Short extenders forcings – doing without preparations.

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We would like to present a way of doing of short extenders forcings without forcing first with a preparation forcings of type $\mathcal{P}'$ of [1]. The main issue with short extenders forcings is to show that κ^{++} and cardinals above it are preserved in the final model. In [1] the preparation forcing (which added a structure with pistes) was used eventually to show κ^{++}-c.c. of the main forcing. A negative side of this preparation forcing is that it is only strategically closed which is not enough in order to preserve large cardinals like a supercompact. Actually it adds a version of the square principle which is incompatible with supercompacts [2].
Carmi Merimovich [5] used for the gap 3 a variation of Velleman's simplified morass [7] instead. κ^{++}-c.c. break down but he was able to show κ^{++}–properness instead. The forcing adding a simplified morass is directed closed enough in order to preserve supercompacts cardinals. Unfortunately generalizations (at least those that we considered) of Merimovich's idea of first adding a simplified morass and then to use a properness instead of a chain condition of the main forcing, run into server difficulties already for Gap 4.

Here we suggest an other way. The main forcing will be used directly over V without a preparation. Actually a simple version of the preparation forcing of [1] will be incorporated directly into the main forcing. Again as in [5] κ^{++}-c.c. will break down and we will show a properness instead.

1 Gap 4.

We deal here with the first new case - Gap 4.
Assume GCH.

1.1 Structures with pistes.

We present here a simple variation of the preparation forcing $\mathcal{P}'$ of [1].

Let us start with the main definition. It will be rather long, but one of the reasons of this is that we will treat each size (there will be three sizes) separately repeating similar properties. We hope that this way the matter will become more clear and intuitive.

Definition 1.1 Let $\delta < \kappa$ be cardinals and δ is a regular. A δ-structure with pistes over κ is a set $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle, \langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ such that the following conditions hold.

Start with requirements on models of the maximal size κ^{+3}. The structure of this models is the simplest one among the three sizes present. In contrast with two other sizes (κ^+, κ^{++}) they are linearly ordered by inclusion.

1. $A^{0\kappa^{+3}} \preccurlyeq \langle H(\chi), \in, \leq \rangle$, for some large enough regular χ (over κ we can take $\chi = \kappa^{+4}$ as well). It will be the largest model of size κ^{+3}.

2. $|A^{0\kappa^{+3}}| = \kappa^{+3}$,

3. $A^{0\kappa^{+3}} \in A^{1\kappa^{+3}}$,

4. $A^{1\kappa^{+3}}$ is a closed chain of at most δ elementary submodels of $A^{0\kappa^{++}}$,

5. each member of $A^{1\kappa^{+3}}$ has cardinality κ^{+3}. It will be convenient to identify sometimes $X \in A^{1\kappa^{+3}}$ with an ordinal $X \cap \kappa^{+4}$.

6. $A^{1\kappa^{+3}lim} \subseteq A^{1\kappa^{+3}}$. We refer to its elements as *potentially limit points*. Require the following:

   (a) if $X \in A^{1\kappa^{+3}lim}$, then X is a successor point of $A^{1\kappa^{+3}}$,

   (b) if $X \in A^{1\kappa^{+3}lim}$, then $\mathrm{cof}(X \cap \kappa^{+4}) = \kappa^{+3}$,
   or $\mathrm{cof}(X \cap \kappa^{+4}) = \kappa^{++}$,
   or $\mathrm{cof}(X \cap \kappa^{+4}) = \kappa^{+}$,

   (c) if $X \in A^{1\kappa^{+3}lim}$, then ${}^{\mathrm{cof}(X\cap\kappa^{+4})>}X \subseteq X$.

   The idea behind $A^{1\kappa^{+3}lim}$ is to provide places where $A^{1\kappa^{+3}}$ can be extended. Note, that in contrast with [1], the set $A^{1\kappa^{+3}}$ has a small cardinality and so its further extensions will be not only end-extensions.

7. If $X \in A^{1\kappa^{+3}lim}$, $\mathrm{cof}(X \cap \kappa^{+4}) = \mu$, for some μ, then there is an increasing continuous chain $\langle X_i \mid i < \mu\rangle$ of elementary submodels of X such that

(a) $\bigcup_{i<\mu} X_i = X$,

(b) $|X_i| = \kappa^{+3}$,

(c) $X_i \in X$,

Turn now to $C^{\kappa^{+3}}$.

8. $\mathrm{dom}(C^{\kappa^{+3}}) = A^{1\kappa^{+3}}$,

9. for every $B \in \mathrm{dom}(C^{\kappa^{+3}})$, $C^{\kappa^{+3}}(B) = (A^{1\kappa^{+3}} \cap B) \cup \{B\}$.
The function $C^{\kappa^{+3}}$ provides just initial segments of $A^{1\kappa^{+3}}$. It is included only in order to provide a similarity with cases of κ^+, κ^{++} in which the corresponding functions are non-trivial.

10. If $X \in A^{1\kappa^{+3}} \setminus A^{1\kappa^{+3}lim}$ is a non-limit model, then ${}^{\kappa^{++}}X \subseteq X$.

11. If $X \in A^{1\kappa^{+3}}$ is a non-limit model, $X \notin A^{1\kappa^{+3}lim}$, $A \in A^{1\kappa^+} \cup A^{1\kappa^{++}}$ and $X \in A$, then all immediate predecessors of X are in A (actually there is at most one immediate predecessor).
Note that we do not require this closure property for $X \in A^{1\kappa^{+3}lim}$ in order to allow further to add new elements below X.

12. If $X \in A^{1\kappa^{+3}}$ is a limit model, $A \in A^{1\kappa^+} \cup A^{1\kappa^{++}}$ and $X \in A$, then

$$X = \bigcup\{Z \in C^{\kappa^{+3}}(X) \mid Z \neq X, Z \in A\}.$$

Note that we do not require that $C^{\kappa^{+3}}(X) \in A$, but rather an unboundedness. The reason is that if we do so then $C^{\kappa^{+3}}(Y)$ for $Y \in A^{1\kappa^{+3}lim} \cap A$, will be in A as well, and then the immediate predecessor of Y will be in A– the thing that we like to avoid.

13. If $A \in A^{1\kappa^+} \cup A^{1\kappa^{++}}$, $X \in A^{1\kappa^{+3}lim}$, $\mathrm{cof}(X \cap \kappa^{+4}) = \kappa^{+3}$ and $X \in A$, then there is an increasing continuous chain $\langle X_i \mid i < \kappa^{+3}\rangle$ of elementary submodels of X such that

(a) $\langle X_i \mid i < \kappa^{+3}\rangle \in A$,

(b) $\bigcup_{i<\kappa^{+3}} X_i = X$,

(c) $|X_i| = \kappa^{+3}$,

(d) $X_i \in X$,

(e) the model $X_A := \bigcup_{i\in A} X_i$ is in $C^{\kappa^{+3}}(X) \cap A^{1\kappa^{+3}lim}$.

Note that

- $A \cap X = A \cap X_A$, since clearly $A \cap X \supseteq A \cap X_A$, and if $Z \in A \cap X$, then for some $i \in A$, $Z \in X_i$, and so $Z \in A \cap X_A$.
- If $\langle X'_i \mid i < \kappa^{+3} \rangle \in A$ is an other chain which satisfies all the conditions above, then $X_A = X'_A$. This follows from the continuity of the chains, unboundedness and elementarity of X.
  In particular, X_A is uniquely definable from X and A.
- If $X_A \subseteq Z \subseteq X$, then $A \cap Z = A \cap X$.

14. As the previous condition but for $\mathrm{cof}(X \cap \kappa^{+4}) = \kappa^{++}$ and $\mathrm{cof}(X \cap \kappa^{+4}) = \kappa^{+}$. The length of the chain of X_i's are changed accordingly.

15. Let $A \in A^{1\kappa^+} \cup A^{1\kappa^{++}}$, $X \in A^{1\kappa^{+3}lim}$ and $X \in A$. If $Z \in C^{\kappa^{+3}}(X_A)$, then there is $Z' \in C^{\kappa^{+3}}(X_A) \cap A$ such that $Z' \supseteq Z$.

16. Let Y be a successor element of $A^{1\kappa^{+3}}$ and Y_0 be its immediate predecessor. If $X \in (A^{1\kappa^+} \cup A^{1\kappa^{++}}) \cap Y$, then

    - $Y_0 \in X$
      or
    - $X \in Y_0$
      or
    - $X \subset Y_0$, $X \notin Y_0$ and then Y_0 is a limit point of $A^{1\kappa^{+3}}$ or Y_0 is a potentially limit point, i.e. $Y_0 \in A^{1\kappa^{+3}lim}$. In addition we require in this situation that also X is a limit point or a potentially limit point of $A^{1\kappa^+}$ or of $A^{1\kappa^{++}}$, and

$$\bigcup\{Z \in C^{\kappa^{+3}}(Y_0) \restriction Y_0 \mid Z \in X\} = Y_0.$$

17. If $X \in A^{1\kappa^+} \cup A^{1\kappa^{++}}$ and $X \not\subseteq A^{0\kappa^{+3}}$, then $A^{0\kappa^{+3}} \in X$.

    Let us state the requirements on $A^{1\kappa^{++}}$. They will be similar to those on $A^{1\kappa^{+3}}$, but the structure of models inside will not be anymore linear.

18. $A^{0\kappa^{++}} \preccurlyeq \langle H(\chi), \in, \leq \rangle$,

19. $|A^{0\kappa^{++}}| = \kappa^{++}$,

20. $A^{0\kappa^{++}} \in A^{1\kappa^{++}}$,

21. $A^{1\kappa^{++}}$ is a set of at most δ elementary submodels of $A^{0\kappa^{++}}$,

22. each element A of $A^{1\kappa^{++}}$ has cardinality κ^{++} and $A \cap \kappa^{+3}$ is an ordinal,

23. if $X, Y \in A^{1\kappa^{++}}$ then $X \in Y$ iff $X \subsetneq Y$,

24. $A^{1\kappa^{++}lim} \subseteq A^{1\kappa^{++}}$. We refer to its elements as *potentially limit points*. Require the following:

    (a) if $X \in A^{1\kappa^{++}lim}$ then it is a successor point of $A^{1\kappa^{++}}$,

    (b) if $X \in A^{1\kappa^{++}lim}$ then $\operatorname{cof}(X \cap \kappa^{+3}) = \kappa^{++}$ or $\operatorname{cof}(X \cap \kappa^{+3}) = \kappa^{+}$,

    (c) if $X \in A^{1\kappa^{++}lim}$ then ${}^{\operatorname{cof}(X\cap\kappa^{+3})>}X \subseteq X$,

    (d) X has at most one immediate predecessor in $A^{1\kappa^{++}}$.

25. $\operatorname{dom}(C^{\kappa^{++}}) = A^{1\kappa^{++}}$,

26. for every $B \in \operatorname{dom}(C^{\kappa^{++}})$, $C^{\kappa^{++}}(B)$ is a closed chain of models in $A^{1\kappa^{++}} \cap (B \cup \{B\})$ such that the following holds:

    (a) $B \in C^{\kappa^{++}}(B)$,

    (b) if $X \in C^{\kappa^{++}}(B)$, then $C^{\kappa^{++}}(X) = \{Y \in C^{\kappa^{++}}(B) \mid Y \in X \cup \{X\}\}$,

    (c) if B has immediate predecessors in $A^{1\kappa^{++}}$, then one of them is in $C^{\kappa^{++}}(B)$,

27. If $X \in A^{1\kappa^{++}} \setminus A^{1\kappa^{++}lim}$ is a non-limit model, then ${}^{\kappa^+}X \subseteq X$.

28. If $X \in A^{1\kappa^{++}}$ is a non-limit model, $X \notin A^{1\kappa^{++}lim}$, $A \in A^{1\kappa^{+}}$ and $X \in A$, then all immediate predecessors of X are in A.
    Note that we do not require this closure property for $X \in A^{1\kappa^{++}lim}$ in order to allow further to add new elements below X.

