : \tau \longleftrightarrow F_1$ are some fixed in advance bijections.

In particular, we will have that $\mathrm{otp}(F_0) = \mathrm{otp}(F_1)$ and $F_0 \cap \tau^* = F_1 \cap \tau^*$.
Note that here we use $C^\rho \restriction A^{1\rho} \cap F_i$ $(i < 2)$. In

the gap 3 case we had only $A^{1\kappa^{++}}$, but it was just an increasing sequence and so served as a replacement of $C^{\kappa^{++}}$ as well.

8. For each $\xi \in s$, if $X \in A^{1\xi}$ and $X \supseteq F_0, F_1$, then $X \supseteq F$.

Define the switch q of p by F_0, F_1, F to be

$$\langle \langle A^{0\xi}, A^{1\xi}, D^\xi \rangle \mid \xi \in s \rangle,$$

where D^ξ, for $\xi \in s \setminus \tau + 1$ is determined by switching in $p \setminus \tau + 1$ below $\max(\sup(F_0 \cap \theta^+), \sup(F_1 \cap \theta^+))$ which turns $C^{\tau^*}(H_1)$ into an initial segment of τ^*-central line. $D^\tau(F) = C^\tau(F_1)^\frown F$ and $D^\tau(A^{0\tau}) = D^\tau(F)^\frown \langle X \in C^\tau(A^{0\tau}) \mid X \supset F \rangle$. The rest is defined in the obvious fashion by taking images under isomorphisms π_{F_0,F_1} etc.
Further let denote such q by $swt(p, F)$.
Note that that it need not be a condition.
Denote by $swt(p, B_1, \ldots, B_n)$ the result of an application of the switch operation n-times: $p_{i+1} = swt(p_i, B_i)$, for each $1 \leq i \leq n$, where $p_1 = p$ and $swt(p, B_1, \ldots, B_n) = p_{n+1}$.

Note that there is no Δ-system type triples in the cardinality θ.

Now we define the preparation forcing $\mathcal{P}'$.

Definition 2.2.4 The set $\mathcal{P}'$ consists of elements of the form

$$\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$$

so that the following hold:

1. $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle \in \mathcal{P}''$,

   We call $C^\tau(A^{0\tau})$ *τ-central line* of $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$.
   The following conditions describe a special way in which $A^{1\tau}$ is generated from the central line, for each $\tau \in s$.

2. Let $B \in A^{1\tau}$. Then $B \in C^\tau(A^{0\tau})$ (i.e. it is on the central line) or there there is a finite sequence $piste(B)$ of models in $\bigcup_{\rho \in s \setminus \tau} A^{1\rho}$ that terminates with B. We call this

sequence *a piste* to B and it will be defined recursively below.
First let us define *blue (easy) pistes* and the set $bp(A^{0\tau})$ of elements of $A^{1\tau}$ reachable by such pistes from $A^{0\tau}$.

Set $C^\tau(A^{0\tau}) \subseteq bp(A^{0\tau})$. If $B \in C^\tau(A^{0\tau})$, then set $piste(B) = \langle B \rangle$.
Suppose now that $A \in bp(A^{0\tau})$ and $bp(A)$ is defined. Again, set $C^\tau(A) \subseteq bp(A^{0\tau})$. If $B \in C^\tau(A) \setminus \{A\}$, then set $piste(B) = piste(A)^\frown \langle B \rangle$.
It is allowed to continue a blue piste via a Δ-system triple.

Blue Piste Continuation–First Continuation.

Suppose now that $A \in bp(A^{0\tau})$, $bp(A)$ is defined and there are models $A_0, A_1 \in A \cap A^{1\tau}$ such that

(a) the triple A_0, A_1, A is of a Δ-system type with respect to $\langle \langle A^{0\xi}, A^{1\xi}, C^\xi \rangle \mid \xi \in s \backslash \tau \rangle$,

(b) $A_0 \in C^\tau(A)$,

Then we add A_1 to $bp(A^{0\tau})$ and $piste(A_1)$ is defined by adding A_1, A_0 and the models witnessing the Δ system to $piste(A)$.

Now let $B \in A^{1\tau}$. We define $piste(B)$ the piste leading to B. If $B \in bp(A^{0\tau})$, then $piste(B)$ was already defined. Suppose that $B \notin bp(A^{0\tau})$. We follow first the blue piste down from $A^{0\tau}$ until reaching the least element A of $bp(A^{0\tau})$ with $B \in A$, i.e. first pick the least element of $C^\tau(A^{0\tau})$ with B inside if it is not of a Δ-system type, then set A to be this element; otherwise, take the immediate predecessor of it which is not on the central line and continue down through it etc.

We continue and define recursively (and using $\in$-induction) in addition sets of models connected by a piste to a given model $T \in \bigcup_{\nu \in s} A^{1\nu}$ which will be denoted by $pc(T)$. Require that $bp(T) \subseteq pc(T)$.

Second Continuation.

There are $\rho \in s \cap A, \rho > \tau$, $T^*, T_0, T_1, T \in A^{1\rho} \cap A$ such that

(a) T^* is on ρ–central line relatively to A, i.e. once we make the switches along the piste up to A which move A to the central line, then T^* is moved there as well; other way to state this: if Z is the largest model of $A^{1\rho} \cap A$, then $T^* \in C^\rho(Z)$. In particular, if A is the first model of the piste or only the first continuation was used on the way to A, then $T^* \in C^\rho(A^{0\rho})$.

(b) $T, T_0, T_1 \in pc(T^*)$.
Note that $T^* \in A$, so we can assume that $pc(T^*)$ is already defined.

(c) $piste(T^*, T) \in A$,

(d) T_0, T_1, T are of a Δ-system type,

(e) There is $E \in bp(A)$ such that

  i. $E \subseteq T_0$,

  ii. there is no $E' \in bp(A)$ with $E \subsetneq E' \subseteq T_0$,

  iii. $B \subseteq \pi_{T_0,T_1}(E)$.

We add all the relevant models above, i.e. $T^*, T_0, T_1, T, E, \pi_{T_0,T_1}(E)$ etc. to $piste(B)$ and $\pi_{T_0,T_1}(E)$ to $pc(A)$. Continue further from $\pi_{T_0,T_1}(E)$.

Note that $\pi_{T_0,T_1}(E)$ is an immediate predecessor of A. Once E is a proper subset of A_0, this produces an immediate predecessor of A of smaller order type.
Also this process applied to different T's and E's may generate a large number of immediate predecessors of A of different order types.

The next continuation is just an iteration of the previous one.

Third Continuation.

There are $n < \omega$ and a sequence $\langle A_i \mid i \leq n \rangle$ of elements of $A^{1\tau} \cap A$ such that

(a) $A_0 \in bp(A)$,

(b) A_1 is obtained from A_0 as in Second Continuation.

(c) For every $i, 2 < i \leq n$, A_i is obtained using A_{i-1}. Namely there are $E^i \in bp(A_{i-1}), \rho_i \in s \cap A, \rho^i > \tau$, $T^{i*}, T^i_0, T^i_1, T^i \in A^{1\rho_i} \cap A$ such that

  i. T^{i*} is on the central line relatively to A,

  ii. $T^i_0, T^i_1, T^i \in pc(T^{i*})$,

  iii. T^i_0, T^i_1, T^i are of a Δ-system type,

  iv. $piste(T^{i*}, T^i) \in A$,

  v. $E^i \subseteq A_{i-1} \cap T^i_0$,

  vi. there is no $E' \in bp(A_{i-1})$ with $E^i \subsetneq E' \subseteq T^i_0$,

  vii. $A_i = \pi_{T^i_0,T^i_1}(E^i)$,

(d) $B \subseteq A_n$

We add all the relevant models above to $piste(B)$ and all models of $\bigcup_{i \leq n} pc(A_i)$ to $pc(A)$.

The next condition insures a kind of symmetry:

3. Let T_0^n, T_1^n, E^n be as in Third Continuation. Then $\pi_{T_0^n, T_1^n}[pc(E^n)] = pc(\pi_{T_0^n, T_1^n}(E^n))$. Moreover $\pi_{T_0^n, T_1^n} \restriction pc(E^n)$ is an isomorphism between the corresponding structures.

   We require that every element of $A^{1\tau}$ is connected to $A^{0\tau}$ by a piste:

4. $A^{1\tau} = pc(A^{0\tau})$, for every $\tau \in s$.

5. Let $F_0, F_1, F \in A^{1\tau}$ be of a Δ-system type, $F_0, F \in C^\tau(A^{0\tau})$. Suppose that $\xi \in s \cap \tau$, $(A^{0\xi})_\tau$ exists and $(A^{0\xi})_\tau \supseteq F_0$. Let $X \in C^\xi(A^{0\xi})$ be the least with $(X)_\tau \supseteq F_0$. Then $(X)_\tau \supseteq F$.
   The meaning of this condition is that it is impossible to have a small model in between models of a Δ-system type of larger cardinality. It will not be very restrictive for our further purposes, since we will be always able to increase first elements of $\mathcal{P}'$ by adding models of cardinality τ at the top, and only then to make a Δ-system type triple.

   The next condition is relevant once inaccessibles are present.

6. Let $F_0, F_1, F \in A^{1\tau}$ be of a Δ-system type, $F_0, F \in C^\tau(A^{0\tau})$. Suppose that $\xi \in s \cap \tau$, $X \in C^\xi(A^{0\xi})$, for some $\rho \in s \setminus \tau$, $(X)_\rho$ exists and $(X)_\rho \supseteq F_0$. Then $(X)_\rho \supseteq F$.

7. (uniqueness) Let $F_0, F_1, F_0', F_1', F \in A^{1\tau}$. If both triples F_0, F_1, F and F_0', F_1', F are of a Δ-system type, then $\{F_0, F_1\} = \{F_0', F_1'\}$.

The following lemma follows directly from the definition.

Lemma 2.2.5 *Let $\langle\langle A^{0\xi}, A^{1\xi}, C^\xi\rangle \mid \xi \in s\rangle \in \mathcal{P}'$. Then $A^{1\theta}$ is a chain.*

Proof. Just note that we have no Δ-system triples in the cardinality θ. Hence each model in $A^{1\theta}$ is on the θ-central line, i.e. on $C^\theta(A^{0\theta})$.
□

We need to allow a possibility to change the component C^τ in elements of $\mathcal{P}'$ and replace one central line by another. It is essential for the definition of an order on $\mathcal{P}'$ given below.

Definition 2.2.6 Let $r, q \in \mathcal{P}'$. Then $r \geq q$ (r is stronger than q) iff there is $p = swt(r, B_1, \ldots, B_n) \in \mathcal{P}'$ for some $B_1, \ldots, B_n$ appearing in r so that the following hold, where

$$p = \langle \langle A^{0\xi}, A^{1\xi}, C^{\xi} \rangle \mid \xi \in s \rangle$$

$$q = \langle \langle B^{0\xi}, B^{1\xi}, D^{\xi} \rangle \mid \xi \in s' \rangle$$

1. $s' \subseteq s$,

2. $B^{0\xi} \in C^{\xi}(A^{0\xi})$, for each $\xi \in s'$,

3. $q = p \restriction \langle B^{0\xi} \mid \xi \in s' \rangle$,
   where $p \restriction \langle B^{0\xi} \mid \xi \in s' \rangle = \langle \langle B^{0\xi}, A^{1\xi} \cap (B^{0\xi} \cup \{B^{0\xi}\}), C^{\xi} \restriction (B^{0\xi} \cup \{B^{0\xi}\}) \rangle \mid \xi \in s' \rangle$,

4. for each $\xi \in s'$ and $X \in C^{\xi}(A^{0\xi}) \setminus C^{\xi}(B^{0\xi}) \quad q \in X$,

5. for each $\xi \in s \setminus s'$ and $X \in C^{\xi}(A^{0\xi}) \quad q \in X$.

The meaning of the last two conditions is that new models over central lines supposed to be above all old ones.

Let $p = \langle \langle A^{0\xi}, A^{1\xi}, C^{\xi} \rangle \mid \xi \in s \rangle \in \mathcal{P}'$ and $\eta \in s$. Set $p \setminus \eta = \langle \langle A^{0\xi}, A^{1\xi}, C^{\xi} \rangle \mid \xi \in s \setminus \eta \rangle$. Define $\mathcal{P}'_{\geq \eta}$ to be the set of all $p \setminus \eta$ for $p \in \mathcal{P}'$.

Lemma 2.2.7 *The function $p \rightarrowtail p \setminus \eta$ projects a dense subset of the forcing $\mathcal{P}'$ onto the forcing $\mathcal{P}'_{\geq \eta}$.*

Remark. Note that we split at η only p's in $\mathcal{P}'$ with η inside s of p. The reason is that in the case of $\eta \notin s$ an extension of $p \setminus \eta$ may include models of cardinality η which for example belong to models of p of cardinalities below η. Such extensions will be incompatible with p.

Proof. The set of p's in $\mathcal{P}'$ with η inside s of p is dense. Denote it by D_η.
Let $p \in D_\eta$ and $q \in \mathcal{P}_{\geq \eta}, q \geq p \setminus \eta$. We need to find $r \in \mathcal{P}', r \geq p$ such that $r \setminus \eta \geq q$. Let us take an equivalent to q condition q' in $\mathcal{P}'_{\geq \eta}$ (a switching of q) with the central lines of q' extending those of $p \setminus \eta$. Then $p^\frown q'$ the combination of p with q' will be in $\mathcal{P}'$, $p^\frown q' \geq p$ and $(p^\frown q') \setminus \eta = q'$.
□

Lemma 2.2.8 *$\mathcal{P}'_{\geq \eta}$ is η^+-strategically closed.*

Proof. We define a winning strategy for the player playing at even stages. Thus suppose $\langle p_j \mid j < i \rangle$, $p_j = \langle \langle A_j^{0\tau}, A_j^{1\tau}, C_j^\tau \rangle \mid \tau \in s_j \rangle$ is a play according to this strategy up to an even stage $i < \eta^+$.

Split into two cases.

Case 1. $i = j + 1$.

Let $p = \langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s = s_j \rangle$ be a switch of p_j which restores $A_{j-1}^{0\tau}$ to τ-th central line, i.e. $A_{j-1}^{0\tau} \in C^\tau(A^{0\tau})$, for every $\tau \in s_{j-1}$.

Then pick an increasing continuous sequence $\langle A_i^{0\tau} \mid \tau \in s \rangle$ such that for every $\tau \in s$

(a) ${}^{\mathrm{cof}(\tau)>}A_i^{0\tau} \subseteq A_i^{0\tau}$,

(b) $\langle p_k \mid k < i \rangle, p, \langle A_i^{0\tau'} \mid \tau' < \tau \rangle \in A_i^{0\tau}$.

Set $p_i = \langle \langle A_i^{0\tau}, A_i^{1\tau}, C_i^\tau \rangle \mid \tau \in s \rangle$, where

$$A_i^{1\tau} = A^{1\tau} \cup \{A_i^{0\tau}\}, C_i^\tau = C^\tau \restriction A^{1\tau} \cup \{\langle A_i^{0\tau}, C^\tau(A^{0\tau}) \cup \{A_i^{0\tau}\} \rangle\}.$$

Case 2. i is a limit ordinal.

Set first

$$s = \text{ the closure of } \bigcup_{j<i} s_j.$$

For every $\tau \in \bigcup_{j<i} s_j$, define

$$A_i^{0\tau} = \bigcup_{j<i} A_j^{0\tau}, A_i^{1\tau} = \bigcup_{j<i} A_j^{1\tau} \cup \{A_i^{0\tau}\},$$

$$C_i^\tau = \bigcup_{j<i, j \text{ is even}} C_j^\tau \cup \{\langle A_i^{0\tau}, \{A_i^{0\tau}\} \cup \bigcup \{C_j^\tau(A_j^{0\tau}) \mid \ j \text{ is even}\} \rangle\}.$$

If $\tau \in s \setminus \bigcup_{j<i} s_j$, then set

$$A_i^{0\tau} = \bigcup_{\tau' \in (\cup_{j<i} s_j) \cap \tau} A_i^{0\tau'},$$

$$A_i^{1\tau} = \{A_i^{0\tau}\} \text{ and } C^\tau(A_i^{0\tau}) = \{\langle A_i^{0\tau}, \{A_i^{0\tau}\} \rangle\}.$$

As an inductive assumption we assume that at each even stage $j < i$, p_j was defined in the same fashion. Then $p_i = \langle \langle A_i^{0\tau}, A_i^{1\tau}, C_i^\tau \rangle \mid \tau \in s \rangle$ will be a condition in $\mathcal{P}'$ stronger than each p_j for $j < i$.

□

If we take $\eta = \theta$, then it is easy to show the following:

Lemma 2.2.9 *$\langle \mathcal{P}'_{\geq\theta}, \leq \rangle$ is θ^+-closed.*

Let $p = \langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$ and $\eta \in s$. Set $p \restriction \eta = \langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s \cap \eta\rangle$.

Let $G(\mathcal{P}'_{\geq\eta})$ be a generic subset of $\mathcal{P}'_{\geq\eta}$. Define $\mathcal{P}'_{<\eta}$ to be the set of all $p \restriction \eta$ for $p \in \mathcal{P}'$ with $p \setminus \eta \in G(\mathcal{P}'_{\geq\eta})$.

Lemma 2.2.10 $\mathcal{P}' \simeq \mathcal{P}'_{\geq\eta} * \underset{\sim}{\mathcal{P}'}_{<\eta}$.

Lemma 2.2.11 *If η is a regular cardinal, then the forcing $\mathcal{P}'_{<\eta}$ satisfies η^+-c.c. in $V^{\mathcal{P}'_{\geq\eta}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \Vdash_{\mathcal{P}'_{\geq\eta}} (\langle \underset{\sim}{p}_\alpha = \langle\langle \underset{\sim}{A}^{0\tau}_\alpha, \underset{\sim}{A}^{1\tau}_\alpha, \underset{\sim}{C}^{\tau}_\alpha\rangle \mid \tau \in \underset{\sim}{s}_\alpha\rangle \mid \alpha < \eta^+\rangle \text{ is an antichain in } \underset{\sim}{\mathcal{P}'}_{<\eta})$$

Without loss of generality we can assume that each $A^{0\tau}_\alpha$ is forced to be a successor model, otherwise just extend conditions by adding one additional models on the top. Define by induction, using Lemma 2.2.8, an increasing sequence $\langle q_\alpha \mid \alpha < \eta^+\rangle$ of elements of $\mathcal{P}'_{\geq\eta}$ and a sequence $\langle p_\alpha \mid \alpha < \eta^+\rangle$, $p_\alpha = \langle\langle A^{0\tau}_\alpha, A^{1\tau}_\alpha, C^{\tau}_\alpha\rangle \mid \tau \in s_\alpha\rangle$ so that for every $\alpha < \eta^+$

$$q_\alpha \Vdash_{\mathcal{P}'_{\geq\eta}} \langle\langle \underset{\sim}{A}^{0\tau}_\alpha, \underset{\sim}{A}^{1\tau}_\alpha, \underset{\sim}{C}^{\tau}_\alpha\rangle \mid \tau \in \underset{\sim}{s}_\alpha\rangle = \check{p}_\alpha \ .$$

For a limit $\alpha < \eta^+$ let $\overline{q}_\alpha$ be an upper bound of $\{q_\beta \mid \beta < \alpha\}$, as defined in Lemma 2.2.8 and q_α be its extension deciding $\underset{\sim}{p}_\alpha$. Also assume that $p_\alpha \in A^{0\eta}(q_\alpha)$, where $A^{0\eta}(q_\alpha)$ is the maximal model of q_α of cardinality η.

Note that the number of possibilities for s_α's is at most η, since if η is an inaccessible, then by Definition 2.2.1(1), $|s_\alpha| < \eta$ and if η is an accessible cardinal, then $\eta = (\eta^-)^+$ (remember that η is a regular cardinal). So $s_\alpha \subseteq \eta^- \cup \{\eta^-\}$. But $2^{\eta^-} = \eta$.
Hence, by shrinking if necessary, we may assume that each $s_\alpha = s^*$, for some $s^* \subseteq \eta$. Let $\eta^* = \max(s^*)$.

Form a Δ-system. By shrinking if necessary assume that for some stationary $S \subseteq \eta^+$ we have the following for every $\alpha < \beta$ in S:

1. $A^{0\eta^*}_\alpha \cap A^{0\eta}(\overline{q}_\alpha) = A^{0\eta^*}_\beta \cap A^{0\eta}(\overline{q}_\beta) \in A^{0\eta}(q_0)$

2. $\langle A^{0\eta^*}_\alpha, \in, \leq, \subseteq, \kappa, C^{\eta^*}_\alpha, f_{A^{0\eta^*}_\alpha}, A^{1\eta^*}_\alpha, q_\alpha \cap A^{0\eta^*}_\alpha\rangle$ and $\langle A^{0\eta^*}_\beta, \in, \leq, \subseteq, \kappa, C^{\eta^*}_\beta, f_{A^{0\eta^*}_\beta}, A^{1\eta^*}_\beta, q_\beta \cap A^{0\eta^*}_\beta\rangle$ are isomorphic over $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta$, i.e. by isomorphism fixing every ordinal below $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta$, where

$$f_{A^{0\eta^*}_\alpha} : \eta^* \longleftrightarrow A^{0\eta^*}_\alpha$$

and

$$f_{A^{0\eta^*}_\beta} : \eta^* \longleftrightarrow A^{0\eta^*}_\beta$$

are the fixed enumerations.

Note that $|A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta| \leq \eta^*$. So we can define a function $h_\alpha : \eta^* \to \eta$, for every $\alpha \in S$, by mapping each $i < \eta^*$ to the order type $A^{0\eta^*}_\alpha \cap \theta^+$ between the i-th element of $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta \cap \theta^+$ and its immediate successor in $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta \cap \theta^+$. The total number of such h_α's is at most η, hence by shrinking if necessary we will get the same function. This will insure the isomorphism which is the identity on $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta \cap \theta^+$ and, hence, on $A^{0\eta^*}_\alpha \cap A^{0\eta^*}_\beta$.

We claim that for $\alpha < \beta$ in S it is possible to extend q_β to a condition forcing compatibility of p_α and p_β. Proceed as follows. Pick A to be an elementary submodel of cardinality η^* with $p_\alpha, p_\beta, q_\beta$ inside.

Then the triple $A^{0\eta^*}_\beta, A^{0\eta^*}_\alpha, A$ is of a Δ-system type relatively to q_β, by (2) above. Use this to construct a condition stronger than both p_α, p_β.

Let $\langle A(\tau) \mid \tau \in s^* \cup s(q_\beta)\rangle$ (where $s(q_\beta)$ denotes the support of q_β) be an increasing and continuous sequence of elementary submodels such that for each $\tau \in s^* \cup s(q_\beta)$ the following hold:

- $p_\alpha, p_\beta, q_\beta, A \in A(\tau)$,
- $|A(\tau)| = \tau$.

Extend q_β to q by adding to it $\langle A(\tau) \mid \tau \in s(q_\beta)\rangle$, as maximal models, i.e. $A^{0\tau}(q) = A(\tau)$. Set $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s^*\rangle$, where

$$A^{0\eta^*} = A(\eta^*), A^{1\eta^*} = A^{1\eta^*}_\alpha \cup A^{1\eta^*}_\beta \cup \{A, A^{0\eta^*}\},$$

$$C^{\eta^*} = C^{\eta^*}_\alpha \cup C^{\eta^*}_\beta \cup \langle A, C^{\eta^*}_\beta(A^{0\eta^*}_\beta)^\frown A\rangle \cup \langle A^{0\eta^*}, C^{\eta^*}_\beta(A^{0\eta^*}_\beta)^\frown A^\frown A^{0\eta^*}\rangle,$$

and for each $\tau \in s^* \cap \eta^*$,

$$A^{0\tau} = A(\tau), A^{1\tau} = A^{1\tau}_\alpha \cup A^{1\tau}_\beta \cup \{A^{0\tau}\},$$

$$C^\tau = C^\tau_\alpha \cup C^\tau_\beta \cup \langle A^{0\tau}, C^\tau_\beta(A^{0\tau}_\beta)^\frown A^{0\tau}\rangle.$$

The triple $A^{0\eta^*}_\beta, A^{0\eta^*}_\alpha, A$ is of a Δ-system type relatively to q, by (2) above. It follows that $\langle p, q\rangle \in \mathcal{P}'$. Thus the condition (2) of Definition 2.2.4 holds since each of $\langle p_\alpha, q\rangle, \langle p_\beta, q\rangle$ satisfies it.

□

Lemma 2.2.12 *Let η, $\kappa < \eta \leq \theta$, be a regular cardinal. Then in $V^{\mathcal{P}'}$ we have $2^\eta = \eta^+$.*

Proof. Fix $N \prec H((2^\lambda)^+)$, for λ large enough, such that $\mathcal{P}' \in N$, $|N| = \eta^+$ and ${}^\eta N \subseteq N$. We find $p^N_{\geq\eta^+} \in \mathcal{P}'_{\geq\eta^+}$ which is N-generic for $\mathcal{P}'_{\geq\eta^+}$, using η^{++}-strategic closure of $\mathcal{P}'_{\geq\eta^+}$. Let $G(\mathcal{P}'_{\geq\eta^+})$ be a generic subset of $\mathcal{P}'_{\geq\eta^+}$ with $p_{\geq\eta^+} \in G(\mathcal{P}'_{\geq\eta^+})$. Then, $N[p_{\geq\eta^+}] \prec V_\lambda[G(\mathcal{P}'_{\geq\eta^+})]$. By Lemma 2.2.11, $\mathcal{P}'_{<\eta^+}$ satisfies η^{++}-c.c in $V[G(\mathcal{P}'_{\geq\eta^+})]$. In particular, $\mathcal{P}'_{=\eta}$ satisfies η^{++}-c.c. Let $G(\mathcal{P}'_{=\eta})$ be a generic subset of $\mathcal{P}'_{=\eta}$ over $V[G(\mathcal{P}'_{\geq\eta^+})]$. Denote $N[p_{\geq\eta^+}]$ by N_1. Then $N_1[N_1 \cap G(\mathcal{P}'_{=\eta})] \prec V_\lambda[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})]$, since each antichain for $\mathcal{P}'_{=\eta}$ has cardinality at most η^+. Hence, if it belongs to N_1 then it is also contained in N_1. Denote $N_1[N_1 \cap G(\mathcal{P}'_{=\eta})]$ by N_2.

Consider $\mathcal{P}'_{<\eta} \cap N_2$. Clearly this is a forcing of cardinality η^+. By Lemma 2.2.11, $\mathcal{P}'_{<\eta}$ satisfies η^+-c.c., so $\mathcal{P}'_{<\eta} \cap N_2$ is a nice suborder of $\mathcal{P}'_{<\eta}$. Thus, let $G \subseteq \mathcal{P}'_{<\eta}$ be generic over $V[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})]$ and $H = G \cap N_2$. Then H is $\mathcal{P}'_{<\eta} \cap N_2$ generic over $V[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})]$, since, if $A \subseteq \mathcal{P}'_{<\eta} \cap N_2$ is a maximal antichain, then A is a maximal antichain also in $\mathcal{P}'_{<\eta}$. This follows due to the fact that N_2 is an elementary submodel closed under η-sequences of its elements. Namely, $|A| \leq \eta$, so $A \in N_2$. Then

$$N_2 \models A \text{ is a maximal antichain in } \mathcal{P}'_{<\eta}.$$

Now, by elementarity, A is a maximal antichain in $\mathcal{P}'_{<\eta}$. So there is $p \in G \cap A$. Finally, $A \subseteq N_2$ implies that $p \in N_2$ and hence $p \in H$.

We claim that each subset of η in $V[G(\mathcal{P}'_{\geq\eta^+})][G(\mathcal{P}'_{=\eta})][G]$ is already in $N_2[G]$. It is enough since $|N_2[G]| = |N| = \eta^+$.

Work in V. The construction below can be preformed above any condition of $\mathcal{P}'$ stronger than $p^N_{\geq\eta^+} \in \mathcal{P}'_{\geq\eta^+}$ (which is needed in order to preserve the elementarity of N in generic extensions). So, by density arguments, we will obtain the desired conclusion.

Let $\underset{\sim}{a}$ be a name of a function from η to 2. Define by induction (using the strategic closure of the forcings and η^+-c.c. of $\mathcal{P}'_{<\eta}$) sequences of ordinals

$$\langle \delta_\beta \mid \beta < \eta \rangle, \langle \gamma(\alpha, \beta) \mid \beta < \eta, \alpha < \delta_\beta \rangle$$

and sequences of conditions

$$\langle p_\beta(\alpha) \mid \alpha < \delta_\beta \rangle (\beta < \eta), \langle p(\beta) \mid \beta < \eta \rangle$$

such that

(1) for each $\beta < \eta$, $\delta_\beta < \eta^+$,

(2) for each $\beta < \eta$, $\langle p_\beta(\alpha)_{\geq\eta} \mid \alpha < \delta_\beta \rangle$ is increasing sequence of elements of $\mathcal{P}'_{\geq\eta}$ and $p(\beta)$ is its upper bound obtained as in the Strategic Closure Lemma 2.2.8,

(3) $p_0(0)_{\geq\eta^+} \geq p^N_{\geq\eta^+}$,

(4) the sequence $\langle p(\beta) \mid \beta < \eta \rangle$ is increasing,

(5) for each $\beta < \eta$ and $\alpha < \delta_\beta$, $p_\beta(\alpha) \Vdash \underset{\sim}{a}(\beta) = \gamma(\alpha, \beta)$,

(6) if for some $p \in \mathcal{P}'$ we have $p \setminus \eta \geq_{\mathcal{P}'_{\geq\eta}} p(\beta)_{\geq\eta}$, then there is $\alpha < \delta_\beta$ such that the conditions $p, p_\beta(\alpha)$ are compatible. (I.e. $\{p_\beta(\alpha)_{<\eta} \mid \alpha < \delta_\beta\}$ is a pre-dense set as forced by $p(\beta)_{\geq\eta}$).

Set $p(\eta)$ to be the upper bound of $\langle p(\beta) \mid \beta < \eta \rangle$ as in the Strategic Closure Lemma 2.2.8. Let L' denote the top model of cardinality η of $p(\eta)$, i.e. $A^{0\eta}(p(\eta))$. By the construction in 2.2.8, we have $\delta_\beta, p(\beta) \in L'$ and $\gamma(\alpha, \beta), p_\beta(\alpha) \in L'$, for each $\beta < \eta$ and $\alpha < \delta_\beta$. Alternatively, we can just extent the model L' to one which includes this sequences. Extend L' further if necessary to a model L in order to include $p(\eta)$ as an element.

Turn for a moment to a generic extension. Let $G(\mathcal{P}'_{\geq\eta^+})$ be a generic subset of $\mathcal{P}'_{\geq\eta^+}$ with $p(\eta) \setminus \eta^+ \in G(\mathcal{P}'_{\geq\eta^+})$. Pick $K \in N$ realizing the same type as those of L in $H(2^\lambda)[G(\mathcal{P}'_{\geq\eta^+})]$ over $N \cap L$. Note that $N \cap L$ is a subset of N of cardinality η and, hence, it is in N.

Let

$$\langle q(\beta) \mid \beta < \eta \rangle, \langle q_\beta(\alpha) \mid \alpha < \delta_\beta \rangle (\beta < \eta)$$

be the sequences corresponding to

$$\langle p_\beta(\alpha) \mid \alpha < \delta_\beta \rangle (\beta < \eta), \langle p(\beta) \mid \beta < \eta \rangle$$

and let $q(\eta)$ correspond to $p(\eta)$. Note that $q(\beta) \setminus \eta^+, q_\beta(\alpha) \setminus \eta^+$ are in $G(\mathcal{P}'_{\geq\eta^+})$, since $p(\beta) \setminus \eta^+, p_\beta(\alpha) \setminus \eta^+$ are in $G(\mathcal{P}'_{\geq\eta^+})$. Then,

$$q(\beta) \setminus \eta^+, q_\beta(\alpha) \setminus \eta^+ \leq_{\mathcal{P}'_{\leq\eta^+}} p^N_{\geq\eta^+},$$

by the choice of $p^N_{\geq\eta^+}$ and since $p^N_{\geq\eta^+} \leq_{\mathcal{P}'_{\leq\eta^+}} p(\eta) \setminus \eta^+ \in G(\mathcal{P}'_{\geq\eta^+})$.

Combine now K, L into one condition making them a splitting point. Let M be a model of cardinality η such that $K, L \in M$. Then the triple L, K, M will be of a Δ-system type relatively to $p(\eta)^\frown L^\frown M$ (which is defined in the obvious fashion with $L \in C^\eta(M)$). Now, we add $q(\eta)^\frown K$ to $p(\eta)^\frown L^\frown M$ and turn this into condition in $\mathcal{P}'$, exactly the same way as it was done at the end of the proof of Lemma 2.2.11. Denote such condition by r.

Define a name $\underset{\sim}{b}$ of a subset of η to be

$$\{\langle q_\beta(\alpha), \gamma(\alpha, \beta)\rangle \mid \alpha < \delta_\beta, \beta < \eta\}.$$

Clearly, $\underset{\sim}{b}$ is in N.

Claim 2.2.12.1 $r \Vdash \underset{\sim}{a} = \underset{\sim}{b}$.