29. If $X \in A^{1\kappa^{++}}$ is a limit model, $A \in A^{1\kappa^{+}}$ and $X \in A$, then

$$X = \bigcup\{Z \in C^{\kappa^{++}}(X) \mid Z \neq X, Z \in A\}.$$

    Note that we do not require that $C^{\kappa^{++}}(X) \in A$, but rather an unboundedness. The reason is that if we do so then $C^{\kappa^{++}}(Y)$ for $Y \in A^{1\kappa^{++}lim} \cap A$, will be in A as well, and then the immediate predecessor of Y will be in A– the thing that we like to avoid.

30. If $A \in A^{1\kappa^{+}}$, $X \in A^{1\kappa^{++}lim}$, $\operatorname{cof}(X \cap \kappa^{+3}) = \kappa^{++}$ and $X \in A$, then there is an increasing continuous chain $\langle X_i \mid i < \kappa^{++}\rangle$ of elementary submodels of X such that

(a) $\langle X_i \mid i < \kappa^{++} \rangle \in A$,

(b) $\bigcup_{i<\kappa^{++}} X_i = X$,

(c) $|X_i| = \kappa^{++}$,

(d) $X_i \in X$,

(e) the model $X_A := \bigcup_{i \in A} X_i$ is in $C^{\kappa^{++}}(X) \cap A^{1\kappa^{++}lim}$.

Note that

- $A \cap X = A \cap X_A$, since clearly $A \cap X \supseteq A \cap X_A$, and if $Z \in A \cap X$, then for some $i \in A$, $Z \in X_i$, and so $Z \in A \cap X_A$.
- If $\langle X'_i \mid i < \kappa^{++} \rangle \in A$ is an other chain which satisfies all the conditions above, then $X_A = X'_A$. This follows from the continuity of the chains, unboundedness and elementarity of X.
  In particular, X_A is uniquely definable from X and A.
- If $X_A \subseteq Z \subseteq X$, then $A \cap Z = A \cap X$.

31. The same as previous condition only with $\mathrm{cof}(X \cap \kappa^{+3}) = \kappa^+$. The length of the chain of X_i's is κ^+.

32. Let $A \in A^{1\kappa^+}$, $X \in A^{1\kappa^{++}lim}$ and $X \in A$. If $Z \in C^{\kappa^{++}}(X_A)$, then there is $Z' \in C^{\kappa^{++}}(X_A) \cap A$ such that $Z' \supseteq Z$.

33. If $X \in A^{1\kappa^{++}}$ is a non-limit model, then either

(a) X is a minimal under $\in$ or equivalently under $\supsetneq$,
or

(b) X has a unique immediate predecessor in $A^{1\kappa^{++}}$,
or

(c) X has exactly two immediate predecessors X_0, X_1 in $A^{1\kappa^{++}}$ and X, X_0, X_1 form a Δ–*system triple* relatively to some $F_0, F_1 \in A^{1\kappa^{+3}}$ which means the following:

i. $F_0 \subsetneq F_1$ (or $F_1 \subsetneq F_0$),
ii. $X_0 \in F_1$ (or $X_1 \in F_0$),
iii. $F_0 \in X_0$ and $F_1 \in X_1$,
iv. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

v. the structures

$$\langle X_0, \in, X_0 \cap A^{1\kappa^{++}}, X_0 \cap A^{1\kappa^{++}lim}, X_0 \cap A^{1\kappa^{+3}}, X_0 \cap A^{1\kappa^{+3}lim},$$

$$(C^{\kappa^{++}} \restriction X_0 \cap A^{1\kappa^{++}}) \cap X_0, (C^{\kappa^{+3}} \restriction X_0 \cap A^{1\kappa^{+3}}) \cap X_0\rangle$$

and

$$\langle X_1, \in, X_1 \cap A^{1\kappa^{++}}, X_1 \cap A^{1\kappa^{++}lim}, X_1 \cap A^{1\kappa^{+3}}, X_1 \cap A^{1\kappa^{+3}lim},$$

$$(C^{\kappa^{++}} \restriction X_1 \cap A^{1\kappa^{++}}) \cap X_1, (C^{\kappa^{+3}} \restriction X_1 \cap A^{1\kappa^{+3}}) \cap X_1\rangle$$

are isomorphic over $X_0 \cap X_1$.

Further we will refer to such X as *a splitting point.*

34. Let Y be a successor element of $A^{1\kappa^{++}}$ with a unique immediate predecessors Y_0. If $X \in A^{1\kappa^{+}} \cap Y$, then

- $Y_0 \in X$
  or
- $X \in Y_0$
  or
- $X \subset Y_0$, $X \notin Y_0$ and then Y_0 is a limit point of $A^{1\kappa^{++}}$ or its potentially limit point. In addition we require in this situation that also X is a limit point of $A^{1\kappa^{+}}$ or its potentially limit point respectively, and, if limit

$$\bigcup\{Z \in C^{\kappa^{++}}(Y_0) \restriction Y_0 \mid Z \in X\} = Y_0.$$

35. If $X \in A^{1\kappa^{++}}$, $Y \in A^{1\kappa^{+}} \cup A^{1\kappa^{++}} \cup A^{1\kappa^{+3}}$ and $Y \in X$, then Y is *a piste reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n\rangle$ of elements of $A^{1\kappa^{++}}$ which we call *a piste leading to* Y such that

(a) $X = X(0)$,

(b) for every $i, 0 < i \leq n$, $X(i) \in C^{\kappa^{++}}(X(i-1))$ or $X(i-1)$ has two immediate successors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^{\kappa^{++}}(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$ or $Y = X(i-1)_1$,

(c) $Y = X(n)$, if $Y \in A^{1\kappa^{++}}$ and if $Y \in A^{1\kappa^{+}} \cup A^{1\kappa^{+3}}$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\kappa^{++}}$.

36. If $A \in A^{1\kappa^+}, X \in A^{1\kappa^{++}}, A \in X$ and X is a splitting point, then $A \in X'$, for some immediate predecessor X' of X.

    So elements of small cardinality are not allowed in between a splitting points and their immediate predecessors.

37. If $X \in A^{1\kappa^+}$ and $X \not\subseteq A^{0\kappa^{++}}$, then $A^{0\kappa^{++}} \in X$.

38. Either $A^{0\kappa^{++}} \in A^{0\kappa^{+3}}$ and then $A^{1\kappa^{++}} \subseteq A^{0\kappa^{+3}}$

    or

    $A^{0\kappa^{+3}} \in A^{0\kappa^{++}}$ and then $A^{1\kappa^{+3}} \setminus \{X_{A^{0\kappa^{++}}} \mid X \in A^{1\kappa^{+3}lim} \cap A^{0\kappa^{++}}\} \subseteq A^{0\kappa^{++}}$,
    or $A^{0\kappa^{++}} \in A^{1\kappa^{++}lim}$, $A^{0\kappa^{+3}} \in A^{1\kappa^{+3}lim}$, $A^{0\kappa^{++}} \subseteq A^{0\kappa^{+3}}$ and $\sup(A^{0\kappa^{++}} \cap \kappa^{+4}) = \sup(A^{0\kappa^{+3}} \cap \kappa^{+4})$.

    Finally let us state the requirements on $A^{1\kappa^+}$.

39. $A^{0\kappa^+} \preccurlyeq \langle H(\chi), \in, \leq \rangle$, for some fixed large enough χ,

40. $|A^{0\kappa^+}| = \kappa^+$,

41. $A^{0\kappa^+} \in A^{1\kappa^+}$,

42. $A^{1\kappa^+}$ is a set of at most δ elementary submodels of $A^{0\kappa^+}$,

43. each element A of $A^{1\kappa^+}$ has cardinality κ^+ and $A \cap \kappa^{++}$ is an ordinal,

44. if $X, Y \in A^{1\kappa^+}$ then $X \in Y$ iff $X \subsetneq Y$,

45. $A^{1\kappa^+lim} \subseteq A^{1\kappa^+}$. We refer to its elements as *potentially limit points*. Require the following:

    (a) if $X \in A^{1\kappa^+lim}$, then it is a successor point of $A^{1\kappa^+}$ and $\mathrm{cof}(X \cap \kappa^{++}) = \kappa^+$,

    (b) X has at most one immediate predecessor in $A^{1\kappa^+}$.

46. $\mathrm{dom}(C^{\kappa^+}) = A^{1\kappa^+}$,

47. for every $B \in \mathrm{dom}(C^{\kappa^+})$, $C^{\kappa^+}(B)$ is a closed chain of models in $A^{1\kappa^+} \cap (B \cup \{B\})$ such that the following holds:

    (a) $B \in C^{\kappa^+}(B)$,

    (b) if $X \in C^{\kappa^+}(B)$, then $C^{\kappa^+}(X) = \{Y \in C^{\kappa^+}(B) \mid Y \in X \cup \{X\}\}$,

(c) if B has immediate predecessors in $A^{1\kappa^+}$, then one of them is in $C^{\kappa^+}(B)$.

48. If $X \in A^{1\kappa^+}$ is a non-limit model, then ${}^{\kappa}X \subseteq X$.

49. If $X \in A^{1\kappa^+}$ is a non-limit model, then either

(a) X is a minimal under $\in$ or equivalently under $\supsetneq$,
or

(b) X has a unique immediate predecessor in $A^{1\kappa^+}$,
or

(c) X has exactly two immediate predecessors X_0, X_1 in $A^{1\kappa^+}$ and then either

i. X, X_0, X_1 form a Δ–*system triple* relatively to some $F_0, F_1 \in A^{1\kappa^{++}}$ which means the following:

A. $F_0 \subsetneq F_1$ (or $F_1 \subsetneq F_0$),

B. $F_0 \in X_0$ and $F_1 \in X_1$,

C. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

D. the structures

$$\langle X_0, \in, X_0 \cap A^{1\kappa^+}, X_0 \cap A^{1\kappa^+ lim}, X_0 \cap A^{1\kappa^{++}}, X_0 \cap A^{1\kappa^{++} lim}, X_0 \cap A^{1\kappa^{+3}}, X_0 \cap A^{1\kappa^{+3} lim},$$

$$(C^{\kappa^+} \restriction X_0 \cap A^{1\kappa^+}) \cap X_0, (C^{\kappa^{++}} \restriction X_0 \cap A^{1\kappa^{++}}) \cap X_0, (C^{\kappa^{+3}} \restriction X_0 \cap A^{1\kappa^{+3}}) \cap X_0 \rangle$$

and

$$\langle X_1, \in, X_1 \cap A^{1\kappa^+}, X_1 \cap A^{1\kappa^+ lim}, X_1 \cap A^{1\kappa^{++}}, X_1 \cap A^{1\kappa^{++} lim}, X_1 \cap A^{1\kappa^{+3}}, X_1 \cap A^{1\kappa^{+3} lim},$$

$$(C^{\kappa^+} \restriction X_1 \cap A^{1\kappa^+}) \cap X_1, (C^{\kappa^{++}} \restriction X_1 \cap A^{1\kappa^{++}}) \cap X_1, (C^{\kappa^{+3}} \restriction X_1 \cap A^{1\kappa^{+3}}) \cap X_1 \rangle$$

are isomorphic over $X_0 \cap X_1$.

Further we will refer to such X as *a splitting point.*

Or

ii. there are $G, G_0, G_1 \in X \cap A^{1\kappa^{++}}$ which form a Δ-system type triple such that

A. $X_0 \in G_0$,

B. $X_1 \in G_1$,

C. $X_1 = \pi_{G_0 G_1}[X_0]$,

D. $\pi_{G_0G_1} \upharpoonright X_0$ is the isomorphism between the structures

$$\langle X_0, \in, X_0 \cap A^{1\kappa^+}, X_0 \cap A^{1\kappa^+ lim}, X_0 \cap A^{1\kappa^{++}}, X_0 \cap A^{1\kappa^{++} lim}, X_0 \cap A^{1\kappa^{+3}}, X_0 \cap A^{1\kappa^{+3} lim},$$

$$(C^{\kappa^+} \upharpoonright X_0 \cap A^{1\kappa^+}) \cap X_0, (C^{\kappa^{++}} \upharpoonright X_0 \cap A^{1\kappa^{++}}) \cap X_0, (C^{\kappa^{+3}} \upharpoonright X_0 \cap A^{1\kappa^{+3}}) \cap X_0 \rangle$$

and

$$\langle X_1, \in, X_1 \cap A^{1\kappa^+}, X_1 \cap A^{1\kappa^+ lim}, X_1 \cap A^{1\kappa^{++}}, X_1 \cap A^{1\kappa^{++} lim}, X_1 \cap A^{1\kappa^{+3}}, X_1 \cap A^{1\kappa^{+3} lim},$$

$$(C^{\kappa^+} \upharpoonright X_1 \cap A^{1\kappa^+}) \cap X_1, (C^{\kappa^{++}} \upharpoonright X_1 \cap A^{1\kappa^{++}}) \cap X_1, (C^{\kappa^{+3}} \upharpoonright X_1 \cap A^{1\kappa^{+3}}) \cap X_1 \rangle$$

. Note $\pi_{G_0G_1}$ is identity on $X_0 \cap X_1$, since it is the identity on $G_0 \cap G_1$ and $X_0 \cap X_1 \subseteq G_0 \cap G_1$.