Proof. Let G be a generic subset of $\mathcal{P}'$ with $r \in G$. Then also $p(\eta)_{\geq\eta}, q(\eta)_{\geq\eta} \in G$. Now, for each $\beta < \eta$ there is $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$ (just otherwise there will be a condition t in G forcing that for some β there is no $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$. Extend it to t' deciding the value $\underset{\sim}{a}(\beta)$. By (6) there is α such that $t', p_\beta(\alpha)$ are compatible). Let $r' \in G$ be a common extension of r and $p_\beta(\alpha)$. Recall that L, K, M is a triple of a Δ-system type in r and the isomorphism π_{LK} moves $p_\beta(\alpha)$ to $q_\beta(\alpha)$. Hence $q_\beta(\alpha) \leq r'$. But then $q_\beta(\alpha) \in G$.
$\square$ of the claim.
$\square$

Remark 2.2.13 The proof of 2.2.12 actually shows that in $V[G(\mathcal{P}_{\geq\eta})]$ the forcing $\mathcal{P}_{<\eta}$ is equivalent to the forcing $N_2 \cap \mathcal{P}_{<\eta}$ of cardinality η^+. Thus, instead of a name $\underset{\sim}{a}$ of a subset of η take a $\mathcal{P}'_{\geq\eta}$-name of a maximal antichain of $\mathcal{P}'_{<\eta}$. By η^+-c.c. of $\mathcal{P}'_{<\eta}$, the antichain has cardinality $\leq \eta$. Using the strategic closure of $\mathcal{P}'_{\geq\eta}$ we produce a condition deciding all the elements of the antichain. Let L be its top model of cardinality η. Find K as in the proof of 2.2.12 and copy the antichain to N_2. Finally, any $N_2 \cap \mathcal{P}_{<\eta}$-generic will intersect this image, which in turn will imply that on the L-side the same happens.
The models of small cardinalities (i.e. $< \eta$) will always be moved from K to L and vice verse by the definition of Continuations, which is much more relaxed here than in [7].

Let us show that $2^\eta = \eta^+$ for singular η's as well. Note that it is possible to deduce this appealing to Core Models arguments (provided that there is no inner model with too large cardinals).

Lemma 2.2.14 *(a) Let η be a singular cardinal in $[\kappa^+, \theta]$. Then in $V^{\mathcal{P}'}$ we have $2^\eta = \eta^+$.*

(b) $V^{\mathcal{P}'}$ satisfies GCH.

Proof. It is enough to prove (a) since then (b) will follow by the previous lemma 2.2.12.
Fix a singular cardinal $\eta \in [\kappa^+, \theta]$. Let N, $p_{\geq\eta^+}, N_1, N_2, \underset{\sim}{a}$ be as in the proof of 2.2.12.
Pick an increasing sequence $\langle \eta_i \mid i < \mathrm{cof}(\eta)\rangle$ of regular cardinals cofinal in η. Let $\langle L_i \mid i < \mathrm{cof}(\eta)\rangle$ be an increasing sequence of elementary submodels of $H((2^\lambda)^+)$ such that

1. $|L_i| = \eta_i$,

2. $L_i \supseteq \eta_i$,

3. ${}^{\eta_i >}L_i \subseteq L_i$,

4. $\langle L_j \mid j < i \rangle \in L_i$,

5. $N, p_{\geq \eta^+}, \underset{\sim}{a} \in L_0$.

Set $L = \bigcup_{i<\mathrm{cof}(\eta)} L_i$.
Now we construct a sequence $\langle p(i) \mid i < \mathrm{cof}(\eta)\rangle$ of elements of $\mathcal{P}'$ such that

1. $p(0) \geq p_{\geq \eta^+}$,

2. $p(i)_{\geq \eta_i}$ is $(L_i, \mathcal{P}')$-generic over $p(i)_{<\eta_i}$, i.e. for any maximal antichain $A \subseteq \mathcal{P}'$ with $A \in L_i$, if some q is in A and is compatible with $p(i)$, then there is $r \geq q, p(i)$ such that for some $r' \leq r$ we have $r' \in A \cap L_i$.

3. $p(j) \restriction \eta_i = p(i)_{<\eta_i}$, for every $j > i$,

4. $p(i) \in L_{i+1}$.

The construction is by recursion and uses that at each $i < \mathrm{cof}(\eta)$ strategic closure of $\mathcal{P}'_{\geq \eta_i}$ together with η_i^+-c.c. of $\mathcal{P}'_{<\eta_i}$.
Now let p be the result of putting $\langle p_i \mid i < \mathrm{cof}(\eta)\rangle$ together as in the strategic closure lemma 2.2.8 with L the top model of cardinality η. Note that if $G \subseteq \mathcal{P}'$ with $p \in G$, then $L[G \cap L] \prec H((2^\lambda)^+)[G]$. Thus, if $A \in L$ is a maximal antichain, then $A \in L_i$ for some $i < \mathrm{cof}(\eta)$ and by (2) above some $r' \in G$ is in $A \cap L_i$.
In particular, $\underset{\sim}{a}$ can be computed correctly inside L. We continue further as in 2.2.11 define K etc.,with p replacing $p(\eta)$ of 2.2.11.
□

2.3 The Intersection Property- beyond 3

We deal here with the intersection properties in context of gaps above 3.
The following definition is a straightforward generalization of the intersection property for gap 3 from Chapter 1.

Definition 2.3.1 Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$, $\tau \leq \rho$ and $A \in A^{1\tau}, B \in A^{1\rho}$. We say that A *satisfies the intersection property with respect to* B or shortly $ip(A, B)$ iff either

1. $B \in pc(A)$, or
2. $A \in pc(B)$, or
3. $A \notin pc(B), B \notin pc(A)$, $\rho = \tau$ and then there are pairwise different ordinals $\eta_1, ..., \eta_n \in s \setminus \rho$ and sets $A_1 \in A^{1\eta_1} \cap pc(A), ..., A_n \in A^{1\eta_n} \cap pc(A), A' \in A^{1\tau} \cap pc(A)$ such that

$$A \cap B = A' \cap A_1 \cap ... \cap A_n,$$

or

4. $A \notin pc(B), B \notin pc(A)$, $\rho > \tau$ and then there are pairwise different ordinals $\eta_1, ..., \eta_n \in s \setminus \rho$ and sets $A_1 \in A^{1\eta_1} \cap pc(A), ..., A_n \in A^{1\eta_n} \cap pc(A)$ such that

$$A \cap B = A \cap A_1 \cap ... \cap A_n.$$

If $\rho = \tau$, then let $ipb(A, B)$ denotes that both $ip(A, B)$ and $ip(B, A)$ hold.

Unfortunately such defined intersection property may break down already at gap 4, as shows an example below.

An example of a failure of the intersection property ip **at gaps above 3.**

Let X_0, X_1, X be a Δ-system triple with witnessing model $F_0 \in X_1$, i.e. $X_1 \cap X_0 = X_1 \cap F_0$. Assume that $X_0 \in F_0$. Suppose that F_0 is a part of an other Δ-system type triple F_0, F_1, F. Let $G_0 \in F_0$ be such that $F_1 \cap F_0 = F_0 \cap G_0$.
Assume that $F_0, F_1, F \in X_1$ and F_0, F, X, X_0, G_0 are on the central lines. Suppose finally that $C^{|X_1|}(X_1) \setminus \{X_1\}$ has (or consists only of) a model A_1 such that A_1 is above F (i.e. $\sup(A_1) > \sup(F)$), $G_0 \in A_1$, but $F \notin A_1$.
Note that in this case $(A_1)_{|F|}$ should exist.
Set $A_0 = \pi_{X_1,X_0}(A_1)$ and $A = \pi_{F_0,F_1}(A_0)$.
We claim that $ip(X_1, A)$ fails. First note that $A \cap |X|^+ = A_0 \cap |X|^+ = A_1 \cap |X|^+$ and $A_1 \cap |X|^+ < X_1 \cap |X|^+$, since $A_1 \in X_1$. So, if $ip(X_1, A)$ holds then A_1 should be one of the models intersection of which witnesses $ip(X_1, A)$, since there is no other models in X_1 of cardinality $|X|$.
Consider $A_1 \cap A$. Clearly $A_1 \cap A = A_1 \cap A \cap F$. Remember that F, F_0 are on the central line,

this central line remains such in A_1 (just $C^{|F|}((A_1)_{|F|})$) and $F \notin A_1$. So $A_1 \cap F = A_1 \cap F_0$. Now $A \cap F_0 = A_0 \cap F_0 \cap F_1$. Hence we have

$$X_1 \cap A \subseteq A_1 \cap A = A_1 \cap A_0 \cap F_0 \cap F_1 = A_1 \cap F_0 \cap F_1,$$

since $A_1 \cap A_0 = A_1 \cap F_0$.
Note that $X_1 \cap A = \pi_{F_0,F_1}[A_1 \cap F_0]$, since $\pi_{F_0 F_1} \in X_1$ and so

$$z \in X_1 \cap A \Leftrightarrow \pi_{F_1 F_0}(z) \in X_1 \cap A_0 = A_1 \cap F_0.$$

We have $G_0 \in (A_1 \cap F_0) \setminus F_1$. Then $\pi_{F_0,F_1}(G_0) \notin F_0$, but it is in $X_1 \cap A$ and $X_1 \cap A \subseteq A_1 \cap F_0 \cap F_1$. Contradiction.

It is possible to have $|A| > |X_1|$ as well here.

We define now and use a weaker notion *wip*.

Definition 2.3.2 Let $A \in A^{1\tau}$. Denote by $pwc_0(A)$ (0- piste connected) the set of all intersections of the form

$$E_0 \cap ... \cap E_n$$

such that $n < \omega$ and for each $i \leq n$, $\quad E_i \in pc((A)_{\eta_i}) \cap (A \cup \{A\})$ for some $\eta_i \in s \cap A \setminus \tau$.
If $Z \in pwc_0(A)$, then Z is of the form $E_0 \cap ... \cap E_n$. Let us call witnessing this sets $E_0, ..., E_n$ a *description* of Z.

Set

$pwc_1(A) = \{\pi_{F_0,F_1}[E] \mid$ for some $\rho \in s \cap A \setminus \tau{+}1$ there is $F \in A \cap pc((A)_\rho)$ such that the triple

F_0, F_1, F is of a Δ-system type, $E \in pwc_0(A)$ and $(E \in F_0$ or $E \subseteq F_0)\}$.

If $Z \in pwc_1(A)$, then Z is of the form $\pi_{F_0,F_1}[E]$. By a description of Z we mean F_0, F_1, F together with a description of E.

Let us take now intersections again, i.e. set

$$pwc_2(A) = \{E_0 \cap ... \cap E_n \mid n < \omega, \text{ for each } i \leq n \quad E_i \in pwc_0(A) \cup pwc_1(A)\}.$$

If $Z \in pwc_0(A)$, then Z is of the form $E_0 \cap ... \cap E_n$. By a description of Z we mean the union of descriptions of $E_0, ..., E_n$.

Continue the definition further by induction taking intersections at even stages and images under Δ-systems isomorphisms at odd stages.
Finally define the set $wpc(A)$ (weakly piste connected) to be

$$\bigcup_{n<\omega} pwc_n(A).$$

Note that $A \supseteq wpc(A)$. Also if $ip(A, B)$, then $A \cap B \in wpc(A)$.

If $Z \in wpc(A)$, then for some

Definition 2.3.3 Let $\langle\langle A^{0\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\rangle \in \mathcal{P}'$, $\tau \leq \rho$ and $A \in A^{1\tau}, B \in A^{1\rho}$. We say that A *satisfies the weak intersection property with respect to* B or shortly $wip(A, B)$ iff

$$A \cap B \in wpc(A).$$

If $\rho = \tau$, then let $wipb(A, B)$ denotes that both $wip(A, B)$ and $wip(B, A)$ hold.

The proof of the next lemma just repeats the proof of the intersection lemma of gap 3 of Chapter 1.

Lemma 2.3.4 *Let* $\langle\{\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\}\rangle \in \mathcal{P}', \tau \in s, A, B \in bp(A^{0\tau})$. *Then* $ipb(A, B)$.

In the general case the following holds:

Lemma 2.3.5 *Let* $\langle\{\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\}\rangle \in \mathcal{P}', \tau_1, \tau_2 \in s, A \in A^{1\tau_1}, B \in A^{1\tau_2}$. *Then*

1. $\tau_1 < \tau_2$ *implies* $wip(A, B)$,
2. $\tau_1 = \tau_2$ *implies* $wipb(A, B)$.

Proof. Without loss of generality we can assume that $B \in A^{0\tau_1}$, just otherwise extend the condition $\langle\{\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\}\rangle$.
Consider the pistes from $A^{0\tau_1}$ leading to A and to B. Let X be the least common point of this pistes using only the first continuation. X must be a successor point.
Case 1. *There are* $X_0, X_1 \in X \cap A^{1|X|}$ *such that the triple* X_0, X_1, X *is a triple of a* Δ*-system type, and say the piste to* A *continues via* X_0 *and those to* B *via* X_1.
Set $A_1 = \pi_{X_0,X_1}[A]$. Let $F_0 \in X_0, F_1 \in X_1$ be the models witnessing that X_0, X_1, X is a Δ-type triple.
Then

$$A \cap B = A \cap X_0 \cap B \cap X_1 = (A \cap F_0) \cap (A_1 \cap B) = (A_1 \cap F_1) \cap (A_1 \cap B).$$

Apply the induction to A_1, B. Then we will have $A_1 \cap B \in wpc(A_1)$. Let E_1 be a description of $A_1 \cap B$ inside $wpc(A_1)$. Denote by E the image of E_1 under π_{X_0,X_1}. Then E will be a description in $wpc(A)$ and $A \cap F_0 \cap \pi_{X_0,X_1}(Z) = A_1 \cap F_1 \cap Z$ for every $Z \in E_1$, by the elementarity of π_{X_0,X_1}. Apply the induction to $A \cap F_0$. Then we will have $A \cap F_0 \in wpc(A)$ and $E \subseteq wpc(A)$, hence the intersection[1] will be in $wpc(A)$ as well. So $A \cap B \in wpc(A)$.

Case 2. *There are $\rho^0, \rho^1 \in (s \setminus \tau) \cap X, T^0 \in C^{\rho^0}((X)_{\rho^0}) \cap X, T^1 \in C^{\rho^1}((X)_{\rho^1}) \cap X$ such that the continuation of the piste to A splits from T^0 and those to B from T^1.*

Let T_0^i, T_1^i be the splitting, i.e. T^i, T_0^i, T_1^i form a Δ-system with T_0^i on the central line, where $i \in 2$.

We do not exclude the possibility that $T^0 = X$ or $T^1 = X$.

Assume that $\sup(T^0) < \sup(T^1)$. Further we will always deal with $wip(B, A)$ once $|B| \leq |A|$, so there will be no need to consider separately also the case when $\sup(T^0) > \sup(T^1)$.

Subcase 2.1. $T^0 \notin T^1$.

Then necessarily, $|T^1| < |T^0|$. Consider $(T^1)_{|T^0|}$. T^1 is on a central line so $T^0 \in C^{|T^0|}((T^1)_{|T^0|})$. Let $Z \in C^{|T^0|}((T^1)_{|T^0|})$ be the least model above T^0 which belongs to T^1. Then

$$T^1 \cap T^0 = T^1 \cap T_0^0 = T^1 \cap Z.$$

Now we have

$$A \cap B = A \cap T^0 \cap T^1 \cap B = A \cap T_0^0 \cap B \cap T_0^0 = B \cap A_0 \cap G_0,$$

where $A_0 = \pi_{T_1^0 T_0^0}(A)$ and $G_0 \in T_0^0$ is a Δ-system witness, i.e. $T_0^0 \cap T_1^0 = T_0^0 \cap G_0$. The induction applied to B, A_0 and B, G_0 gives $wip(B, A)$, once $\tau_1 = \tau_2$.

Let us deduce $wip(A, B)$. Apply the induction toA_0, B. There is a set $E_0 \in wpc(A_0)$ such that $A_0 \cap E_0 = A_0 \cap B$. So

$$A \cap B = A_0 \cap E_0 \cap G_0 = A \cap A_0 \cap E_0 \cap G_0.$$

Set $G = \pi_{T_0^0 T_1^0}(G_0)$ and let $E \in wpc(A)$ be the set which description (in the process of constructing $wpc(A)$) will be the image of the description of E_0. By elementarity $\pi_{T_0^0 T_1^0}(E_0) = E$. The elementarity and the fact that $\pi_{T_0^0 T_1^0}$ is the identity on $T_0^0 \cap T_1^0$ imply

$$z \in A \cap A_0 \cap E_0 \cap G_0 \Leftrightarrow z = \pi_{T_0^0 T_1^0}(z) \in A \cap A_0 \cap E \cap G.$$

So $A \cap A_0 \cap E_0 \cap G_0 = A \cap A_0 \cap E \cap G = A \cap E \cap G$. Apply the induction to A, G. We obtain $Y \in wpc(A)$ such that $A \cap A_0 \cap E \cap G = Y$. Then also $A \cap B = Y$ and we are done.

[1]here and further let us identify E with the set in $wpc(A)$ which it describes.

Subcase 2.2. $T^0 \in T^1$.

Sub-Subcase 2.2.1. $T^0 \in T^1_0$.

Then we just repeat the argument of Case 1.

Sub-Subcase 2.2.2. $T^0 \notin T^1_0$ *and* $T^0 \notin T^1_1$.

Note that $(T^1_1)_{|T^0|}$ is on the central line of $|T^0|$, since $|T^1| < |T^0|$ and T^1_0, T^1_1, T^1 is a Δ-system triple. If $T^0 \in (T^1_1)_{|T^0|}$, then as in Subcase 2.1 we have

$$T^1_1 \cap T^0 = T^1_1 \cap T^0_0 = T^1_1 \cap Z,$$

where $Z \in C^{|T^0|}((T^1_1)_{|T^0|})$ is the least model above T^0 which belongs to T^1_1. The argument of Subcase 2.1 applies now.

Suppose that $T^0 \notin (T^1_1)_{|T^0|}$, then $(T^1_1)_{|T^0|} \in T^0$. Then also $(T^1_1)_{|T^0|} \in T^0_0$. This implies $T^1_1 \in T^0_0$. Hence

$$A \cap B = B \cap T^1_1 \cap T^0_0 \cap A = B \cap T^0_0 \cap A = B \cap A_0 \cap G_0,$$

where as before $A_0 = \pi_{T^0_1 T^0_0}(A)$ and $G_0 \in T^0_0$ is a Δ-system witness, i.e. $T^0_0 \cap T^0_1 = T^0_0 \cap G_0$. Now we continue as in Subcase 2.1.

Sub-Subcase 2.2.3. $T^0 \notin T^1_0$ *and* $T^0 \in T^1_1$.

Sub-Sub-Subcase 2.2.3.1. $T^0 \notin T^1_0$ *and* $T^0 \in B$.

Let us show first $wip(B, A)$, once it makes sense. Apply the induction to B, A_0. Let $E \in wpc(B)$ be so that $B \cap A_0 = E$. Now, $T^0 \in B$ implies that $T^0_0, T^0_1 \in B$ and also $\pi_{T^0_0 T^0_1} \in B$. Then

$$z \in A \cap B \Leftrightarrow \pi_{T^0_1 T^0_0}(z) \in A_0 \cap B = E.$$

Set $E' = \pi_{T^0_0 T^0_1}(E)$. Clearly, $E' \in wpc(B)$ and $A \cap B = E'$, which means $wip(B, A)$.

Turn now to $wip(A, B)$. Let us apply the induction to A_0, B and find $E_0 \in wpc(A_0)$ such that $B \cap A_0 = E_0$. Set $E = \pi_{T^0_0 T^0_1}(E_0)$. Then $E \in wpc(A)$. We claim that $A \cap B = E$. Thus

$$z \in A \cap B \Leftrightarrow \pi_{T^0_1 T^0_0}(z) \in A_0 \cap B = E_0 \Leftrightarrow z \in E.$$

Sub-Sub-Subcase 2.2.3.2. $T^0 \notin T^1_0$ *and* $T^0 \notin B$.

Consider $(B)_{|T^0|}$. If $T^0 \in (B)_{|T^0|}$, then the argument of Sub-Subcase 2.2.2 works here.

If $(B)_{|T^0|} \neq \emptyset$ and $(B)_{|T^0|} \in T^0$, then $B \in T^0$. This implies that $B \in T^0_0$ or $B \in T^0_1$, since between models forming a Δ-system there is no models of a small cardinality. Now if $B \in T^0_0$, then we apply the induction to A_0, B. So there is $E_0 \in wpc(A_0)$ such that $A_0 \cap B = E_0$. Set $E = \pi_{T^0_0 T^0_1}(E_0)$ and let $G_1 \in T^0_1$ be such that $T^0_0 \cap T^0_1 = T^0_1 \cap G_1$. Then

$$A \cap B = A \cap T_0^0 \cap B = A \cap A_0 \cap G_1 \cap B = A \cap E \cap G_1,$$

and we are done.
If $B \in T_1^0$, then consider $B_0 = \pi_{T_1^0 T_0^0}(B)$. It is simpler than B, since T_0^0 on the central line. Apply the induction to A_0, B_0 and then move the result back by $\pi_{T_1^0 T_0^0}$.

Suppose now that $T^0 \notin (B)_{|T^0|}$ and $(B)_{|T^0|} \notin T^0$ or $(B)_{|T^0|} = \emptyset$ but $B \notin T^0$. Then the piste from T_1^1 must split at some point of cardinality $\geq |T^0|$ above T^0. Let S be such point and S_0, S_1 its immediate predecessors forming a Δ-system with S_0 on the central line and $B \in S_1$. Now $S_0 \supseteq T^0$, hence

$$A \cap B = A \cap S_0 \cap S_1 \cap B = A \cap B_0 \cap M_0,$$

where $B_0 = \pi_{S_1 S_0}(B)$ and $M_0 \in S_0$ is such that $S_0 \cap S_1 = S_0 \cap M_0$. Apply the induction to A, B_0 and A, M_0. This shows $wip(A, B)$. The property $wip(B, A)$, when applies, can be shown by using in addition $\pi_{S_0 S_1}$.

2.4 Suitable structures and assignment functions– beyond 3

We address first the new splitting possibility, which is crucial for GCH and does not appear in the gap 2, 3 cases.

Definition 2.4.1 Let $\nu < \xi < \mu$ be cardinals, A, X, Y_0, Y_1, Y be models, $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$. We call triples F_0, F_1, F and A_0', A_0, A_1 *splitting triples* over A, X, Y_0, Y_1, Y inside C_ν, C_ξ iff

1. $|A_0| = \nu$,

2. $|Y_0| = \xi$,

3. $|X| = \mu$,

4. $A_0, A_0', A_1 \in C_\nu$,

5. $Y_0, Y_1, Y, F_0, F_1, F \in C_\xi$,

6. $F_0, F_1 \in F$,

7. F_0, F_1 are isomorphic over $F_0 \cap F_1$,

8. $F_0, F_1, F \in A_1$,

9. $X \in F_1$,

10. $F_0 \cap F_1 = F_1 \cap X$,

11. $A_0 \in F_0$,

12. $A_1 \cap A_0 = A_1 \cap F_0$,

13. A_1, A_0 are isomorphic over $A_1 \cap A_0$,

14. $A_0' = \pi_{F_0,F_1}(A_0)$,

15. $A \subseteq A_0'$,

16. $Y_0 = \pi_{F_0,F_1}(\pi_{A_1,A_0}(F_0)), Y_1 = \pi_{F_0,F_1}(\pi_{A_1,A_0}(F_1)), Y = \pi_{F_0,F_1}(\pi_{A_1,A_0}(F))$.
Note that $A_0 \cap A_1 = A_0 \cap \pi_{A_1,A_0}(F_0)$, since $\alpha \in A_0 \cap A_1$ iff $\alpha \in A_1 \cap F_0$ iff $\pi_{A_1,A_0}(\alpha) \in A_0 \cap \pi_{A_1,A_0}(F_0)$, but for $\alpha \in A_0 \cap A_1, \pi_{A_1,A_0}(\alpha) = \alpha$.
Then $A_0' \cap A_1 = A_1 \cap F_1 = A_0' \cap Y_0$, since $\pi_{F_1,F_0} \in A_1$. Hence Y is a model which corresponds to F_0 in A_0'.

Normally, we will have $|A_0| < |F|$ and $|X| = |F|^*$.

Lemma 2.4.2 *Suppose that all the models of Definition 2.4.1 are members of a condition in $\mathcal{P}'$. Then $Y_0 \in A$ implies $Y_1, Y, X \in A$.*

Proof. Set $A_1^* = \pi_{A_0,A_1}(\pi_{F_1,F_0}(A))$. If $Y_0 \in A$, then $\pi_{F_1,F_0}(Y_0) \in \pi_{F_1,F_0}(A)$, and hence $\pi_{A_0,A_1}(\pi_{F_1,F_0}(Y_0)) = F_0 \in A_1^*$. Then $F \in A_1^*$, since there are no models of small cardinality between F_0 and F. Hence, $F_1 \in A_1^*$. So, their pre-images Y and Y_1 are in A.
Now, there is $G_0 \in F_0 \cap A_1^*$ such that $F_0 \cap F_1 = F_0 \cap G_0$. Then $G_0 \in A_1 \cap F_0 = A_0 \cap A_1$. Moreover, $G_0 \in A_1^* \cap F_0 = A_0^* \cap A_1^*$, where $A_0^* = \pi_{F_1,F_0}(A)$.
Set $G_1 = \pi_{F_0,F_1}(G_0)$. Then $G_1 \in A \cap A_1^*$ and $F_0 \cap F_1 = F_1 \cap G_1$, i.e. $G_1 = X$ and $X \in A$.
□

Lemma 2.4.3 *(Existence of splitting triples). Let $\mu > \xi > \nu$ be regular cardinals in $[\kappa^+, \theta]$. Then for every closed unbounded sets $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ there is a closed unbounded $C_\mu \subseteq \mathcal{P}_{\mu^+}(H(\theta^+))$ such that for every model $X \in C_\mu$, with ${}^\xi X \subseteq X$, there are*

$Y_0, Y_1, Y \in C_\xi, {}^\nu Y_0 \subseteq Y_0, {}^\nu Y_1 \subseteq Y_1, {}^\nu Y \subseteq Y$ *so that for every model* A *with* $|A| \leq \nu$ *there are splitting triples over* A, X, Y_0, Y_1, Y *inside* C_ν, C_ξ.

Proof. Suppose otherwise. Then there are clubs $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ such that for every club $C_\mu \subseteq \mathcal{P}_{\mu^+}(H(\theta^+))$ there is a model $X \in C_\mu$ so that for every models $Y_0, Y_1, Y \in C_\xi$ there is a model $A(X, Y_0, Y_1, Y)$ without splitting triples over $A(X, Y_0, Y_1, Y), X, Y_0, Y_1, Y$ inside C_ν, C_ξ.
Let $C_\nu \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), C_\xi \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ be such clubs. Define a function

$$I : \mathcal{P}_{\mu^+}(H(\theta^+)) \times C_\xi \times C_\xi \times C_\xi \to \mathcal{P}_{\nu^+}(H(\theta^+))$$

by setting $I(X, Y_0, Y_1, Y)$ to be the least model $A \in \mathcal{P}_{\nu^+}(H(\theta^+))$ without splitting triples over $A(X, Y_0, Y_1, Y), X, Y$ inside C_ν, C_ξ, if there is one and 0 otherwise.
Fix functions $h_\nu : [H(\theta^+)]^{<\omega} \to \mathcal{P}_{\nu^+}(H(\theta^+)), h_\xi : [H(\theta^+)]^{<\omega} \to \mathcal{P}_{\xi^+}(H(\theta^+))$ such that

$$C_\nu \supseteq \{t \in \mathcal{P}_{\nu^+}(H(\theta^+)) \mid h_\nu(e) \subseteq t \text{ whenever } e \in [t]^{<\omega}\},$$

$$C_\xi \supseteq \{t \in \mathcal{P}_{\xi^+}(H(\theta^+)) \mid h_\xi(e) \subseteq t \text{ whenever } e \in [t]^{<\omega}\}.$$

Turn to submodels of $\langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ for λ much bigger than θ. Consider

$$C = \{Z \in \mathcal{P}_{\mu^+}(H(\lambda^{+5})) \mid Z \prec \langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle\}.$$

Then

$$C \restriction H(\theta^+) = \{Z \cap H(\theta^+) \mid Z \in C\}$$

contains a club in $\mathcal{P}_{\mu^+}(H(\theta^+))$. Let C_μ be such a club. Pick $X \in C_\mu, {}^\xi X \subseteq X$, to be a counterexample.
Find $X^* \in C$ with $X^* \cap H(\theta^+) = X$. Note that X^* may be not closed under ξ-sequences of its elements (even $\sup(X^* \cap \theta^{++})$ can have cofinality ω).
Let $F_1^* \prec \langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ be a model of cardinality ξ, closed under ν-sequences of its elements and with X^* inside. Then $F_1 = F_1^* \cap H(\theta^+)$ is closed under h_ξ and hence $F_1 \in C_\xi$. Let F_0^* be obtained from F_1^* via a reflection to X^*. Here $F_1^* \cap X^*$ need not be an element of X^* due the possible lack of closure, but $F_1 \cap X$ is in $X = X^* \cap H(\theta^+)$, since ${}^\xi X \subseteq X$. We pick $F_0^* \prec \langle H(\lambda^{+4}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ to be a model realizing the same type as F_1^* over $F_1 \cap X$. So F_1^*, F_0^* are isomorphic by the isomorphism which is the identity over $F_1 \cap X$, but probably not the identity over $F_1^* \cap F_0^*$.

Let $F^* \prec \langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ be a model with F_0^*, F_1^* inside and closed under ν-sequences of its elements. Pick now $A_1^* \prec \langle H(\lambda^{+5}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$ to be a model of

cardinality ν with $F_0^*, F_1^*, F^*, X^* \in A_1^*$. Reflect A_1^* to F_0^*. Let $A_0^* \subseteq F_0^* \cap H(\lambda^{+3})$ be a result. Then $A_0^* \prec \langle H(\lambda^{+3}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$, the isomorphism $\pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}$ is the identity on $A_1^* \cap H(\theta^+) \cap A_0^*$ and $A_1^* \cap H(\theta^+) \cap F_0^* = A_1^* \cap A_0^* \cap H(\theta^+)$.
Set $A_0'^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(A_0^*)$. Then, $A_0'^* \prec \langle H(\lambda^{+3}), \in, <, \theta^+, h_\nu, h_\xi, I\rangle$, since $A_0^* \prec F_0^* \cap H(\lambda^{+3})$ and $F_0^* \simeq F_1^* \cap H(\lambda^{+4})$. This implies in particular that $A_0' = A_0'^* \cap H(\theta^+)$ is in C_ν and $A_0'^*$ is closed under I.
Set $F_0^{0*} = \pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}(F_0^* \cap H(\lambda^{+3})), F_1^{0*} = \pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}(F_1^* \cap H(\lambda^{+3}))$
and $F^{0*} = \pi_{A_1^* \cap H(\lambda^{+3}), A_0^*}(F^* \cap H(\lambda^{+3}))$.
Move these models to $A_0'^*$. Thus let $Y_0^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(F_0^{0*}), Y_1^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(F_1^{0*})$ and $Y^* = \pi_{F_0^*, F_1^* \cap H(\lambda^{+4})}(F^{0*})$. Then $Y_0^*, Y_1^*, Y^* \in A_0'^*$.
Define $F_0 = F_0^* \cap H(\theta^+), F_1 = F_1^* \cap H(\theta^+), F = F^* \cap H(\theta^+), Y_0 = Y_0^* \cap H(\theta^+), Y_1 = Y_1^* \cap H(\theta^+), Y = Y^* \cap H(\theta^+), A_0 = A_0^* \cap H(\theta^+)$ etc. Then $X, Y_0, Y_1, Y \in A_0'$, since $X \in A_1 \cap F_1 = A_1 \cap A_0'$ (the last equality holds because $A_1 \cap F_0 = A_1 \cap A_0$ and $\pi_{F_0, F_1} \in A_1$). The models A_0', A_0, A_1 are in C_ν, since they are closed under h_ν. Similarly $F_0, F_1, F, Y_0, Y_1, Y \in C_\xi$.
Finally, $A_0'^*$ is closed under I and $X, Y_0, Y_1, Y \in A_0'^*$, hence $I(X, Y_0, Y_1, Y) \in A_0'^*$. By the choice of X, Y_0, Y_1, Y, $I(X, Y_0, Y_1, Y)$ must be a model without splitting triples over $I(X, Y_0, Y_1, Y), X, Y_0, Y_1, X$ inside C_ν, C_ξ. But $F_0, F_1, F \in C_\xi$ and $A_0', A_0, A_1 \in C_\nu$ are splitting triples over $I(X, Y_0, Y_1, Y), X, Y_0, Y_1, Y$. Contradiction.
□

Lemma 2.4.4 *Suppose that X, Y_0, Y_1, Y satisfy the conclusion of Lemma 2.4.3 and they are in M for a model $M \in C_\nu$. Then there are splitting triples $A_0', A_0, A_1, F_0, F_1, F$ over M, X, Y_0, Y_1, Y with $A_0' = M$.*

Proof. Let $A_0', A_0, A_1, F_0, F_1, F$ be any splitting triples over M, X, Y_0, Y_1, Y. Consider $M_0 = \pi_{F_1, F_0}(M)$ and $M_1 = \pi_{A_0, A_1}(M_0)$. Then, $F_0, F_1, F \in M_1$, since $F_0 = \pi_{A_0, A_1}(\pi_{F_1, F_0}(Y_0)), F_1 = \pi_{A_0, A_1}(\pi_{F_1, F_0}(Y_1)), F = \pi_{A_0, A_1}(\pi_{F_1, F_0}(Y))$.
So, we can replace A_0' by M, A_0 by M_0 and A_1 by M_1. Hence M, M_0, M_1, F_0, F_1, F will be splitting triples over M, X, Y_0, Y_1, Y.
□

For every cardinal $\mu \in [\kappa^+, \theta]$ we define a closed unbounded subset C_μ of $\mathcal{P}_{\mu^+}(H(\theta^+))$ by induction as follows: $C_{\kappa^+} = \mathcal{P}_{\kappa^{++}}(H(\theta^+))$,
$C_{\kappa^{++}} = \mathcal{P}_{\kappa^{+3}}(H(\theta^+))$,
if μ is a limit cardinal, then
$C_\mu = \mathcal{P}_{\mu^+}(H(\theta^+))$,

if μ is a successor cardinal, then let C_μ be the intersection of the clubs given by Lemma 2.4.3 for each $\nu < \xi < \mu$.