Further we will refer to such X as *a splitting point of higher order.*

50. If $X \in A^{1\kappa^+}$, $Y \in A^{1\kappa^+} \cup A^{1\kappa^{++}} \cup A^{1\kappa^{+3}}$ and $Y \in X$, then Y is *a piste reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n \rangle$ of elements of $A^{1\kappa^+}$ which we call *a piste leading to* Y such that
   (a) $X = X(0)$,
   (b) for every $i, 0 < i \leq n$, $X(i) \in C^{\kappa^+}(X(i-1))$ or $X(i-1)$ has two immediate successors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^{\kappa^+}(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$ or $Y = X(i-1)_1$,
   (c) $Y = X(n)$, if $Y \in A^{1\kappa^+}$ and if $Y \in A^{1\kappa^{++}} \cup A^{1\kappa^{+3}}$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\kappa^+}$.

51. Either $A^{0\kappa^+} \in A^{0\kappa^{+3}}$ and then $A^{1\kappa^+} \subseteq A^{0\kappa^{+3}}$
or $A^{0\kappa^{+3}} \in A^{0\kappa^+}$ and then $A^{1\kappa^{+3}} \setminus \{X_{A^{0\kappa^+}} \mid X \in A^{1\kappa^{+3} lim} \cap A^{0\kappa^+}\} \subseteq A^{0\kappa^+}$,
or $A^{0\kappa^+} \in A^{1\kappa^+ lim}$, $A^{0\kappa^{+3}} \in A^{1\kappa^{+3} lim}$, $A^{0\kappa^+} \subseteq A^{0\kappa^{+3}}$
and $\sup(A^{0\kappa^+} \cap \kappa^{+4}) = \sup(A^{0\kappa^{+3}} \cap \kappa^{+4})$.

52. Either $A^{0\kappa^+} \in A^{0\kappa^{++}}$ and then $A^{0\kappa^+} \subseteq A^{0\kappa^{++}}$
or $A^{0\kappa^{++}} \in A^{0\kappa^+}$ and then $A^{1\kappa^{++}} \setminus \{X_{A^{0\kappa^+}} \mid X \in A^{1\kappa^{++} lim} \cap A^{0\kappa^+}\} \subseteq A^{0\kappa^+}$,
or $A^{0\kappa^+} \in A^{1\kappa^+ lim}$, $A^{0\kappa^{++}} \in A^{1\kappa^{++} lim}$, $A^{0\kappa^+} \subseteq A^{0\kappa^{++}}$
and $\sup(A^{0\kappa^+} \cap \kappa^{+4}) = \sup(A^{0\kappa^{++}} \cap \kappa^{+4})$.

53. It is allowed that $A^{1\kappa^{+i}} = \emptyset$, for $i \in \{1, 2, 3\}$.

Remark 1.2 1. $<\delta$–structure with pistes over κ is defined the same way only requiring that the cardinality δ is replaced by cardinality less than δ.

2. It is possible to define a structure without pistes by requiring directness below limit models. This way it will be a direct generalization of Merimovich's fake morass [5].

3. In contrast with [1] pistes for models of different cardinalities need not go into the same direction here.
   Thus for example it is possible to have a Δ-system type triple X, X_0, X_1 with $X_0 \in C^{\kappa^{++}}$ and models A, B of cardinality κ^+ such that $B \in C^{\kappa^+}(A), B \in X_1 \setminus X_0$ and $X \in A$.

4. Also in we do not require here that once X, X_0, X_1 is a Δ-system type triple of models of cardinality κ^{++}, $A \in X_0$ of cardinality κ^+, then the image of A under the isomorphism $\pi_{X_0 X_1}$ of X_0, X_1 is in $A^{1\kappa^+}$.
   Such requirement complicated the matters a lot and was crucial in [1] since without it after forcing the preparation $2^{\kappa^{++}}$ will be κ^{+4}. Here this does not matter since there will be no preparation forcing at all. Still, a weaker property 1.1(49(c)ii) of this type seems still to be needed for properness (as well as for a chain condition).
   So the pistes used here are bit more complicated than the blue pistes of [1]. Let us refer to them as *red pistes*.

Let us define the intersection property.

Definition 1.3 (Models of size κ^{++}). Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle, \langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ be a δ-structure with pistes over κ.

Let $A, B \in A^{1\kappa^{++}}$. By $ip(A, B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then either

   - there is $X \in A \cap A^{1\kappa^{+3}}$ such that $A \cap B = A \cap X$,
     or

- there are $X \in A \cap A^{1\kappa^{+3}}, A' \in A \cap A^{1\kappa^{++}}$ such that $A \cap B = A \cap A' \cap X$.

Definition 1.4 (Model of size κ^+ with a model of size κ^{++}). Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle, \langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ be a δ-structure with pistes over κ.

Let $A \in A^{1\kappa^+}, B \in A^{1\kappa^{++}}$. By $ip(A,B)$ we mean the following:

1. $B \in A$,

   or

2. $A \subset B$,

   or

3. $B \notin A$, $A \not\subset B$ and then either

   - there is $B' \in A \cap A^{1\kappa^{++}}$ such that $A \cap B = A \cap B'$,

     or

   - there is $X \in A \cap A^{1\kappa^{+3}}$ such that $A \cap B = A \cap X$,

     or

   - there are $B' \in A \cap A^{1\kappa^{++}}, X \in A \cap A^{1\kappa^{+3}}$ such that $A \cap B = A \cap B' \cap X$.

Definition 1.5 (Intersection with models of size κ^{+3}). Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle, \langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ be a δ-structure with pistes over κ.

Let $A \in A^{1\kappa^+}$ or $A \in A^{1\kappa^{++}}$ and $Y \in A^{1\kappa^{+3}}$. By $ip(A,B)$ we mean the following:

1. $Y \in A$

   or

2. $A \subset Y$

   or

3. there is $Y' \in A \cap A^{1\kappa^{+3}}$ such that $A \cap Y = A \cap Y'$.

Definition 1.6 (Models of size κ^+). Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle, \langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ be a δ-structure with pistes over κ.

Let $A, B \in A^{1\kappa^+}$. By $ip(A,B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then either

   - there is $X \in A \cap A^{1\kappa^{++}}$ such that $A \cap B = A \cap X$,
     or
   - there are $X \in A \cap A^{1\kappa^{++}}, A' \in A \cap A^{1\kappa^{+}}$ such that $A \cap B = A \cap A' \cap X$,
     or
   - there are $Y \in A \cap A^{1\kappa^{+3}}, X \in A \cap A^{1\kappa^{++}}, A' \in A \cap A^{1\kappa^{+}}$ such that $A \cap B = A \cap A' \cap X \cap Y$.

If both $ip(A, B)$ and $ip(B, A)$ hold, then we denote this by $ipb(A, B)$.

Lemma 1.7 *Let $\langle\langle A^{0\kappa^{+}}, A^{1\kappa^{+}}, A^{1\kappa^{+}lim}, C^{\kappa^{+}}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle$, $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ be a δ-structure with pistes over κ. Assume $A \in A^{1\kappa^{++}}$ and $X \in A^{1\kappa^{+3}}$. Then $ip(A, X)$.*

Proof. Assume that $A \notin X$ and $X \notin A$. Let us split the proof into two cases.

Case 1. $X \in A^{0\kappa^{++}}$.

Consider the pistes from $A^{0\kappa^{++}}$ to X and to A. Let B be the last common point of this pistes. Then B is a successor model. Let B_0 be its immediate predecessor such that $A \in B_0 \cup \{B_0\}$. Then $X \notin B_0$. Also $X \not\supseteq B_0$, by the assumption.

Subcase 1.1. There are elements of $A^{1\kappa^{+3}} \cap B_0$ above X.

Let Z be the least like this. Then Z must be in $A^{1\kappa^{+3}lim}$. Thus if $Z \notin A^{1\kappa^{+3}lim}$ is a successor point of $A^{1\kappa^{+3}}$, then by 1.1(11) its immediate predecessor is in B_0. If Z is a limit point of $A^{1\kappa^{+3}}$, then by 1.1(12), Z cannot be the least.

Consider Z_{B_0} of 1.1(14). Then $X \not\subseteq Z_{B_0}$, by 1.1(15). So $Z_{B_0} \subseteq X \subseteq Z$. Then, by 1.1(14) $B_0 \cap X = B_0 \cap Z$. Hence

$$A \cap X = A \cap B_0 \cap X = A \cap B_0 \cap Z = A \cap Z,$$

and we can apply the induction to A, Z, since the common part of pistes to them is longer and so the last common model is smaller.

Subcase 1.2. There are no elements of $A^{1\kappa^{+3}} \cap B_0$ above X.
Assume first that there are elements of $A^{1\kappa^{+3}}$ which include B_0. Pick Z to be the least such. Clearly $Z \supset X$, since $X \not\supseteq B_0$. Then, by 1.1(16), Z should be a limit model and there should be elements $A^{0\kappa^{+3}} \cap B_0$ above X. Contradiction.

Hence there are no elements of $A^{1\kappa^{+3}}$ which include B_0. In particular, $B_0 \not\subseteq A^{0\kappa^{+3}}$. But then 1.1(17) implies that $A^{0\kappa^{+3}} \in B_0$, which gives the contradiction, since clearly $X \subseteq A^{0\kappa^{+3}}$.

Case 2. $X \notin A^{0\kappa^{++}}$.
If $A^{0\kappa^{+3}} \in A^{0\kappa^{++}}$ and $X = A^{0\kappa^{+3}}_{A^{0\kappa^{++}}}$, then

$$A \cap X = A \cap A^{0\kappa^{++}} \cap X = A \cap A^{0\kappa^{++}} \cap A^{0\kappa^{+3}}_{A^{0\kappa^{++}}} = A \cap A^{0\kappa^{++}} \cap A^{0\kappa^{+3}} = A \cap A^{0\kappa^{+3}},$$

and we are in the situation of Case 1.

So, then $A^{0\kappa^{++}} \in A^{0\kappa^{+3}}$. Pick the least element $Z \in A^{1\kappa^{+3}}$ which includes $A^{0\kappa^{++}}$. If $X \supseteq A^{0\kappa^{++}}$ then $X \supseteq A$. So, $X \not\supseteq A^{0\kappa^{++}}$. Hence $X \in Z$. Then by 1.1(16), $A^{0\kappa^{++}} \cap A^{1\kappa^{+3}} \neq \emptyset$. Let $Y \in A^{0\kappa^{++}} \cap A^{1\kappa^{+3}}$ be the least element which includes X. By 1.1(11,12), Y should be in $A^{1\kappa^{+3}lim}$. Then $A^{0\kappa^{++}} \cap X = A^{0\kappa^{++}} \cap Y$, as was pointed out in Subcase 1.1. But

$$A \cap X = A \cap A^{0\kappa^{++}} \cap X = A \cap A^{0\kappa^{++}} \cap Y = A \cap Y,$$

and $Y \in A^{0\kappa^{++}}$. So we are in the situation considered in Case 1.
□

Lemma 1.8 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a* δ*-structure with pistes over* κ*. Assume* $A \in A^{1\kappa^{++}}, B \in A^{1\kappa^{++}}$*. Then* $ipb(A, B)$*.*

Proof. Induction on pistes length.
□

Lemma 1.9 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a* δ*-structure with pistes over* κ*. Assume* $A \in A^{1\kappa^+}$ *and* $X \in A^{1\kappa^{+3}}$*. Then* $ip(A, X)$*.*

The proof repeats those of Lemma 1.7.