Definition 2.4.5 A model M of a regular cardinality ν is called a *reliable model* iff

1. $M \cap H(\theta^+) \in C_\nu$,

2. for every regular cardinals $\xi, \mu \in M, \nu < \xi < \mu$, for every clubs $E \subseteq \mathcal{P}_{\nu^+}(H(\theta^+)), D \subseteq \mathcal{P}_{\xi^+}(H(\theta^+))$ in M and there is a club $C \subseteq \mathcal{P}_{\mu^+}(H(\theta^+)), C \subseteq C_\mu, C \in M$ such that for every $X \in C \cap M$ there are $Y_0, Y_1, Y \in D \cap M$ which satisfy the conclusion of Lemma 2.4.3 with E and D.

Definition 2.4.6 A structure $\mathfrak{X} = \langle X, E, C \in, \subseteq \rangle$, where $E \subseteq [X]^2$ and $C \subseteq [X]^3$ is called *suitable structure* iff there is $p(\mathfrak{X}) = \langle \langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), C^\tau(\mathfrak{X}) \rangle \mid \tau \in s(\mathfrak{X}) \rangle \in \mathcal{P}'$ such that

1. $X = A^{0\kappa^+}(\mathfrak{X})$,

2. $s(\mathfrak{X}) \in X$,

3. $s(\mathfrak{X}) \subseteq X$,

4. $\langle a, b \rangle \in E$ iff $a \in s(\mathfrak{X})$ and $b \in A^{1a}(\mathfrak{X})$,

5. $\langle a, b, d \rangle \in C$ iff $a \in s(\mathfrak{X}), b \in A^{1a}(\mathfrak{X})$ and $d \in C^a(\mathfrak{X})(b)$.

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.

Definition 2.4.7 A suitable structure $\mathfrak{X} = \langle X, E, C \in, \subseteq \rangle$ is called *suitable generic structure* iff there is $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle \in G(\mathcal{P}')$ such that

1. $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \setminus \{\kappa^+\} \rangle \in A^{0\kappa^+}$.
   In particular $s \in A^{0\kappa^+}$. Note that s may have cardinality above κ^+ (which is not a case in a suitable structure) and so s not necessary is contained in $A^{0\kappa^+}$.

2. $\mathfrak{X}$ is a substructure (not necessarily elementary) of the suitable structure generated by $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$, i.e. $\langle A^{0\kappa^+}, \{\langle \tau, B \rangle \mid \tau \in s, B \in A^{1\tau}\}, \{\langle \tau, B, D \rangle \mid \tau \in s, B \in A^{1\tau}, D \in C^\tau(B)\}$,

3. $X \in C^{\kappa^+}(A^{0\kappa^+})$,

4. $p(\mathfrak{X})$ and $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$ agree about the pistes to members of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$. In other words we require that all the elements of pistes in $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$ to elements of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$ are in X.

5. If $A \in A^{1\tau}(\mathfrak{X})$, for some $\tau \in s(\mathfrak{X})$, then either A it is of one of the first three types of Definition 2.2.4(2) inside $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$ or the models witnessing that it is of the forth type appear in $\mathfrak{X}$ as well.

Note that, as a condition in $\mathcal{P}'$, $p(\mathfrak{X})$ need not be weaker than $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$, and hence it need not be in $G(\mathcal{P}')$.
Note also, that any stronger condition $\langle\langle B^{0\tau}, B^{1\tau}, D^{\tau}\rangle \mid \tau \in r\rangle \in G(\mathcal{P}')$ such that

- $\langle\langle B^{0\tau}, B^{1\tau}, D^{\tau}\rangle \mid \tau \in r \setminus \{\kappa^+\}\rangle \in B^{0\kappa^+}$,
  and

- $C^{\tau}(A^{0\tau})$ is an initial segment of $D^{\tau}(B^{0\tau})$, for each $\tau \in s$

will witness that $\mathfrak{X}$ is a suitable generic structure.

Fix $n < \omega$. We define an analog $\mathcal{P}'_n$ of $\mathcal{P}'$ on the level n just replacing κ by κ_n^{+n} and θ by some λ_n big enough (λ_n a Mahlo will be more than enough; we can use for the gap 4 case $\lambda_n = \kappa_n^{+n+4}$, etc). An assignment function a_n will be an isomorphism between a suitable generic structure of cardinality less than κ_n over κ and a suitable structure over κ_n^{+n}.

Define Q_{n0}.

Definition 2.4.8 Let Q_{n0} be the set of the triples $\langle a, A, f\rangle$ so that:

1. f is partial function from θ^+ to κ_n of cardinality at most κ

2. a is an isomorphism between a suitable generic structure $\mathfrak{X}$ of cardinality less than κ_n and a suitable structure $\mathfrak{X}'$ in $\mathcal{P}'_n$ so that

   (a) every model in $\mathfrak{X}'$ is a reliable model,

   (b) X' is above every model which appears in $A^{1\tau}(\mathfrak{X}')$ for some $\tau \in s(\mathfrak{X}') \setminus \{\kappa^+\}$ and also those in $A^{1\kappa^+}(\mathfrak{X}') \setminus \{X'\}$ in the order $\leq_{E_n}$ of the extender E_n, (or actually, after codding X' by an ordinal),

   (c) if $t \in \bigcup\{A^{1\tau}(\mathfrak{X}') \mid \tau \in s(\mathfrak{X}')\}$, then for some $k, 2 < k < \omega$,
   $t \prec H(\chi^{+k})$, with χ big enough fixed in advance. (Alternatively, may be to work with subsets of λ_n only and further require it is a restriction of such model to λ_n.)

We deal with elementary submodels of $H(\chi^{+k})$, instead of those of $H(\lambda_n)$.
Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of a model $t \in X \cup Y$ will be an elementary submodel of $H(\chi^{+k})$.
The way to compare such models $t_1 \prec H(\chi^{+k_1}), t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{n,X'}$,

4. for every ordinals α, β, γ which code models in $\bigcup\{A^{1\tau}(\mathfrak{X}') \mid \tau \in s(\mathfrak{X}')\}$ we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \quad \text{implies}$$
$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho))$$

for every $\rho \in \pi``_{X',\alpha}(A)$.

5. For every ordinals $\alpha < \beta$ which code models in $\bigcup\{A^{1\tau}(\mathfrak{X}') \mid \tau \in s(\mathfrak{X}')\}$, for every $\rho \in A$ we have

$$\pi^{E_n}_{X',\alpha}(\rho) < \pi^{E_n}_{X',\beta}(\rho).$$

Define a partial order on Q_{n0} as follows.

Definition 2.4.9 Let $\langle a, A, f\rangle$ and $\langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $a \supseteq b$,

2. $f \supseteq g$,

3. $\pi_{\max(\mathrm{rng}(a)),\max(\mathrm{rng}(b))}``A \subseteq B$,

4. $\mathrm{dom}(f) \cap A^{1\theta}(\mathrm{dom}(b)) = \mathrm{dom}(g) \cap A^{1\theta}(\mathrm{dom}(b))$, where $A^{1\theta}(\mathrm{dom}(b))$ is the set of ordinals of the suitable structure on which b is defined.
   Note that here we do not require disjointness of the domain of g and of $A^{1\theta}(\mathrm{dom}(b))$, but as it will follow from the further definition of non-direct extension, the value given by g will be those that eventually counts.

Definition 2.4.10 Q_{n1} consists of all partial functions $f : \theta^+ \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

Definition 2.4.11 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq_n^* = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f \rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown \nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{rng}(a)),a(\alpha)}(\nu) \mid \alpha \in A^{1\theta}(\mathrm{dom}(a)) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $A^{1\theta}(\mathrm{dom}(a))$ are used to produce non direct extensions, the rest of models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq_n^* q$ or $p \in Q_{n1}, q = \langle b, B, g \rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown \nu$.

Definition 2.4.12 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) for every $n < \omega$, $\quad p_n \in Q_n$,

(2) there is $\ell(p) < \omega$ such that

  (i) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,

  (ii) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$,

  (iii) there is $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle \in G(\mathcal{P}')$ which witnesses that $\mathrm{dom}(a_n(p))$ is a suitable generic structure (i.e. $\mathrm{dom}(a_n(p))$ and $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$ satisfy 2.4.7), simultaneously for every $n, l(p) \leq n < \omega$.

(3) For every $n \geq m \geq \ell(p)$, $\quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,

(4) for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite. (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).

(5) For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.

Next lemma which allows to extend elements of $\mathcal{P}$ is crucial.

Lemma 2.4.13 *Let $p \in \mathcal{P}$ and $\langle \langle B^{0\tau}, B^{1\tau}, D^\tau \rangle \mid \tau \in r \rangle \in G(\mathcal{P}')$. Then*

1. *for every $t \in \bigcup\{B^{1\tau} \mid \tau \in r\}$ there is $q \geq^* p$ such that $t \in \mathrm{dom}(a_n(q))$ for all but finitely many n's;*

2. *for every* $A \in B^{1\kappa^+}$ *there is* $q \geq^* p$ *such that* $A \in \mathrm{dom}(a_n(q))$ *for all but finitely many* n*'s. Moreover, if* $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \geq \langle\langle B^{0\tau}, B^{1\tau}, D^\tau\rangle \mid \tau \in r\rangle$ *witnesses a generic suitability of* p *and* $A \in C^{\kappa^+}(A^{0\kappa^+})$*, then the addition of* A *does not require adding of ordinals and the only models that probably will be added together with* A *are its images under* Δ*-system type isomorphisms for triples in* p*.*

Proof. The proof follows the proof of this lemma in a gap 3 case. Let us concentrate on the new possibility of splitting. Namely given triples $A_0', A_0, A_1 \in A$ and F_0, F_1, F as in the last case of Definition 2.2.4 (Second, Third continuations) with A_0', A and F_1, F on the central lines (other possibilities are as in a gap 3 case), we would like to add A_0, A_1, F_0. Denote by $\hat{A}$ the largest model of $C^{|A|}(A_0') \setminus \{A_0'\}$ which is in p, if such a model exists. Suppose that it exists. If it does not exist then the argument is similar and simpler. Consider $X \in F_1 \cap A^{1|F_1|^*}$ such that $F_0 \cap F_1 = F_1 \cap X$ and $Y_0, Y_1, Y \in A^{1|F_1|^*}$ as in Definition 2.4.1. Then $X, Y_0, Y_1, Y \in A_0'$. Using the induction we can assume that X already appears in p. Now apply Lemma 2.4.3 to $X^* = a_n(X)$ and appropriate C (C will depend on $a_n(\hat{A})$ and its place relatively to Y_0, Y_1, Y) and find models Y_0^*, Y_1^*, Y^* satisfying the conclusion of this lemma and which can be added to $\mathrm{rng}(a_n)$ as images of Y_0, Y_1, Y. Assume that already $a_n(Y_0) = Y_0^*, a_n(Y_1) = Y_1^*$ and $a_n(Y) = Y^*$. Pick now inside $A^* = a_n(A)$ splitting triples F_0^*, F_1^*, F^* and $A_0'^*, A_0^*, A_1^*$ over $a_n(A_0'), X^*, Y_0^*, Y_1^*, Y^*$. By Lemma 2.4.4, we can assume that $A'^* = a_n(A_0')$. Add these models to $\mathrm{rng}(a_n)$ as images of the corresponding models over κ. Finally extend a_n further by adding the images under isomorphisms corresponding to Δ-system types.

We need the following property:

if $A \in A^{0\kappa^+} \cap \mathrm{dom}(a_n)$, for some $n \geq \ell(p)$ big enough, and $B \in \max(\mathrm{dom}(a_n))$ is a model which is reachable by a piste from A, then

(1) it is possible to extend a_n to b_n by adding B, probably in addition also models which belong to A and then taking isomorphic images.

(2) Let $A \in \mathrm{dom}(a_n)$, B a model added to $\mathrm{dom}(a_n)$ and $\tilde{B}$ is an isomorphic image of B which belongs to A, then $b_n(\tilde{B}) \in a_n(A)$ as well as all the models of the piste from A to $\tilde{B}$, where b_n denotes the extension of a_n obtained by adding B and taking isomorphic images.

This means basically that for adding such B we should take care only of models which are in A. The images of the rest of models with B inside will have the image of B inside automatically.

The proof is similar to the gap 3 case in Chapter 1 and uses weak intersection property of Chapter 2.3.

□

As in the Gap 3, we have the following:

Lemma 2.4.14 *$\mathcal{P}' * Q_{n0}$ is $< \kappa_n$-strategically closed.*

Lemma 2.4.15 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 2.4.16 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Define $\to$ on $\mathcal{P}$ similar to those of [1] or [3].

Lemma 2.4.17 *$\langle \mathcal{P}, \to \rangle$ satisfies κ^{++}-c.c.*

Combining the previous lemmas together, we obtain the following:

Theorem 2.4.18 *$V^{\mathcal{P}' * \langle \mathcal{P}, \to \rangle}$ is a cardinal preserving extension of V which satisfies $2^\kappa = \theta^+$.*

Chapter 3

Preserving Strong Cardinals

We will need to make some minor changes in the previous settings made in Chapter 2. Thus, first it will be convenient to increase a bit a set of conditions by allowing to remove some maximal models (i.e. $A^{0\alpha}$) from elements of $\mathcal{P}'$. This way the original $\mathcal{P}'$ will be dense in the new one, so from the forcing point of view nothing changes. Second, we like to deal with elementarity. In Chapter 2, we had $H(\theta^+)$ and considered its elementary submodels. We would like to deal instead with $H(\theta)$ and its elementary submodels, for regular (or even inaccessible) θ's. Note that once embeddings $j : V \to M$ are around, $j(H(\theta)) = (H(j(\theta)))^M$ may differ from $H(\theta)$ even if θ is not moved. So being elementary in sense of M will differ from being elementary in sense of V. We suggest below two ways to overcome this difficulty. The first one will be to assume that θ is a 2^θ-supercompact cardinal. Consider the following set

$$S = \{\alpha < \theta | \alpha \text{ is a superstrong cardinal with target } \theta$$

(i.e. there is $i : V \to M$, $crit(i) = \alpha$, $i(\alpha) = \theta$ and $M \supseteq V_\theta)\}$.

It is stationary (actually of measure one for a normal measure over θ), see for example Kanamori [11], 26.11.

Now, $V_\alpha \prec V_\theta$ for every $\alpha \in S$. Hence, $V_\alpha \prec V_\beta$ for every $\alpha < \beta$, $\alpha, \beta \in S$. Also the following holds:

Lemma 3.0.19 *Let $\alpha \in S$ and $i : V \to N$ is an ultrapower by an (α, ν)-extender for some $\nu \leq \theta$. which is a part of superstrong (α, θ)-extender with target θ. Then $V_\beta \prec (V_{i(\alpha)})^N$ for every $\beta \in S$, $\alpha \leq \beta < \nu$.*

Proof. Let $j : V \to M$ be the ultrapower by a superstrong (α, θ)-extender with target θ

extending the used (α, ν)-extender. Then the following diagram is commutative

$$\begin{array}{ccc} & & M \\ & \overset{j}{\nearrow} & \\ V & & \uparrow k \\ & \overset{i}{\searrow} & \\ & & N \end{array}$$

where k is defined in the obvious fashion.

Now, $k((V_{i(\alpha)})^N) = V_{j(\alpha)} = V_\theta$. Also $k(\beta) = \beta$ and $V_\beta \prec V_\theta$. Hence, $V_\beta \prec (V_{i(\alpha)})^N$.

□

Note also that by elementarity $(V_{i(\alpha)})^N \prec (V_{i(\theta)})^N = (V_\theta)^N$.

The second way will be to deal with just subsets (closed enough) and Σ_1 elementarity. Using this approach there will be no need in supercompacts cardinals- thus strongs alone suffice.

Lemma 3.0.20 *Suppose that $V_\delta \prec_{\Sigma_1} V_\theta$, α is δ-strong and $j : V \to M$ be an elementary embedding such that*

- *$M \supseteq V_\delta$*
- *$j(\theta) = \theta$.*

Then $V_\delta \prec_{\Sigma_1} (V_\theta)^M$.

Proof. Just note that

$$V_\delta \subset (V_\theta)^M \subset V_\theta.$$

Models $V_\theta, (V_\theta)^M$ agree about Σ_0 formulas. So each Σ_1 formula with parameters from $(V_\theta)^M$ true in $(V_\theta)^M$ is also true in V_θ. But $V_\delta \prec_{\Sigma_1} V_\theta$, hence $V_\delta \prec_{\Sigma_1} (V_\theta)^M$.

□

Let us make some changes in the definition of the preparation forcing $\mathcal{P}' = \mathcal{P}'(\theta)$.

Definition 3.0.21 (Changes in the definition of $\mathcal{P}'$).

1. V_τ is allowed to be an element of $A^{1\tau}$, for $\tau \in s$ which is an inaccessible and $V_\tau \prec_{\Sigma_1} V_\theta$. Note that V_τ will be the least (under $\in, \subseteq$) member of $A^{1\tau}$, since τ is an element and contained in any other member of $A^{1\tau}$.

2. If $\tau \in s$ is a critical point of an elementary embedding $j : V \to M, {}^\tau M \subseteq M$ which is strong enough and the previous condition is satisfied,
   then it is allowed to have $j''A \in A^{1\tau}$, for $A \in A^{1\tau}$, provided the intersection properties are respected, i.e.:

   - $B \in \bigcup_{\rho \in s \setminus \tau} A^{1\rho} \Rightarrow ip(j''A, B)$,
   - $B \in A^{1\tau} \Rightarrow ipb(j''A, B)$,
   - $B \in \bigcup_{\rho \in s \cap \tau} A^{1\rho} \Rightarrow ip(B, j''A)$.

   Note that allowing sets of the form $j''A$ in $A^{1\tau}$, we allow also as well a new Continuation in the definition of the set of models piste connected to a given model (see Chapter 2).

3. Let τ be as in 1 above.
   It is convenient to allow in the present context conditions having two top models for each $\alpha \in s \cap \tau$. However conditions with a single top model in every cardinality will remain dense.
   Thus

$$\langle\langle A^{00\xi}, A^{01\xi}\rangle, A^{1\xi}, C^\xi\rangle \mid \xi \in s \cap \tau\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle \mid \nu \in s\setminus\tau\rangle \in \mathcal{P}'$$

   provided both

$$\langle\langle A^{00\xi}, A^{1\xi}, C^\xi\rangle \mid \xi \in s \cap \tau\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle \mid \nu \in s\setminus\tau\rangle,$$

$$\langle\langle A^{01\xi}, A^{1\xi}, C^\xi\rangle \mid \xi \in s \cap \tau\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle \mid \nu \in s\setminus\tau\rangle$$

   are in $\mathcal{P}'$ and

   - $A^{00\xi} \in V_\tau$, for each $\xi \in s \cap \tau$,
   - the structures

     $\langle A^{00\max(s\cap\tau)}, <, \in, \subseteq, \kappa, \langle A^{00\xi}, A^{1\xi}\cap(A^{00\xi}\cup\{A^{00\xi}\}), C^\xi \restriction (A^{00\xi}\cup\{A^{00\xi}\}) \mid \xi \in s\cap\tau\rangle\rangle$

     and

     $\langle A^{01\max(s\cap\tau)}, <, \in, \subseteq, \kappa, \langle A^{01\xi}, A^{1\xi}\cap(A^{01\xi}\cup\{A^{01\xi}\}), C^\xi \restriction (A^{01\xi}\cup\{A^{01\xi}\}) \mid \xi \in s\cap\tau\rangle\rangle$

     are isomorphic over $V_\tau \cap A^{01\max(s\cap\tau)}$.

Let $p^0 = \langle\langle A^{00\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\cap\tau\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^{\nu}\rangle \mid \nu \in s\backslash\tau\rangle$ and $p^1 = \langle\langle A^{01\xi}, A^{1\xi}, C^{\xi}\rangle \mid \xi \in s\cap\tau\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^{\nu}\rangle \mid \nu \in s\backslash\tau\rangle$ be as in the previous item and $q \in \mathcal{P}'(\tau)$ be an extension of p^0.

The next condition allows to combine p^1 and q into a condition in $\mathcal{P}'$ which is stronger than both q, p^1.

4. Let $\langle B_\xi \mid \xi \in s\rangle$ be an increasing continuous sequence such that for every $\xi \in s$ the following holds:

   (a) $B_\xi \prec V_\theta$,

   (b) $|B_\xi| = \xi$,

   (c) $q, p^1 \in B_{\kappa^+}$.

   Then
$$r = \langle\langle A^{0\xi}(r), A^{1\xi}(r), C^{\xi}(r)\rangle \mid \xi \in s\rangle \in \mathcal{P}'$$
   and it is stronger than both q, p^1, where

   - $A^{0\xi}(r) = B_\xi$,
   - $A^{1\xi}(r) = A^{1\xi} \cup \{B_\xi\}$, if $\xi \in s\backslash\tau$ and $A^{1\xi}(r) = A^{1\xi} \cup \{B_\xi\} \cup A^{1\xi}(q)$, if $\xi \in s\cap\tau$
   - $C^{\xi}(r) = C^{\xi} \cup \langle B_\xi, C^{\xi\frown}B^{\xi}\rangle$, if $\xi \in s\backslash\tau$ and $C^{\xi}(r) = C^{\xi} \cup C^{\xi}(q) \cup \langle B_\xi, C^{\xi}(q)^\frown B^{\xi}\rangle$, if $\xi \in s\cap\tau$.

□ of the changes.

The crucial observation will be that $\mathcal{P}'$ breaks at each $\alpha \in S$ (or just for each $\alpha < \theta$ which is Mahlo and has δ's as in 3.0.20) into forcing $\mathcal{P}'(\alpha)$ which deals with elementary submodels (or just closed enough subsets) of V_α and $\mathcal{P}'_{\geq\alpha}$ which breaks in turn into $\mathcal{P}'_{>\alpha} * \mathcal{P}'_{\{\alpha\}} * Q_\alpha$.

Define $\mathcal{P}'(\alpha)$ the same way as $\mathcal{P}'$ but only with V_α replacing V_θ. Thus in this notation $\mathcal{P}'$ is actually $\mathcal{P}'(\theta)$.

Lemma 3.0.22 *Suppose that α is a Mahlo cardinal. Then $\mathcal{P}'(\alpha)$ satisfies α -c.c.*

Proof. Let $\langle p_\beta | \beta < \alpha\rangle$ be a sequence of conditions in $\mathcal{P}'(\alpha)$, $p_\beta = \langle\langle A^{0\tau}(p_\beta), A^{1\tau}(p_\beta), C^{\tau}(p_\beta)\rangle | \tau \in s(p_\beta)\rangle, \beta < \alpha$.
Consider their supports sequence $\langle s(p_\beta) | \beta < \alpha\rangle$. Recall that supports are of the Easton form. Hence we can find a stationary $X \subseteq \alpha$ and s such that $\langle s(p_\beta) | \beta \in X\rangle$ forms a Δ-system with kernel s. Moreover,

- each $\beta \in X$ is inaccessible
- $s(p_\beta) \cap \beta = s$
- if $\gamma < \beta$ is also in X then for each $\tau \in s(p_\gamma)$, then $A^{0\tau}(p_\gamma) \subset V_\beta$.

This implies that

$$A \subset V_\beta \subseteq B,$$

whenever $\gamma < \beta$ in X, $A \in A^{1\tau}, \tau \in s(p_\gamma)$ and $B \in A^{1\rho}, \rho \in s(p_\beta)\backslash s$.

Shrinking X more, if necessary we can insure that for each $\gamma, \beta \in X$ the following two structures

$$\langle A^{0\max(s)}(p_\beta), <, \in, \subseteq, \kappa, A^{0\max(s)}(p_\beta) \cap p_\beta \rangle$$

and

$$\langle A^{0\max(s)}(p_\gamma), <, \in, \subseteq \kappa, A^{0\max(s)}(p_\gamma) \cap p_\gamma \rangle$$

are isomorphic over $A^{0\max(s)}(p_\beta) \cap A^{0\max(s)}(p_\gamma)$.

Note that $A^{0\tau}(p_\beta)$'s may have elements above β.

Now we claim that such p_β and p_γ are compatible, say $\gamma < \beta$. The proof repeats the corresponding argument in Chapter 2. Note that models of cardinalities in $s_\gamma \backslash s$ should be added between models of p_β of cardinalities in s and those including them of cardinalities in $s(p_\beta)\backslash s$. In order to this, we work over the center line of p_β to add models which include p_γ as a member and then such setting via isomorphisms.
□

Lemma 3.0.23 *Suppose that α is a Mahlo cardinal and $V_\alpha \prec V_\theta$. Then $\mathcal{P}' \gtrdot \mathcal{P}'(\alpha)$.*

Proof. Consider $\mathcal{P}' \cap V_\alpha$. By the definition of the preparation forcing Chapter 2 we have $\mathcal{P}'(\alpha) = \mathcal{P}' \cap V_\alpha$. The cardinal α is an inaccessible. Hence ${}^{\alpha>}V_\alpha \subseteq V_\alpha$. In particular, each antichain of $\mathcal{P}'(\alpha)$ is in V_α, by the previous lemma. Hence , if $H \subseteq \mathcal{P}'(\alpha)$ is $\mathcal{P}'(\alpha)$-generic over V_α, then H will be full $\mathcal{P}'(\alpha)$-generic.
Note that $\mathcal{P}'(\alpha)$ is definable in V_α and using the same formula that defines $\mathcal{P}'$ in V_θ.
Let $A \subseteq \mathcal{P}'(\alpha)$ be a maximal antichain. Then $|A| < \alpha$ and, so $A \in V_\alpha$. In addition,

$$V_\alpha \models A \text{ is a maximal antichain in } \mathcal{P}'.$$

Then, by elementarity,

$$V_\theta \models A \text{ is a maximal antichain in } \mathcal{P}'.$$

So, $G \cap A \neq \emptyset$, for any generic $G \subseteq \mathcal{P}'$. Also, $V_\alpha[G \cap V_\alpha] \prec V_\theta[G]$.
□

By the lemma above $\mathcal{P}'$ projects to $\mathcal{P}'(\alpha)$. We prefer to deal with an explicit projection rather then with the projection defined via the corresponding Boolean algebras. In order to define an explicit projection we consider the following dense subset of $\mathcal{P}'$:

$$D = \{\langle\langle A^{00\tau}, A^{01\tau}\rangle, A^{1\tau}, C^\tau\rangle \mid \tau \in s \cap \alpha\rangle^\frown \langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle \mid \nu \in s \backslash \alpha\rangle \in \mathcal{P}' \quad |$$

$$\alpha \in s \& \forall \tau \in s \cap \alpha \quad A^{00\tau} \in V_\alpha \text{ and the structure}$$

$$\langle A^{00\max(s\cap\alpha)}, <, \in, \subseteq, \kappa, \langle A^{00\tau}, A^{1\tau} \cap (A^{00\tau} \cup \{A^{00\tau}\}), C^\tau \restriction (A^{00\tau} \cup \{A^{00\tau}\})\rangle | \tau \in s \cap \alpha\rangle\rangle$$

is isomorphic to

$$\langle A^{01\max(s\cap\alpha)}, <, \in, \subseteq, \kappa, \langle A^{01\tau}, A^{1\tau} \cap (A^{01\tau} \cup \{A^{01\tau}\}), C^\tau \restriction (A^{01\tau} \cup \{A^{01\tau}\})\rangle | \tau \in s \cap \alpha\rangle\rangle$$

over $V_\alpha \cap A^{01\max(s\cap\alpha)}\}$.

Here is the point where we prefer to allow two top models ($A^{00\tau}, A^{01\tau}$, $\tau \in s \cap \alpha$) instead of a single one. Using $V_\alpha \prec_{\Sigma_1} V_\theta$ it is easy to extend any standard (i.e. with single top model in each cardinality) condition in $\mathcal{P}'$ to one in D. We need just to intersect its part consisting of models of cardinality below α with V_α and then using elementarity of V_α to find inside V_α something isomorphic over this intersection.

Now, once we have $p = \langle\langle A^{00\tau}, A^{01\tau}\rangle, A^{1\tau}, C^\tau\rangle | \tau \in s \cap \alpha\rangle^\frown \langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle | \nu \in s \backslash \alpha\rangle \in D$, then define $\sigma(p)$ to $\mathcal{P}'(\alpha)$ to be

$$\langle A^{00\tau}, A^{1\tau} \cap \mathcal{P}(A^{00\tau}), C^\tau \restriction \mathcal{P}(A^{00\tau}))\rangle | \tau \in s \cap \alpha\rangle^\frown \langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle | \nu \in s \backslash \alpha\rangle.$$

Let us check that such defined σ is indeed a projection map.

Lemma 3.0.24 *The map σ is a projection map from D to $\mathcal{P}'(\alpha)$.*

Proof. Let $p \in D$ be as above and $q \in \mathcal{P}'(\alpha)$ be an extension of $\sigma(p)$. Pick increasing continuous sequence $\langle B_\tau | \tau \in s\rangle$ such that for each $\tau \in s$ the following holds:

1. $B_\tau \prec V_\theta$
2. $|B_\tau| = \tau$

3. $p, q \in B_{\kappa^+}$.

Now let $r = \langle\langle A^{0\tau}(r), A^{1\tau}(r), C^\tau(r)\rangle | \tau \in s\rangle$ be defined as follows:

- $A^{0\tau}(r) = B_\tau$
- $A^{1\tau}(r) = A^{1\tau} \cup \{B_\tau\}$, if $\tau \in s\backslash\alpha$ and $A^{1\tau}(r) = A^{1\tau} \cup \{B_\tau\} \cup A^{1\tau}(q)$, if $\tau \in s \cap \alpha$
- $C^\tau(r) = C^\tau \cup \langle B_\tau, C^{\tau\frown} B^\tau\rangle$, if $\tau \in s\backslash\alpha$ and $C^\tau(r) = C^\tau \cup C^\tau(q) \cup \langle B_\tau, C^\tau(q)^\frown B^\tau\rangle$, if $\tau \in s \cap \alpha$.

Then r is an element of $\mathcal{P}'$ stronger than both p and q. Note that the situation as here was specially allowed in 3.0.21(4) in contrast with the parallel definition of Chapter 2. It remains to extend r to some $r' \in D$ and then to take $\sigma(r')$ which will be above q.
□

Lemma 3.0.25 *Suppose that α is a Mahlo cardinal and $V_\alpha \prec V_\theta$. Let $\gamma < \alpha$ be a regular cardinal. Then $\mathcal{P}'_{\geq\gamma} \gtrdot \mathcal{P}'(\alpha)_{\geq\gamma}$.*

The proof repeats those of Lemma 3.0.23.

Note that $\mathcal{P}'_{\geq\alpha}$ does not add new sets of cardinalities $\leq \alpha$ and $\mathcal{P}' = \mathcal{P}'_{\geq\alpha} * \mathcal{P}'_{<\alpha}$.

Lemma 3.0.26 *Let $V_\alpha \prec V_\theta$, α be a Mahlo and $\delta < \alpha$ be a regular. Then $\mathcal{P}' = \mathcal{P}'_{\geq\delta} * (\mathcal{P}'(\alpha))_{<\delta}$.*

Proof. Pick $M \prec V_\alpha$, $\delta^+ \subseteq M$ and $|M| = \delta^+$. By 3.0.23, we have $\mathcal{P}'_{\geq\delta} \gtrdot (\mathcal{P}'(\alpha))_{\geq\delta}$. Note that $M \cap \mathcal{P}' = M \cap \mathcal{P}'(\alpha)$, since $M \prec V_\alpha$. Pick $p \in V_\alpha \cap \mathcal{P}'_{\geq\delta^+}$ to be $(\mathcal{P}'_{\geq\delta^+}, M)$-generic. Then $p \in (\mathcal{P}'(\alpha))_{\geq\delta^+}$ and it is $((\mathcal{P}'(\alpha))_{\geq\delta^+}, M)$-generic. Pick now $G_{\geq\delta^+} \subseteq \mathcal{P}'_{\geq\delta^+}$ generic with $p \in G_{\geq\delta^+}$ and $G_{=\delta} \subseteq \mathcal{P}'_{=\delta}$ generic over $V[G_{\geq\delta^+}]$. Recall that $\mathcal{P}'_{=\delta}$ satisfies δ^{++}-c.c. Hence each antichain of $\mathcal{P}'_{=\delta}$ which belongs to $M[p]$ will be contained in $M[p]$. But $(G_{\geq\delta^+} * G_{=\delta}) \cap V_\alpha$ is $\mathcal{P}'_{\geq\delta}$ -generic over V_α, by 3.0.23. So $G_{=\delta} \cap M[p]$ will be $(\mathcal{P}'_{=\delta}, M[p])$-generic. Denote $G_{=\delta} \cap M[p]$ by G_M. Then $M[p, G_M] \prec V_\alpha[G_{\geq\delta^+} * G_{=\delta} \cap V_\alpha] \prec V_\theta[G_{\geq\delta^+}, G_{=\delta}]$.

Let us turn now to $\mathcal{P}'_{<\delta}$. By 2.2.12,2.2.13, $\mathcal{P}'_{<\delta}$ in $V_\theta[G_{\geq\delta}, G_{=\delta}]$ is equivalent to $\mathcal{P}'_{<\delta} \cap M[p, G_M]$. But $M[p, G_M] \prec V_\alpha[G_{\geq\delta^+} * G_{=\delta} \cap V_\alpha]$. Hence, $\mathcal{P}'_{<\delta} \cap M[p, G_M]$ is just the same as $(\mathcal{P}'(\alpha))_{<\delta} \cap M[p, G_M]$. But this last forcing is equivalent to $(\mathcal{P}'(\alpha))_{<\delta}$. So we are done.