Lemma 1.10 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a* δ*-structure with pistes over* κ*. Assume* $A \in A^{1\kappa^+}$ *and* $X \in A^{1\kappa^{++}}$*. Then* $ip(A, X)$*.*

Proof. Assume that $A \notin X$ and $X \notin A$.Let us split the proof into two cases.

Case 1. $X \in A^{0\kappa^+}$.

Consider the pistes from $A^{0\kappa^+}$ to X and to A. Let B be the last common point of this pistes. Then B is a successor model. Let B_0 be its immediate predecessor such that $A \in B_0 \cup \{B_0\}$. Then $X \notin B_0$. Also $X \not\supseteq B_0$, by the assumption.

Subcase 1.1. There are elements of $A^{1\kappa^{++}} \cap B_0$ above X.

Let Z be the least like this. Then Z must be in $A^{1\kappa^{++}lim}$. Thus if $Z \notin A^{1\kappa^{++}lim}$ is a successor point of $A^{1\kappa^{+3}}$, then by 1.1(28) all of its immediate predecessors are in B_0. If Z is a limit point of $A^{1\kappa^{++}}$, then by 1.1(29), Z cannot be the least.

Consider Z_{B_0} of 1.1(30). Then $X \not\subseteq Z_{B_0}$, by 1.1(32).

If $Z_{B_0} \subseteq X$, then, by 1.1(30) $B_0 \cap X = B_0 \cap Z$. Hence

$$A \cap X = A \cap B_0 \cap X = A \cap B_0 \cap Z = A \cap Z,$$

and we can apply the induction to A, Z, since the common part of pistes to them is longer and so the last common model is smaller.

Suppose now that $Z_{B_0} \not\subseteq X$. Apply $ip(Z_{B_0}, X)$ and find $Y \in Z_{B_0} \cap A^{0\kappa^{+3}}$, $Z'_{B_0} \in (Z_{B_0} \cup \{Z_{B_0}\}) \cap A^{0\kappa^{++}}$ such that $Z_{B_0} \cap X = Z'_{B_0} \cap Y$. Then

$$A \cap X = A \cap Z \cap X = A \cap Z_{B_0} \cap X = A \cap Z'_{B_0} \cap Y.$$

If $Z'_{B_0} = Z_{B_0}$, then $A \cap Z'_{B_0} = A \cap Z$ and the induction applies. If $Z'_{B_0} \in Z_{B_0}$, then, by 1.1(32), we can apply induction to A and Z'_{B_0}.

Subcase 1.2. There are no elements of $A^{1\kappa^{++}} \cap B_0$ above X.

Assume first that there are elements of $A^{1\kappa^{++}}$ which include B_0. Pick Z to be the least such. Clearly $X \not\supseteq Z$, since $X \not\supseteq B_0$. If $Z \supset X$, then, by 1.1(34,36), Z should be a limit model and there should be elements $A^{0\kappa^{++}} \cap B_0$ above X. Contradiction.

Now use $ip(Z, X)$. There are $Z' \in Z \cup \{Z\}, Y \in A^{1\kappa^{+3}}$ such that $Z \cap X = Z' \cap Y$. Then

$$A \cap X = A \cap Z \cap X = A \cap Z' \cap Y.$$

If $Z' = Z$, then $A \cap Z' \cap Y = A \cap Y$ and $ip(A, Y)$ applies. If $Z' \in Z$, then $Z' \subset Z$ and then there will be elements of $A^{0\kappa^{++}} \cap B_0$ above Z', by 1.1(34,36). So we are in the situation of Subcase 1.1.

Suppose now that there are no elements of $A^{1\kappa^{++}}$ which include B_0. But then 1.1(52) gives the contradiction.

Case 2. $X \notin A^{0\kappa^+}$.
If $A^{0\kappa^{++}} \in A^{0\kappa^+}$ and $X = A^{0\kappa^{++}}_{A^{0\kappa^+}}$, then

$$A \cap X = A \cap A^{0\kappa^+} \cap X = A \cap A^{0\kappa^+} \cap A^{0\kappa^{++}}_{A^{0\kappa^+}} = A \cap A^{0\kappa^+} \cap A^{0\kappa^{++}} = A \cap A^{0\kappa^{++}},$$

and we are in the situation of Case 1.

So, then $A^{0\kappa^+} \in A^{0\kappa^{++}}$, by 1.1(52). Pick the least element $Z \in A^{1\kappa^{++}}$ which includes $A^{0\kappa^+}$. If $X \supseteq A^{0\kappa^+}$ then $X \supseteq A$. So, $X \not\supseteq A^{0\kappa^+}$. Clearly, $X \not\supseteq Z$. If $X \in Z$, then, by 1.1(16), $A^{0\kappa^+} \cap A^{1\kappa^{++}} \neq \emptyset$. Let $T \in A^{0\kappa^+} \cap A^{1\kappa^{++}}$ be the least element which includes X. By 1.1(28,29), T should be in $A^{1\kappa^{++}lim}$. Then $A^{0\kappa^+} \cap X = A^{0\kappa^+} \cap T$,
or $A^{0\kappa^+} \cap X = A^{0\kappa^+} \cap T' \cap Y$, for some $Y \in A^{1\kappa^{+3}} \cap T_{A^{0\kappa^+}}, T' \in T_{A^{0\kappa^+}}$ as was pointed out in Subcase 1.1. In the former case we have

$$A \cap X = A \cap A^{0\kappa^+} \cap X = A \cap A^{0\kappa^+} \cap T = A \cap T,$$

and $T \in A^{0\kappa^+}$. So we are in the situation considered in Case 1.
In the later case–

$$A \cap X = A \cap A^{0\kappa^+} \cap X = A \cap A^{0\kappa^+} \cap T' \cap Y = A \cap T' \cap Y.$$

By 1.1(32), there will be $P \in A^{0\kappa^+} \cap C^{\kappa^{++}}(T_{A^{0\kappa^+}})$ which includes T'. Hence we in the situation considered in Subcase 1.1 with B_0 replaced by $A^{0\kappa^+}$ and X by T'.

Suppose now that $X \notin Z$. Apply $ip(Z, X)$. There are $Z' \in Z \cup \{Z\}, Y \in A^{1\kappa^{+3}}$ such that $Z \cap X = Z' \cap Y$. Then

$$A \cap X = A \cap A^{0\kappa^+} \cap Z \cap X = A \cap Z' \cap Y.$$

If $Z' = Z$, then $A \cap Z' \cap Y = A \cap Y$ and $ip(A, Y)$ applies.
If $Z' \in Z$, then we are in the situation considered in the previous paragraph with X replaced by Z'.
□

Lemma 1.11 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a δ-structure with pistes over κ. Assume $A \in A^{1\kappa^+}, B \in A^{1\kappa^+}$. Then $ipb(A, B)$.*

Proof. Induction on pistes length. Let us only check the point related to red piste. Thus suppose that X is the last common point of pistes from $A^{0\kappa^+}$ to A and to B, and suppose that

X is a splitting point of higher order. Let X_0, X_1 be its immediate predecessors, G, G_0, G_1 be a Δ-system triple in $X \cap A^{1\kappa^{++}}$ which witness this. Suppose that $A \in X_0 \cup \{X_0\}$ and $B \in X_1 \cup \{X_1\}$. Then

$$A \cap B = A \cap X_0 \cap B \cap X_1 = A \cap G_0 \cap G_1 \cap B.$$

There is $Y_0 \in G_0 \cap A^{1\kappa^{+3}}$ such that $G_0 \cap G_1 = G_0 \cap Y_0$. Set $B_0 = \pi_{G_1 G_0}[B]$. Then $B \cap G_0 = B_0 \cap Y_0$, since $\pi_{G_1 G_0} \restriction G_0 \cap G_1$ is the identity. So,

$$A \cap B = A \cap G_0 \cap B_0 = A \cap B_0 \cap Y_0.$$

The induction applies to A, B_0.
$\square$

Lemma 1.12 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++} lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3} lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a* δ*-structure with pistes over* κ*. Suppose that* $A \in A^{1\kappa^+}$ *is a non-limit point and* $A \cap A^{1\kappa^{++}} \neq \emptyset$*. Then there is* $X \in A \cap A^{1\kappa^{++}}$ *which includes every element of* $A \cap A^{1\kappa^{++}}$*.*

Proof. If there is no elements of $A^{1\kappa^{++}}$ which include A, then $A^{0\kappa^{++}} \in A$, by 1.1(37), and we are done. Otherwise let as pick $Z \in A^{1\kappa^{++}}$ to be a least which (under inclusion or just the least point of the piste leading to A) includes A. Then Z must be a successor point, since A is a successor. So, by 1.1(36), Z has a unique predecessor Z_0. Now, by 1.1(34), since A is non-limit we must have $Z_0 \in A$.
$\square$

Next two lemmas are similar.

Lemma 1.13 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++} lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3} lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a* δ*-structure with pistes over* κ*. Suppose that* $A \in A^{1\kappa^+}$ *is a non-limit point and* $A \cap A^{1\kappa^{+3}} \neq \emptyset$*. Then there is* $X \in A \cap A^{1\kappa^{+3}}$ *which includes every element of* $A \cap A^{1\kappa^{+3}}$*.*

Lemma 1.14 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++} lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3} lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a* δ*-structure with pistes over* κ*. Suppose that* $A \in A^{1\kappa^{++}}$ *is a non-limit point and* $A \cap A^{1\kappa^{+3}} \neq \emptyset$*. Then there is* $X \in A \cap A^{1\kappa^{+3}}$ *which includes every element of* $A \cap A^{1\kappa^{+3}}$*.*

Notation. Denote the set of δ-structures with pistes over κ by $\mathcal{P}_{\kappa\delta}$ and similar the set of $<\delta$-structures with pistes over κ by $\mathcal{P}_{\kappa,<\delta}$.

Let us define a partial order over $\mathcal{P}_{\kappa\delta}$ ($\mathcal{P}_{\kappa,<\delta}$).

Definition 1.15 Let
$p_0 = \langle\langle A_0^{0\kappa^+}, A_0^{1\kappa^+}, A_0^{1\kappa^+lim}, C_0^{\kappa^+}\rangle, \langle A_0^{0\kappa^{++}}, A_0^{1\kappa^{++}}, A_0^{1\kappa^{++}lim}, C_0^{\kappa^{++}}\rangle, \langle A_0^{0\kappa^{+3}}, A_0^{1\kappa^{+3}}, A_0^{1\kappa^{+3}lim}, C_0^{\kappa^{+3}}\rangle\rangle$,
$p_1 = \langle\langle A_1^{0\kappa^+}, A_1^{1\kappa^+}, A_1^{1\kappa^+lim}, C_1^{\kappa^+}\rangle, \langle A_1^{0\kappa^{++}}, A_1^{1\kappa^{++}}, A_1^{1\kappa^{++}lim}, C_1^{\kappa^{++}}\rangle, \langle A_1^{0\kappa^{+3}}, A_1^{1\kappa^{+3}}, A_1^{1\kappa^{+3}lim}, C_1^{\kappa^{+3}}\rangle\rangle$
be in $\mathcal{P}_{\kappa\delta}$. Then $p_0 \leq p_1$ (p_1 extends p_0) iff

1. $A_0^{1\kappa^{+i}} \subseteq A_1^{1\kappa^{+i}}$, for every $i \in \{1, 2, 3\}$,

2. let $A \in A_0^{1\kappa^{+i}}$, for some $i \in \{1, 2, 3\}$, then $A \in A_0^{1\kappa^{+i}lim}$ iff $A \in A_1^{1\kappa^{+i}lim}$.
   The next item deals with a property called switching in [1]. In the present context it is much simpler due to simplicity of splittings and since we do not require that pistes of different cardinalities go the same way.

3. For every $A \in A_0^{1\kappa^{+i}}$, $C_0^{\kappa^{+i}}(A) \subseteq C_1^{\kappa^{+i}}(A)$, for every $i \in \{1, 2, 3\}$,
   or for some i's, $i \in \{1, 2, 3\}$ there are finitely many splitting (or generalized splitting) points $B(0), ..., B(k) \in A_0^{1\kappa^{+i}}$ with $B(j)', B''(j)$ the immediate predecessors of $B(j)$ $(j \leq k)$ such that

   (a) $B(j)' \in C_0^{\kappa^{+i}}(B(j))$,

   (b) $B(j)'' \in C_1^{\kappa^{+i}}(B(j))$.

4. If $A \in A_0^{1\kappa^{+i}}$ is a successor point and it is not in $A_0^{1\kappa^{+i}\,\mathrm{lim}}$, then A has the same immediate predecessors in $A_1^{1\kappa^{+i}}$.

So, by 1.15(4), potentially limit points are the only places where not end-extensions are allowed.

Notation. Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++}lim}, C^{\kappa^{++}}\rangle,$
$\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3}lim}, C^{\kappa^{+3}}\rangle\rangle$ be a δ-structure with pistes over κ.
Let $A \in A^{1\kappa^+} \cup A^{1\kappa^{++}} \cup A^{1\kappa^{++}}$.

1. Denote by $(A)_{\kappa^{+i}}, i \in \{1, 2, 3\}$ the maximal $B \in (A^{1\kappa^{+i}} \cap (A \cup \{A\}))$, if such B exists. Note that by 1.12,1.13,1.14, if A is a non–limit model and $A \in A^{1\kappa^+}$ then both $(A)_{\kappa^{++}}, (A)_{\kappa^{+3}}$ exist, and if $A \in A^{1\kappa^{++}}$, then $(A)_{\kappa^{+3}}$ exists.

2. Suppose that $(A)_{\kappa^{+i}}$ exists, for each $i, i \in \{1,2,3\}$. Denote then by $p \restriction A$ the set

$$\langle\langle (A)_{\kappa^+}, A^{1\kappa^+} \cap A, A^{1\kappa^+ lim} \cap A, (C^{\kappa^+} \restriction A^{1\kappa^+} \cap A) \cap A\rangle,$$
$$\langle (A)_{\kappa^{++}}, A^{1\kappa^{++}} \cap A, A^{1\kappa^{++} lim} \cap A, (C^{\kappa^{++}} \restriction A^{1\kappa^+} \cap A) \cap A\rangle,$$
$$(A)_{\kappa^{+3}}, A^{1\kappa^{+3}} \cap A, A^{1\kappa^{+3} lim} \cap A, (C^{\kappa^{+3}} \restriction A^{1\kappa^{+3}} \cap A) \cap A\rangle\rangle.$$

Lemma 1.16 *Let* $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, A^{1\kappa^+ lim}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, A^{1\kappa^{++} lim}, C^{\kappa^{++}}\rangle,$ $\langle A^{0\kappa^{+3}}, A^{1\kappa^{+3}}, A^{1\kappa^{+3} lim}, C^{\kappa^{+3}}\rangle\rangle$ *be a* δ*-structure with pistes over* κ*. Suppose that* $A \in A^{1\kappa^+} \cup A^{1\kappa^{++}} \cup A^{1\kappa^{++}}$ *is a non-limit point. If* $(A)_{\kappa^{+i}}, i \in \{1,2,3\}$ *exist, then* $p \restriction A$ *is in* $\mathcal{P}_{\kappa\delta}$ *and* $p \restriction A \leq p$*.*

Proof. Follows from 1.1, 1.15.
□

1.2 Suitable structures.

We reorganize here the structures with pistes of the previous section in order to allow isomorphisms of them over different cardinals.

Definition 1.17 Let $\delta < \kappa$ be cardinals and δ is a regular. A structure $\mathfrak{X} = \langle X, E, E^{lim}, C, \in, \subseteq \rangle$, where $E \subseteq [X]^2$ and $C \subseteq [X]^3$ is called a δ-suitable (or $< \delta$) structure with pistes over κ iff there is
$p(\mathfrak{X}) = \langle\langle A^{0\kappa^+}(\mathfrak{X}), A^{1\kappa^+}(\mathfrak{X}), A^{1\kappa^+ lim}(\mathfrak{X}), C^{\kappa^+}(\mathfrak{X})\rangle, \langle A^{0\kappa^{++}}(\mathfrak{X}), A^{1\kappa^{++}}(\mathfrak{X}), A^{1\kappa^{++} lim}(\mathfrak{X}), C^{\kappa^{++}}(\mathfrak{X})\rangle,$ $\langle A^{0\kappa^{+3}}(\mathfrak{X}), A^{1\kappa^{+3}}(\mathfrak{X}), A^{1\kappa^{+3} lim}(\mathfrak{X}), C^{\kappa^{+3}}(\mathfrak{X})\rangle\rangle$ a δ-structure (or $< \delta$) with pistes over κ such that

1. $X = A^{0i(X)}$, where $i(X) \in \{\kappa^+, \kappa^{++}, \kappa^{+3}\}$ is such that if $j \in \{\kappa^+, \kappa^{++}, \kappa^{+3}\}$, then $A^{0j} \in X$ or $A^{0j} \subseteq X$,
2. $\langle a, b\rangle \in E$ iff $a \in \{\kappa^+, \kappa^{++}, \kappa^{+3}\}$ and $b \in A^{1a}(\mathfrak{X})$,
3. $\langle a, b\rangle \in E^{lim}$ iff $a \in \{\kappa^+, \kappa^{++}, \kappa^{+3}\}$ and $b \in A^{1a lim}(\mathfrak{X})$,
4. $\langle a, b, d\rangle \in C$ iff $a \in \{\kappa^+, \kappa^{++}, \kappa^{+3}\}, b \in A^{1a}(\mathfrak{X})$ and $d \in C^a(\mathfrak{X})(b)$.

Let us refer to $\mathfrak{X}$ for shortness as a δ-suitable (or $< \delta$) structure once κ is fixed.

Note that $p(\mathfrak{X})$ is uniquely defined from $\mathfrak{X}$ and from $p \in \mathcal{P}_{\kappa\delta}$ it is easy to define a δ-suitable structure.

Definition 1.18 Let $\mathfrak{X}, \mathfrak{Y}$ be δ-suitable structures. Set $\mathfrak{X} \leq \mathfrak{Y}$ iff $p(\mathfrak{X}) \leq p(\mathfrak{Y})$.

1.3 Forcing conditions.

Let κ be a limit of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega \rangle$ with each κ_n being $\kappa_n^{+n+2+4} + 1$–strong as witnessed by an extender E_n.

For every $n < \omega$ define Q_{n0}.

Definition 1.19 Let Q_{n0} be the set of the triples $\langle a, A, f \rangle$ so that:

1. f is a partial function from κ^{+4} to κ_n of cardinality at most κ,

2. a is an isomorphism between a $< \kappa_n$–suitable structure $\mathfrak{X}$ over κ and a $< \kappa_n$–suitable structure $\mathfrak{X}'$over κ_n^{+n} such that

   (a) X' is above every model which appears in $A^{1\tau}(\mathfrak{X}')\backslash\{X'\}$, for some $\tau \in \{\kappa^+, \kappa^{++}, \kappa^{+3}\}$ in the order $\leq_{E_n}$, (or actually after codding X' by an ordinal),

   (b) if $t \in A^{1\kappa^+}(\mathfrak{X}') \cup A^{1\kappa^{++}}(\mathfrak{X}') \cup A^{1\kappa^{+3}}(\mathfrak{X}')$, then for some $k, 2 < k < \omega$, $t \prec H(\chi^{+k})$, with χ big enough fixed in advance.
   Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image t of a model from X will be elementary submodel of $H(\chi^{+k})$.
   The way to compare such models $t_1 \prec H(\chi^{+k_1})$, $t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
   move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{nX'}$,

4. for every ordinals α, β, γ which code models in $A^{1\kappa^+}(\mathfrak{X}') \cup A^{1\kappa^{++}}(\mathfrak{X}') \cup A^{1\kappa^{+3}}(\mathfrak{X}')$, we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \text{ implies}$$
$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho)),$$

   for every $\rho \in \pi''_{X'\alpha} A$.

Definition 1.20 Let $\langle a, A, f \rangle, \langle b, B, g \rangle$ be in Q_{n0}. Set $\langle a, A, f \rangle \geq_{n0} \langle b, B, g \rangle$ iff

1. $\mathrm{dom}(a) \geq \mathrm{dom}(b)$,

2. $\mathrm{ran}(a) \geq \mathrm{ran}(b)$,

3. $a \supseteq b$,

4. $f \supseteq g$,

5. $\pi^{E_n}_{\max(\mathrm{ran}(a)),\max(\mathrm{ran}(b)}$"$A \subseteq B$.

Definition 1.21 Q_{n1} consists of all partial functions $f : \kappa^{+3} \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

Definition 1.22 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq^*_n = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f \rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown \nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{ran}(a)),a(\alpha)}(\nu) \mid \alpha \in A^{1\kappa^{+3}}(\mathrm{dom}(a)) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $A^{1\kappa^{+3}}(\mathrm{dom}(a))$ are used to produce non direct extensions, the rest of models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq^*_n q$ or $p \in Q_{n1}, q = \langle b, B, g \rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown \nu$.

Definition 1.23 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

1. for every $n < \omega$, $\quad p_n \in Q_n$,
2. there is $\ell(p) < \omega$ such that
   (a) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,
   (b) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$,
   (c) if $\ell(p) \leq n \leq m$, then $\mathrm{dom}(a_n) \leq \mathrm{dom}(a_m)$,
   (d) if $\ell(p) \leq n \leq m$, then $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$.
3. For every $n \geq m \geq \ell(p)$, $\quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,
4. for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).
5. For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.

6. There is a κ-structure with pistes $\mathfrak{p}$ over κ such that

   (a) $\mathfrak{p} \geq \mathrm{dom}(a_n)$, for every $n, \ell(p) \leq n < \omega$,

   (b) if a model A appears in $\mathfrak{p}$, then A appears in $\mathrm{dom}(a_n)$ for some $n, \ell(p) \leq n < \omega$ (and then in a final segment of them),

   (c) $\max(\mathrm{dom}(a_n)) = \max(\mathfrak{p})$ (actually this follows from the previous condition).

Note that $\mathfrak{p}$ of 1.23(6) is uniquely determined by p. Let us refer to it further as the κ-structure with pistes over κ of p.

Lemma 1.24 *$\langle Q_{n0}, \leq_{n0} \rangle$ is $< \kappa_n$-strategically closed.*

Lemma 1.25 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 1.26 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Lemma 1.27 *Let $p \in \mathcal{P}$ and $\alpha < \kappa^{+4}$, then there are $q \geq^* p$ and $\beta, \alpha < \beta < \kappa^{+4}$ such that $\beta = M \cap \kappa^{+4}$, for some M which appears in Q.*

Proof. Pick some $M \prec H(\kappa^{+4})$ of size κ^{+3} which is above the maximal model of $\mathfrak{p}$ (say $\mathfrak{p} \in M$) and such that $M \cap \kappa^{+4} > \alpha$. Add it to p. Let q be the resulting condition. Then it is as desired.

□

The next lemma follows now:

Lemma 1.28 *Let G be a generic subset of $\langle \mathcal{P}, \leq \rangle$. Then in $V[G]$ there are $\mathrm{cof}((\kappa^{+4})^V)$–many ω–sequences of ordinals below κ.*

Define $\rightarrow$ on $\mathcal{P}$ as in [1].

κ^{++}–c.c., κ^{+3}–c.c. and even κ^{+4}–c.c. break down here for the forcing $\langle \mathcal{P}, \rightarrow \rangle$. Following C. Merimovich [5] we replace them by properness.

1.4 Properness.

The following basic definition is due to S. Shelah [6]:

Definition 1.29 Let $\eta > \omega$ be a regular cardinal and P a forcing notion. P is called η–proper iff for every $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \eta, {}^{\eta>}M \subseteq M$,

$P, p \in M$ there is $p' \geq_P p$ such that for every dense open $D \subseteq P, D \in M$, $p' \Vdash$ "$D \cap \underset{\sim}{G} \cap M \neq \emptyset$." Such p' is called (M, P)-generic.