□

Lemma 3.0.27 *Let $V_\alpha \prec V_\theta$, α be a Mahlo and $\delta < \alpha$ be a regular. Then $\mathcal{P}' = \mathcal{P}'(\alpha)_{\geq\delta} * (Q \times (\mathcal{P}'(\alpha))_{<\delta})$.*

Proof. By Lemma 3.0.25, $\mathcal{P}'_{\geq\delta} \gtrdot \mathcal{P}'(\alpha)_{\geq\delta}$. So let $\mathcal{P}'_{\geq\delta} = \mathcal{P}'(\alpha)_{\geq\delta} * Q$, for some Q. Now, $\mathcal{P}'(\alpha) = \mathcal{P}'(\alpha)_{\geq\delta} * \mathcal{P}'(\alpha)_{<\delta}$. By Lemma 3.0.26 we have $\mathcal{P}' = \mathcal{P}'_{\geq\delta} * (\mathcal{P}'(\alpha))_{<\delta}$. Hence

$$\mathcal{P}' = \mathcal{P}'(\alpha)_{\geq\delta} * Q * (\mathcal{P}'(\alpha))_{<\delta}.$$

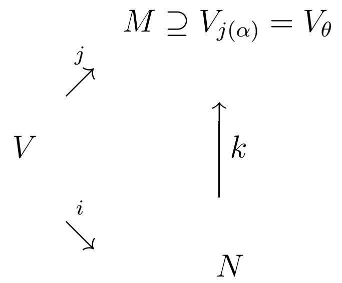
But Q does not add new bounded subsets to α. So this can be written as follows:

$$\mathcal{P}' = \mathcal{P}'(\alpha)_{\geq\delta} * (Q \times (\mathcal{P}'(\alpha))_{<\delta}).$$

□

Recall that $\mathcal{P}'_{\geq\alpha} * (\mathcal{P}'_{<\alpha})_{\geq\beta}$ is β-strategically closed, $\mathcal{P}'(\alpha)_{<\beta}$ satisfies β^+-c.c. and is actually isomorphic to a forcing of cardinality β^+, by 2.2.12.

Lemma 3.0.28 *Let $\alpha \in S$, $\delta < \theta$, $(S \cap \delta)\backslash\alpha + 1 \neq \emptyset$ and*



be a commutative diagram with N being the ultrapower by an (α, δ)-extender. Then i extends to

$$\hat{i} : V^{\mathcal{P}'} \longrightarrow N^{i(\mathcal{P}')}$$

Alternatively, using only strongs we can show that the following analog of this lemma holds:

Lemma 3.0.29 *Suppose that*

1. *$\rho < \theta$ is a Mahlo cardinal*

2. *$V_\rho \prec_{\Sigma_1} V_\theta$*

3. *α is ρ-strong, as witnessed by $j : V \to M \supset V_\rho$*

4. *$\delta, \alpha < \delta < \rho$ is a regular cardinal*

5. *there is $\mu, \alpha < \mu < \delta$ such that $V_\mu \prec V_\rho$.*

Let

$$\begin{array}{ccc} & & M \supseteq V_\rho \\ & \overset{j}{\nearrow} & \\ V & & \uparrow k \\ & \overset{i}{\searrow} & \\ & & N \end{array}$$

be a commutative diagram with N being the ultrapower by an (α, δ)-extender derived from j, such that $\rho = k(\xi)$, for some ξ. Then i extends to

$$\hat{i} : V^{\mathcal{P}'} \to N^{i(\mathcal{P}')}.$$

The proofs of both lemmas are very similar. We concentrate on the proof of 3.0.28 and state the minor changes needed for those of 3.0.29.

Proof. Note that by the definition of forcings $\mathcal{P}'(\xi)$ we have $\mathcal{P}' = \mathcal{P}'(\theta)$. Also, $i(\theta) = \theta$, since θ is an inaccessible. In N, hence $i(\mathcal{P}') = (\mathcal{P}'(i(\theta)))^N = (\mathcal{P}'(\theta))^N$. We split first $(\mathcal{P}'(\theta))^N$ into $(\mathcal{P}'(i(\alpha)) \times ((\mathcal{P}'(\theta)_{\geq i(\alpha)}) * (\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq \alpha}))^N$.

Let us deal first with $(\mathcal{P}'(i(\alpha)))^N$. Note that $V_\delta \subseteq N$. We split in N the forcing $\mathcal{P}'(i(\alpha))$ into $\mathcal{P}'(i(\alpha))_{\geq \delta} * \mathcal{P}'(i(\alpha))_{<\delta}$. The part $\mathcal{P}'(i(\alpha))_{\geq \delta}$ is δ^+-strategically closed. The extender used to form N has no generators above δ, so standard methods apply. Thus, we can find an N^*-generic set for $(\mathcal{P}'(i_{N^*}(\alpha))_{\geq \delta})^{N^*}$ move it then to N and in this way obtain an N-generic set for $(\mathcal{P}'(i(\alpha))_{\geq \delta})^N$, where N^* is the ultrapower by the measure $U = \{X \subseteq \alpha^2 \mid (\alpha, \delta) \in i(X)\}$. For 3.0.29, we include also ξ, i.e. $U = \{X \subseteq \alpha^3 \mid (\alpha, \delta, \xi) \in i(X)\}$.
Denote the corresponding embedding by i^* and those of N^* into N by k^*. Then we obtain the following commutative diagram:

$$\begin{array}{cccl} & & M & \supseteq V_{j(\alpha)} = V_\theta \quad \text{(or just } M \supseteq V_\rho \text{ in 3.0.29)} \\ & \overset{j}{\nearrow} & \uparrow k & \\ V & \overset{i}{\longrightarrow} & N & \\ & \overset{i^*}{\searrow} & \uparrow k^* & \\ & & N^* & \simeq V^\alpha / U \end{array}$$

Let δ^* be the preimage of δ under k^* (and ξ^* the preimage of ξ). Use α^+-strategic closure of $\mathcal{P}'(i^*(\alpha))_{\geq (\delta^*)}$ to build an N^*-generic subset of $(\mathcal{P}'(i^*(\alpha))_{\geq \delta^*})^{N^*}$. Then move it by k^* to obtain an N-generic subset of $(\mathcal{P}'(i(\alpha))_{\geq \delta})^N$.

We deal now with $(\mathcal{P}'(i(\alpha))_{<\delta})^N$. Let $A^* \in N^*$ be an elementary submodel of $(V_{i^*(\alpha)})^{N^*}$ (or of $(V_{\xi^*})^{N^*}$ in 3.0.29) of cardinality $((\delta^*)^+)^{N^*}$ closed under δ^*-sequences. Let $A \in N$ be $k^*(A^*)$. Then it is an elementary submodel of $(V_{i(\alpha)})^N$ of cardinality $(\delta^+)^N$ closed under δ-sequences. Let $k(A) = B$. Then, B will be an elementary submodel of $(V_{j(\alpha)})^M = V_{j(\alpha)}$ (or of $(V_\rho)^M = V_\rho$ correspondingly) of cardinality δ^+. Recall that $k \restriction (\delta^+)^N = id$, $|(\delta^+)^N| = \delta, cf((\delta^+)^N) = \alpha^+$ and $k((\delta^+)^N) = \delta^+$.

Pick in N^* a condition $r_1 \in \mathcal{P}'(i^*(\alpha))_{\geq(\delta^*)^+}$ which is A^*-generic. Let G^* be an N^*-generic subset of $(\mathcal{P}'(i^*(\alpha))_{\geq\delta^*})^{N^*}$ with $r_1 \in G^*$, built using the α^+ strategic closure of the forcing.

Moving to N we set $q_1 = k^*(r_1)$. Then $q_1 \in \mathcal{P}'(i(\alpha))_{\geq\delta^+}$ will be A-generic. Set $p_1 = k(q_1)$. Then, by elementarity, p_1 will be B-generic for the real $\mathcal{P}'(j(\alpha))_{\geq\delta^+}$.

Let r_2 be $G^* \cap A^*[r_1]$ and q_2 be generated by $k^{*\prime\prime}r_2$. Then q_2 will be $(A, \mathcal{P}'(i(\alpha))_{\{\delta\}})$-generic set (remember that $\mathcal{P}'(i(\alpha))_{\{\delta\}}$ is δ^+-strategically closed).

Consider $k''q_2$. It contains an increasing cofinal subset of size α^+ - the image of r_2 under $k \circ k^*$. Now, $k''A \in B$, since ${}^\delta B \subseteq B$, by elementarity. Let $p_2 \in \mathcal{P}'(j(\alpha))_{\{\delta\}}$ be the union of conditions in $k''q_2$. It exists, due to this cofinal subset of size α^+.

Chose a generic over M (or,the same V) with (p_1, p_2) inside. Let $\tilde{p}_2$ be a $(B[p_1], \mathcal{P}'(j(\alpha))_{\{\delta\}})$-generic over M with $p_2 \in \tilde{p}_2$. Then $k \restriction A$ extends to an elementary embedding

$$\tilde{k} : A[q_1, q_2] \to B[p_1, \tilde{p}_2] \ .$$

By 2.2.12,2.2.13, $\mathcal{P}'(j(\alpha))_{<\delta}$ is equivalent to $\mathcal{P}'(j(\alpha))_{<\delta} \cap B[p_1, \tilde{p}_2]$ and the same is true in N replacing $B[p_1, \tilde{p}_2]$ by $A[q_1, q_2]$. Also, by 2.2.11, $\mathcal{P}'(j(\alpha))_{<\delta}$ satisfies δ^+-c.c. Hence $\tilde{k}$ will move maximal antichains to maximal antichains. This allows us to obtain $(\mathcal{P}'(i(\alpha)))^N_{<\delta}$ -generic set from $\mathcal{P}'(j(\alpha))_{<\delta}$ -generic one, just intersect the last one with $\tilde{k}''A[q_1, \tilde{q}_2]$ and pull back the result to N using $\tilde{k}^{-1}$.

Putting together now the parts above and below δ we will obtain an N generic subset $G_{i(\alpha)}$ of $(\mathcal{P}'(i(\alpha)))^N$.

Let us turn now to the forcing $(\mathcal{P}'(\theta))^N$ and also deal with the master condition part.

Let $\mu \in (S \cap \delta) \backslash (\alpha + 1)$ (or in 3.0.29, let μ we as in (5), i.e. $V_\mu \prec V_\rho$). We pick in V an elementary submodel $A \prec V_\mu \prec V_\theta$ (or V_ρ) of cardinality α^+ and closed under α-sequences of its elements. Let p be $\mathcal{P}'_{\geq\alpha^+}$-generic over A. It exists since $\mathcal{P}'_{\geq\alpha^+}$ is α^{++}-strategically closed. Fix an increasing continuous sequence $\langle A_\nu \mid \nu < \alpha^+ \rangle$ of elementary submodels of A each of cardinality α, $\langle A_\xi \mid \xi \leq \nu \rangle \in A_{\nu+1}$ and $V_\alpha \in A_0$. Without loss of generality for each $\nu < \alpha^+$ we may assume that $A_\nu[p \cap A_\nu] \prec A[p]$. Consider now the forcing $\mathcal{P}'_{=\alpha}$. It satisfies α^{++} -c.c. Hence each antichain in $\mathcal{P}'_{=\alpha}$ that belongs to $A[p]$ is contained in $A[p]$. Now working inside

A it is easy to see for each $\xi < \alpha^+$ the set of conditions q in $\mathcal{P}'_{=\alpha}$ having A_ν for some ν, $\xi < \nu < \alpha^+$, as the maximal model, i.e. $A^{0\alpha}(q) = A_\nu$ is dense. Let us use $G_{i(\alpha)} \cap \mathcal{P}'_{=\alpha}(\delta^*)$ to produce $\mathcal{P}'_{=\alpha}$ -generic over A. Note that the set

$$T = \{\nu < \alpha^+ \mid A_\nu \text{ is the maximal model of a condition in this generic set}\}$$

is unbounded. Actually, using α^+ -strategic closure of $\mathcal{P}'_{=\alpha}$ it is not hard to see that T is stationry and fat.

Consider in N models

$$B = i(A), B_{i(\nu)} = i(A_\nu),\ B[i(p)], B_{i(\nu)}[i(p) \cap B_{i(\nu)}]\ .$$

We have $\cup(i''\alpha^+) = i(\alpha^+)$, hence

$$B = \bigcup_{\nu<\alpha^+} B_{i(\nu)} \text{ and } B[i(p)] = \bigcup_{\nu<\alpha^+} B_{i(\nu)}[i(p) \cap B_{i(\nu)}]\ .$$

Now we fix a list $\langle E_\nu \mid \nu < \alpha^+ \rangle$ of dense open subsets of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\delta})^N$ in $B[i(p)]$ which are the images of all dense open subsets coming from the ultrapower by the normal measure of the extender i. Note that the forcing under the consideration is δ^+-strategically closed (in N) and the generators of i are below δ, so this can be done.

For each $\nu < \alpha^+$ let E'_ν be the dense open subset of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\alpha})^N$ obtained from E_ν by adding to each $q \in E_\nu$ models of cardinalities in the interval $[\alpha, \delta]$, i.e. $q^\frown r \in E'_\nu$ iff $q \in E_\nu$, $q^\frown r \in ((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\alpha})^N$ and r consists of models of cardinalities in the interval $[\alpha, \delta]$. We may assume that E_ν (and hence also E'_ν) is in $B_{i(\nu)}[i(p) \cap B_{i(\nu)}]$, just removing some of B_ν's if necessary.

Recall that $G_{i(\alpha)}$ is an N -generic subset of $(\mathcal{P}'(i(\alpha)))^N$ constructed above. Our next task will be to consider the projection of $(\mathcal{P}'(\theta))^N_{\geq\alpha}$ over $G_{i(\alpha)}$ and to claim that certain elements are in $(\mathcal{P}'(\theta))^N_{\geq\alpha}/G_{i(\alpha)}$.

Claim 3.0.28.1 *For each $\nu \in T$ of cofinality α we have $i''A_\nu \in (\mathcal{P}'(\theta))^N_{\geq\alpha}/G_{i(\alpha)}$.*

Remark Note that $(G_{i(\alpha)})_{\geq\alpha} \cap A_\nu$ is a condition in $\mathcal{P}'$ (or just in $(\mathcal{P}'(i(\alpha)))^N$), due to 3.0.21. Our interest is in $((G_{i(\alpha)})_{\geq\alpha} \cap A_\nu)^\frown A_\nu$. By putting in $i''A_\nu$ we actually add all of $i''(((G_{i(\alpha)})_{\geq\alpha} \cap A_\nu)^\frown A_\nu)$. The claim basically deals with it rather then only with $i''A_\nu$.

Proof. Consider $C^\alpha(A_\nu) \restriction A_\nu$. It is a closed unbounded sequence in A_ν and since $cof(\nu) = \alpha$, it has a cofinal subsequence $\langle A_{\nu,\beta} | \beta < \alpha \rangle$. Apply i. Then $i(\langle A_{\nu,\beta} | \beta < \alpha \rangle)$ will be a cofinal subsequence of $C^{i(\alpha)}(B_\nu) = i(C^\alpha(A_\nu))$. Denote $i(\langle A_{\nu,\beta} | \beta < \alpha \rangle)$ by $\langle B_{\nu,\beta} | \beta < i(\alpha) \rangle$. Clearly, $i''A_\nu \subset B_{\nu,\alpha}$.

It is enough to show that $i''A_\nu$ is compatible with every element of $G_{i(\alpha)}$. Note that models of cardinalities $\geq \alpha$ are mapped to generic set over N for $(\mathcal{P}'(\theta))_{\geq i(\alpha)}$, just this set is generated by such images. Hence there is no problems with the images (i.e. $i(X)$) of elements of $A_\nu \cap (G_{i(\alpha)})_{\geq \alpha}$. We need only to take care of $i''X$ for $X \in (A_\nu \cap (G_{i(\alpha)})_{=\alpha}) \cup \{A_\nu\}$.

Pick any element q of $(\mathcal{P}'(i(\alpha))^N$ with A_ν inside. Assume also that A_ν is on the central line of q. Consider $i(q)$. It will consists of models of cardinalities below α and those of cardinalities at least $i(\alpha)$ (remember that each condition has Easton support). Also B_ν appears in $i(q)$ on the central line. We would like to find a common extension of q and $i(q)$ which includes $i''A_\nu$. Proceed as follows. Pick first some $\beta^*, \alpha < \beta^* < i(\alpha)$, such that B_{β^*} is a unique immediate predecessor of B_{β^*+1} and there is no models of cardinalities above $i(\alpha)$ (and so, no models at all) in between. Using elementarity and density argument it is possible to find such β^*. Now inside B_{ν,β^*} we pick an increasing continuous sequence $\langle X_\tau | \tau \in s(q)\rangle$ of models (elementary or Σ_1-elementary in B_{ν,β^*}) such that $q, i''A_\nu, i(q) \cap B_{\nu,\beta^*+1} \in X_{\kappa^+}$. Then $q^\frown i''A_\nu{}^\frown\langle X_\tau | \tau \in s(q)\rangle^\frown i(q)$ will be as desired.
$\square$ of the claim.

Let ν_0 be the first element of T of cofinality α. Consider $A_{\nu_0}{}^\frown i''A_{\nu_0}$. By Claim 3.0.28.1, $A_{\nu_0}{}^\frown i''A_{\nu_0} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$. Now inside B_{ν_0} we extend $A_{\nu_0}{}^\frown i''A_{\nu_0}$ to a condition q_0 in E'_0 with the projection to $(\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ inside $G_{i(\alpha)}$.

Claim 3.0.28.2 $q_0{}^\frown B_{\nu_0} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$.

Proof. Again we need to show that $q_0{}^\frown B_{\nu_0}$ is compatible with every element of $G_{i(\alpha)}$. Let $t \in G_{i(\alpha)}$ There is a common extension q of q_0 and t with projection in $G_{i(\alpha)}$, since $q_0 \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$. By elementarity, we can find such q inside B_{ν_0}. Thus

$$(\mathcal{P}'(i(\alpha)))^N \subseteq (V_{i(\alpha)})^N \subseteq B_{\nu_0}$$

and, hence

$$B_{\nu_0}[G_{i(\alpha)}] \prec B[G_{i(\alpha)}] \prec (V_\theta[G_{i(\alpha)}])^N.$$

Also, $B_{\nu_0}[G_{i(\alpha)}] \cap (\mathcal{P}'(\theta))^N = B_{\nu_0} \cap (\mathcal{P}'(\theta))^N$.

Consider $q^\frown B_{\nu_0}$. It is almost a condition in $(\mathcal{P}'(\theta))^N$ only with maximal models missing for lot of cardinalities. Extend it to some $r \in (\mathcal{P}'(\theta))^N$ for which the projection to $(\mathcal{P}'(\theta))^N$ is defined. Then $r \geq q$ implies that the projection r' of r is above the one of q. But then $r' \geq t$ in $(\mathcal{P}'(i(\alpha)))^N$. This means in particular that $q_0{}^\frown B_{\nu_0}$ is compatible with t.
$\square$ of the claim.

We proceed similar at each successor stage. Thus, if for $\xi < \alpha^+$, q_ξ, B_{ν_ξ} are defined $q_\xi \subseteq B_{\nu_\xi}$ and $q_\xi{}^\frown B_{\nu_\xi} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$, then we pick $\nu_{\xi+1}$ to be the least element of T above ν_ξ such that $cof(\nu_{\xi+1}) = \alpha$ and $A_{\nu_\xi} \in C^\alpha(A_{\nu_{\xi+1}})$. As in Claim 3.0.28.1, we will have $q = A_{\nu_{\xi+1}}{}^\frown q_\xi{}^\frown B_{\nu_\xi} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$.
Now inside $B_{\nu_{\xi+1}}$ we extend q to a condition $q_{\xi+1}$ in $E'_{\xi+1}$ with the projection to $(\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ inside $G_{i(\alpha)}$. Then, as in Claim 3.0.28.2, we will have $q_{\xi+1}{}^\frown B_{\nu_{\xi+1}} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$.

Let us turn to limit stages of the construction. Assume that ξ is a limit ordinal. Let $\nu_\xi = \cup_{\tau<\xi}\nu_\tau, \nu_{\xi+1}$ be the first element of $T\backslash\nu_\xi + 1$ of cofinality α and $q'_\xi = \cup\{q_\tau | \tau < \xi\}$. This q'_ξ is just the formal union of all q_τ's constructed at the previous stages. We do not take unions of the maximal models of q_τ's etc. Let q''_ξ be obtained from q'_ξ by adding $i''A_{\nu_{\xi+1}}$ and, if A_{ν_ξ} is in a condition in $G_{i(\alpha)}$, then also $i''A_{\nu_\xi}$.

Claim 3.0.28 *q''_ξ projects to an element of $G_{i(\alpha)}$.*

Proof. Let us show that for each $t_1 \in G_{i(\alpha)}$ above the projection of q'_ξ the following holds:

if $t \in (\mathcal{P}'(i(\alpha))^N_{\geq\alpha}$ and $t \geq t_1$, then there is $q \geq q''_\xi$ with the projection to $(\mathcal{P}'(i(\alpha))^N_{\geq\alpha}$ stronger than t.

Let $t_1 \leq t$ be as above. Then initial segments of q''_ξ project below t. Just q'_ξ projects to a condition in $G_{i(\alpha)}$ below $t_1 \leq t$. Also, the addition of $i''A_{\nu_{\xi+1}}, i''A_{\nu_\xi}$ is above $i(\alpha)$. So we can find a common extension $r \in B_{i(\nu_{\xi+1})}$ of t and q''_ξ. Using the elementarity of $V^N_{i(\alpha)}$, find $r' \in (V_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha))_{\geq\alpha})^N$ realizing the same type as r over $r \cap V^N_{i(\alpha)}$. Finally, let q be obtained from $r \cup r'$ by adding the maximal models including those of both r, r' and this models via $C^\rho(q)$'s to those of r'. Then the projection of q to $(\mathcal{P}'(i(\alpha))^N_{\geq\alpha}$ is $r' \geq t$ and we are done.

□ of the claim.

Now we extend q''_ξ to $q_\xi \in E_\xi$ in $B_{i(\nu_{\xi+1})}$ with the projection to $(\mathcal{P}'(i(\alpha))^N_{\geq\alpha}$ inside $G_{i(\alpha)}$.

This completes the construction.

Consider finally the resulting sequence $\langle q_\nu \mid \nu < \alpha^+\rangle$. Let $\langle q^*_\nu \mid \nu < \alpha^+\rangle$ be the sequence obtained from it by removing from each q_ν models of cardinalities below δ^+. Then, $q^*_\nu \in E_\nu$ for every $\nu < \alpha^+$. Hence $\langle q^*_\nu \mid \nu < \alpha^+\rangle$ generates a $B[i(p)]$-generic subset of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\delta^+})^N$. By the construction, the projections of q^*_ν's to $((\mathcal{P}'(i(\alpha)))_{\geq\delta^+})^N$ are in $G_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha)_{\geq\delta^+})^N$. The same is true (again by the construction) for q_ν's, i.e. projections to $((\mathcal{P}'(i(\alpha)))_{\geq\alpha})^N$ are in $G_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha)))_{\geq\alpha})^N$. Then q_ν's will be in $B[i(p)]$-generic subset of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\alpha})^N$ generated by $G_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha))_{\geq\alpha})^N$ and $\langle q^*_\nu \mid \nu < \alpha^+\rangle$. Moreover, models $i''(A_\nu)$ appear in q_ν's. Each $r \in \mathcal{P}'_{<\alpha}$ which is inside some A_ν will be moved by i to $i(r) \in (\mathcal{P}'(\theta)_{<\alpha})^N$ inside $i''A_\nu$. But $i''A_\nu$ is a model inside a condition in generic set, so $i(r)$ is such as well. Hence

images of elements from $G_{i(\alpha)} \cap \mathcal{P}'_{<\alpha}$ are in the constructed this way N-generic subset of $(\mathcal{P}'(\theta)_{<\alpha})^N$. So we are done.
$\square$ of the lemma.

Chapter 4

Dropping cofinalities-gap 3-single drop

Our aim is to present constructions in which $2^\kappa = \kappa^{+3}$ and the cofinality κ^{++} drops down, i.e. the generator $b_{\kappa^{++}}$ for the cofinality κ^{++} is far apart from $b_{\kappa^{+3}}$. Note that in the usual constructions of models with a singular strong limit cardinal κ with $2^\kappa = \kappa^{+3}$, like Silver-Prikry, Extender Based Prikry etc. (see [2]), we have

$$b_{\kappa^{++}} \supseteq \{\eta^- \mid \eta \in b_{\kappa^{+3}}\},$$

where η^- denotes the immediate predecessor of η.

4.1 Preliminary Settings

Let $\lambda_0 < \kappa_0 < \lambda_1 < \kappa_1 < ... < \lambda_n < \kappa_n < ...., n < \omega$ be a sequence of cardinals such that for each $n < \omega$

- λ_n is λ_n^{+n+2} - strong as witnessed by an extender E_{λ_n}
- κ_n is κ_n^{+n+2} - strong as witnessed by an extender E_{κ_n}

Set $\kappa = \bigcup_{n<\omega} \kappa_n$.

Let us denote by $\pi_{\lambda_n,\alpha,\beta}$ the projection map of the extender E_{λ_n} and by $\pi_{\kappa_n,\alpha,\beta}$ those of E_{κ_n}, see [2] for the definitions.

Force with the forcing $\mathcal{P}'$ of Chapter 1. Let $G(\mathcal{P}')$ be a generic subset.

4.2 Models and types

The main difference in present setting from those of [1], Chapter 1 will be due to the fact that the cardinalities of models in the range of a condition (i.e. in suitable structures over κ_n's) may be smaller than the number of existing types. So any such model may contain only a limited number of types. We would like to insure that it will be still sufficiently large.

Fix $n < \omega$. Set $\delta_n = \kappa_n^{+n+2}$. Fix using GCH an enumeration $\langle a_\alpha \mid \alpha < \kappa_n \rangle$ of $[\kappa_n]^{<\kappa_n}$ so that for every successor cardinal $\delta < \kappa_n$ the initial segment $\langle a_\alpha \mid \alpha < \delta \rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_n(\langle a_\alpha \mid \alpha < \kappa_n \rangle) = \langle a_\alpha \mid \alpha < j_n(\kappa_n) \rangle$ where j_n is the canonical embedding of the (κ_n, δ_n)-extender E_{κ_n}. Then $\langle a_\alpha \mid \alpha < \delta_n \rangle$ will enumerate $[\delta_n]^{\leq \delta_n}$ and we fix this enumeration. For each $k \leq \omega$ consider a structure

$$\mathfrak{A}_{n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, E_{\kappa_n}, E_{\lambda_n}, \lambda_n, \kappa_n, \delta_n,$$

$$\chi, \langle a_\alpha \mid \alpha < \delta_n \rangle, 0, 1, \ldots, \alpha, \ldots \mid \alpha < \kappa_n^{+k} \rangle$$

in the appropriate language $\mathcal{L}_{n,k}$ with a large enough regular cardinal χ.

Remark 4.2.1 It is possible to use κ_n^{++} here (as well as in [1]) instead of κ_n^{+k}. The point is that there are only κ_n^{++} many ultrafilters over κ_n and we would like that equivalent conditions use the same ultrafilter. The only parameter that need to vary is k in $H(\chi^{+k})$.

Let $\mathcal{L}'_{n,k}$ be the expansion of $\mathcal{L}_{n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less or equal than δ_n let $\mathfrak{A}_{n,k,a}$ be the expansion of $\mathfrak{A}_{n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less or equal than δ_n. Denote by $tp_{n,k}(b)$ the $\mathcal{L}_{n,k}$-type realized by b in $\mathfrak{A}_{n,k}$. Further we identify it with the ordinal coding it and refer to it as the k-type of b. Let $tp_{n,k}(a, b)$ be a the $\mathcal{L}'_{n,k}$-type realized by b in $\mathfrak{A}_{n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [1].

4.3 The main forcing

We will use λ_n's ($n < \omega$) to generate ω-sequences corresponding to ordinals below κ^{++} in the same way as it was done in [1].
The treatment of κ^{+3} will be parallel to those of Chapter 1, but with major changes due to the lack of cardinals between κ_n and κ_n^{+n+2} that correspond to κ^{++}. Here κ_n^{+n+2} will correspond to κ^{+3} and λ_n^{+n+2} to κ^{++}. Recall that in Chapter 1, κ_n^{+n+3} corresponds to κ^{+3} and κ_n^{+n+2} to κ^{++}.

We will use suitable and suitable generic structures over κ as defined in Chapter 1. The corresponding structures over κ_n's will be rather names related to choices made over λ_n's.

Fix $n < \omega$.

Let give first the following preliminary definition:

Definition 4.3.1 Let η be a cardinal less than λ_n. A suitable structure $\langle\langle X, Y\rangle, C, \in, \subseteq\rangle$ at the level κ_n (see Chapter 1)is called *an η-suitable* iff each element of X (i.e. each model) has cardinality η.

Note that in Chapter 1 models at the level κ_n have cardinality κ_n^{+n+2}. Here it drops below $\lambda_n < \kappa_n$.

Definition 4.3.2 Let Q_{n0} be the set consisting of pairs of triples

$$q = \langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle$$

so that:

1. f is partial function from κ^{+2} to λ_n of cardinality at most κ

2. There is a suitable generic structure $\langle\langle X, Y\rangle, C, \in, \subseteq\rangle$ of cardinality less than λ_n (not κ_n, as in Chapter 1), such that

   a is an order preserving function from the set $\{Z \cap \kappa^{++} \mid Z \in X\}$ to λ_n^{+n+2}.
   Note that by Chapter 1, the set $\{Z \cap \kappa^{++} \mid Z \in X\}$ is a closed subset of κ^{++}.

3. $a(\max(X) \cap \kappa^{++}) = \max(\text{rng}(a))$ is above all the elements of $\text{rng}(a)$ in the order of the extender E_{λ_n}.

4. $\text{dom}(a) \cap \text{dom}(f) = \emptyset$.

5. $A \in E_{\lambda_n, \max(\mathrm{rng}(a))}$.

6. $\min(A) > |X|$.

7. For every ordinals $\alpha, \beta, \gamma \in \mathrm{rng}(a)$ and $\rho \in \pi''_{\lambda_n, \max \mathrm{rng}(a), \alpha}(A)$

$$\begin{aligned} &\alpha \geq_{E_{\lambda_n}} \beta \geq_{E_{\lambda_n}} \gamma \quad \text{implies} \\ &\pi_{\lambda_n, \alpha, \gamma}(\rho) = \pi_{\lambda_n, \beta, \gamma}(\pi_{\lambda_n, \alpha, \beta}(\rho)). \end{aligned}$$

8. For every $\alpha > \beta$ in $\mathrm{rng}(a)$ and $\rho \in A$

$$\pi_{\lambda_n, \max \mathrm{rng}(a), \alpha}(\rho) > \pi_{\lambda_n, \max \mathrm{rng}(a), \beta}(\rho).$$

Let us turn now to the second component of a condition, i. e. to $\langle \underset{\sim}{b}, \underset{\sim}{B}, g \rangle$.

9. g is a function from κ^{+3} to κ_n of cardinality at most κ

10. $\underset{\sim}{b}$ is a name, depending on $\langle a, A \rangle$. For each $\eta \in A$ the interpretation $\underset{\sim}{b}[\eta]$ of $\underset{\sim}{b}$ according to η satisfies the following conditions.

    (a) There is an $((\eta)^0)^{+n+1}$-suitable structure [1] $\langle \langle X_\eta, Y_\eta \rangle, C_\eta, \in, \subseteq \rangle$ at the level κ_n such that

    i. $\underset{\sim}{b}[\eta]$ is the isomorphism between $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ and $\langle \langle X_\eta, Y_\eta \rangle, C_\eta, \in, \subseteq \rangle$,

    ii. for every $Z \in X$ we have

$$\pi_{\lambda_n, \max(\mathrm{rng}(a)), a(Z \cap \kappa^{++})}(\eta) = \underset{\sim}{b}[\eta](Z) \cap ((\eta)^0)^{+n+2}.$$

    In particular,

$$\eta = \underset{\sim}{b}[\eta](\max(X)) \cap ((\eta)^0)^{+n+2}.$$

    Further let us identify between $\underset{\sim}{b}[\eta](Z)$ and $\underset{\sim}{b}(Z)[\eta]$.
    Note that the domain of $\underset{\sim}{b}$ is X and this does not depend on η.

    (b) (Dependence) Let $Z \in X$. Then $\underset{\sim}{b}[\eta](Z)$ depends on the value of the one element Prikry forcing with the measure $a(Z \cap \kappa^{++})$ over λ_n. More precisely: let

$$A(Z) = \pi_{\lambda_n, \max \mathrm{rng}(a), a(Z \cap \kappa^{++})}{}'' A,$$

[1] η^0, as usual, denotes the projection of η to the normal measure of the extender, i.e. $\eta^0 = \pi_{\lambda_n, \max(\mathrm{rng}(a)), \lambda_n}(\eta)$.

then each choice of an element from $A(Z)$ already decides $\underset{\sim}{b}(Z)$, i.e. whenever $\eta_1, \eta_2 \in A$ and

$$\pi_{\lambda_n, \max \operatorname{rng}(a), a(Z\cap\kappa^{++})}(\eta_1) = \pi_{\lambda_n, \max \operatorname{rng}(a), a(Z\cap\kappa^{++})}(\eta_2)$$

we have

$$\underset{\sim}{b}(Z)[\eta_1] = \underset{\sim}{b}(Z)[\eta_2].$$

Further let us denote by $\eta(Z)$ the projection of η to $a(Z \cap \kappa^{++})$, i.e. $\pi_{\lambda_n, \max \operatorname{rng}(a), a(Z\cap\kappa^{++})}(\eta)$, for each $\eta \in A$,.

So $\underset{\sim}{b}(Z)$ depends only on members of $A(Z)$ rather than those of A.

The next condition is crucial for the κ^{++}-c.c. of the forcing.

(c) (Inclusion condition)

Let $\eta, \eta' \in A, \eta < \eta'$. Then

- $\underset{\sim}{b}(\max(X))[\eta] \in \underset{\sim}{b}(\max(X))[\eta']$,
- if $Z \in X \cap C(\max(X))$ and

$$\pi_{\lambda_n, \max \operatorname{rng}(a), a(Z\cap\kappa^{++})}(\eta') > \eta,$$

then either

$$\underset{\sim}{b}(\max(X))[\eta] \in \underset{\sim}{b}(Z)[\eta']$$

or

the k-type realized by $\underset{\sim}{b}(\max(X))[\eta] \cap H(\chi^{+k})$ is in $\underset{\sim}{b}(Z)[\eta']$, where $k < \omega$ is the least such that $\underset{\sim}{b}(Z)[\eta'] \subseteq H(\chi^{+k+1})$.

The same holds over any element of $\underset{\sim}{b}(Z)[\eta']$, i.e. $tp_k(z, \underset{\sim}{b}(\max(X))[\eta] \cap H(\chi^{+k})) \in \underset{\sim}{b}(Z)[\eta']$, for any $z \in \underset{\sim}{b}(Z)[\eta']$.

We require in addition that this $k > 2$.

Let us allow the above also if $\underset{\sim}{b}(Z)[\eta'] \subseteq H(\chi^{+\omega})$. In this case we take k to be any natural number above 2 and require that once we go up to the higher levels then corresponding k's increase (with n).

We cannot in general require only that

$$\underset{\sim}{b}(\max(X))[\eta] \in \underset{\sim}{b}(Z)[\eta']$$

since the sequence C of a new generic suitable structure may go not through the old maximal model. But still having the type inside Z will suffice.

Note that given $\eta' \in A$ the number of possibilities for $\eta \in \eta' \cap A$ is bounded by $(\eta'^0)^{+n+1}$, as $\eta' < (\eta'^0)^{+n+2}$.

(d) $Y \cap \operatorname{dom}(g) = \emptyset$.

(e) For every $\alpha \in Y$ and $\eta \in A$

  i. $\underset{\sim}{b}[\eta](\alpha)$ is a model of cardinality κ_n^{+n+1},

  ii. $\kappa_n^{+n+1} \subseteq \underset{\sim}{b}[\eta](\alpha)$,

  iii. $\operatorname{cof}(\sup(\underset{\sim}{b}[\eta](\alpha) \cap \kappa_n^{+n+2})) = (\eta^0)^{+n+2}$.

  Note that all the cardinals $\kappa_n, ..., \kappa_n^{+n+1}$ will correspond here to κ^+. So, we need to drop down to to the indiscernible $(\eta^0)^{+n+2}$ for λ_n^{+n+2} in order to get to κ^{++}.

(f) For every $\eta \in A$

$$\underset{\sim}{B}[\eta] \in E_{\kappa_n, \max(X_\eta)}.$$

(g) For every $\eta \in A$ and ordinals α, β, γ which are elements of $\operatorname{rng}(\underset{\sim}{b})[\eta]$ (i.e. actually the ordinals coding models in $\operatorname{rng}(\underset{\sim}{b})[\eta]$) we have

$$\alpha \geq_{E_{\kappa_n}} \beta \geq_{E_{\kappa_n}} \gamma \quad \text{implies}$$
$$\pi_{\kappa_n,\alpha,\gamma}(\rho) = \pi_{\kappa_n,\beta,\gamma}(\pi_{\kappa_n,\alpha,\beta}(\rho))$$

  for every $\rho \in \pi''_{\kappa_n, \max \operatorname{rng}(\underset{\sim}{b}[\eta]),\alpha}(\underset{\sim}{B}[\eta])$,

(h) for every $Z \in \operatorname{dom}(\underset{\sim}{b}_n)$ there is $k(Z) \leq \omega$ such that for every $\eta \in A$ we have

$$\underset{\sim}{b}_n(Z)[\eta] \prec H(\chi^{+k(Z)}).$$

  Note that it is easy to arrange this condition just by shrinking A. Thus for each $\eta \in A$ there is $k_\eta \leq \omega$ such that $\underset{\sim}{b}_n(Z)[\eta] \prec H(\chi^{+k_\eta})$. Now pick $A_Z \subseteq A$ in $E_{\lambda_n, \max(\operatorname{rng}(a))}$ and $k(Z) \leq \omega$ such that for every $\eta \in A_Z$, $k_\eta = k(Z)$. Finally replace A by $\bigcap\{A_Z \mid Z \in \operatorname{dom}(\underset{\sim}{b}_n)\}$. Note this intersection is still in $E_{\lambda_n, \max(\operatorname{rng}(a))}$, since $|\operatorname{dom}(\underset{\sim}{b}_n)| < \lambda_n$.

Definition 4.3.3 Suppose that $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle$ and $\langle\langle a', A', f'\rangle, \langle \underset{\sim}{b}', \underset{\sim}{B}', g'\rangle\rangle$ are two elements of Q_{n0}. Define

$$\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle \geq_{Q_{n0}} \langle\langle a', A', f'\rangle, \langle \underset{\sim}{b}', \underset{\sim}{B}', g'\rangle\rangle$$

iff

1. $f \supseteq f'$

2. $g \supseteq g'$

3. $a \supseteq a'$

4. $\pi''_{\lambda_n,\max(a),\max(a')} A \subseteq A'$

5. for every $\nu \in A$ we have

$$\underset{\sim}{b}[\nu] \supseteq \underset{\sim}{b}'[\pi_{\lambda_n,\max(a),\max(a')}(\nu)].$$

This means just that the empty condition of one element Prikry forcing forces the inclusion.

6. for every $\nu \in A$ we have

$$\pi''_{\kappa_n,\max(\underset{\sim}{b}[\nu]),\max(\underset{\sim}{b}'[\pi_{\lambda_n,\max(a),\max(a')}(\nu)])} \underset{\sim}{B}[\nu] \subseteq \underset{\sim}{B}'[\pi_{\lambda_n,\max(a),\max(a')}(\nu)]$$

We define now Q_{n1} and $\langle Q_n, \leq_n, \leq_n^* \rangle$ similar to those of Chapter 1.

Definition 4.3.4 Q_{n1} consists of pairs $\langle f, g\rangle$ such that

1. f is a partial function from κ^{++} to λ_n of cardinality at most κ
2. g is a partial function from κ^{+3} to κ_n of cardinality at most κ

Q_{n1} is ordered by extension. Denote this order by $\leq_1$.
So, it is basically the Cohen forcing for adding κ^{+3} Cohen subsets to κ^+.

Definition 4.3.5 Set $Q_n = Q_{n0} \cup Q_{n1}$. Define $\leq_n^* = \leq_{Q_{n0}} \cup \leq_{Q_{n1}}$.

Define now a natural projection to the first coordinate:

Definition 4.3.6 Let $p \in Q_n$. Set $(p)_0 = p$, if $p \in Q_{n1}$ and let $(p)_0 = \langle a, A, f\rangle$, if $p \in Q_{n0}$ is of the form $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle$.

Let $(Q_n)_0 = \{(p)_0 \mid p \in Q_n\}$.

Definition 4.3.7 Let $p, q \in Q_n$. Then $p \leq_n q$ iff either

1. $p \leq_n^* q$

   or

2. $p = \langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle \in Q_{n0}$, $q = \langle e, h\rangle \in Q_{n1}$ and the following hold:

   (a) $e \supseteq f$

(b) $h \supseteq g$

(c) $\mathrm{dom}(e) \supseteq \mathrm{dom}(a)$

(d) $e(\max(\mathrm{dom}(a))) \in A$

(e) for every $\beta \in \mathrm{dom}(a), e(\beta) = \pi_{\lambda_n, a(\max(\mathrm{dom}(a)), a(\beta)}(e(\max(\mathrm{dom}(a)))$

(f) $\mathrm{dom}(h) \supseteq \mathrm{dom}(\underset{\sim}{b})$

(g) $h(\max(\mathrm{dom}(\underset{\sim}{b})) \in \underset{\sim}{B}[e(\max(\mathrm{dom}(a))]$.
I.e., we use $e(\max(\mathrm{dom}(a))$ in order to interpret $\underset{\sim}{B}$. Note that by 2d above, it is inside A and so the interpretation makes sense.

(h) for every $\beta \in \mathrm{dom}(\underset{\sim}{b})$

$$h(\beta) = \pi_{\kappa_n, \max(\mathrm{rng}(\underset{\sim}{b}[\nu])), \underset{\sim}{b}(\beta)[\nu]}(h(\max(\mathrm{dom}(\underset{\sim}{b})),$$

where $\nu = e(\max(\mathrm{dom}(a)))$. Recall that we code models by ordinals.

Definition 4.3.8 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

1. for every $n < \omega$, $p_n \in Q_n$

2. there is $\ell(p) < \omega$ such that

   (a) for every $n < \ell(p)$, $p_n \in Q_{n1}$

   (b) for every $n \geq \ell(p)$, $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b_n}, \underset{\sim}{B_n}, g_n \rangle \rangle \in Q_{n0}$

   (c) for every $n, m \geq \ell(p)$, $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$ and $\max(\mathrm{dom}(\underset{\sim}{b_n})) = \max(\mathrm{dom}(\underset{\sim}{b_m}))$

   (d) for every $n \geq m \geq \ell(p)$, $\mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$ and $\mathrm{dom}(\underset{\sim}{b_m})) \subseteq \mathrm{dom}(\underset{\sim}{b_n})$

   (e) for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ the following holds:
   for each $k < \omega$ the set

   $$\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$$

   is finite.

   (f) for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(\underset{\sim}{b_n})$ the following holds:
   for each $k < \omega$ the set

   $$\{m < \omega \mid \exists \nu \in A_m(\neg(\underset{\sim}{b_m}(X)[\nu] \cap H(\chi^{+k}) \prec H(\chi^{+k})))\}$$

   is finite.

We define the orders $\leq, \leq^*$ as in [3].

Definition 4.3.9 Let $p = \langle p_n | n < \omega \rangle, q = \langle q_n | n < \omega \rangle$ be in $\mathcal{P}$. Define

1. $p \geq q$ iff for each $n < \omega, p_n \geq_n q_n$
2. $p \geq^* q$ iff for each $n < \omega, p_n \geq^*_n q_n$

Definition 4.3.10 Let $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$. Set $(p)_0 = \langle (p_n)_0 \mid n < \omega \rangle$.
Define $(\mathcal{P})_0 = \{(p)_0 \mid p \in \mathcal{P}\}$.

Finally, the equivalence relation $\longleftrightarrow$ and the order $\to$ are defined on $(\mathcal{P})_0$ exactly as it was done in [1], or in Chapter 1. We extend $\to$ to $\mathcal{P}$ as follows:

Definition 4.3.11 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$. Set $q \to p$ iff

1. $(q)_0 \to (p)_0$
2. $\ell(p) \geq \ell(q)$
3. for every $n < \ell(p), p_n$ extends q_n
4. for every $n \geq \ell(p)$, let $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$ and $q_n = \langle \langle a'_n, A'_n, f'_n \rangle, \langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n, g'_n \rangle \rangle$. Require the following:
   (a) $g_n \supseteq g'_n$
   (b) there is $\underset{\sim}{b}''_n$ such that for every $\nu \in A_n$ the following holds:
      i. $\underset{\sim}{b}_n[\nu]$ extends $\underset{\sim}{b}''_n[\nu']$
      ii. $\mathrm{dom}(\underset{\sim}{b}'_n) = \mathrm{dom}(\underset{\sim}{b}''_n)$
      iii. $\pi''_{\kappa_n, \max(\underset{\sim}{b}_n[\nu]), \max(\underset{\sim}{b}'_n[\nu'])} \underset{\sim}{B}_n[\nu] \subseteq \underset{\sim}{B}'_n[\nu']$,
      where $\nu' = \pi_{\lambda_n, \max(\mathrm{rng}(a_n)), \xi}(\nu)$ and $\xi = a_n(\max(\mathrm{dom}(a'_n))$
      iv. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'] \longleftrightarrow_{k_n} \mathrm{rng}(\underset{\sim}{b}''_n)[\nu']$, where ν' is as above and k_n is the k_n's member of a nondecreasing sequence converging to the infinity.
      v. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'] \upharpoonright \kappa_n^{+n+1} = \mathrm{rng}(\underset{\sim}{b}''_n)[\nu'] \upharpoonright \kappa_n^{+n+1}$

   Here is the main difference between $\to$ in the present context and those of [1] etc. In the present context we deal with assignment functions b_n's which act over κ_n's but are of cardinalities below κ_n's (as well as the models in $\mathrm{rng}(b_n)$ which are images of those of cardinality κ^+). Thus, assume that n is fixed and

$Z = b_n(\max(\mathrm{dom}(b_n))$, where $b_n = \underset{\sim}{b}_n[\eta]$ is the interpretation according to some $\eta < \lambda_n < \kappa_n$. Then $|Z| = (\eta^0)^{+n+1}$. Now if we like to realize types inside Z, as it was done usually in [1] etc., it may be just impossible since Z is too small and so does not contain all the types.
The way suggested here in order to overcome this difficulty, will be to use 4.3.2(10c) together with the above definition. It turns out that once working with names it is still possible to prove κ^{++}-c.c. of the final forcing $\langle \mathcal{P}, \rightarrow \rangle$. It will be done in 4.4.5.

4.4 Basic Lemmas

In this section we state basic lemmas for the forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$. Most of the proofs just repeat those of Chapter 1 with minor adjustments.

Lemma 4.4.1 *Let $p \in \mathcal{P}$ and $\langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$. Then*

1. *for every $\alpha \in B^{1\kappa^{++}}$ there is $q \geq^* p$ such that $\alpha \in \mathrm{dom}(\underset{\sim}{b}_n(q))$ for all but finitely many n's;*

2. *for every $A \in B^{1\kappa^+}$ there is $q \geq^* p$ such that $A \cap \kappa^{++} \in \mathrm{dom}(a_n)(q)$ and $A \in \mathrm{dom}(\underset{\sim}{b}_n(q))$ for all but finitely many n's. Moreover, if $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \geq \langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle$ witnesses a generic suitability of p and $A \in C^{\kappa^+}(A^{0\kappa^+})$, then the addition of A does not require adding of ordinals and the only models that probably will be added together with A are its images under Δ-system type isomorphisms for triples in p.*

Lemma 4.4.2 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_0 \rangle$ does not add new sequences of ordinals of the length $< \lambda_n$, i.e. it is (λ_n, ∞) – distributive.*

Lemma 4.4.3 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \lambda_0$.*

Lemma 4.4.4 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Let us turn now to the main lemma in the present context:

Lemma 4.4.5 *$\langle \mathcal{P}, \rightarrow \rangle$ satisfies κ^{++}-c.c.*

Proof.

Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++}\rangle$ be a name of an antichain of the length κ^{++}. Using the strategic closure of $\underset{\sim}{\mathcal{P}}'$, we find an increasing sequence

$$\langle\langle\langle A^{0\kappa^+}_\alpha, A^{1\kappa^+}_\alpha, C^{\kappa^+}_\alpha\rangle, A^{1\kappa^{++}}_\alpha\rangle \mid \alpha < \kappa^{++}\rangle$$

of elements of $\mathcal{P}'$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ so that for every $\alpha < \kappa^{++}$ the following hold:

(a) $\langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle \Vdash \underset{\sim}{p}_\alpha \leq \check{p}_\alpha$,

(b) $\bigcup_{\beta<\alpha} A^{0\kappa^+}_\beta = A^{0\kappa^+}_\alpha$, if α is a limit ordinal,

(c) ${}^\kappa A^{0\kappa^+}_{\alpha+1} \subseteq A^{0\kappa^+}_{\alpha+1}$,

(d) $A^{0\kappa^+}_{\alpha+1}$ is a successor model,

(e) $\langle A^{1\kappa^+}_\beta \mid \beta < \alpha\rangle \in A^{0\kappa^+}_{\alpha+1}$,

(f) for every $\alpha \leq \beta < \kappa^{++}$ we have

$$C^{\kappa^+}_\alpha(A^{0\kappa^+}_\alpha) \text{ is an initial segment of } C^{\kappa^+}_\beta(A^{0\kappa^+}_\beta),$$

(g) $p_\alpha = \langle p_{\alpha n} \mid n < \omega\rangle$,

(h) for every $n \geq l(p_\alpha)$

- $A^{0\kappa^+}_{\alpha+1} \cap \kappa^{++}$ is the maximal ordinal of $\mathrm{dom}(a_{\alpha n})$ and $A^{0\kappa^+}_\alpha \cap \kappa^{++} \in \mathrm{dom}(a_{\alpha n})$,
- $A^{0\kappa^+}_{\alpha+1}$ is the maximal model of $\mathrm{dom}(\underset{\sim}{b}_{\alpha n})$ and $A^{0\kappa^+}_\alpha \in \mathrm{dom}(\underset{\sim}{b}_{\alpha n})$,

where $p_{\alpha n} = \langle\langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle, \langle \underset{\sim}{b}_{\alpha n}, \underset{\sim}{B}_{\alpha n}, g_{\alpha n}\rangle\rangle$.
Actually this condition is the reason for not requiring the equality in (a) above.

Let $p_{\alpha n} = \langle\langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle, \langle \underset{\sim}{b}_{\alpha n}, \underset{\sim}{B}_{\alpha n}, g_{\alpha n}\rangle\rangle$ for every $\alpha < \kappa^{++}$ and $n \geq l(p_\alpha)$.
Let $\alpha < \kappa^{++}$. Fix some

$$\langle\langle B^{0\kappa^+}_{\alpha+1}, B^{1\kappa^+}_{\alpha+1}, D^{\kappa^+}_{\alpha+1}\rangle, B^{1\kappa^{++}}_{\alpha+1}\rangle \leq_{\mathcal{P}'} \langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle$$

which witnesses a generic suitability of structure $\mathrm{dom}(\underset{\sim}{b}_{\alpha n})$ for each $n, l(p_\alpha) \leq n < \omega$, as in Definition 4.3.2. Note that $B^{0\kappa^+}_{\alpha+1}$ need not be in $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$ and even if it does, then

$D^{\kappa^+}_{\alpha+1}(B^{0\kappa^+}_{\alpha+1})$ need not be an initial segment of $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$. By the definition of the order $\leq_{\mathcal{P}'}$ there are $m < \omega$ and $E_1, ..., E_m \in A^{1\kappa^+}_{\alpha+1}$ such that

$$swt(\langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle, E_1, ..., E_m) \text{ and } \langle\langle B^{0\kappa^+}_{\alpha+1}, B^{1\kappa^+}_{\alpha+1}, D^{\kappa^+}_{\alpha+1}\rangle, B^{1\kappa^{++}}_{\alpha+1}\rangle$$

are as in the definition of the order of $\mathcal{P}'$ (Chapter 1, 1.15).
By Lemma 4.4.1 it is possible to add all $E_i (i = 1, ..., m)$ to $\mathrm{dom}(a_{\alpha n})$, for a final segment of n's. By adding and taking non-direct extension if necessary, we can assume that E_i's are already in $\mathrm{dom}(a_{\alpha n})$, for every $n \geq l(p_\alpha)$.
Now we can apply the opposite switch (i.e. the one starting with E_m, then E_{m-1}, ...,and finally E_1) to $\mathrm{dom}(a_{\alpha n})$ (and the corresponding to it under $a_{\alpha n}$ to $\mathrm{rng}(a_{\alpha n})$). Denote the result still by $a_{\alpha n}$.
Finally, $\langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle$ will witness a generic suitability of structure $\mathrm{dom}(a_{\alpha n})$ for each $n, l(p_\alpha) \leq n < \omega$.

In particular, we have now that the central line of $\mathrm{dom}(a_{\alpha n})$ is a part of $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$ and $A^{0\kappa^+}_{\alpha}$ is on it, for every $n, l(p_\alpha) \leq n < \omega$.

Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^{++}$ the following holds:

(1) $\ell = \ell(p_\alpha) = \ell(p_\beta)$,

(2) for every $n < \ell$, $p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1} i.e. $p_{\alpha n} \cup p_{\beta n}$ is a function,

(3) for every n, $\ell \leq n < \omega$, $\langle \mathrm{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++}\rangle$ and $\langle \mathrm{dom}(g_{\alpha n}) \mid \alpha < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$,

(4) for every n, $\omega > n \geq \ell$, $\mathrm{rng}(a_{\alpha n}) = \mathrm{rng}(a_{\beta n})$,

(5) for every n, $\omega > n \geq \ell$, $A_{\alpha n} = A_{\beta n}$,

(6) for every n, $\omega > n \geq \ell, \eta \in A_{\alpha n}$, $\mathrm{rng}\ \underset{\sim}{b}_{\alpha n}[\eta] = \mathrm{rng}\ \underset{\sim}{b}_{\beta n}[\eta]$,

(7) for every n, $\omega > n \geq \ell, \eta \in A_{\alpha n}$, $\underset{\sim}{B}_{\alpha n}[\eta] = \underset{\sim}{B}_{\beta n}[\eta]$,

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let α be in S. For each $n, \ell \leq n < \omega$, there will be $\beta(\alpha, n) < \alpha$ such that

$$\mathrm{dom}(\underset{\sim}{b}_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} \subseteq A^{0\kappa^+}_{\beta(\alpha,n)}.$$

Just recall that $\operatorname{dom}(\underset{\sim}{b}_{\alpha n})$ is not actually a name and $|\operatorname{dom}(\underset{\sim}{b}_{\alpha n})| < \lambda_n$. Shrink S to a stationary subset S^* so that for some $\alpha^* < \min S^*$ of cofinality κ^+ we will have $\beta(\alpha, n) < \alpha^*$, whenever $\alpha \in S^*, \ell \leq n < \omega$. Now, the cardinality of $A^{0\kappa^+}_{\alpha^*}$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\alpha, \beta \in S^*, \ell \leq n < \omega$

$$\operatorname{dom}(\underset{\sim}{b}_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} = \operatorname{dom}(\underset{\sim}{b}_{\beta n}) \cap A^{0\kappa^+}_{\beta}.$$

Let us add $A^{0\kappa^+}_{\alpha^*}$ to each p_α with $\alpha \in S^*$.

By 4.4.1(2), we can add it without adding ordinals and the only other models that probably were added are the images of $A^{0\kappa^+}_{\alpha^*}$ under Δ-system type isomorphisms. Denote the result for simplicity by p_α as well.

Let now $\beta < \alpha$ be ordinals in S^*. We claim that p_β and p_α are compatible in $\langle \mathcal{P}, \rightarrow \rangle$. First extend p_α by adding $A^{0\kappa^+}_{\beta+2}$. This will not add other additional models or ordinals except the images of $A^{0\kappa^+}_{\beta+2}$ under isomorphisms to p_α, as was remarked above.

Let p be the resulting extension. Denote p_β by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$ and $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$. Let $q_n = \langle \langle a'_n, A'_n, f'_n \rangle, \langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n, g'_n \rangle \rangle$. Without loss of generality we may assume that the ordinal $a_n(A^{0\kappa^+}_{\beta+2} \cap \kappa^{++})$ is k_n-good with $k_n \geq 5$. Just increase n if necessary.

Realize the $k_n - 1$-type of $\operatorname{rng}(a'_n)$ below $a_n(A^{0\kappa^+}_{\beta+2} \cap \kappa^{++})$ over $a_n((A^{0\kappa^+}_{\beta+2} \cap \kappa^{++}) \cap \operatorname{dom}(a_n))$, i.e. above the common part on κ^{++}. Denote the ordinal corresponding to $\max(\operatorname{rng}(a'_n))$ in this realization by δ'. Note that $a_n(A^{0\kappa^+}_{\alpha+1} \cap \kappa^{++})$ and δ' have the same projection to the common part $a_n((A^{0\kappa^+}_{\beta+2} \cap \kappa^{++}) \cap \operatorname{dom}(a_n))$.

Fix now $\eta \in A_n$. Set $\eta' = \pi_{\lambda_n, \max(\operatorname{rng}(a_n)), \delta'}(\eta)$.

Consider $\underset{\sim}{b}_n(A^{0\kappa^+}_{\beta+2})[\eta]$. Again we can assume that it is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$ (and k_n does not depend on η). Now we have

$$\eta' = \pi_{\lambda_n, \max(\operatorname{rng}(a_n)), \delta'}(\eta) < \eta \text{ and } A^{0\kappa^+}_{\beta+2} \in C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1}).$$

Hence, by Definition 4.3.2(10c), the $k_n - 1$-type realized by $\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\eta']$ is in $\underset{\sim}{b}_n(A^{0\kappa^+}_{\beta+2})[\eta]$, as well as the $k_n - 1$-type realized by $\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\eta']$ over $\underset{\sim}{b}''_n(A^{0\kappa^+}_{\beta+2} \cap \operatorname{dom}(\underset{\sim}{b}_n))[\eta]$, i.e. the common part of of the conditions. Realize the $k_n - 1$-type of $\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\eta']$ over $\underset{\sim}{b}''_n(A^{0\kappa^+}_{\beta+2} \cap \operatorname{dom}(\underset{\sim}{b}_n))[\eta]$ in $\underset{\sim}{b}_n(A^{0\kappa^+}_{\beta+2})[\eta]$.

Doing the above for each $\eta \in A_n$ will produce a condition $p^*_n \geq p_n$ with $q_n \rightarrow p^*_n$ as in Chapter 1.

□

4.5 The resulting PCF structure

Force with $\langle \mathcal{P}, \to \rangle$. Let $G(\mathcal{P})$ be a generic set. By the lemmas above no cardinals are collapsed. Let $\langle \nu_n \mid n < \omega \rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\lambda_n} \mid n < \omega \rangle$ and $\langle \rho_n \mid n < \omega \rangle$ those for $\langle E_{\kappa_n} \mid n < \omega \rangle$.[2] We can deduce now the following conclusion:

Theorem 4.5.1 *The following hold in $V[G(\mathcal{P}'(\theta)) * G(\mathcal{P})]$:*

(1) $\mathrm{cof}(\prod_{n<\omega} \nu_n^{+n+2}/$ *finite* $) = \kappa^{++}$.

(2) $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+2}/$ *finite* $) = \kappa^{+3}$. *Moreover, there is a scale* $\langle H_\tau \mid \tau < \kappa^{+3} \rangle$ *in* $\prod_{n<\omega} \rho_n^{+n+2}/$ *finite with the following special property:*

$(*)$ *for every* $\tau < \kappa^{+3}$,

(a) if $\mathrm{cof}(\tau) = \kappa^{++}$, *then* H_τ *is an exact upper bound of* $\langle H_\mu \mid \mu < \tau \rangle$ *and for all but finitely many* $n < \omega$, $\mathrm{cof}(H_\tau(n)) = \nu_n^{+n+2}$;

(b) if $\mathrm{cof}(\tau) < \kappa^{++}$, *then for all but finitely many* $n < \omega$, $\mathrm{cof}(H_\tau(n)) < \nu_n^{+n+2}$.

(3) For every unbounded subset a *of* κ *consisting of regular cardinals and disjoint to both* $\{\nu_n^{+n+2} \mid n < \omega\}$ *and* $\{\rho_n^{+n+2} \mid n < \omega\}$, *for every ultrafilter* D *over* a *which includes all co-bounded subsets of* κ *we have*

$$\mathrm{cof}(\prod a/D) = \kappa^+$$

Proof. Items (1) and (2) follow easily from the construction. Thus, for (1), take the increasing (under the inclusion) enumeration $\langle X_\tau | \tau < \kappa^{++} \rangle$ of the chain of models given by $G(\mathcal{P}'(\kappa^{++}))$. Define a scale of functions $\langle F_\tau \mid \tau < \kappa^{++} \rangle$ in the product $\prod_{n<\omega} \nu_n^{+n+2}$ as follows: let for each $\tau < \kappa^{++}$

$$F'_\tau(n) = f_n(X_\tau), \text{ if } f_n(X_\tau) < \nu_n^{+n+2}$$

and

$$F'_\tau(n) = 0, \text{ otherwise},$$

where for some $p = \langle p_k | k < \omega \rangle \in G(\mathcal{P})$ with $\ell(p) > n$ we have f_n as the first coordinate of p_n. Now let $\langle F_\tau | \tau < \kappa^{++} \rangle$ be the subsequence of $\langle F'_\tau | \tau < \kappa^{++} \rangle$ consisting of all F'_τ which are not in V. [3]

[2] See [2] or [1] for more information on such sequences.
[3] By arguments of [2] or [1] this is a scale.

Similarly, for (2), take the increasing (under the inclusion) enumeration $\langle Y_\tau | \tau < \kappa^{+3}\rangle$ of the chain of models of cardinality κ^{++} given by $G(\mathcal{P}')$. Define a scale of functions $\langle H_\tau \mid \tau < \kappa^{+3}\rangle$ in the product $\prod_{n<\omega} \rho_n^{+n+2}$ as follows:

$$H'_\tau(n) = g_n(Y_\tau), \text{ if } g_n(Y_\tau) < \rho_n^{+n+2}$$

and

$$H'_\tau(n) = 0, \text{ otherwise},$$

where for some $p = \langle p_k | k < \omega\rangle \in G(\mathcal{P})$ with $\ell(p) > n$ we have g_n as the second coordinate of p_n. Let $\langle H_\tau \mid \tau < \kappa^{+3}\rangle$ be the subsequence of $\langle H'_\tau \mid \tau < \kappa^{+3}\rangle$ consisting of all H'_τ's which are not in V. [4] The scale $\langle H_\tau \mid \tau < \kappa^{+3}\rangle$ in $\prod_{n<\omega} \rho_n^{+n+2}/$ finite satisfies the property $(*)$ by the construction.

Let us turn to (3) which requires a more delicate analyses of the forcing $\langle \mathcal{P}, \to\rangle$. We deal with

$$\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+1}/\text{ finite }).$$

The rest of cases are similar or just standard. The crucial observation here is that given $\langle\langle a_n, A_n, f_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle \in Q_{n0}$, for some $n < \omega$, it is impossible to change $\mathrm{rng}(b_n)[\nu] \restriction \kappa_n^{+n+1}$ by passing to an equivalent condition, for any $\nu \in A_n$. Just the definition 4.3.11(4(b)v) explicitly requires this.
This means, in particular that

$$\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+1}/\text{ finite }) = \mathrm{cof}(\prod_{n<\omega} \kappa_n^{+n+1}/\text{ finite }),$$

where the connection is provided by $\underset{\sim}{b}_n$'s. But note that the cofinality of the last product is κ^+, since every function there can be bounded by an old function. So we are done.
□

4.6 Dropping cofinalities-gap 3 with infinite repetitions

We continue here to study dropping cofinalities.
Let as before,

$\lambda_0 < \kappa_0 < \lambda_1 < \kappa_1 < ... < \lambda_n < \kappa_n < ...(n < \omega)$ be an increasing sequence of cardinals with a limit κ.

Assume the following:

[4] Again, by arguments of [2] or [1] this is a scale.

- κ_n is κ_n^{+n+2} - strong as witnessed by an extender E_{κ_n}, for every $n < \omega$
- λ_n is λ_n^{+n+2} - strong as witnessed by an extender E_{λ_n}, for every $n < \omega$

Our aim will be to make $2^\kappa = \kappa^{+3}$, but so that for each $n < \omega$ the cofinality over κ_n that corresponds to κ^{++} may drop down to each of λ_m's with $m \leq n$. In particular we will allow a drop to λ_0 at each level $n < \omega$.
Recall that in the previous section, the drop down from κ_n was only to λ_n. By Theorem 4.5.1, there was a scale $\langle H_\tau \mid \tau < \kappa^{+3} \rangle$
in $\prod_{n<\omega} \rho_n^{+n+2}/$ finite with the following special property:
(*) for every $\tau < \kappa^{+3}$,

1. if $\text{cof}(\tau) = \kappa^{++}$, then H_τ is an exact upper bound of $\langle H_\mu \mid \mu < \tau \rangle$ and for all but finitely many $n < \omega$, $\text{cof}(H_\tau(n)) = \nu_n^{+n+2}$;
2. if $\text{cof}(\tau) < \kappa^{++}$, then for all but finitely many $n < \omega$, $\text{cof}(H_\tau(n)) < \nu_n^{+n+2}$.

Where ν_n's and ρ_n's are indiscernibles which correspond to the normal measures of extenders E_{λ_n}'s and E_{κ_n}'s respectively.
Here we would like to allow more freedom and to produce a scale $\langle H^*_\tau \mid \tau < \kappa^{+3} \rangle$
in $\prod_{n<\omega} \rho_n^{+n+2}/$ finite such that
(**) for every $\tau < \kappa^{+3}$,

1. if $\text{cof}(\tau) = \kappa^{++}$, then H^*_τ is an exact upper bound of $\langle H^*_\mu \mid \mu < \tau \rangle$ and for all but finitely many $n < \omega$, $\text{cof}(H^*_\tau(n)) = \nu_m^{+m+2}$, for some $m \leq n$;
2. if $\text{cof}(\tau) < \kappa^{++}$, then for all but finitely many $n < \omega$, $\text{cof}(H^*_\tau(n)) < \nu_n^{+n+2}$;
3. for every converging to infinity sequence $\langle m_k \mid k < \omega \rangle$ (not necessary increasing), there are unboundedly many $\tau < \kappa^{+3}$ of cofinality κ^{++} such that for all but finitely many $n < \omega$, $\text{cof}(H^*_\tau(n)) = \nu_{m_n}^{+m_n+2}$.

It is a bit simpler probably to consider the setting with only a drop to λ_0 occurs infinitely many times, i.e. for every $m > 0$ a drop to λ_m occurs only at κ_m. Then λ_0 will be used infinitely many times and all the rest only once. It will be possible then to make a non-direct extension at λ_0 and this will bring the situation basically to the usual dropping cofinality forcing.