The following is obvious:

Lemma 1.30 *If P is η-proper, then it preserves η^+.*

Our tusk will be to prove the following three lemmas:

Lemma 1.31 *$\langle \mathcal{P}, \rightarrow \rangle$ is κ^+-proper.*

Lemma 1.32 *$\langle \mathcal{P}, \rightarrow \rangle$ is κ^{++}-proper.*

Lemma 1.33 *$\langle \mathcal{P}, \rightarrow \rangle$ is κ^{+3}-proper.*

Proof of 1.31. Let $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \kappa^+, {}^\kappa M \subseteq M$, $P, p \in M$.

Let $\tilde{M}$ be a model of cardinality κ^{++} which is the union of a chain of models $\langle M_i \mid i < \kappa^+ \rangle$ such that

- $M_i \in M$,
- $M_i \prec H(\kappa^{+4})$,
- $\cup_{i<\kappa^+} M_i \cap M = M \cap H(\kappa^{+4})$.

Let $\tilde{\tilde{M}}$ be a model of cardinality κ^{+3} which is the union of a chain of models $\langle \tilde{M}_i \mid i < \kappa^+ \rangle$ such that

- $\tilde{M}_i \in M$,
- $\tilde{M}_i \prec H(\kappa^{+4})$,
- $M_i \subseteq \tilde{M}_i$,
- $\cup_{i<\kappa^{++}} \tilde{M}_i \cap M = M \cap H(\kappa^{+4})$.

Set $M' := M \cap H(\kappa^{+4})$.

Then $M' \subseteq \tilde{M} \subseteq \tilde{\tilde{M}}$ and $\sup(M' \cap \kappa^{+4}) = \sup(\tilde{M} \cap \kappa^{+4}) = \sup(\tilde{\tilde{M}} \cap \kappa^{+4})$.

Extend p by adding M',$\tilde{M}$ and $\tilde{\tilde{M}}$ as the largest models and also make them potentially limit points.

The role of $\tilde{M}$ and $\tilde{\tilde{M}}$ is to separate points of cardinalities κ^{++}, κ^{+3} which will be added

below in M from those above M. This is needed in order to satisfy 1.1(16). In the final stage of the argument after moving from M outside one may need points of of cardinalities κ^{++}, κ^{+3} in order to satisfy 1.1(16) and $\tilde{M}$,$\tilde{\tilde{M}}$ are such points. M' insures 1.1(34).
Let p' be the resulting condition. We claim that p' is (M, P)–generic.
Let $q \geq p'$ and $D \in M$ be a dense open. Let us show that there is an element of $D \cap M$ which is compatible with q. Consider $\mathfrak{q}$ the κ–structure with pistes over κ of q. Now, $\mathfrak{q} \restriction M'$ is κ–structure with pistes over κ, by 1.16, since $(M')_{\kappa^{++}}, (M')_{\kappa^{+3}}$ exist by 1.12,1.13.
Pick some $M'' \prec H(\kappa^{+4})$ of size κ^+, $M'' \in M'$ and such that $\mathfrak{q} \restriction M'$ with M' removed is in M''. Add M'' to $\mathfrak{q} \restriction M'$. It is possible, since M' is a potentially limit model. Denote the result by $\mathfrak{q}'$ and a corresponding condition by q' (i.e. we extend q in order to incorporate M'').
Set $\mathfrak{q}'' = \mathfrak{q}' \restriction M''$. Then, as above it is a κ–structure with pistes over κ. Let $q'' \in M$ be a corresponding condition. Pick $r \in M \cap D$ above q''. Combine r with q passing to an equivalent condition and moving models under isomorphisms of splitting points if necessary. The result will be as desired.
$\square$

Proof of 1.32.
Let $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \kappa^{++}, {}^{\kappa^+}M \subseteq M$, $P, p \in M$. Let $\tilde{M}$ be a model of cardinality κ^{+3} which is the union of a chain of models $\langle M_i \mid i < \kappa^{++} \rangle$ such that

- $M_i \in M$,
- $M_i \prec H(\kappa^{+4})$,
- $\cup_{i<\kappa^{++}} M_i \cap M = M \cap H(\kappa^{+4})$.

Consider $M' := M \cap H(\kappa^{+4})$. Extend p by adding M' and $\tilde{M}$ as the largest models and also make them potentially limit points.
The role of $\tilde{M}$ is to separate points of cardinality κ^{+3} which will be added below in M from those above M. This is needed in order to satisfy 1.1(16). In the final stage of the argument after moving from M outside one may need points of cardinality κ^{+3} in order to satisfy 1.1(16) and $\tilde{M}$ is such a point. M' insures 1.1(34). [1]

[1]Note that it is possible to have an extension of p' in which there is A of cardinality κ^+, $M', \tilde{M} \in A$ such that A is not potentially limit point. Moreover it has an immediate predecessor $A_0 \in M$. Still this does not prevent further extensions of p' which contain models B of cardinality κ^+ with $A_0 \in B \in M'$. Just reflections of A (or bigger models) to M' and then creation of Δ–system triples can be used for this purpose, as it will be done further in the proof.

Let p' be the resulting condition. We claim that p' is (M, P)–generic.
Let $q \geq p'$ and $D \in M$ be a dense open. Extending if necessary, we can assume that $q \in D$. Let us show that some condition in $D \cap M$ which is compatible with q.
Consider $\mathfrak{q}$ the κ–structure with pistes over κ of q. Extending if necessary, we can assume that $A^{0\kappa^+}(\mathfrak{q})$ is the maximal model of $\mathfrak{q}$. Consider also $\mathfrak{q} \restriction M'$. Note that it need not be κ–structure with pistes over κ, since there may be no single maximal model of size κ^+ inside. Let us reflect $A^{0\kappa^+}(\mathfrak{q})$ and q down to M over $A^{0\kappa^+}(\mathfrak{q}) \cap M$, i.e. we pick some $A' \in M$ and q' which realizes the same k-type (for some $k < \omega$ sufficiently big) over $A^{0\kappa^+}(\mathfrak{q}) \cap M$ as $A^{0\kappa^+}(\mathfrak{q})$ and q do in a rich enough language which includes D as well. [2] In particular $q' \in D \cap M$.
Now q' is compatible with q. Just pick some model A of cardinality κ^+ which includes all relevant information, i.e. $A^{0\kappa^+}(\mathfrak{q}), A', q, q', M'$ etc. The triple $A, A^{0\kappa^+}(\mathfrak{q}), A'$ will form a Δ–system triple relatively to M' and the model which corresponds to M' in A'. Combine q, q' together adding A as the maximal model and replacing models in the range of q by equivalent ones in order to fit with the range of q'.
□
Proof of 1.33. The argument repeats those of 1.32. Just M is picked of cardinality κ^{+3}, there is no need in $\tilde{M}$ and here so called red pistes apply 1.1(49(c)ii).
□

2 Arbitrary gaps.

We will extend here the setting of the previous section from gap 4 to an arbitrary gap.

2.1 Structures with pistes–arbitrary gaps.

Assume GCH.

Definition 2.1 Let $\delta < \eta < \theta$ be regular cardinals.
δ (or $< \delta$) structure with pistes over η of the length θ (our main application will be to the case when $\eta = \kappa^+$, so let us steak below to this situation) is a set $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ such that

1. (Support) s is a closed set of cardinals from the interval $[\kappa^+, \theta]$ (and once $\theta < \aleph_{\kappa^{++}}$ we can restrict ourself to regular cardinals only) satisfying the following:

[2] We follow here a suggestion by Carmi Merimovich to include D into the language which simplifies the original argument considerably.

(a) $|s| \leq \delta$, (or $|s| < \delta$ in case of $< \delta$–structure),

(b) $\kappa^+, \theta \in s$,

(c) if $\rho^+ \in s$ and $\rho > \kappa$, then $\rho \in s$,

(d) if $\rho \in s$ is singular, then s is unbounded in ρ and $\rho^+ \in s$.

2. (Models) For every $\tau \in s$ the following holds:

(a) $A^{0\tau} \preccurlyeq \langle H(\chi), \in, \leq \rangle$,

(b) $|A^{0\tau}| = \tau$,

(c) $A^{0\tau} \in A^{1\tau}$,

(d) $A^{1\tau}$ is a set of at most δ (or less than δ in case of $< \delta$–structure) elementary submodels of $A^{0\tau}$,

(e) each element A of $A^{1\tau}$ has cardinality τ, $A \supseteq \tau$ and $A \cap \tau^+$ is an ordinal,

(f) if $X, Y \in A^{1\tau}$ then $X \in Y$ iff $X \subsetneq Y$,

(g) (Potentially limit points) $A^{1\tau lim} \subseteq A^{1\tau}$. We refer to its elements as *potentially limit points*. Require the following:

i. if τ is a regular cardinal and $X \in A^{1\tau lim}$ then it is a successor point of $A^{1\tau}$ and $\kappa^+ \leq \mathrm{cof}(X \cap \tau^+) \leq \tau$,

ii. ${}^{\mathrm{cof}(X \cap \tau^+)>}X \subseteq X$,

iii. X has at most one immediate predecessor in $A^{1\tau}$.

iv. there is an increasing continuous chain $\langle X_i \mid i < \mathrm{cof}(X \cap \tau^+) \rangle$ of elementary submodels of X such that

A. $\bigcup_{i < \mathrm{cof}(X \cap \tau^+)} X_i = X$,

B. $|X_i| = \tau$,

C. $X_i \in X$.

(h) (Piste function) $\mathrm{dom}(C^\tau) = A^{1\tau}$,

(i) for every $B \in \mathrm{dom}(C^\tau)$, $C^\tau(B)$ is a closed chain of models in $A^{1\tau} \cap (B \cup \{B\})$ such that the following holds:

i. $B \in C^\tau(B)$,

ii. if $X \in C^\tau(B)$, then $C^\tau(X) = \{Y \in C^\tau(B) \mid Y \in X \cup \{X\}\}$,

iii. if B has immediate predecessors in $A^{1\tau}$, then one of them is in $C^\tau(B)$,

(j) If $X \in A^{1\tau} \setminus A^{1\tau lim}$ is a non-limit model, then ${}^{\mathrm{cof}(\tau)>}X \subseteq X$.

(k) If $X \in A^{1\tau}$ is a non-limit model, $X \notin A^{1\tau lim}$, $A \in A^{1\tau'}$, for some $\tau' \in s, \tau' \neq \tau$, and $X \in A$, then all immediate predecessors of X are in A.
Note that we do not require this closure property for $X \in A^{1\tau lim}$ in order to allow further to add new elements below X.

(l) If $X \in A^{1\tau}$ is a limit model, $A \in A^{1\tau'}$, for some $\tau' \in s, \tau' < \tau$, and $X \in A$, then

$$X = \bigcup\{Z \in C^{\tau}(X) \mid Z \neq X, Z \in A\}.$$

Note that we do not require that $C^{\tau}(X) \in A$, but rather an unboundedness. The reason is that if we do so then $C^{\tau}(Y)$ for $Y \in A^{1\tau lim} \cap A$, will be in A as well, and then the immediate predecessor of Y will be in A– the thing that we like to avoid.

(m) If τ is a regular cardinal, $A \in A^{1\tau'}$, for some $\tau' \in s, \tau' < \tau$, $X \in A^{1\tau lim}$ and $X \in A$, then there is an increasing continuous chain $\langle X_i \mid i < \mathrm{cof}(X \cap \tau^+)\rangle$ of elementary submodels of X such that

i. $\langle X_i \mid i < \mathrm{cof}(X \cap \tau^+)\rangle \in A$,
ii. $\bigcup_{i<\mathrm{cof}(X\cap\tau^+)} X_i = X$,
iii. $|X_i| = \tau$,
iv. $X_i \in X$,
v. the model $X_A := \bigcup_{i\in A} X_i$ is in $C^{\tau}(X) \cap A^{1\tau lim}$.

Note that

- $A \cap X = A \cap X_A$, since clearly $A \cap X \supseteq A \cap X_A$, and if $Z \in A \cap X$, then for some $i \in A$, $Z \in X_i$, and so $Z \in A \cap X_A$.
- If $\langle X'_i \mid i < \mathrm{cof}(X \cap \tau^+)\rangle \in A$ is an other chain which satisfies all the conditions above, then $X_A = X'_A$. This follows from the continuity of the chains, unboundedness and elementarity of X.
  In particular, X_A is uniquely definable from X and A.
- If $X_A \subseteq Z \subseteq X$, then $A \cap Z = A \cap X$.

(n) Let τ is a regular cardinal, $A \in A^{1\tau'}$, for some $\tau' \in s, \tau' < \tau$, $X \in A^{1\tau lim}$ and $X \in A$. If $Z \in C^{\tau}(X_A)$, then there is $Z' \in C^{\tau}(X_A) \cap A$ such that $Z' \supseteq Z$.

(o) If $X \in A^{1\tau}$ is a non-limit model, then either

i. X is a minimal under $\in$ or equivalently under $\supsetneqq$,

or

ii. X has a unique immediate predecessor in $A^{1\tau}$,

or

iii. X has exactly two immediate predecessors X_0, X_1 in $A^{1\tau}$ and X, X_0, X_1 form a Δ–*system triple* relatively to some $F_0, F_1 \in A^{1\tau^*}$, $\tau^* = \min(s \setminus \tau + 1)$, which means the following:

A. $F_0 \subsetneqq F_1$ (or $F_1 \subsetneqq F_0$),

B. $X_0 \in F_1$ (or $X_1 \in F_0$),

C. $F_0 \in X_0$ and $F_1 \in X_1$,

D. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

E. the structures

$$\langle X_0, \in, \langle X_0 \cap A^{1\rho}, X_0 \cap A^{1\rho lim}, (C^\rho \restriction X_0 \cap A^{1\rho}) \cap X_0 \mid \rho \in (s \setminus \tau) \cap X_0 \rangle\rangle$$

and

$$\langle X_1, \in, \langle X_1 \cap A^{1\rho}, X_1 \cap A^{1\rho lim}, (C^\rho \restriction X_1 \cap A^{1\rho}) \cap X_1 \mid \rho \in (s \setminus \tau) \cap X_1 \rangle\rangle$$

are isomorphic over $X_0 \cap X_1$.

F. $X \in A^{0\tau^*}$.

Further we will refer to such X as *a splitting point.*

Or

iv. there are $G, G_0, G_1 \in X \cap A^{1\mu}$, for some $\mu \in s \setminus \min(s \setminus \tau + 1)$, which form a Δ-type triple with witnessing models in X such that

A. $X_0 \in G_0$,

B. $X_1 \in G_1$,

C. $X_1 = \pi_{G_0 G_1}[X_0]$.

D. $X \in A^{0\mu}$,

E. $X \in A^{0\mu^*}$, where $\mu^* = \min(s \setminus \mu + 1)$.

Further we will refer to such X as *a splitting point of higher order.*

(p) Let Y be a successor element of $A^{1\tau}$ with a unique immediate predecessors Y_0. If $X \in A^{1\tau'} \cap Y$, for some $\tau' \in s, \tau' < \tau$ and $\tau \in X$, then

i. $Y_0 \in X$ and then $X \in A^{1\tau' lim}$ implies that also $Y \in A^{1\tau lim}$.

We did not require that there is no overlapping of potentially limit point of small cardinality with non-limit point of higher cardinality in the gap 4 case. It is possible to do without to do without this once $\theta < \kappa^{+\delta}$. Non-existence of such overlapping was crucial for the properness arguments, see Lemma 1.32. It was arranged easily since there was only three possible sizes of models involved. Here the number of possible sizes may be much bigger than δ.

Or

ii. $X \in Y_0$

or

iii. $X \subset Y_0$, $X \notin Y_0$ and then Y_0 is a limit point of $A^{1\tau}$ or its potentially limit point. In addition we require in this situation that also X is a limit point of $A^{1\tau'}$ or its potentially limit point accordingly, and

$$\bigcup\{Z \in C^\tau(Y_0) \restriction Y_0 \mid Z \in X\} = Y_0.$$

(q) Let Y be a successor element of $A^{1\tau}$ with a unique immediate predecessors Y_0. If $X \in A^{1\tau'} \cap Y$, for some $\tau' \in s, \tau' < \tau$ and $\tau \notin X$, then

i. $X \in Y_0$,

or

ii. $X \subset Y_0$, $X \notin Y_0$ and then Y_0 is a limit point of $A^{1\tau}$ or its potentially limit point. In addition we require in this situation that also X is a limit point of $A^{1\tau'}$ or its potentially limit point accordingly, and

$$\bigcup\{Z \in C^\tau(Y_0) \restriction Y_0 \mid Z \in X\} = Y_0.$$

Or

iii. There are $\mu < \tau, \mu \in X \cap s$ and an increasing continuous sequence $\langle Y_0(\alpha) \mid \alpha \in Card \cap [\mu, \eta]\rangle \in X$, where $\eta = \min(X \cap s \setminus \tau)$ such that

A. $Y_0(\alpha) \in A^{1\alpha}$, if $\alpha \in s$,

B. $Y_0(\tau) = Y_0$,

C. $\bigcup\{(Y_0(\alpha))_X \mid \alpha \in Card \cap [\mu, \eta] \cap X \setminus \{\eta\}\}$ is in $A^{1\sup(Card \cap [\mu,\eta] \cap X \setminus \{\eta\})}$.

Note that then the sequence $\langle (Y_0(\alpha))_X \mid \alpha \in Card \cap [\mu, \eta]\rangle$ (defined as in 2m) is continuous as well and $X \cap Y_0 = X \cap Y_0(\eta) = \bigcup\{(Y_0(\alpha))_X \mid \alpha \in Card \cap [\mu, \eta] \cap X \setminus \{\eta\}\}$.

Require again here that $X \in A^{1\tau' lim}$ implies that also $Y \in A^{1\tau lim}$.

(r) If $X \in A^{1\tau'}$, for some $\tau' \in s, \tau' < \tau$, and $X \not\subseteq A^{0\tau}$, then $A^{0\tau} \in X$.

(s) If $X \in A^{1\tau}$, $Y \in \bigcup_{\rho \in s} A^{1\rho}$ and $Y \in X$, then Y is *a piste reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n \rangle$ of elements of $A^{1\tau}$ which we call *a piste leading to* Y such that

i. $X = X(0)$,

ii. for every $i, 0 < i \leq n$, $X(i) \in C^\tau(X(i-1))$ or $X(i-1)$ has two immediate successors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^\tau(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$ or $Y = X(i-1)_1$,

iii. $Y = X(n)$, if $Y \in A^{1\tau}$ and if $Y \in A^{1\rho}$, for some $\rho \neq \tau$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\tau}$.

(t) If $A \in A^{1\tau'}$, $\tau' \in s, \tau' < \tau$, $X \in A^{1\tau}, A \in X$ and X is a splitting point, then $A \in X'$, for some immediate predecessor X' of X.
So elements of small cardinality are not allowed in between a splitting points and their immediate predecessors.

3. Let $\eta < \rho, \eta, \rho \in s$ and $Z \in A^{1\eta}$.
If $Z \notin A^{0\rho}$, then $A^{0\rho} \in Z$ and $A^{0\rho} \setminus \{X_Z \mid X \in A^{1\tau lim} \cap Z\} \subseteq Z$.

4. Let $\tau' < \tau, \tau', \tau \in s$. Then either $A^{0\tau'} \in A^{0\tau}$ and then $A^{1\tau'} \subseteq A^{0\tau}$
or $A^{0\tau} \in A^{0\tau'}$ and then $A^{1\tau} \setminus \{X_{A^{0\tau'}} \mid X \in A^{1\tau lim} \cap A^{0\tau'}\} \subseteq A^{0\tau'}$

or $A^{0\tau} \in A^{1\tau lim}$, $A^{0\tau'} \in A^{1\tau' lim}$, $A^{0\tau'} \subseteq A^{0\tau}$ and $\sup(A^{0\tau'} \cap \theta) = \sup(A^{0\tau} \cap \theta)$.

5. There is a regular $\tau \in s$ such that for every $\rho \in s, \rho \neq \tau$ we have
$A^{0\rho} \in A^{0\tau}$ or $A^{0\tau} \in A^{1\tau lim}, A^{0\rho} \in A^{1\rho lim}$ and then

(a) $\sup(A^{0\rho} \cap \theta) = \sup(A^{0\tau} \cap \theta)$,

(b) if $\tau < \rho$, then $A^{0\tau} \subseteq A^{0\rho}$,

(c) if $\rho < \tau$, then $A^{0\rho} \subseteq A^{0\tau}$.

6. It is allowed that $A^{1\tau} = \emptyset$, for $\tau \in s$.

Let us define the intersection property.

Definition 2.2 (Models of different sizes). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau \rangle \mid \tau \in s\rangle$ be a δ structure with pistes over κ of the length θ.

Let $A \in A^{1\tau}, B \in A^{1\rho}$ and $\tau < \rho$. By $ip(A, B)$ we mean the following:

1. $B \in A$,
   or

2. $A \subset B$,
   or

3. $B \notin A$, $A \not\subset B$ and then

   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \rho) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

Definition 2.3 (Models of a same size). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a δ structure with pistes over κ of the length θ.

Let $A, B \in A^{1\tau}$. By $ip(A,B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then

   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \tau) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

If both $ip(A,B)$ and $ip(B,A)$ hold, then we denote this by $ipb(A,B)$.

Lemma 2.4 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a δ structure with pistes over κ of the length θ. Assume $A \in A^{1\tau}, B \in A^{1\rho}$, for some $\tau \leq \rho, \tau, \rho \in s$. Then $ip(A,B)$ and if $\tau = \rho$, then also $ipb(A,B)$.*

Proof. Assume that $A \neq B, A \notin B$ and $B \notin A$. If $A \notin A^{0\rho}$, then, by 2.1(3), $A \supset A^{1\rho}$ and $B \in A$.
So suppose that $A \in A^{0\rho}$.
Let $X \in A^{1\rho}$ be a least element of $A^{1\rho}$ which includes both A and B.
Let us assume first that X is a splitting point. Proceed by induction on rank (X).
So $A \cap B = A \cap B_0 \cap H_0$, for some $H_0 \in X \cap A^{1\eta}$, $\eta \in s \setminus \rho + 1$. Consider a least model Z of $A^{1\eta}$ which includes X. Then $H_0 \in Z$ and it must have a unique immediate predecessor.

Denote it Z_0. Then $Z_0 \in X$ and $Z_0 \supseteq H_0$. $Z \supseteq X$ implies $A \in Z$. Then $Z_0 \in A$ or $A \in Z_0$. In the later case the induction applies to A, H_0, since rank$(X) >$rank(Z_0).

Suppose now that X does not split. Let X_0 be its immediate predecessor. Then $B = X_0$ or $B \in X_0$. If $A \in X_0$ then $B = X_0$ is impossible by the initial assumptions and $B \in X_0$ will contradict the minimality of X.

Suppose that $X_0 \in A$. Then $B \neq X_0$, and hence $B \in X_0$. Let $Z \in A \cap A^{1\rho}$ be a least model with $B \in Z$ (piste from Z to B as far as it runs in A). Then $Z \in A^{1\rho lim}$.

Consider Z_A of 2.1(2m). Then $B \not\subseteq Z_A$, by 2.1(32).

If $Z_A \subseteq B$, then, by 2.1(2m) $B \cap A = A \cap Z_A = A \cap Z$.

Suppose now that $Z_A \not\subseteq B$. It is enough to show $ip(B, Z_A)$, since $A \cap B = A \cap Z_A \cap B$ and once the intersection with B is replaced by intersections with members of Z_A–induction can be applied.

Apply $ip(Z_A, B)$ (the induction applies to $\langle Z_A, B\rangle$, since the rank of Z is smaller than the rank of X) and find $\eta_1 < ... < \eta_m$ in $(s \setminus \rho) \cap Z_A$ and $Z_1 \in A^{1\eta_1} \cap Z_A, ..., Z_m \in A^{1\eta_m} \cap Z_A$ such that $Z_A \cap B = Z_A \cap Z_1 \cap ... \cap Z_m$.