If each of λ_n's appears infinitely many times, then the previous trick of taking non-direct extension over λ_0 will not work. Just we cannot take non-direct extensions at infinitely many places.

All λ_n's will correspond to κ^{++} and κ_n's to κ^{+3}. More precisely indiscernibles for λ_n^{+n+2}'s and for κ_n^{+n+2}'s will correspond to κ^{++} and to κ^{+3} respectively.

It is possible, using the same method, to replace n dropping points $\langle \lambda_m \mid m \leq n \rangle$ for κ_n with any finite number. If one likes to replace it with infinitely many, i.e. some strong enough cardinals $\langle \lambda_{nm} \mid m < \omega \rangle$ additional assumption are needed on $\Lambda_n := \bigcup_{m<\omega} \lambda_{nm}$ even if $\kappa_n < \lambda_{n+1,0}$. Just otherwise the indiscernibles for λ_{nm}^{+n+2}'s $(m < \omega)$ will correspond to Λ_n^+ and Λ_n^+'s correspond to κ^+ and everything breaks down. The problem here is that once a non-direct extension is taken over κ_n, then $\langle \lambda_{nm} \mid m < \omega \rangle$ starts to be isolated from the part above κ_n. Some connection of this sort is needed for example in order to show the Prikry condition. Existence of strong enough cardinals between Λ_n and κ_{n+1} allows to generalize the present arguments to such situation. This will not be elaborated here.

Let us turn now to the forcing notions.

Force first with the preparation forcing $\mathcal{P}'(\kappa^{+3})$ followed by $\mathcal{P}'(\kappa^{++})$ of Chapter 1. It is possible instead of forcing with $\mathcal{P}'(\kappa^{++})$ just to take the projection of a generic for $\mathcal{P}'(\kappa^{+3})$ to κ^{++}, i.e. intersect each model there with κ^{++}.
Let $G(\kappa^{+3})$ and $G(\kappa^{++})$ be the corresponding generic subsets. Work in $V[G(\kappa^{+3}), G(\kappa^{++})]$. We shall redefine the forcing of Section 4.3.

The first small change (actually relevant to all short extenders forcings) will be as follows. Given a condition $p = \langle p_n | n < \omega \rangle$, $p_n = \langle a_n, A_n, f_n \rangle$, for each $n \geq \ell(p)$. We require that

1. for each $n < \ell(p)$, if $X \in \mathrm{dom}(p_n)$, then starting with some $m > n$ we have $X \in \mathrm{dom}(a_m)$.

2. for each $n \geq \ell(p)$, if $X \in \mathrm{dom}(f_n)$, then starting with some $m > n$ we have $X \in \mathrm{dom}(a_m)$.

This change prevents appearance of old ω sequences among those produced by a generic object.
Recall that the cardinalities of $\mathrm{dom}(p_n)$ and $\mathrm{dom}(f_n)$ are at most κ, as well as the cardinality of $\bigcup_{\ell(p) \leq k < \omega} \mathrm{dom}(a_k)$. So it is always possible to spread κ-many things among $\mathrm{dom}(a_k)$'s.

Fix some $n < \omega$.

Definition 4.6.1 Let Q_{n0} be the set consisting of pairs

$$\langle \langle \langle a_m, A_m, f_m \rangle \mid m \leq n \rangle, \langle \underset{\sim}{b}, \underset{\sim}{b}', \neg \underset{\sim}{b}', \underset{\sim}{B}, g \rangle \rangle$$

so that:

for each $m \leq n$

1. f_m is partial function from κ^{+2} to λ_m of cardinality at most κ.

2. There is a suitable generic structure $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ for $G(\kappa^{++})$ of cardinality less than λ_m such that a_m is an order preserving from $X \cup Y$ (actually here for κ^+ we have that X itself consists of ordinals and Y is unneeded) into λ_m^{+m+2}.

3. $a_m(\max(X)) = \max(\mathrm{rng}(a_m))$ is above all the elements of $\mathrm{rng}(a_m)$ in the order of the extender E_{λ_m}.

4. $\mathrm{dom}(a_m) \cap \mathrm{dom}(f_m) = \emptyset$.

5. $A_m \in E_{\lambda_m, a_m(\max(X))}$

6. $\min(A_m) > |\operatorname{dom}(a_m)|$

7. for every $\alpha, \beta, \gamma \in \operatorname{rng}(a_m)$ we have

$$\alpha \geq_{E_{\lambda_m}} \beta \geq_{E_{\lambda_m}} \gamma \quad \text{implies}$$
$$\pi_{\lambda_m,\alpha,\gamma}(\rho) = \pi_{\lambda_m,\beta,\gamma}(\pi_{\lambda_m,\alpha,\beta}(\rho))$$

for every $\rho \in \pi``_{\lambda_m, \max \operatorname{rng}(a_m),\alpha}(A_m)$.

8. For every $\alpha > \beta$ in $\operatorname{rng}(a_m)$ and $\rho \in A_m$

$$\pi_{\lambda_m, \max(\operatorname{rng}(a_m)),\alpha}(\rho) > \pi_{\lambda_m, \max(\operatorname{rng}(a_m)),\beta}(\rho).$$

Let us turn now to the second component of a condition, i.e. to $\langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle$. Main differences (and complications) appear in this part-namely in the assignment function $\underset{\sim}{b}$.

9. $\underset{\sim}{b}$ is a name, depending on $\langle\langle a_m, A_m\rangle \mid m \leq n\rangle$, of a partial function of cardinality less or equal than $\bigcup_{m\leq n} \lambda_m$.

   The following conditions are satisfied:

   (a) (Domain) There is a generic suitable structure $\langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ for $G(\kappa^{+3})$ of cardinality less than or equal than $\bigcup_{m\leq n} \lambda_m$ such that

      i. $\operatorname{dom}(\underset{\sim}{b}) = X \cup Y$;

      ii. for every $Z \in X$ there is $m \leq n$ such that $Z \cap \kappa^{++}$ is in $\operatorname{dom}(a_m)$.

   (b) (Maximal model) $\max(X)$ is a maximal (under $\in$) model in $\operatorname{dom}(\underset{\sim}{b})$
   Here its image will not necessary be the maximal model of the range. The complication is due to the fact that now we will have models of different sizes in the range of b and $\max(X)$ may correspond to a model of a size λ_m for some $m < n$. In this case the following situation will be allowed:

$$Z_1 \in Z_2 \in Z \text{ and } Z_1 \notin Z.$$

   The choice of a size of $\underset{\sim}{b}(A)$, for $A \in \operatorname{dom}(\underset{\sim}{b}) \cap X$, will be determined by $A \cap \kappa^{++}$. Thus we require that there is a splitting of κ^{++} into intervals (determined by the condition) which rule the correspondence of sizes.

(c) (Splitting into intervals) There is a disjoint partition of κ^{++} into intervals $\langle s_i \mid i < \delta \rangle$, for some $\delta < \lambda_n$, such that

- for each $i < \delta$ there is a unique $m(i) \leq n$ which is the index of $\lambda_{m(i)}$ corresponding to size of models in the sense of the next condition;
- let $A \in \mathrm{dom}(\underset{\sim}{b})$ be a model of cardinality κ^+ on the central line. Denote by $i(A)$ the unique $i < \delta$ such that $A \cap \kappa^{++}$ belongs to s_i. $\underset{\sim}{b}(A)$ will be a model of size less than $\lambda_{m(i(A))}$.

(d) (More on the maximal model) The name $\underset{\sim}{b}(A)$ can depend on all $\langle \lambda_k \mid k \leq n \rangle$. But we require that the image of the maximal model $\max(X)$ depends only on $\lambda_{m(i(A^{0\kappa^+}))}$. This is needed for the chain condition argument.

Note that $\underset{\sim}{b}(\max(X))$ need not even contain every model of $\mathrm{rng}(\underset{\sim}{b})$ even of cardinality $\lambda_{m(i(\max(X)))}$.

On the other hand models in the range (of a bigger size) can refer to ones of a smaller sizes and then the last stop to be names for λ_i's corresponding to their size.

Suppose for simplicity that $n = 1$. Then we have only λ_0 and λ_1. So models over κ_1 can have sizes $< \lambda_0$ and $< \lambda_1$.

A non-pure extension over λ_0 results in a condition over κ_1 in which b acts in order preserving (i.e. $\in$ preserving) fashion only over the intervals corresponding to λ_1. The behavior on the intervals which correspond to λ_0 is like the function g of the condition, i.e. no order preservation is required. The number of models then is not κ like in g, but rather $< \lambda_1$.

If a non-pure extension is made over λ_1, then we make such extension also over λ_0, κ_0 and κ_1.

(e) (Weak order preserving condition) Let A, B be in $\mathrm{dom}(\underset{\sim}{b})$. If $A \in B$, then $\underset{\sim}{b}(A) \in \underset{\sim}{b}(B)$ (forced by the empty condition) or there is $C \in B, C \in \mathrm{dom}(\underset{\sim}{b})$, $C \cap \kappa^{++}$ in an interval corresponding to λ_1 and $\underset{\sim}{b}(A) \in \underset{\sim}{b}(C) \in \underset{\sim}{b}(B)$ (forced by the empty condition).

(f) Suppose that $Z \in \mathrm{dom}(\underset{\sim}{b})$ and $Z \cap \kappa^{++}$ in the interval corresponding to λ_1. Then $\underset{\sim}{b}(Z)$ depends only on λ_1, i.e. it is a name in the forcing over λ_1 without the forcing over λ_0.

(g) (Inclusion condition 1)

Suppose that $\max(X) \cap \kappa^{++}$ is in the interval corresponding to λ_1.
Let $\eta, \eta' \in A_1, \eta < \eta'$. Then

- $\underset{\sim}{b}(\max(X))[\eta] \in \underset{\sim}{b}(\max(X))[\eta']$,
- if $Z \in X \cap C(\max(X))$, $Z \cap \kappa^{++}$ in the interval corresponding to λ_1 and

$$\pi_{\lambda_n, \max \operatorname{rng}(a), a(Z \cap \kappa^{++})}(\eta') > \eta,$$

then either

$$\underset{\sim}{b}(\max(X))[\eta] \in \underset{\sim}{b}(Z)[\eta']$$

or

the k-type realized by $\underset{\sim}{b}(\max(X))[\eta] \cap H(\chi^{+k})$ is in $\underset{\sim}{b}(Z)[\eta']$, where $k < \omega$ is the least such that $\underset{\sim}{b}(Z)[\eta'] \subseteq H(\chi^{+k+1})$.

The same holds over any element of $\underset{\sim}{b}(Z)[\eta']$, i.e. $tp_k(z, \underset{\sim}{b}(\max(X))[\eta] \cap H(\chi^{+k})) \in \underset{\sim}{b}(Z)[\eta']$, for any $z \in \underset{\sim}{b}(Z)[\eta']$.

We require in addition that this $k > 2$.

Let us allow the above also if $\underset{\sim}{b}(Z)[\eta'] \subseteq H(\chi^{+\omega})$. In this case we take k to be any natural number above 2 and require that once we go up to the higher levels then corresponding k's increase (with n).

We cannot in general require only that

$$\underset{\sim}{b}(\max(X))[\eta] \in \underset{\sim}{b}(Z)[\eta']$$

since the sequence C of a new generic suitable structure may go not through the old maximal model. But still having the type inside Z will suffice.

Note that given $\eta' \in A_1$ the number of possibilities for $\eta \in \eta' \cap A_1$ is bounded by $(\eta'^0)^{+n+1}$, as $\eta' < (\eta'^0)^{+n+2}$.

(h) If $Z \in X$ and $Z \cap \kappa^{++}$ is in the interval corresponding to λ_0, then either

i. $\underset{\sim}{b}(Z)$ does not depend on λ_1.
In this case we require that there is no models corresponding to λ_1 below Z.
Or

ii. there is $Z' \in X \cap C(Z)$ with $Z' \cap \kappa^{++}$ in the interval corresponding to λ_1 such that the empty condition forces "$\underset{\sim}{b}(Z') \in \underset{\sim}{b}(Z)$". Moreover, if $\rho_1, \rho_2 \in A_1$ and $\pi_{\lambda_1, \max(\operatorname{rng}(a_1)), a(Z' \cap \kappa^{++})}(\rho_1) = \pi_{\lambda_1, \max(\operatorname{rng}(a_1)), a(Z' \cap \kappa^{++})}(\rho_2)$, then for every $\nu \in A_0$ we have $\underset{\sim}{b}(Z)[\nu, \rho_1] = \underset{\sim}{b}(Z)[\nu, \rho_2]$.

The intuition behind this condition is that the number of types inside a model of cardinality below λ_0 is too small to include all the types that may be generated by picking different indiscernibles for λ_1. In order to insure the chain condition will put together two conditions say by realizing the type of one with smaller index (in the increasing chain of κ^{++}-many) inside an other with a bigger index. This done as follows: first a model from the central line above the common part which includes the low condition and below the upper one (except the kernel)should be added. If Z is inside it, then it is in the common part and the condition below takes care of such situation. If Z is not inside, then such model is in Z (since they are on the central line). In this situation, it is enough to preserve on the side of the images the order ($\in$) only in a weak sense (see the corresponding condition above). This is easy since the cardinality of the image of Z is above λ_0.
If there were not only λ_0 and λ_1, rather an infinite sequence, then still it possible to play with this *weak* order preservation. Thus once we deal with a model of certain cardinality (i.e. all its interpretations in given cardinality) take a model of a bigger cardinality which includes all such interpretations, and as a name depends only on indiscernibles corresponding to its cardinality. Now we are able to add a model of smaller cardinality as a name which depends only on the last model of this bigger cardinality.

(i) (Inclusion condition 0)

Suppose that $\max(X) \cap \kappa^{++}$ in the interval corresponding to λ_0.
Let $\eta, \eta' \in A_0, \eta < \eta', \xi \in A_1$. Then

- $\underset{\sim}{b}(\max(X))[\eta, \xi] \in \underset{\sim}{b}(\max(X))[\eta', \xi]$,
- if $Z \in X \cap C(\max(X))$, $Z \cap \kappa^{++}$ in the interval corresponding to λ_0, the maximal model in $C(\max(X))$ which corresponds to λ_1 belongs to Z and

$$\pi_{\lambda_n, \max \operatorname{rng}(a_0), a(Z\cap\kappa^{++})}(\eta') > \eta,$$

then either

$$\underset{\sim}{b}(\max(X))[\eta, \xi] \in \underset{\sim}{b}(Z)[\eta', \xi]$$

or
the k-type realized by $\underset{\sim}{b}(\max(X))[\eta, \xi] \cap H(\chi^{+k})$ is in $\underset{\sim}{b}(Z)[\eta', \xi]$, where $k < \omega$ is the least such that $\underset{\sim}{b}(Z)[\eta', \xi] \subseteq H(\chi^{+k+1})$.
The same holds over any element of $\underset{\sim}{b}(Z)[\eta', \xi]$, i.e. $tp_k(z, \underset{\sim}{b}(\max(X))[\eta, \xi] \cap$

$H(\chi^{+k})) \in \underset{\sim}{b}(Z)[\eta', \xi]$, for any $z \in \underset{\sim}{b}(Z)[\eta', \xi]$.

We require in addition that this $k > 2$.

Let us allow the above also if $\underset{\sim}{b}(Z)[\eta', \xi] \subseteq H(\chi^{+\omega})$. In this case we take k to be any natural number above 2 and require that once we go up to the higher levels then corresponding k's increase (with n).

(j) $Y \cap \mathrm{dom}(g) = \emptyset$.

(k) For every $\alpha \in Y$ and $\eta \in A_1, \xi \in A_0$

i. $\underset{\sim}{b}[\xi, \eta](\alpha)$ is a model of cardinality κ_n^{+n+1},

ii. $\kappa_n^{+n+1} \subseteq \underset{\sim}{b}[\xi, \eta](\alpha)$,

iii. $\mathrm{cof}(\sup(\underset{\sim}{b}[\xi, \eta](\alpha) \cap \kappa_n^{+n+2})) = (\eta^0)^{+n+2}$ or it is $(\xi^0)^{+n+2}$.

Note that all the cardinals $\kappa_n, ..., \kappa_n^{+n+1}$ will correspond here to κ^+. So, we need to drop down to the indiscernibles $(\xi^0)^{+n+2}$ or $(\eta^0)^{+n+2}$ in order to get to κ^{++}.

Next, let us address $\underset{\sim}{b}'$. First we explain the purpose of introducing it.

Thus let $\alpha \in Y \cap \mathrm{dom}(\underset{\sim}{b})$, $\mathrm{cof}(\alpha) = \kappa^{++}$ and $\underset{\sim}{b}(\alpha)$ is forced to have a cofinality which corresponds to the indiscernible for λ_0^{+n+2} (similarly, it may be a model from a part of the partition corresponding to λ_0). In such a case $\mathrm{dom}(\underset{\sim}{b})$ may contain increasing sequences of models of each size κ^+ with unions bounded in α. On the other hand the images are unbounded in $\underset{\sim}{b}(\alpha)$. Note that the order preserving implies that $\mathrm{dom}(\underset{\sim}{b})$ may contain at most one of such sequences. Now, it is necessary for the chain condition of the forcing to be able to put together two conditions both with such α inside but having in their b's different sequences of the type above.

Also suppose that a non-pure extension was made at λ_0 and the central piste reflects into α. We need to allow a possibility of being unbounded in α, in order to keep λ_1 closure of the forcing. Assume that we have a condition of this type (i.e. models corresponding to λ_1 are unbounded in α) and we like to extend the central piste. How to reflect an extension to α? In this case we just move the previous cofinal in α sequence (or its part that interferes with the new one) to b' and replace it by a new one.

So the role of b' is to allow to keep such different sequences inside a single condition. Further in the definition of extensions we will allow to change b by replacing the sequence to α which is inside b by one from b'.

At the next level $n + 1$ all models which appear in b' will be required to appear in dom(b) of $n+1$-th level, and so will have different images.

(l) $\mathrm{dom}(\underset{\sim}{b}') \subseteq X \cup Y$.

i. If $\alpha \in \mathrm{dom}(\underset{\sim}{b}')$, then for each $\xi \in A_0, \eta \in A_1$ we have $\mathrm{cof}(\sup(\underset{\sim}{b}[\xi,\eta](\alpha) \cap \kappa_n^{+n+2})) = (\xi^0)^{+n+2}$.

ii. If $Z \in \mathrm{dom}(\underset{\sim}{b}')$, then for each $\xi \in A_0, \eta \in A_1$ we have $\mathrm{cof}(\sup(\underset{\sim}{b}[\xi,\eta](Z) \cap \kappa_n^{+n+2})) < (\xi^0)^{+n+2}$.

iii. For every $w \in \mathrm{dom}(\underset{\sim}{b}')$ (an ordinal or a model),

A. $\underset{\sim}{b}'(w)$ is a function with domain a tree of sequences of models each of them belongs to $(X \cup Y) \cap w$.

B. $\mathrm{rng}(\underset{\sim}{b}'(w))$ is a tree of models over κ_n with the set of their sup's unbounded in $\underset{\sim}{b}(w)$.

C. $\mathrm{dom}(\underset{\sim}{b}) \cap \bigcup_{w \in \mathrm{dom}(\underset{\sim}{b}')} \mathrm{dom}(\underset{\sim}{b}'(w)) = \emptyset$.
This condition insures that a same model does not correspond to different ones via b together with b'.

D. If $\underset{\sim}{b} \restriction w$ is replaced by one of branches of $\underset{\sim}{b}'(w)$ (i.e. if t is such a branch, then we take the last model Z of the central line of $\mathrm{dom}(\underset{\sim}{b} \restriction w)$ which is below the first model of the central line of t and replace $\underset{\sim}{b} \restriction w$ by $(\underset{\sim}{b} \restriction Z)^\frown t$), then the changed b will share the requirements on b stated above.

E. Suppose that w is not limit of elements of dom(b). Let w^* be its immediate predecessor. Then for every $\xi, \xi' \in A_0, \eta, \eta' \in A_1$ with $\xi < \xi'$ and $\eta \leq \eta'$ we require the following:
if $\underset{\sim}{b}(w^*)[\xi,\eta] = \underset{\sim}{b}(w^*)[\xi',\eta']$, then every $t \in \mathrm{dom}(\underset{\sim}{b}'(w)[\xi,\eta])$ we have $t \in \underset{\sim}{b}'(w)[\xi',\eta']$ and $(\underset{\sim}{b}'(w)[\xi,\eta])(t) = (\underset{\sim}{b}'(w)[\xi',\eta'])(t)$.
If w is the least element of dom(b), then every $t \in \mathrm{dom}(\underset{\sim}{b}'(w)[\xi,\eta])$ we have $t \in \underset{\sim}{b}'(w)[\xi',\eta']$ and $(\underset{\sim}{b}'(w)[\xi,\eta])(t) = (\underset{\sim}{b}'(w)[\xi',\eta'])(t)$.
Similar if w' is any element dom(b), $t \in \mathrm{dom}(\underset{\sim}{b}'(w')[\xi,\eta])$ and $w \in w' \cap \mathrm{dom}(b)$ has an immediate predecessor w^* (or just the least element in which case w^* is unneeded) such that $\underset{\sim}{b}(w^*)[\xi,\eta] = \underset{\sim}{b}(w^*)[\xi',\eta']$, then $t \in \underset{\sim}{b}'(w')[\xi',\eta']$ and $(\underset{\sim}{b}'(w')[\xi,\eta])(t \restriction \sup(w)) = (\underset{\sim}{b}'(w')[\xi',\eta'])(t \restriction \sup(w))$.

This property allows us to prove κ^{++}-c.c. of the final forcing. Thus we arrange first the situation where $\xi < \xi'$ and $\eta \leq \eta'$ just as in the ordinary gap-2 forcing. Now over κ_n's run the argument of 4.4. The crucial point in it is to replace the value which the condition with a bigger index obtain with ξ, and in the present situation with ξ, η, by the condition with a smaller index. This should be done carefully over a common part of this conditions. The item above insures that the relevant common part does not change once the condition with a bigger index is evaluated according to ξ', η'.

Let us describe an additional relevant situation. It may occur as well proving the chain condition once at one of the components say in λ_0 a non-direct extension was made.

Thus over κ_n we may have the following two condition:

both consist of an initial segment corresponding (via the splitting) to λ_0 having last models Z and Z' correspondingly and models M and M' above for λ_1. Z and Z' are different and say $Z \in Z'$, the same with M and M' (i.e. $M \in M'$). Assume that they have a common part which is over κ_n is unbounded in both Z and Z'. Now a non-direct extension was made over λ_0, so we no way to put Z and Z' together. Still there is a need to combine two such conditions together without taking a non-direct extension over λ_1 or κ_n. The way of doing this will be as follows. We either move Z' to b' (of the extension of the second (')-condition) and replace it in b by Z together with M, where M is added as already was described in the case of only direct extensions. Or, alternatively, we keep Z' and add Z with M to b'.

(m) $\neg \underset{\sim}{b}'$ is a subset of $X \cup Y$ such that for each $\xi \in A_0, \eta \in A_1$ we have

i. $\mathrm{dom}(\underset{\sim}{b}'[\xi, \eta]) \cap \neg \underset{\sim}{b}'[\xi, \eta] = \emptyset$,

ii. for any $\xi' \in A_0, \eta' \in A_1, \xi' > \xi, \eta' \geq \eta$ and $t \in \mathrm{dom}(\underset{\sim}{b}'[\xi', \eta']) \setminus \mathrm{dom}(\underset{\sim}{b}'[\xi, \eta])$ we require $t \in \neg \underset{\sim}{b}'[\xi, \eta]$.

The reason for introducing $\neg \underset{\sim}{b}'$ is to insure the Prikry condition of the forcing. Thus once running the standard argument for showing the Prikry condition we take (at each level) non direct extensions and then combine them together into a direct one. Without the above requirement it is possible to extend a given forcing condition non directly by picking some $\xi \in A_0, \eta \in A_1$ and then to extend further by adding to $\mathrm{dom}(\underset{\sim}{b}'[\xi, \eta])$ some t which belongs to $\mathrm{dom}(\underset{\sim}{b}'[\xi', \eta'])$ for some $\xi' \in A_0, \eta' \in A_1, \xi' > \xi, \eta' \geq \eta$. Such addition may now contradict 9(l)iiiE.

(n) g is the usual one (i.e. as in all short extenders forcings), it is a function from κ^{+3} to κ_n of cardinality at most κ.

Q_{n1} is defined in the usual fashion, only we have here more functions.

Definition 4.6.2 Q_{n1} consists of sequences $\langle\langle f_m \mid m \leq n\rangle, g\rangle$ such that

1. for every $m \leq n$, f_m is a partial function from κ^{++} to λ_m of cardinality at most κ,

2. g is a partial function from κ^{+3} to κ_n of cardinality at most κ.

We have here intermediate non direct extensions between Q_{n0} and Q_{n1}. Just it is possible to take a non direct extension at each of λ_m's ($m \leq n$).

Turn now to the definition of the main forcing.

Definition 4.6.3 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega\rangle$ so that

1. for every $n < \omega$, $p_n \in Q_n$

2. there is $\ell(p) < \omega$ such that

   (a) for every $n < \ell(p)$, $p_n \in Q_{n1}$

   (b) for every $n \geq \ell(p)$, $p_n = \langle\langle\langle a_{nm}, A_{nm}, f_{nm}\rangle \mid m \leq n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{b}'_n, \neg\underset{\sim}{b}'_n, \underset{\sim}{B}_n, g_n\rangle\rangle \in Q_{n0}$

   (c) for every $n, n' \geq \ell(p)$,$m \leq \min(n, n')$ $\max(\mathrm{dom}(a_{nm})) = \max(\mathrm{dom}(a_{n'm}))$ and $\max(\mathrm{dom}(\underset{\sim}{b}_n)) = \max(\mathrm{dom}(\underset{\sim}{b}_{n'}))$

   (d) for every $n \geq n' \geq \ell(p)$, for every $m \leq n, m' \leq n'$, if $m' \leq m$, then $\mathrm{dom}(a_{n'm'}) \subseteq \mathrm{dom}(a_{nm})$

   (e) for every n, m, $\ell(p) \leq n < \omega, m \leq n$, and $X \in \mathrm{dom}(a_{nm})$ the following holds: for each $k < \omega$ the set

   $$\{(n', m') \in \omega^2 \mid \neg(a_{n'm'}(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$$

   is finite.

   Turn now to $\underset{\sim}{b}_n$'s. Assume that $n \geq \ell(p)$.

   (f) For every $A \in \mathrm{dom}(\underset{\sim}{b}_n) \cup \mathrm{dom}(\underset{\sim}{b}'_n)$ we require that the cardinality of $\underset{\sim}{b}_{n+1}(A)$ (or of $\underset{\sim}{b}'_{n+1}(A)$, if $A \in \mathrm{dom}(\underset{\sim}{b}'_n)$) is above those of $\underset{\sim}{b}_n(A)$ (or those of $\underset{\sim}{b}'_n(A)$) respectively).

(g) If α is as above in (9l) at the level n, then the sequences (i.e. models, ordinals of their domains) of $\underset{\sim}{b}'_n$ for α are incorporated *together* (i.e. in order preserving fashion) inside $\underset{\sim}{b}_{n+1}$ or alternatively inside $\underset{\sim}{b}'_{n+1}$.

The rest of the requirements are similar to those of 4.4 with obvious adaptations.

Let us define the order $\leq^*$.

Definition 4.6.4 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$. Set $q \leq^* p$ iff

1. $\ell(q) = \ell(p)$,
2. for every $n < \ell(p), q_n \leq_{Q_{n1}} p_n$,
3. for every $n, \ell(p) \leq n < \omega$, the following holds:
   $q_n = \langle \langle \langle a(q)_{nm}, A(q)_{nm}, f(q)_{nm} \rangle \mid m \leq n \rangle, \langle \underset{\sim}{b}(q)_n, \underset{\sim}{b}(q)'_n, \neg \underset{\sim}{b}(q)'_n, \underset{\sim}{B}(q)_n, g(q)_n \rangle \rangle \leq_{Q_{n0}}$
   $p_n = \langle \langle \langle a(p)_{nm}, A(p)_{nm}, f(p)_{nm} \rangle \mid m \leq n \rangle, \langle \underset{\sim}{b}(p)_n, \underset{\sim}{b}(p)'_n, \neg \underset{\sim}{b}(p)'_n, \underset{\sim}{B}(p)_n, g(p)_n \rangle \rangle$, where $\leq_{Q_{n0}}$ is defined in the usual fashion with two additions:
   - the partition used in q_n can be refined (we allow to combine intervals together). Note that it does not case problems with the chain condition of the forcing since the number of elements used to define partitions is less than λ_n and we would like to have κ^{++}-c.c.
   - $\underset{\sim}{b}(p)'_n$ extends $\underset{\sim}{b}(q)'_n$,
   - $\neg \underset{\sim}{b}(p)'_n$ extends $\neg \underset{\sim}{b}(q)'_n$

The orders $\leq$ and $\rightarrow$ on $\mathcal{P}$ are defined now as in 4.4.
Let us turn to the main issue- the chain condition.

Lemma 4.6.5 *The forcing $\langle \mathcal{P}, \rightarrow \rangle$ satisfies κ^{++}-c.c.*

Proof. The proof mainly repeats the corresponding proof in 4.4. An additional point used in order to show the compatibility was explained in 4.6.1(9(l)iiiE).
□

Force with $\langle \mathcal{P}, \rightarrow \rangle$. Let $G(\mathcal{P})$ be a generic set. By the lemmas above no cardinals are collapsed. Let $\langle \nu_n \mid n < \omega \rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\lambda_n} \mid n < \omega \rangle$ and $\langle \rho_n \mid n < \omega \rangle$ those for $\langle E_{\kappa_n} \mid n < \omega \rangle$.

Then the following analog of 4.5.1 holds:

Theorem 4.6.6 *The following hold in $V[G(\mathcal{P}'(\theta)) * G(\mathcal{P})]$:*

(1) $\mathrm{cof}(\prod_{n<\omega} \nu_n^{+n+2}/ \textit{ finite }) = \kappa^{++}$.

(2) $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+2}/ \textit{ finite }) = \kappa^{+3}$.
Moreover, there is a scale $\langle H^*_\tau \mid \tau < \kappa^{+3}\rangle$ in $\prod_{n<\omega} \rho_n^{+n+2}/$ finite with the following special property:
$(**)$ *for every $\tau < \kappa^{+3}$,*

- *(a) if $\mathrm{cof}(\tau) = \kappa^{++}$, then H^*_τ is an exact upper bound of $\langle H^*_\mu \mid \mu < \tau\rangle$ and for all but finitely many $n < \omega$, $\mathrm{cof}(H^*_\tau(n)) = \nu_m^{+m+2}$, for some $m \leq n$;*
- *(b) if $\mathrm{cof}(\tau) < \kappa^{++}$, then for all but finitely many $n < \omega$, $\mathrm{cof}(H^*_\tau(n)) < \nu_n^{+n+2}$;*
- *(c) for every converging to infinity sequence $\langle m_k \mid k < \omega\rangle$ (not necessary increasing), there are unboundedly many $\tau < \kappa^{+3}$ of cofinality κ^{++} such that for all but finitely many $n < \omega$, $\mathrm{cof}(H^*_\tau(n)) = \nu_{m_n}^{+m_n+2}$.*

(3) For every unbounded subset a of κ consisting of regular cardinals and disjoint to both $\{\nu_n^{+n+2} \mid n < \omega\}$ and $\{\rho_n^{+n+2} \mid n < \omega\}$, for every ultrafilter D over a which includes all co-bounded subsets of κ we have

$$\mathrm{cof}(\prod a/D) = \kappa^+.$$

Chapter 5

Gaps from optimal assumptions

Our aim here will be to present constructions in which the power of a singular cardinal may be arbitrary large starting from the weakest possible assumption:

$$\exists\kappa\forall n < \omega\exists\alpha < \kappa(o(\alpha) = \alpha^{+n}).$$

5.1 $\aleph_1$–gap and infinitely many drops in cofinalities

5.1.1 Preliminary settings

Let κ be a singular cardinal of cofinality ω such that for each $\gamma < \kappa$ and $n < \omega$ there is $\alpha, \gamma < \alpha < \kappa$, such that $o(\alpha) = \alpha^{+n}$. We fix a sequence of cardinals $\kappa_0 < \kappa_1 < ... < \kappa_n < ..., n < \omega$ so that

- $\bigcup_{n<\omega} \kappa_n = \kappa$
- for every $n < \omega$, κ_n is κ_n^{+n+2} - strong as witnessed by an extender E_{κ_n}
- for every $n < \omega$, the normal measure of E_{κ_n} concentrates on

  τ's which are $\tau^{+n+2} + \omega_1$ - strong as witnessed by a coherent sequence of extenders $\langle E_{\tau\xi} | \xi < \omega_1\rangle$

Fix also an increasing sequence $\langle\lambda_n | n < \omega\rangle$ such that

- $\lambda_0 < \kappa_0$
- $\kappa_{n-1} < \lambda_n < \kappa_n$, for every $n, 0 < n < \omega$
- for every $n < \omega$, λ_n is λ_n^{+n+2} - strong as witnessed by an extender E_{λ_n}

Our aim will be to make $2^\kappa = \kappa^{+\omega_1+1}$. There is nothing special here in choosing ω_1. The same construction will work if we replace everywhere ω_1 by an ordinal $\eta, \eta < \lambda_0$. Actually, replacing the original λ_0 by a bigger one, we can deal similar with any $\eta < \kappa$. Note that for finite η's our assumption is not anymore optimal and for countable η's the result was already known, see the detailed discussion in [5].

Force first with the preparation forcing $\mathcal{P}'(\kappa^{+\omega_1+1})$ of Chapter 2 with $\theta = \kappa^{+\omega_1+1}$. Then the main forcing will produce the following PCF structure. We assign to κ^{++} at a level n the indiscernible η_n^{+n+2}, where η_n is the indiscernible for the normal measure of the extender E_{λ_n}. The correspondence between regular cardinals in the interval $[\kappa^{+3}, \kappa^{+\omega_1+1}]$ will be as follows: we assign to $\kappa^{+\omega_1+1}$ at a level n the indiscernible ρ_n^{+n+2}, where ρ_n is the indiscernible for the normal measure of the extender E_{κ_n}. Let $\langle \rho_{n\alpha} | \alpha < \omega_1 \rangle$ be the Magidor sequence corresponding to the normal measures of E_{ρ_n} the one used in the extender based Magidor forcing (see [13]) to change cofinality of ρ_n to ω_1. For every $\alpha, 1 < \alpha < \omega_1$, we assign $\rho_{n\alpha}^{+n+2}$ to $\kappa^{+\alpha+1}$.
The role of λ_n's is to produce the first drop. It is possible to incorporate them into Magidor sequences as their first elements. It seems a bit more convenient to separate the first drop.

5.1.2 Projections

Let $p = \langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in [\kappa^+, \kappa^{+\omega_1+1}] \cap \text{Cardinals} \rangle \in \mathcal{P}'(\kappa^{+\omega_1+1})$. Suppose that η is a regular cardinal in $[\kappa^+, \kappa^{+\omega_1+1}]$. We would like to define $p \restriction \eta$ - *the restriction of* p *to* η. Thus if $A \in A^{1\tau}, \tau \in [\kappa^+, \eta] \cap \text{Cardinals}$, then set $A \restriction \eta = A \cap H(\eta^+)$. Define

$$A^{1\tau} \restriction \eta = \{A \restriction \eta \mid A \in A^{1\tau}\}, C^\tau \restriction \eta = \{A \restriction \eta \mid A \in C^\tau\},$$

for every $\tau \in [\kappa^+, \eta] \cap \text{Cardinals}$ and

$$p \restriction \eta = \langle \langle A^{0\tau} \restriction \eta, A^{1\tau} \restriction \eta, C^\tau \restriction \eta \rangle \mid \tau \in [\kappa^+, \eta] \cap \text{Cardinals} \rangle.$$

Lemma 5.1.1 *Suppose that* η *is a regular cardinal in* $[\kappa^+, \kappa^{+\omega_1+1}]$ *and* $p \in \mathcal{P}'(\kappa^{+\omega_1+1})$. *Then* $p \restriction \eta \in \mathcal{P}'(\eta)$.

Proof. Let $p = \langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in [\kappa^+, \kappa^{+\omega_1+1}] \cap \text{Cardinals} \rangle$. Note that if $A, B \prec H(\kappa^{+\omega_1+2}), A, B \supseteq \kappa^+, |A|, |B| \leq \eta$ and $A \in B$, then $A \restriction \eta \in B \restriction \eta$. Thus, if $\eta = \kappa^{+\omega_1+1}$, then $A \restriction \eta = A, B \restriction \eta = B$. If $\eta < \kappa^{+\omega_1+1}$, then $H(\eta^+) \in B$, which implies $A \cap H(\eta^+) \in B$, but $|A| \leq \eta$, hence $A \cap H(\eta^+) \in H(\eta^+)$.
This implies that $\langle C^\tau \restriction \eta \mid \tau \in [\kappa^+, \eta] \cap \text{Cardinals} \rangle \in \mathcal{P}'''(\eta)$, i.e. the restriction of central

lines of p satisfies the definition of central lines of $\mathcal{P}'(\eta)$, see Definition 2.2.1(Chapter 2). The rest follows now by induction on complexity of pistes.
$\square$

Note that if G is a generic subset of $\mathcal{P}'(\kappa^{+\omega_1+1})$, then $G \restriction \eta := \{p \restriction \eta \mid p \in G\}$ is never generic for $\mathcal{P}'(\eta)$, since it is possible to extend an arbitrary condition $q \in \mathcal{P}'(\eta)$ to one with a maximal model (say those of cardinality κ^+) which is not of the form $A \cap H(\eta^+)$ for any $A \prec H(\kappa^{+\omega_1+2})$.

It is possible under the same lines to deal with arbitrary regular θ instead of $\kappa^{+\omega_1+1}$. The following holds:

Lemma 5.1.2 *Let $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}'(\theta)$ and $\eta \in s$ is a regular cardinal. Then $p \restriction \eta \in \mathcal{P}'(\eta)$.*

Define now restrictions of suitable structures.
Let $\mathfrak{X} = \langle X, E, C, \in, \subseteq \rangle$ be a suitable structure, $p(\mathfrak{X}) = \langle\langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), C^\tau(\mathfrak{X}) \mid \tau \in s(\mathfrak{X})\rangle$ the corresponding condition in $\mathcal{P}'(\theta)$(see Definition 2.4.6)and η a regular cardinal in $s(\mathfrak{X})$. Set $\mathfrak{X} \restriction \eta$ to be the suitable structure generated by $p(\mathfrak{X}) \restriction \eta$.

5.2 Level n

Fix $G(\mathcal{P}'(\kappa^{+\omega_1+1}))$ be a generic subset of $\mathcal{P}'(\kappa^{+\omega_1+1})$.
Let $n < \omega$. We describe the forcing used at the level n of the construction.

Definition 5.2.1 Let Q_{n0} be the set consisting of pairs of triples $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, B, g\rangle\rangle$ so that:

1. f is partial function from κ^{+2} to λ_n of cardinality at most κ.

2. There is a suitable generic structure $\mathfrak{X} = \langle X, E, C, \in, \subseteq \rangle$ with the corresponding condition $p(\mathfrak{X}) = \langle\langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), C^\tau(\mathfrak{X}) \mid \tau \in s(\mathfrak{X})\rangle$ and $|\bigcup_{\tau \in s(\mathfrak{X})} A^{1\tau}(\mathfrak{X})| < \lambda_n$ (this is basically the total number of structures in $\mathfrak{X}$), such that
a is an order preserving function from the set $\{Z \cap \kappa^{++} \mid Z \in C^{\kappa^+}(\mathfrak{X})\}$ to λ_n^{+n+2}.

3. $a(X \cap \kappa^{++}) = \max(\text{rng}(a))$ is above all the elements of $\text{rng}(a)$ in the order of the extender E_{λ_n}.

4. $\text{dom}(a) \cap \text{dom}(f) = \emptyset$

5. $A \in E_{\lambda_n, a(\max(a))}$

6. $\min(A) > |\bigcup_{\tau \in s(\mathfrak{X})} A^{1\tau}(\mathfrak{X})|$.

7. for every ordinals α, β, γ in $\mathrm{rng}(a)$ we have

$$\alpha \geq_{E_{\lambda_n}} \beta \geq_{E_{\lambda_n}} \gamma \quad \text{implies}$$
$$\pi_{\lambda_n,\alpha,\gamma}(\rho) = \pi_{\lambda_n,\beta,\gamma}(\pi_{\lambda_n,\alpha,\beta}(\rho))$$

for every $\rho \in \pi''_{\lambda_n, \max \mathrm{rng}(a), \alpha}(A)$.

Let us turn now to the second component of a condition, i.e. to $\langle \underset{\sim}{b}, B, g\rangle$.

8. g is a partial function from $\kappa^{+\omega_1+1}$ to V_{κ_n} of cardinality at most κ.
It will be further convenient to view it as a sequence of functions $\langle g_\alpha \mid 1 < \alpha \leq \omega_1\rangle$.
This function as usual is needed to hide the actual correspondence once a non-direct extension was made. Here, once a non-direct extension was used somewhere over λ_n or in the Extender Based Magidor forcing over κ_n, we will make one which chooses $\rho_{n\omega_1}$ (the ω_1-th, the largest element of the Magidor sequence for κ_n) and from this point g will be combined with $\underset{\sim}{b}$ hiding the information.

9. $B = \langle B_\alpha \mid 1 < \alpha \leq \omega_1\rangle$ such that for every $\alpha, 1 < \alpha \leq \omega_1$, we have

   (a) $B_\alpha \in E_{\kappa_n,\alpha,\zeta_\alpha}$, i.e. it is a set of ζ_α-th measure one in the extender $E_{\kappa_n,\alpha}$, [1] where ζ_α is large enough (in the order of the extender $E_{\kappa_n,\alpha}$) to include all the possibilities for $\mathrm{rng}(\underset{\sim}{b}_\alpha)$ which will be defined below.
   Note that in Chapter 4 we used names sets of measure one instead. It is possible to do the same there and to work with actual sets just choosing large enough ζ's.

10. $\underset{\sim}{b} = \langle \underset{\sim}{b}_\alpha \mid 1 < \alpha \leq \omega_1\rangle$ is a name, depending on $\langle a, A\rangle$, of a partial functions b_α of cardinality less than λ_n.
For each $\alpha, 1 < \alpha \leq \omega_1$, and $\eta \in A$ the interpretation $\underset{\sim}{b}_\alpha[\eta]$ of $\underset{\sim}{b}_\alpha$ according to η satisfies the following conditions.

    (a) There is a suitable structure $\mathfrak{X}_{\alpha\eta}$ at the level κ_n (actually at the α-th member $\rho_{n\alpha}$ of the Magidor sequence at level n), such that

[1] in the Merimovich paper [13] a single set is used instead of a sequence B here, but since we deal only with a small relatively to κ number of extenders, the present setting is equivalent to those of [13].

i. $\underset{\sim}{b}_\alpha[\eta]$ is an isomorphism between $\mathfrak{X} \restriction \kappa^{+\alpha+1}$ and $\mathfrak{X}_{\alpha\eta}$,

ii. for every $Z \in A^{1\kappa^+}(\mathfrak{X} \restriction \kappa^{+\alpha+1})$ we have

$$\pi_{\lambda_n,\max(\mathrm{rng}(a)),a(Z\cap\kappa^{++})}(\eta) = \underset{\sim}{b}_\alpha[\eta](Z) \cap ((\eta)^0)^{+n+2}.$$

In particular,

$$\eta = \underset{\sim}{b}_\alpha[\eta](A^{0\kappa^+}(\mathfrak{X} \restriction \kappa^{+\alpha+1})) \cap ((\eta)^0)^{+n+2}.$$

iii. for every $\tau, 1 < \tau \leq \alpha$ and $Z \in A^{1\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1})$ we have $\underset{\sim}{b}_\alpha[\eta](Z)$ a name of a structure of cardinality $\underset{\sim}{\rho}_{n\tau}^{+n+2}$ (where $\rho_{n\tau}$ is the τ-th element of the Magidor sequence) depending only on the the τ-th member of the Magidor sequence at level n which is not yet determined.
If say we have some $\beta, 1 \leq \beta \leq \tau$ and $Y \in A^{1\kappa^{+\beta+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1})$ and $Z \in Y$, then the name $\underset{\sim}{b}_\alpha[\eta](Z)$ should belong to to any interpretation of $\underset{\sim}{b}_\alpha[\eta](Y)$, according to the β-th member of the Magidor sequence, if $\beta > 1$ and $\underset{\sim}{b}_\alpha[\eta](Z) \in \underset{\sim}{b}_\alpha[\eta](Y)$, if $\beta = 1$.
Note that, say with $\beta = 1$, the set of possible values of $\underset{\sim}{\rho}_\tau$ has cardinality κ_n which is much larger than the size of the model $\underset{\sim}{b}_\alpha[\eta](Y)$ (which is below λ_n). Only the name of the image of Z is inside $\underset{\sim}{b}_\alpha[\eta](Y)$, but by the elementarity also the Extender Based Magidor forcing is inside. So, after preforming the forcing the model $\underset{\sim}{b}_\alpha[\eta](Y)$ expends to one which includes the Magidor sequence and the interpretation of the image of Z.
This way the following connection will be established:
κ^{++} to η_n^{+n+2}, where η_n is an indiscernible (one element Prikry) for λ_n;
$\kappa^{+\tau+1}$ to $\rho_{n\tau}^{+n+2}$, for every $\tau, 1 < \tau \leq \alpha$.

(b) (Dependence) Let $Z \in A^{1\kappa^{+\tau+1}}$ for some $\tau, 1 < \tau \leq \alpha$. There are $\beta_1 < \beta_2 < ... < \beta_k < \alpha$ such that

- $\tau \in \{2, \beta_1, ..., \beta_k\}$.
- $\underset{\sim}{b}_\alpha(Z)$ depends on the values of $a(Z \cap \kappa^{++})$ over λ_n and $\underset{\sim}{b}_{\beta_i}(Z \restriction \kappa^{+\beta_i+1})$, $i = 1, ..., k$.
- If $Z = A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1})$, then $\underset{\sim}{b}_\alpha(Z)$ depends only on the values of $a(Z \cap \kappa^{++})$ in case $\tau = 2$ and on $a(Z \cap \kappa^{++}), \underset{\sim}{b}_\tau(Z \restriction \kappa^{+\tau+1})$, if $\tau > 2$.

(c) (Inclusion condition for cardinality κ^{++}) Let $\eta, \eta' \in A, \eta < \eta'$. Then

- $\underset{\sim}{b}_\alpha(A^{0\kappa^{++}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta] \in \underset{\sim}{b}_\alpha(A^{0\kappa^{++}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta']$,

- if $Z \in C^{\kappa^{++}}(\mathfrak{X} \restriction \kappa^{+\alpha+1})$ and

$$\pi_{\lambda_n, \max \mathrm{rng}(a), a(Z \cap \kappa^{++})}(\eta') > \eta,$$

  then either

$$\underset{\sim}{b}_\alpha(A^{0\kappa^{++}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta] \in \underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m]$$

  or
  the k-type realized by $\underset{\sim}{b}_\alpha(A^{0\kappa^{++}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta] \cap H(\chi^{+k})$ is in $\underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m]$, where $\nu_1, ..., \nu_m$ are the elements of the Magidor sequence on which Z depends and $k < \omega$ is the least such that $\underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m] \subseteq H(\chi^{+k+1})$.
  The same holds over any element of $\underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m]$, i.e. $tp_k(z, \underset{\sim}{b}_\alpha(A^{0\kappa^{++}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta] \cap H(\chi^{+k})) \in \underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m]$, for any $z \in \underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m]$.
  We require in addition that this $k > 2$.
  Let us allow the above also if $\underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m] \subseteq H(\chi^{+\omega})$. In this case we take k to be any natural number above 2 and require that once we go up to the higher levels then corresponding k's increase (with n).

We cannot in general require only that

$$\underset{\sim}{b}_\alpha(A^{0\kappa^{++}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta] \in \underset{\sim}{b}_\alpha(Z)[\eta', \nu_1, ..., \nu_m]$$

since the sequence C of a new generic suitable structure may go not through the old maximal model. But still having the type inside Z will suffice.

Note that given $\eta' \in A$ the number of possibilities for $\eta \in \eta' \cap A$ is bounded by $(\eta'^0)^{+n+1}$, as $\eta' < (\eta'^0)^{+n+2}$.

(d) (Inclusion condition for cardinalities above κ^{++}) Let $\tau, 1 < \tau \leq \alpha$ be an ordinal. We formulate a condition similar to the one above, but for structures of size $\kappa^{+\tau+1}$. Let $\eta, \eta' \in A, \eta < \eta', \rho, \rho' \in B_\tau, \rho < \rho'$. Then

- $\underset{\sim}{b}_\alpha(A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta, \rho] \in \underset{\sim}{b}_\alpha(A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta, \rho']$, we mean by this the interpretations according the values of a τ-th member of the Magidor sequence (i.e. $\rho = \underset{\sim}{\rho}_{n\tau}$ or $\rho' = \underset{\sim}{\rho}_{n\tau}$).
- $\underset{\sim}{b}_\alpha(A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta, \rho] \in \underset{\sim}{b}_\alpha(A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta', \rho']$.
- If $Z \in C^{\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1})$ and

$$\pi_{\lambda_n, \max \mathrm{rng}(b_\tau), b(Z \restriction \kappa^{+\tau+1})}(\rho') > \rho,$$

then either

$$\underset{\sim}{b}_\alpha(A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta,\rho] \in \underset{\sim}{b}_\alpha(Z)[\eta,\rho',\nu_1,...,\nu_m]$$

or

the k-type realized by $\underset{\sim}{b}_\alpha(A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta,\rho] \cap H(\chi^{+k})$ is in $\underset{\sim}{b}_\alpha(Z)[\eta,\rho',\nu_1,...,\nu_m]$, where $\nu_1,...,\nu_m$ are the rest of elements of the Magidor sequence on which Z depends and $k<\omega$ is the least such that $\underset{\sim}{b}_\alpha(Z)[\eta,\rho',\nu_1,...,\nu_m] \subseteq H(\chi^{+k+1})$.

The same holds over any element of $\underset{\sim}{b}_\alpha(Z)[\eta,\rho',\nu_1,...,\nu_m]$, i.e. $tp_k(z, \underset{\sim}{b}_\alpha(A^{0\kappa^{+\tau+1}}(\mathfrak{X} \restriction \kappa^{+\alpha+1}))[\eta,\rho] \cap H(\chi^{+k})) \in \underset{\sim}{b}_\alpha(Z)[\eta,\rho',\nu_1,...,\nu_m]$, for any $z \in \underset{\sim}{b}_\alpha(Z)[\eta,\rho',\nu_1,...,\nu_m]$.

We require in addition that this $k>2$.

Let us allow the above also if $\underset{\sim}{b}_\alpha(Z)[\eta,\rho',\nu_1,...,\nu_m] \subseteq H(\chi^{+\omega})$. In this case we take k to be any natural number above 2 and require that once we go up to the higher levels then corresponding k's increase (with n).

- The previous item with η replaced in $\underset{\sim}{b}_\alpha(Z)$ by η'.

11. For each $\alpha, 1<\alpha\leq\omega_1$, the following holds:

$$\{\tau<\kappa^{+\alpha+1} \mid \tau \in \mathrm{dom}(\underset{\sim}{b}_\alpha)\} \cap \mathrm{dom}(g) = \emptyset.$$

12. For every $\alpha, 1<\alpha\leq\omega_1$, $\nu \in A, \xi_1 \in B_{\beta_1}, \xi_2 \in B_{\beta_2}, \xi_3 \in B_{\beta_3}$, with $\beta_1,\beta_2,\beta_3<\alpha$, and every ordinals μ,ρ,η which are elements of $\mathrm{rng}(\underset{\sim}{b}_\alpha)[\nu,\xi_1,\xi_2,\xi_3]$ or actually the ordinals coding models in $\mathrm{rng}(\underset{\sim}{b}_\alpha)[\nu,\xi_1,\xi_2,\xi_3]$ we have

$$\mu \geq_{E_{\kappa_n,\alpha}} \rho \geq_{E_{\kappa_n,\alpha}} \eta \quad \text{implies}$$
$$\pi_{\kappa_n,\mu,\eta}(\delta) = \pi_{\kappa_n,\rho,\eta}(\pi_{\kappa_n,\mu,\rho}(\delta))$$

for every $\delta \in \pi''_{\kappa_n,\max\mathrm{rng}(\underset{\sim}{b}_\alpha[\nu,\xi_1,\xi_2,\xi_3]),\mu}(B_\alpha) \setminus \max(\xi_1,\xi_2,\xi_3)$.

We define now Q_{n1} and $\langle Q_n, \leq_n, \leq_n^* \rangle$ similar to the corresponding notions of Chapter 4. The only new point here is that the Extender Based Magidor forcing is used here instead of a trivial one element Prikry forcing in Chapter 4.

Definition 5.2.2 Suppose that $\langle\langle a,A,f\rangle, \langle \underset{\sim}{b}, B, g\rangle\rangle$ and $\langle\langle a',A',f'\rangle, \langle \underset{\sim}{b}', B', g'\rangle\rangle$ are two elements of Q_{n0}. Define

$$\langle\langle a,A,f\rangle, \langle \underset{\sim}{b}, B, g\rangle\rangle \geq_{Q_{n0}} \langle\langle a',A',f'\rangle, \langle \underset{\sim}{b}', B', g'\rangle\rangle$$

iff

1. $f \supseteq f'$.

2. For each $\alpha, 1 < \alpha \le \omega_1$,
$$g_\alpha \supseteq g'_\alpha.$$

3. $a \supseteq a'$.

4. $\pi''_{\lambda_n, \max(a), \max(a')} A \subseteq A'$.

5. For each $\alpha, 1 < \alpha \le \omega_1$,
$$\underset{\sim}{b}_\alpha \text{ extends } \underset{\sim}{b}'_\alpha,$$
according to the appropriate projections of measure one sets. This means just that the empty condition of (one element Prikry forcing followed by Extender Based Magidor) forces the inclusion.

6. For each $\alpha, 1 < \alpha \le \omega_1$, we have
$$\pi''_{\kappa_n, \alpha, \zeta_{n\alpha}, \zeta'_{n\alpha}}[B_\alpha] \subseteq B'_\alpha,$$
where $\zeta_{n\alpha}, \zeta'_{n\alpha}$ denote the measures of the extender $E_{\kappa_n, \alpha}$ to which B_α and B'_α belong.

Definition 5.2.3 Q_{n1} consists of triples $\langle f, g, t\rangle$ such that

1. f is a partial function from κ^{++} to λ_n of cardinality at most κ,

2. t is a condition in the Extender Based Magidor forcing of the length ω_1 over some $\rho(t), \lambda_n < \rho(t) < \kappa_n$.

3. $g = \langle g_\alpha \mid 1 < \alpha \le \omega_1\rangle$.
For each $\alpha, 1 < \alpha \le \omega_1$, the following holds: g_α is function from $\kappa^{+\alpha+1}$ of cardinality at most κ such that for each $\xi \in \text{dom}(g_\alpha)$ we have $g_\alpha(\xi) = \langle \rho, \underset{\sim}{\nu}\rangle$, for some $\rho < \kappa_n$ and a name in the extender based Magidor forcing over ρ corresponding to the α-th member of the Magidor sequence.
Again, $\underset{\sim}{\nu}$ can be viewed as void if this forcing is undefined or does not have μ-th sequence.
Note that we do not require that necessarily $\rho = \rho(t)$.

Define a partial order $\le_1$ over Q_{n1}.

Definition 5.2.4 Let $\langle f, g, t\rangle, \langle f', g', t'\rangle \in Q_{n1}$. Then $\langle f, g, t\rangle \le_1 \langle f', g', t'\rangle$ iff

1. $f' \supseteq f$,

2. $g'_\alpha \supseteq g_\alpha$, for each $\alpha, 1 < \alpha \leq \omega_1$,

3. t' extends t in the Extender Based Magidor forcing.

Definition 5.2.5 Set $Q_n = Q_{n0} \cup Q_{n1}$. Define $\leq^*_n = \leq_{Q_{n0}} \cup \leq_{Q_{n1}}$.

Define now a natural projection to the first coordinate:

Definition 5.2.6 Let $p \in Q_n$. Set $(p)_0 = p$, if $p \in Q_{n1}$ and let $(p)_0 = \langle a, A, f\rangle$, if $p \in Q_{n0}$ is of the form $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, B, g\rangle\rangle$.

Let $(Q_n)_0 = \{(p)_0 \mid p \in Q_n\}$.

Definition 5.2.7 Let $p, q \in Q_n$. Then $p \leq_n q$ iff either

1. $p \leq^*_n q$

   or

2. $p = \langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, B, g\rangle\rangle \in Q_{n0}$, $q = \langle e, h, t\rangle \in Q_{n1}$ and the following hold:

   (a) $e \supseteq f$

   (b) $h = \langle h_\alpha \mid 1 < \alpha \leq \omega_1\rangle$ and for each $\alpha, 1 < \alpha \leq \omega_1$ we have $h_\alpha \supseteq g_\alpha$

   (c) t extends the Extender Based Magidor part of p, decides $\rho_{n\omega_1}$ and $\rho(t) = \rho_{n\omega_1}$

   (d) $\mathrm{dom}(e) \supseteq \mathrm{dom}(a)$

   (e) $e(\max(\mathrm{dom}(a))) \in A$

   (f) for every $\beta \in \mathrm{dom}(a), e(\beta) = \pi_{\lambda_n, a(\max(\mathrm{dom}(a))), a(\beta)}(e(\max(\mathrm{dom}(a)))$

   (g) for every $\alpha, 1 < \alpha \leq \omega_1$ we have $\mathrm{dom}(h_\alpha) \supseteq \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap A^{1\kappa^{+\alpha+1}}(\kappa^{+\alpha+1})$

   (h) for every $\alpha, 1 < \alpha \leq \omega_1$ we require that

   - for every $\beta \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap A^{1\kappa^{+\alpha+1}}(\kappa^{+\alpha+1})$

     $$h_\alpha(\beta) = \langle \rho(t), \text{ the interpretation of } \underset{\sim}{b}_\alpha(\beta) \text{ after } \rho(t) = \rho_{n\omega_1}$$

     and probably other members of the Magidor sequence are determined in $t\rangle$

Definition 5.2.8 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega\rangle$ so that

1. for every $n < \omega$, $p_n \in Q_n$

2. there is $\ell(p) < \omega$ such that

   (a) for every $n < \ell(p)$, $p_n \in Q_{n1}$

   (b) for every $n \geq \ell(p)$, $p_n = \langle\langle a_n, A_n, f_n\rangle, \langle \underset{\sim}{b}_n, B_n, g_n\rangle\rangle \in Q_{n0}$

   (c) for every $n, m \geq \ell(p)$, $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$ and $\max(\mathrm{dom}(\underset{\sim}{b}_n)) = \max(\mathrm{dom}(\underset{\sim}{b}_m))$

   (d) for every $n \geq m \geq \ell(p)$, $\mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$ and $\mathrm{dom}(\underset{\sim}{b}_m)) \subseteq \mathrm{dom}(\underset{\sim}{b}_n)$

   (e) for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ the following holds:
   for each $k < \omega$ the set

$$\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$$

   is finite.

   (f) for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(\underset{\sim}{b}_n)$ the following holds:
   for each $k < \omega$ the set

$$\{m < \omega \mid \exists\vec{\nu}\exists\alpha, 1 < \alpha \leq \omega_1, \underset{\sim}{b}_{m\alpha}[\vec{\nu}] \text{ is defined, and } (\neg(\underset{\sim}{b}_{m\alpha}(X)[\vec{\nu}] \cap H(\chi^{+k}) \prec H(\chi^{+k})))\}$$

   is finite.

We define the orders $\leq, \leq^*$ as in the previous chapters.

Definition 5.2.9 Let $p = \langle p_n \mid n < \omega\rangle, q = \langle q_n \mid n < \omega\rangle$ be in $\mathcal{P}$. Define

1. $p \geq q$ iff for each $n < \omega, p_n \geq_n q_n$

2. $p \geq^* q$ iff for each $n < \omega, p_n \geq^*_n q_n$

Definition 5.2.10 Let $p = \langle p_n \mid n < \omega\rangle \in \mathcal{P}$. Set $(p)_0 = \langle (p_n)_0 \mid n < \omega\rangle$.
Define $(\mathcal{P})_0 = \{(p)_0 \mid p \in \mathcal{P}\}$.

Finally, the equivalence relation $\longleftrightarrow$ and the order $\rightarrow$ are defined on $(\mathcal{P})_0$ exactly as it was done in Chapter 4. We extend $\rightarrow$ to $\mathcal{P}$ as follows:

Definition 5.2.11 Let $p = \langle p_n \mid n < \omega\rangle, q = \langle q_n \mid n < \omega\rangle \in \mathcal{P}$. Set $q \rightarrow p$ iff

1. $(q)_0 \rightarrow (p)_0$

2. $\ell(p) \geq \ell(q)$

3. for every $n < \ell(p), p_n$ extends q_n

4. for every $n \geq \ell(p)$, let $p_n = \langle\langle a_n, A_n, f_n\rangle, \langle \underset{\sim}{b}_n, B_n, g_n\rangle\rangle$ and $q_n = \langle\langle a'_n, A'_n, f'_n\rangle, \langle \underset{\sim}{b}'_n, B'_n, g'_n\rangle\rangle$. Require the following:

   (a) $g_n \supseteq g'_n$

   (b) there is $\underset{\sim}{b}''_n = \langle \underset{\sim}{b}_{n\alpha} \mid 1 < \alpha \leq \omega_1\rangle$ such that for every $\alpha, 1 < \alpha \leq \omega_1$,$\nu, \nu' \in A_n$ the following holds:

      i. $\mathrm{dom}(\underset{\sim}{b}'_{n\alpha}) = \mathrm{dom}(\underset{\sim}{b}''_{n\alpha})$

      ii. $\pi''_{\kappa_n,\alpha,\zeta(B_{n\alpha}),\zeta(B'_{n\alpha})} B_{n\alpha} \subseteq B'_{n\alpha}$,
      where $\zeta(B_{n\alpha}), \zeta(B'_{n\alpha})$ the indexes of the measures of $E_{n\alpha}$ to which $B_{n\alpha}$ and $B'_{n\alpha}$ belong.

      iii. $\underset{\sim}{b}_n[\nu]$ extends $\underset{\sim}{b}''_n[\nu']$ and for each $\vec{\nu}$ and its projection $\vec{\nu'}$ we have

      $$\underset{\sim}{b}_n[\nu^\frown\vec{\nu}] \text{ extends } \underset{\sim}{b}''{}_n[\nu'^\frown\vec{\nu'}]$$

      iv. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'^\frown\vec{\nu'}] \longleftrightarrow_{k_n} \mathrm{rng}(\underset{\sim}{b}''_n)[\nu'^\frown\vec{\nu'}]$, where $\nu', \vec{\nu'}$ are as above and k_n is the k_n's member of a nondecreasing sequence converging to the infinity.

      v. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'^\frown\vec{\nu'}] \restriction \kappa^{+n+1} = \mathrm{rng}(\underset{\sim}{b}''_n)[\nu'^\frown\vec{\nu'}] \restriction \kappa^{+n+1}$.

5.3 Basic Lemmas

In this section we study the properties of the forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$ defined in the previous section.

Lemma 5.3.1 *Let $p = \langle p_k \mid k < \omega\rangle \in \mathcal{P}$, $p_k = \langle\langle a_k, A_k, f_k\rangle, \langle \underset{\sim}{b_k}, B_k, g_k\rangle\rangle$ for $k \geq \ell(p)$ and X be a model appearing in an element of $G(\mathcal{P}'(\kappa^{+\omega_1+1}))$. Suppose that*

(a) $X \notin \bigcup_{\ell(p)\leq k<\omega} \mathrm{dom}(\underset{\sim}{b}_k) \cup \mathrm{dom}(g_k)$

(b) X is a successor model or if it is a limit one with $\mathrm{cof}(otp_{|X|}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega\rangle$, $q_k = \langle\langle a'_k, A'_k, f'_k\rangle, \langle \underset{\sim}{b}'_k, B'_k, g_k\rangle\rangle$ for $k \geq \ell(q)$, of p so that starting with some $n \geq \ell(q)$ we have $X \restriction \kappa^{++} \in \mathrm{dom}(a'_k)$ and $X \restriction \kappa^{+\alpha+1} \in \mathrm{dom}(\underset{\sim}{b}'_{k\alpha})$ for each $k \geq n, 1 < \alpha \leq \omega_1$.

The proof is similar to those of the corresponding lemma in Chapters 1, 2.

Next three lemmas transfer directly from Chapters 1, 2.

Lemma 5.3.2 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_0 \rangle$ does not add new sequences of ordinals of the length $< \lambda_n$, i.e. it is (λ_n, ∞) – distributive.*

Lemma 5.3.3 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \lambda_0$.*

Lemma 5.3.4 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Let us turn now to the chain condition lemma. The proof will be similar to those of 4.4.5.

Lemma 5.3.5 *$\langle \mathcal{P}, \to \rangle$ satisfies κ^{++}-c.c.*

Proof.

Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\zeta \mid \zeta < \kappa^{++} \rangle$ be a name of an antichain of the length κ^{++}. Using the strategic closure of $\underset{\sim}{\mathcal{P}}'$, we find an increasing sequence

$$\langle \langle \langle A^{0\tau}_\zeta, A^{1\tau}_\zeta, C^\tau_\zeta \rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}] \rangle \mid \zeta < \kappa^{++} \rangle$$

of elements of $\mathcal{P}'(\kappa^{+\omega_1+1})$ and a sequence $\langle p_\zeta \mid \zeta < \kappa^{++} \rangle$ so that for every $\zeta < \kappa^{++}$ the following hold:

(a) $\langle \langle A^{0\tau}_{\zeta+1}, A^{1\tau}_{\zeta+1}, C^\tau_{\zeta+1} \rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}] \rangle \Vdash \underset{\sim}{p}_\zeta \leq \check{p}_\zeta$,

(b) $\bigcup_{\beta<\zeta} A^{0\tau}_\beta = A^{0\tau}_\zeta$, if ζ is a limit ordinal and $\tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]$,

(c) ${}^{\tau>}A^{0\tau}_{\zeta+1} \subseteq A^{0\tau}_{\zeta+1}$,

(d) $A^{0\tau}_{\zeta+1}$ is a successor model,

(e) $\langle \langle \langle A^{0\tau'}_\mu, A^{1\tau'}_\mu, C^{\tau'}_\mu \rangle \mid \tau' \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}] \rangle \mid \mu < \zeta \rangle \in A^{0\tau}_{\zeta+1}$, for every $\tau \in [\kappa^+, \kappa^{+\omega_1+1}]$,

(f) for every $\zeta \leq \beta < \kappa^{++}, \tau \in [\kappa^+, \kappa^{+\omega_1+1}]$ we have

$$C^\tau_\zeta(A^{0\tau}_\zeta) \text{ is an initial segment of } C^\tau_\beta(A^{0\tau}_\beta),$$

(g) $p_\zeta = \langle p_{\zeta n} \mid n < \omega \rangle$,

(h) for every $n \geq l(p_\zeta)$

- $A^{0\kappa^+}_{\zeta+1} \cap \kappa^{++}$ is the maximal ordinal of $\mathrm{dom}(a_{\zeta n})$ and $A^{0\kappa^+}_\zeta \cap \kappa^{++} \in \mathrm{dom}(a_{\zeta n})$,
- $A^{0\kappa^+}_{\zeta+1}$ is the maximal model of $\mathrm{dom}(\underset{\sim}{b}_{\zeta n})$ and $A^{0\kappa^+}_\zeta \in \mathrm{dom}(\underset{\sim}{b}_{\zeta n})$,

where $p_{\zeta n} = \langle\langle a_{\zeta n}, A_{\zeta n}, f_{\zeta n}\rangle, \langle \underset{\sim}{b}_{\zeta n}, B_{\zeta n}, g_{\zeta n}\rangle\rangle$.

Actually this condition is the reason for not requiring the equality in (a) above.

Let $p_{\zeta n} = \langle\langle a_{\zeta n}, A_{\zeta n}, f_{\zeta n}\rangle, \langle \underset{\sim}{b}_{\alpha n}, B_{\alpha n}, g_{\alpha n}\rangle\rangle$ for every $\zeta < \kappa^{++}$ and $n \geq\ l(p_\zeta)$.
Let $\zeta < \kappa^{++}$. Fix some

$$\langle\langle B^{0\tau}_{\zeta+1}, B^{1\tau}_{\zeta+1}, D^{\tau}_{\zeta+1}\rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]\rangle \leq_{\mathcal{P}'} \langle\langle A^{0\tau}_{\zeta+1}, A^{1\tau}_{\zeta+1}, C^{\tau}_{\zeta+1}\rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]\rangle$$

which witnesses a generic suitability of structure $\mathrm{dom}(\underset{\sim}{b}_{\zeta n})$ for each $n, l(p_\zeta) \leq n < \omega$, as in Definition 5.2.1, i.e. $\langle\langle B^{0\tau}_{\zeta+1}, B^{1\tau}_{\zeta+1}, D^{\tau}_{\zeta+1}\rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]\rangle = p(\mathfrak{X})$, for some suitable generic structure $\mathfrak{X}$. Note that $B^{0\tau}_{\zeta+1}$ need not be in $C^{\tau}_{\zeta+1}(A^{0\tau}_{\zeta+1})$ and even if it does, then $D^{\tau}_{\zeta+1}(B^{0\tau}_{\zeta+1})$ need not be an initial segment of $C^{\tau}_{\zeta+1}(A^{0\tau}_{\zeta+1})$. By the definition of the order $\leq_{\mathcal{P}'}$ there are $m < \omega$ and $E_1, ..., E_m \in \bigcup_{\tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]} A^{1\tau}_{\zeta+1}$ such that

$$swt(\langle\langle A^{0\tau}_{\zeta+1}, A^{1\tau}_{\zeta+1}, C^{\tau}_{\zeta+1}\rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]\rangle, E_1, ..., E_m) \text{ and}$$

$$\langle\langle B^{0\tau}_{\zeta+1}, B^{1\tau}_{\zeta+1}, D^{\tau}_{\zeta+1}\rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]\rangle$$

are as in the definition of the order of $\mathcal{P}'$ (Chapter 1,1.1.15, Chapter 2,2.2.6).
By Lemma 5.3.1 it is possible to add all $E_i (i = 1, ..., m)$ to $\mathrm{dom}(\underset{\sim}{b}_{\zeta n})$, for a final segment of n's. By adding and taking non-direct extension if necessary, we can assume that E_i's are already in $\mathrm{dom}(\underset{\sim}{b}_{\zeta n})$, for every $n \geq l(p_\zeta)$.
Now we can apply the opposite switch (i.e. the one starting with E_m, then E_{m-1}, ...,and finally E_1) to $\mathrm{dom}(\underset{\sim}{b}_{\zeta n})$ (and the corresponding to it under $b_{\zeta n}$ to $\mathrm{rng}(\underset{\sim}{b}_{\zeta n})$). Denote the result still by $\underset{\sim}{b}_{\zeta n}$.
Finally, $\langle\langle A^{0\tau}_{\zeta+1}, A^{1\tau}_{\zeta+1}, C^{\tau}_{\zeta+1}\rangle \mid \tau \in Reg \cap [\kappa^+, \kappa^{+\omega_1+1}]\rangle$ will witness a generic suitability of structure $\mathrm{dom}(\underset{\sim}{b}_{\zeta n})$ for each $n, l(p_\zeta) \leq n < \omega$.
In particular, we have now that the central line of $\mathrm{dom}(\underset{\sim}{b}_{\zeta n, \tau})$ is a part of $C^{\tau}_{\zeta+1}(A^{0\tau}_{\zeta+1})$ and $A^{0\tau}_{\zeta}$ is on it, for every $n, l(p_\zeta) \leq n < \omega$.

Shrinking if necessary, we assume that for all $\zeta, \xi < \kappa^{++}$ the following holds:

(1) $\ell = \ell(p_\zeta) = \ell(p_\xi)$,

(2) for every $n < \ell$, $p_{\zeta n}$ and $p_{\xi n}$ are compatible in Q_{n1},

(3) for every n, $\ell \leq n < \omega$, $\langle \mathrm{dom}(f_{\zeta n}) \mid \zeta < \kappa^{++}\rangle$ and $\langle \mathrm{dom}(g_{\zeta n}) \mid \zeta < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$,

(4) for every n, $\omega > n \geq \ell$, $\mathrm{rng}(a_{\zeta n}) = \mathrm{rng}(a_{\xi n})$,

(5) for every n, $\omega > n \geq \ell$, $\quad A_{\zeta n} = A_{\xi n}$,

(6) for every n, $\omega > n \geq \ell$, $\quad B_{\zeta n} = B_{\xi n}$,

(7) for every n, $\omega > n \geq \ell, \eta \in A_{\zeta n}$ and $\vec{\nu}$ from appropriate $B_{\zeta n}$'s we have $\text{rng}\, \underset{\sim}{b}_{\zeta n}[\eta, \vec{\nu}] = \text{rng}\, \underset{\sim}{b}_{\xi n}[\eta, \vec{\nu}]$.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let ζ be in S. For each $n, \ell \leq n < \omega$, there will be $\beta(\zeta, n) < \zeta$ such that

$$\text{dom}(\underset{\sim}{b}_{\zeta n}) \cap A_{\zeta}^{0\kappa^+} \subseteq A_{\beta(\zeta,n)}^{0\kappa^+}.$$

Just recall that $\text{dom}(\underset{\sim}{b}_{\zeta n})$ is not actually a name and $|\text{dom}(\underset{\sim}{b}_{\zeta n})| < \lambda_n$. Shrink S to a stationary subset S^* so that for some $\zeta^* < \min S^*$ of cofinality κ^+ we will have $\beta(\zeta, n) < \zeta^*$, whenever $\zeta \in S^*, \ell \leq n < \omega$. Now, the cardinality of $A_{\zeta^*}^{0\kappa^+}$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\zeta, \xi \in S^*, \ell \leq n < \omega$

$$\text{dom}(\underset{\sim}{b}_{\zeta n}) \cap A_{\zeta}^{0\kappa^+} = \text{dom}(\underset{\sim}{b}_{\xi n}) \cap A_{\xi}^{0\kappa^+}.$$

Let us add $A_{\zeta^*}^{0\kappa^+}$ to each p_ζ with $\zeta \in S^*$.

By 5.3.1, we can add it without adding ordinals and the only other models that probably were added are the images of $A_{\zeta^*}^{0\kappa^+}$ under Δ-system type isomorphisms. Denote the result for simplicity by p_ζ as well.

Let now $\xi < \zeta$ be ordinals in S^*. We claim that p_ξ and p_ζ are compatible in $\langle \mathcal{P}, \rightarrow \rangle$. First extend p_ζ by adding $A_{\xi+2}^{0\kappa^+}$. This will not add other additional models or ordinals except the images of $A_{\xi+2}^{0\kappa^+}$ under isomorphisms to p_ζ, as was remarked above.

Let p be the resulting extension. Denote p_ξ by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$ and $p_n = \langle\langle a_n, A_n, f_n\rangle, \langle \underset{\sim}{b}_n, B_n, g_n\rangle\rangle$. Let $q_n = \langle\langle a'_n, A'_n, f'_n\rangle, \langle \underset{\sim}{b}'_n, B'{}_n, g'_n\rangle\rangle$. Without loss of generality we may assume that the ordinal $a_n(A_{\xi+2}^{0\kappa^+} \cap \kappa^{++})$ is k_n-good with $k_n \geq 5$. Just increase n if necessary.

Realize the $k_n - 1$-type of $\text{rng}(a'_n)$ below $a_n(A_{\xi+2}^{0\kappa^+} \cap \kappa^{++})$ over $a_n((A_{\xi+2}^{0\kappa^+} \cap \kappa^{++}) \cap \text{dom}(a_n))$, i.e. above the common part on κ^{++}. Denote the ordinal corresponding to $\max(\text{rng}(a'_n))$ in this realization by δ'. Note that $a_n(A_{\zeta+1}^{0\kappa^+} \cap \kappa^{++})$ and δ' have the same projection to the common part $a_n((A_{\xi+2}^{0\kappa^+} \cap \kappa^{++}) \cap \text{dom}(a_n))$.

Fix now $\eta \in A_n$. Set $\eta' = \pi_{\lambda_n, \max(\text{rng}(a_n)), \delta'}(\eta)$.

Consider first $\underset{\sim}{b}_{n,2}(A_{\xi+2}^{0\kappa^+} \restriction \kappa^{+3})[\eta]$. Recall that $\underset{\sim}{b}_{n,2}$ depends only on on the one element

Prikry forcing over λ_n and here only on a choice of an element from A_n.
Again we can assume that $\underset{\sim}{b}_{n,2}(A^{0\kappa^+}_{\xi+2} \restriction \kappa^{+3})[\eta]$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$ (and k_n does not depend on η). Now we have

$$\eta' = \pi_{\lambda_n, \max(\operatorname{rng}(a_n)), \delta'}(\eta) < \eta \text{ and } A^{0\kappa^+}_{\xi+2} \in C^{\kappa^+}(A^{0\kappa^+}_{\zeta+1}).$$

Hence, by Definition 5.2.1(10c,10d), the k_n-1-type realized by $\underset{\sim}{b}_{n,2}(A^{0\kappa^+}_{\zeta+1})[\eta']$ is in $\underset{\sim}{b}_{n,2}(A^{0\kappa^+}_{\xi+2})[\eta]$, as well as the $k_n - 1$-type realized by $\underset{\sim}{b}_{n,2}(A^{0\kappa^+}_{\zeta+1})[\eta']$ over $\underset{\sim}{b}''_{n,2}(A^{0\kappa^+}_{\xi+2} \cap \operatorname{dom}(\underset{\sim}{b}_{n,2}))[\eta]$, i.e. the common part of of the conditions. Realize the $k_n - 1$-type of $\underset{\sim}{b}_{n,2}(A^{0\kappa^+}_{\zeta+1})[\eta']$ over $\underset{\sim}{b}''_{n,2}(A^{0\kappa^+}_{\xi+2} \cap \operatorname{dom}(\underset{\sim}{b}_{n,2}))[\eta]$ in $\underset{\sim}{b}_{n,2}(A^{0\kappa^+}_{\xi+2})[\eta]$.

Doing the above for each $\eta \in A_n$ will result in compatibility of the second components i.e. $\underset{\sim}{b}_{n,2}$'s.
Next we deal with $\underset{\sim}{b}_{n,3}$ and use the compatibility of a_n's and $\underset{\sim}{b}_{n,2}$'s. Continue by induction all the way to $\underset{\sim}{b}_{n,\omega_1}$. This eventually will produce a condition $p^*_n \geq p_n$ with $q_n \to p^*_n$ as in Chapter 1.
□

Force with $\langle \mathcal{P}, \to \rangle$. Let $G(\mathcal{P})$ be a generic set. By the lemmas above no cardinals are collapsed. Let $\langle \eta_n \mid n < \omega \rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\lambda_n} \mid n < \omega \rangle$ and $\langle \rho_{n\alpha} \mid 1 < \alpha \leq \omega_1 \rangle$, for each $n < \omega$, the Magidor sequence for the normal measures of E_{κ_n}. We can deduce now the following conclusion:

Theorem 5.3.6 *The following hold in* $V[G(\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})), G(\mathcal{P})]$:

(1) $\operatorname{cof}(\prod_{n<\omega} \eta_n^{+n+2} / \textit{ finite }) = \kappa^{++}$

(2) for each $\alpha, 1 < \alpha \leq \omega_1$,

$$\operatorname{cof}(\prod_{n<\omega} \rho_{n\alpha}^{+n+2} / \textit{ finite }) = \kappa^{+\alpha+1}$$

The proof follows easily from the construction.

5.4 Arbitrary Gaps with GCH below

In this section we would like using arguments under similar lines to obtain an arbitrary gap starting with κ such that

$$\text{for every } \tau < \kappa \text{ exists } \alpha < \kappa \quad o(\alpha) = \alpha^{+\tau}.$$

Assume that $\mathrm{cof}(\kappa) = \omega$. Obviously, the least κ as above must have the cofinality ω. As usual we assume GCH in the ground model. Fix a regular cardinal $\theta > \kappa$. Our purpose will be to make $2^\kappa = \theta^+$ without adding any new bounded subsets to κ. In particular GCH will hold below κ in the final model. Note that by [5] our initial assumption is the necessary one.

Force the preparation forcing $\mathcal{P}'(\theta)$ of Chapter 2.

Fix an increasing unbounded in κ sequence of cardinals $\langle \kappa_n | n < \omega \rangle$ such that

- κ_0 is κ_0^{+7}-strong witnessed by an extender E_0
- κ_n is $\kappa_n^{+\kappa_{n-1}^{+\kappa_{n-1}}}$- strong witnessed by an extender E_n, for each $n, 0 < n < \omega$.

We use $E_0 \restriction \kappa_0^{+3}$ at the level 0. θ^+ is attached to κ_0^{+3}. Let μ be a cardinal in the interval $[\kappa^+, \theta]$. In a typical condition models of size μ will be attached to those of size κ_0^{++}. In particular models of different sizes over κ will be connected to models of the same size over κ_0. It would not be problematic, since a non-direct extension will be taken over κ_0.

Next we will use indiscernibles created over κ_0 to rule the attachment to the level 1. Thus, given an indiscernible ρ which corresponds to one of models (a model will be picked generically, leaving a bit of freedom this way) connected to those of cardinality μ over κ, we consider the cardinal $\kappa_1^{+\rho+2}$ and connect it with μ over the level 1.

This way different cardinals, say $\mu, \mu' < \theta$, that were connected at level 0 to ordinals of the same cardinality, will be connected at level 1 to different cardinals $\kappa_1^{+\rho+2}$ and $\kappa_1^{+\rho'+2}$, where ρ' denotes the indiscernible from level 0 which corresponds to μ' there.

Let $\kappa_1^{+\kappa_0+2}$ correspond to θ^+. The conditions of the level 1 are similar to those of Chapter 4 only λ_1 is replaced here by κ_0. In particular size of domain of $\underset{\sim}{b}_1$ is less than κ_0 and the cofinalities drop at this level below κ_0.

At the next level- level 2 we use the indiscernible ρ_1 for $\kappa_1^{+\rho+2}$ as the cardinality of models which correspond to those of cardinality μ at this level. μ as a cardinal will be connected with $\kappa_2^{+\rho_1+2}$. Again let θ^+ correspond to $\kappa_2^{+\kappa_1^{+\kappa_0+2}+2}$. An indiscernible ρ_2 for a model of size ρ_1 will be used at the next level (level 3) to determine the cardinal $\kappa_3^{+\rho_2+2}$ which will correspond to μ there. The conditions of the level 2 are similar to those of Chapter 4 only λ_2

is replaced here by κ_1. In particular size of domain of $\underset{\sim}{b}_2$ is less than κ_1 and the cofinalities drop at this level below κ_1.
Continue to further levels in the same fashion.

At the level 0 we use $E_0 \upharpoonright \kappa_0^{+3}$. Set η_0 to be κ_0^{+3} and $\eta_1 = \kappa_1^{+\kappa_0+2}$. We force with $E_1 \upharpoonright \eta_1$ over κ_1. The size of condition at this level will be below κ_0. Let $n > 1$. Set $\eta_n = \kappa_n^{+\eta_{n-1}+2}$. We use $E_n \upharpoonright \eta_n$, over κ_n. The size of condition at this level will be below κ_{n-1}.

Let us explain the idea behind the above and the necessity of this kind of approach. At each level n, starting with level 1, we have two types of objects: cardinal above κ_n and models of sizes below κ_{n-1}. The indiscernibles for such cardinals will rule sizes of models to be used at the next level. This is needed in order to guarantee a degree of completeness of the main forcing. The use of models of small cardinalities (and droppings in cofinalities that comes with it) is supposed to compensate the number of cardinals at level n that we are allowed to use (which is smaller than κ_n). Thus, even if models Z of size κ_n were allowed, then all possible cardinals will be inside Z. Assuming that number of cardinals between κ and θ is κ^{++} or beyond, it is unclear what to do with models over κ (say of size κ^+) which do not contain all the cardinals of the interval $[\kappa^+, \theta]$, since no Z as above can correspond to such models.

One complication here (relatively to the constructions of Chapter 2 and those of the previous section with $2^\kappa = \kappa^{+\theta}$, for $\theta < \kappa$) is that we do not have λ_n's which separate κ_n's one from an other. Namely, once a non-direct extension was made over some λ_n, then the same was done over the corresponding κ_n, as well. But we cannot do this here, since once a non-direct extension over κ_0 was made -the same must be done over κ_1 and then over κ_2, etc.
Here we will allow to make a non-direct extension at κ_n without making it at κ_{n+1}. In this setting not only rng($\underset{\sim}{b}_{n+1}$) will be a name, but also dom($\underset{\sim}{b}_{n+1}$). This change is required now in order to show the Prikry condition. Thus, after picking a non-direct extension at a level κ_n, we need be able to keep a possible extension at the level κ_{n+1}. The number of models allowed at this level is below κ_n. So, we need to deal with such names, in order to accumulate together all the extensions at the level $n+1$ according to all possible non-direct extensions at the level n.
Still we keep the maximal model explicit and not in a form of a name.
We need to make one change relatively to previous constructions with dropping cofinalities. Thus, during the standard argument for showing the Prikry condition, say we go through all the possibilities (non-directly) over a level n. Now, over the next level $n+1$, it may be a

need to add new models (direct extensions here) in order to decide a given statement. But we do not have a control now over projection of such models to the level n. In previous setting - once a non-direct extension was made over λ_n then such an extension was made over κ_n as well and there was no further connection to $\lambda_{n+1}, \kappa_{n+1}$. So here we may get types of models that do not appear inside other relevant models (remember that sizes of models used are small, and so not every type is inside). As a consequence of this the chain condition argument stops to work.
Let us suggest a way that allows to overcome this difficulty. It is possible to use it in the previous construction with dropping cofinalities as well.

Let $\mathcal{M}$ be an elementary submodel of $\langle H(\chi^{+\omega}, <, ...\rangle$ (for χ regular large enough which contains all relevant information) of size κ_n and which is a union of an elementary chain $\langle \mathcal{M}_\alpha \mid \alpha < \kappa_n \rangle$ such that for every $\alpha < \kappa_n$

1. $|\mathcal{M}_\alpha| < \kappa_n$,
2. $\langle \mathcal{M}_\beta \mid \beta \leq \alpha \rangle \in \mathcal{M}_{\alpha+1}$.

Now, as images of models of cardinality κ^+ let us use only models $\mathcal{M}_\alpha, \alpha < \kappa_n$, and models which realize similar types to the types of models of this sequence. Images of models of bigger size will be models which are elements of models of the sequence or of models which realize similar types to the types of models of the sequence. As before a choice of η in a set of measure one for a maximal coordinate over κ_n will determine the interpretation of $\underset{\sim}{b}_{n+1}$, i.e. $\underset{\sim}{b}_{n+1}[\eta]$ which a real function and not a name. Once $\eta < \eta'$, let us require that also an index $\alpha(\eta)$ which corresponds to the maximal model over κ_{n+1} according to η is less than $\alpha(\eta')$.
Note that this restriction to the sequence $\langle \mathcal{M}_\alpha \mid \alpha < \kappa_n \rangle$ and similar types is not actually very strict. Namely, the following holds:

Lemma 5.4.1 *Let $\alpha < \kappa_n$. Then*

$$\mathcal{M}_\alpha \models \forall k < \omega \forall x \exists y (x \in y \wedge (\forall \delta < \chi \exists y' (\sup(y' \cap \chi) > \delta \wedge tp_k(y') = tp_k(y)))).$$

Proof. Suppose otherwise. Then

$$\mathcal{M}_\alpha \models \exists k < \omega \exists x \forall y (x \in y \to (\exists \delta < \chi \neg \exists y' (\sup(y' \cap \chi) > \delta \wedge tp_k(y') = tp_k(y)))).$$

Pick $k < \omega$ and $x \in \mathcal{M}_\alpha$ witnessing this. By elementarity then

$$H(\chi^{+\omega}) \models \forall y (x \in y \to (\exists \delta < \chi \neg \exists y' (\sup(y' \cap \chi) > \delta \wedge tp_k(y') = tp_k(y)))).$$

Consider the set

$$Z := \{tp_k(a) \mid x \in a\}$$

of all k-types over x. The size of Z is bounded below χ. This means that the same type should appear χ-many times. Which is impossible. Contradiction.

□

In particular, the following holds:

Lemma 5.4.2 *For each $k < \omega$ and $x \in \mathcal{M}_\alpha$ the k-type of $\mathcal{M}_\alpha \cap H(\chi^{+k})$ over x appears unboundedly often below χ.*

Proof. Let $k, 0 < k < \omega$ and $x \in \mathcal{M}_\alpha$. Pick $y \in \mathcal{M}_\alpha$ as in the previous lemma. Then

$$H(\chi^{+\omega}) \vDash \forall \delta < \chi \exists y' (\sup(y' \cap \chi) > \delta \wedge tp_k(y') = tp_k(y))).$$

Fix $\delta < \chi$. Find y' with $\sup(y' \cap \chi) > \delta$ which realizes the same k-type as y over x. Let t be the $k-1$-type realized by $\mathcal{M}_\alpha$ over y. Then there will be t' which realizes the same $k-1$-type over y'. Hence t' will realize the same $k-1$-type as those of $\mathcal{M}_\alpha$.

□

Note that on the other hand not every measure of the extender E_{n+1} is in $\mathcal{M}_\alpha$, since its cardinality is just too small.

The next point will be to show κ^{++}-c.c. of the final forcing. The argument mostly repeats the corresponding one of Chapter 4. The only new element here will be the starting level for compatibility of two conditions, i.e. the one above which the extension will be a direct extension.

Thus, suppose we have two conditions $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle$ of the same length. For simplicity let $\ell(p) = \ell(q) = 1$. Also suppose that the ranges at each $n > 0$ are the same and the domains form a Δ-system as in 5.3.5. Let $\eta < \xi$ and $A_\eta^{0\kappa^+}, A_\xi^{0\kappa^+}$ etc. be as in the proof of 5.3.5. We find a model X (not necessary in $A_\xi^{0\kappa^+}$) which realizes the same type (more precisely, by one less) over $\mathrm{rng}(b_1^\eta \restriction A_\eta^{0\kappa^+}) = \mathrm{rng}(b_1^\xi \restriction A_\xi^{0\kappa^+})$ as $b_1^\xi(A_\xi^{0\kappa^+})$ does but has a smaller supremum. Note that $\ell(p) = 1$, so at the level 0 we have a non- direct extension of say the weakest condition. Hence, at the level 1, the real models and not the names are used (i.e. the interpretations of the names according the generic object at the level 0). It is possible to insure the existence of such model X as follows:

we just can assume without loss of generality that $\sup(b_1^\xi(A_\xi^{0\kappa^+}))$ is an elementary submodel M(of cardinality $\kappa_1^{+\aleph_{\rho_0+3}+2}$) of some $H(\chi^{+k+1})$ with $k > 5$ or so. Then the k-type of $b_1^\xi(A_\xi^{0\kappa^+})$ over $\mathrm{rng}(b_1^\xi \restriction A_\xi^{0\kappa^+})$ is in M. Using elementarity, it is easy to argue that there will be unboundedly many in $\sup(M)$ models realizing this type over $\mathrm{rng}(b_1^\xi \restriction A_\xi^{0\kappa^+})$. Then pick X to be one of them.

Now the measure of the extender E_1 corresponding to X will be the same as those for $b_1^\xi(A_\xi^{0\kappa^+})$. Let B_1 be the set of measure one for this measure in the conditions (note that it is the same set in both. We increase the maximal coordinate in order to catch X in the new maximal coordinate. Let C_1 be a set of measure one for this coordinate with corresponding projections to $b_1^\xi(A_\xi^{0\kappa^+})$ and to X inside B_1. Pick some $\nu \in C_1$. Let ν' and ν'' be its projections to $b_1^\xi(A_\xi^{0\kappa^+})$ and to X respectively. Then $\nu'' < \nu'$, but further projections to the common part are the same, since it is inside both $b_1^\xi(A_\xi^{0\kappa^+})$ and $b_1^\eta(A_\eta^{0\kappa^+})$.

The model X serves as $b_1^\eta(A_\eta^{0\kappa^+})$ on the ξ's side. So we replace $b_1^\eta(A_\eta^{0\kappa^+})$ by an equivalent model. It is likely impossible to put things together at the level 1 in a direct extension fashion due to a quite arbitrary place of X relatively to $b_1^\xi(A_\xi^{0\kappa^+})$. But there is no need here to make a direct extension. So we will take a non-direct.

It may be (and typically would be once θ is much larger than κ) that $A_\eta^{0\kappa^+}$ and $A_\xi^{0\kappa^+}$ have inside models of different cardinalities which are in domains of b_1^η, b_1^ξ. The images of such models should be of different cardinalities as well (at least starting from some level). The number of available cardinals over κ_1 is $\kappa_1^{+\aleph_{\rho_0^{+3}}+3}$. The size of X (as those of $b_1^\xi(A_\xi^{0\kappa^+})$) is only ρ_0^+. Hence most of cardinals are outside.

The choice of X insures that at the next level (level 2) and up cardinalities will be different from those that appear in $b_1^\xi(A_\xi^{0\kappa^+}) \setminus X$.

There may be a need to apply a similar procedure at level 2, since the cardinals from level 1 corresponding to those over κ are determined at level 0 (remember that we assumed $\ell(p) = 1$). It is possible that same cardinal over level 1 corresponds to different cardinals over κ. So we need to split between them at level 2, as it was done above.

Finally, given such ν' and ν'' we go to the next level (level 2) and put together the conditions at this level in the usual fashion, as it was done in Chapter 4 and in Lemma 5.3.5. No further non-direct extension (beyond level 2) is needed here anymore.

5.5 Arbitrary Gaps from weakest assumptions

In the present section we would like to use ideas of Section 5.1 in order to blow up the power of κ to $\kappa^{+\theta+1}$ for arbitrary θ. Start with a singular cardinal κ of cofinality ω such that for each $\gamma < \kappa$ and $n < \omega$ there is $\alpha, \gamma < \alpha < \kappa$, such that $o(\alpha) = \alpha^{+n}$. The present assumption is optimal by [6] and it is clearly weaker than those used in Section 5.4(for each $\gamma < \kappa$ and $\tau < \omega$ there is $\alpha, \gamma < \alpha < \kappa$, such that $o(\alpha) = \alpha^{+\tau}$) which is in turn optimal once GCH holds below κ, by [5].

We fix a sequence of cardinals $\kappa_0 < \kappa_1 < ... < \kappa_n < ..., n < \omega$ so that

- $\bigcup_{n<\omega} \kappa_n = \kappa$
- for every $0 < n < \omega$, κ_n is κ_n^{+n+2} - strong, as witnessed by an extender E_{κ_n}
- for every $n < \omega$, the normal measure of E_{κ_n} concentrates on

  τ's which are $\tau^{+n+2} + \eta(\tau, n)$ - strong as witnessed by a coherent sequence of extenders $\langle E_{\tau\xi} \mid \xi < \eta(\tau, n)\rangle$ of τ^{+n+2}-extenders , where $\eta(\tau, n)$ is an ordinal above τ^{+n+2} which is a repeat point (see C. Merimovich [13]).
  Such a length insures, by [13], that τ will remain a measurable after the Extender Based Radin forcing with the sequence $\langle E_{\tau\xi} \mid \xi < \eta(\tau, n)\rangle$.

Fix also an increasing sequence $\langle \lambda_n \mid n < \omega\rangle$ such that

- $\lambda_0 < \kappa_0$
- $\kappa_{n-1} < \lambda_n < \kappa_n$, for every $n, 0 < n < \omega$
- for every $n < \omega$, λ_n is λ_n^{+n+2} - strong as witnessed by an extender E_{λ_n}

We proceed as in 5.2. Instead of the Extender Based Magidor forcing, the Extender Based Magidor-Radin forcing of [13] is used. This leaves plenty of Mahlo cardinals of the form $\rho_{n\alpha}$ below $\rho_{n\theta}$ (the elements of the generic Magidor-Radin sequence at level n). So for any regular cardinal $\mu \leq \theta$ there will be enough possibilities of a form $\rho_{n\alpha}^{+n+2}$ to connect with μ in a way similar to those of 5.2.

5.6 Down to the first fixed point of the $\aleph$–function

It is possible to incorporate collapse in the construction of Section 5.4 in a fashion of [4] and to turn κ into the first fixed point of the $\aleph$–function. Basically what is needed is to collapse all unused cardinals.

Set η_0 to be κ_0^{+3} and $\eta_1 = \kappa_1^{+\kappa_0+2}$. For $n > 1$ set $\eta_n = \kappa_n^{+\eta_{n-1}+2}$.

Denote by ρ_n the indiscernible for the normal measure of E_n. Then the indiscernible which corresponds to θ^+ on level n will be $\rho_n^{+\eta_{n-1}+2}$, if $n > 1$, ρ_0^{+3}, if $n = 0$ and $\rho_1^{+\kappa_0+2}$ if $n = 1$.

Now the relevant collapses will turn ρ_0 into $\aleph_1$, will preserve $\rho_0^+, \rho_0^{++}, \rho_0^{+3}, \rho_0^{+4}$. Then κ_0 will be turned into the immediate successor of ρ_0^{+4}, all the cardinals $\kappa_0^+, \kappa_0^{++}, \kappa_0^{+3}, \kappa_0^{+4}$ will be preserved. Next, ρ_1 will be turned into the immediate successor of κ_0^{+4}, the cardinals in the interval $[\rho_1, \rho_1^{+\kappa_0+3}]$ will be preserved. Then κ_1 will become the immediate successor of $\rho_1^{+\kappa_0+3}$ and all the cardinals in the interval $[\kappa_1, \eta_1^+]$ will be preserved. ρ_2 will be turned into the immediate successor of η_1^+. All the cardinals of the interval $[\rho_2, \rho_2^{+\eta_1+3}]$ will be preserved, κ_2 will be turned into the immediate successor of $\rho_2^{+\eta_1+3}$, all the cardinals of the interval $[\kappa_2, \eta_2^+]$ will be preserved, and so on.

Let us turn now to a modification of the construction of Section 5.5 which will allow to finish with the least fixed point of the $\aleph$–function. Note that here we will lose GCH below κ (and even SCH will break down below). We would like to blow up the power of κ to $\kappa^{+\theta+1}$ for arbitrary θ starting with a singular cardinal κ of cofinality ω such that for each $\gamma < \kappa$ and $n < \omega$ there is $\alpha, \gamma < \alpha < \kappa$, such that $o(\alpha) = \alpha^{+n}$. In contrast to Section 5.5, we do not intend to create too long Magidor or Radin sequences, since having too many indiscernibles will prevent collapses needed in order to turn κ into the first fixed point.

The present assumption is optimal by [6] and it is clearly weaker than those used in Section 5.4(for each $\gamma < \kappa$ and $\tau < \omega$ there is $\alpha, \gamma < \alpha < \kappa$, such that $o(\alpha) = \alpha^{+\tau}$) which is in turn optimal once GCH holds below κ, by [5].

We fix a sequence of cardinals $\kappa_0 < \kappa_1 < ... < \kappa_n < ..., n < \omega$ so that

- $\bigcup_{n<\omega} \kappa_n = \kappa$
- for every $0 < n < \omega$, κ_n is κ_n^{+n+2} - strong, as witnessed by an extender E_{κ_n}
- for every $n < \omega$, the normal measure of E_{κ_n} concentrates on

  τ's which are $\tau^{+n+2} + \eta(\tau, n)$ - strong as witnessed by a coherent sequence of extenders $\langle E_{\tau\xi} \mid \xi < \eta(\tau, n)\rangle$ of τ^{+n+2}-extenders , where $\eta(\tau, n)$ is κ_{n-1}, if $n > 0$ and $\eta(\tau, n) = 1$, if $n = 0$.

The idea behind is like this:
once an indiscernible ρ_n for the normal measure of the extender E_{κ_n} was picked, then we use the coherent sequence $\langle E_{\rho_n \xi} \mid \xi < \eta(\tau, n) \rangle$ in order to change its cofinality to $\eta(\tau, n)$ using the Magidor Extender Based forcing (blowing its power to ρ_n^{+n+2}, as well). We have here κ_{n-1} (say $n > 0$) cardinals available along the Magidor sequence.

Now we use the method of Section 5.4. Just instead of κ_{n-1} over the level n placed "horizontally" there, we have them placed "vertically" here, i.e. instead of ordinals between κ_n and $\kappa_n^{+\kappa_{n-1}}$, we use members of the Magidor sequence of ρ_n.

Finally, since a relatively small number of indiscernibles are generated in the process, it is possible to define collapses as in the beginning of the section and turn κ into the first repeat point of the $\aleph$–function. Note that SCH will break at each ρ_n, $0 < n < \omega$.

Chapter 6

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