Then

$$A \cap B = A \cap Z \cap B = A \cap Z_A \cap B = A \cap Z_A \cap Z_1 \cap ... \cap Z_m = A \cap Z \cap Z_1 \cap ... \cap Z_m.$$

By 2.1(2n), we can apply induction to A and $Z_1, ..., Z_m$.

Consider now the last possibility when $\rho \notin A$ and the case 2.1(2(q)iii) holds. Then there are $\mu < \rho, \mu \in A \cap s$ and an increasing continuous sequence $\langle X_0(\alpha) \mid \alpha \in Card \cap [\mu, \eta]\rangle \in A$, where $\eta = \min(A \cap s \setminus \rho)$ such that

1. $X_0(\alpha) \in A^{1\alpha}$, if $\alpha \in s$,

2. $X_0(\tau) = X_0$,

Also the sequence $\langle (X_0(\alpha))_A \mid \alpha \in Card \cap [\mu, \eta]\rangle$ is continuous and $A \cap X_0 = A \cap X_0(\eta) = \bigcup\{(X_0(\alpha))_A \mid \alpha \in Card \cap [\mu, \eta] \cap A \setminus \{\eta\}\}$.

Now, if $B \supseteq \bigcup\{(X_0(\alpha))_A \mid \alpha \in Card \cap [\mu, \eta] \cap A \setminus \{\eta\}\}$, then we are done. Suppose that $B \not\supseteq \bigcup\{(X_0(\alpha))_A \mid \alpha \in Card \cap [\mu, \eta] \cap A \setminus \{\eta\}\}$. Denote $\bigcup\{(X_0(\alpha))_A \mid \alpha \in Card \cap [\mu, \eta] \cap A \setminus \{\eta\}\}$ by Y. Apply $ip(Y, B)$. The induction applies to $\langle Y, B\rangle$, since the rank of X_0 is smaller than the rank of X. Find $\eta_1 < ... < \eta_m$ in $(s \setminus \rho) \cap Y$ and $Z_1 \in A^{1\eta_1} \cap Y, ..., Z_m \in A^{1\eta_m} \cap Y$ such that $Y \cap B = Y \cap Z_1 \cap ... \cap Z_m$.

Then

$$A \cap B = A \cap X_0(\eta) \cap B = A \cap Y \cap B = A \cap Y \cap Z_1 \cap ... \cap Z_m = A \cap Z \cap Z_1 \cap ... \cap Z_m.$$

By 2.1(2n), we can apply induction to A and $Z_1, ..., Z_m$.
$\square$

Lemma 2.5 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a δ structure with pistes over κ of the length θ. Suppose that $\tau, \rho \in s, \tau < \rho$, $A \in A^{1\tau}$ is a non-limit point and $A \cap A^{1\rho} \neq \emptyset$. Then there is $X \in A \cap A^{1\rho}$ which includes every element of $A \cap A^{1\rho}$.*

Proof. If there is no elements of $A^{1\rho}$ which include A, then $A^{0\rho} \in A$, by 2.1(3), and we are done. Otherwise let us pick $Z \in A^{1\rho}$ to be a least which (under inclusion or just the least point of the piste leading to A) includes A. Then Z must be a successor point, since A is a successor. So, by 2.1(2t), Z has a unique predecessor Z_0. Now, by 2.1(2p), since A is non-limit we must have $Z_0 \in A$.
$\square$

Notation. Denote the set of δ structure with pistes over κ of the length θ by $\mathcal{P}_{\theta\kappa\delta}$, and similar the set of $< \delta$-structures with pistes over κ by $\mathcal{P}_{\theta\kappa<\delta}$.
Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle \in \mathcal{P}_{\theta\kappa\delta}$ (or in $\mathcal{P}_{\theta\kappa<\delta}$).
Denote further $A^{0\tau}$ by $A^{0\tau}(p)$, $A^{1\tau}$ by $A^{0\tau}(p)$, $A^{1\tau lim}$ by $A^{1\tau lim}(p)$, C^{τ} by $C^{\tau}(p)$ and s by $s(p)$.
Call s the support of p.

Let us define a partial order over $\mathcal{P}_{\theta\kappa\delta}$ ($\mathcal{P}_{\theta\kappa<\delta}$).

Definition 2.6 Let
$p_0 = \langle\langle A_0^{0\tau}, A_0^{1\tau}, A_0^{1\tau lim}, C_0^{\tau}\rangle \mid \tau \in s_0\rangle$, $p_1 = \langle\langle A_1^{0\tau}, A_1^{1\tau}, A_1^{1\tau lim}, C_1^{\tau}\rangle \mid \tau \in s_1\rangle$ be a δ structure with pistes over κ of the length θ. Then $p_0 \leq p_1$ (p_1 extends p_0) iff

1. $s_0 \subseteq s_1$,

2. $A_0^{1\tau} \subseteq A_1^{1\tau}$, for every $\tau \in s_0$,

3. let $A \in A_0^{1\tau}$, then $A \in A_0^{1\tau lim}$ iff $A \in A_1^{1\tau lim}$.
   The next item deals with a property called switching in [1]. In the present context it is much simpler due to simplicity of splittings and since we do not require that pistes of different cardinalities go the same way.

4. For every $A \in A_0^{1\tau}$, $C_0^{\tau}(A) \subseteq C_1^{\tau}(A)$,
   or
   there are finitely many splitting (or generalized splitting) points $B(0), ..., B(k) \in A_0^{1\tau}$ with $B(j)', B''(j)$ the immediate predecessors of $B(j)$ ($j \leq k$) such that

(a) $B(j)' \in C_0^\tau(B(j))$,

(b) $B(j)'' \in C_1^\tau(B(j))$.

5. if $A \in A_0^{1\tau}$ is a successor point and it is not in $A_0^{1\tau\,\mathrm{lim}}$, then A has the same immediate predecessors in $A_1^{1\tau}$.

So, by 2.6(5), potentially limit points are the only places where not end-extensions are allowed.

Remark 2.7 We are not going to force with $\mathcal{P}_{\theta\kappa\delta}$ or with $\mathcal{P}_{\theta\kappa<\delta}$, but rather to use them as domains of conditions of a further forcing. However, the forcing with it may be of an interest. Thus, as was stated in the beginning of Definition 2.1, a regular cardinal η can be used instead of κ^+, and, for example $\mathcal{P}_{\eta,\omega,<\omega}$ may be of an interest on its own since the forcing with it will add a club subset to $\aleph_{\omega+1}$ by finite conditions which runs away from every countable set in the ground model.
Let $G \subseteq \mathcal{P}_{\eta,\omega,<\omega}$ be a generic. The argument that cardinals are preserved in $V[G]$ is a bit easier version of one for the main forcing in the next section. Let us find a club $C \subseteq \eta$ which does not include any countable set of V. Proceed as follows. Pick some $A \in A^{1\eta lim}(p)$ for some $p \in G$. Let $E \subseteq A \cap \eta^+$ be a club in V of order type η. Set

$$F = \{B \cap \eta^+ \mid \exists q \geq p, q \in G \text{ such that } B \in A \cap (A^{1\eta}(q) \setminus A^{1\eta lim}(q))\}.$$

Then F is an unbounded subset of $A \cap \eta^+$, by density arguments since A is a potentially limit model. Let F' be the closure of F. Set $C = E \cap F'$. We claim that it is as desired. Thus suppose that $x \in V$ is a countable subset of E. Let as argue that $x \not\subseteq C$. Work in V. Let $q \geq p$ be a condition. q is finite, so we can extend it to some q' by adding models B, B_0 in $A \cap (A^{1\eta}(q') \setminus A^{1\eta lim}(q'))$ such that B_0 is the unique immediate predecessor of B and $B_0 \cap \eta^+ < \sup(x) < B \cap \eta^+$. Then now elements of F will be able to entre the interval $(B_0 \cap \eta^+, B \cap \eta^+)$. Hence q' will force that C does not contain x.

Notation. Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a δ structure with pistes over κ of the length θ.
Let $A \in \bigcup_{\tau\in s} A^{1\tau}$.

1. Denote by $(A)_\rho, \rho \in s$ the maximal $B \in (A^{1\rho} \cap (A \cup \{A\}))$, if such B exists.
   Note that by 2.5, if A is a non–limit model and $A \cap A^{1\rho} \neq \emptyset$ then $(A)_\rho$ exists.

2. Suppose that $(A)_\rho$ exists, for $\rho \in s$. Denote then by $p \restriction A$ the set $\langle\langle (A)_\rho, A^{1\rho} \cap A, A^{1\rho lim} \cap A, (C^\rho \restriction A^{1\rho} \cap A) \cap A\rangle \mid \rho \in A \cap s\rangle$.

Lemma 2.8 *$p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a δ structure with pistes over κ of the length θ.*
Let $A \in \bigcup_{\tau\in s} A^{1\tau}$. Suppose that $A \in \bigcup_{\tau\in s} A^{1\tau}$ is a non-limit point. If $(A)_\rho, \rho \in s \cap A$ exist, then $p \restriction A$ is in $\mathcal{P}_{\theta\kappa\delta}$ and $p \restriction A \leq p$.

Proof. Follows from 2.1, 2.6.
□

2.2 Suitable structures – arbitrary gaps.

We reorganize here the structures with pistes of the previous section in order to allow isomorphisms of them over different cardinals.

Definition 2.9 Let $\delta < \kappa < \theta$ be cardinals and δ, θ is a regular. A structure $\mathfrak{X} = \langle X, E, E^{lim}, C, S, \in, \subseteq \rangle$, where $E \subseteq [X]^2$ and $C \subseteq [X]^3$ is called a δ-suitable (or $< \delta$) structure with pistes over κ of the length θ iff there is a δ structure with pistes over κ of the length θ
$p(\mathfrak{X}) = \langle\langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), A^{1\tau lim}(\mathfrak{X}), C^{\tau}(\mathfrak{X})\rangle \mid \tau \in s(\mathfrak{X})\rangle$ such that

1. $X = A^{0\eta}(\mathfrak{X})$, where $\eta \in s(\mathfrak{X})$ is such that for every $\tau \in s(\mathfrak{X})$ we have then $A^{0\tau}(\mathfrak{X}) \in X$ or $A^{0\tau}(\mathfrak{X}) \subseteq X$,
2. $S = s(\mathfrak{X})$,
3. $\langle a, b\rangle \in E$ iff $a \in S$ and $b \in A^{1a}(\mathfrak{X})$,
4. $\langle a, b\rangle \in E^{lim}$ iff $a \in S$ and $b \in A^{1alim}(\mathfrak{X})$,
5. $\langle a, b, d\rangle \in C$ iff $a \in S, b \in A^{1a}(\mathfrak{X})$ and $d \in C^{a}(\mathfrak{X})(b)$.

Let us refer to $\mathfrak{X}$ for shortness as a a δ-suitable (or $< \delta$) structure once κ, θ are fixed.

Note that $p(\mathfrak{X})$ is uniquely defined from $\mathfrak{X}$. Also, it is easy to define a δ-suitable structure from $p \in \mathcal{P}_{\kappa\delta\theta}$.

Definition 2.10 Let $\mathfrak{X}, \mathfrak{Y}$ be δ-suitable structures. Set $\mathfrak{X} \leq \mathfrak{Y}$ iff $p(\mathfrak{X}) \leq p(\mathfrak{Y})$.

2.3 Forcing conditions–arbitrary gaps.

Let κ be a limit of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega\rangle$ with each κ_n being strong up to the least Mahlo cardinal λ_n above κ_n as witnessed by an extender E_n.

For every $n < \omega$ define Q_{n0}.

Definition 2.11 Let Q_{n0} be the set of the triples $\langle a, A, f\rangle$ so that:

1. f is a partial function from θ^+ to κ_n of cardinality at most κ,

2. a is an isomorphism between a $< \kappa_n$–suitable structure $\mathfrak{X}$ over κ of the length θ and a $< \kappa_n$–suitable structure $\mathfrak{X}'$over κ_n^{+n} of the length λ_n such that

   (a) X' is above every model which appears in $(\bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')) \setminus \{X'\}$, in the order $\leq_{E_n}$, (or actually after codding X' by an ordinal),

   (b) if $t \in A^{1\kappa^+}(\mathfrak{X}') \cup A^{1\kappa^{++}}(\mathfrak{X}') \cup A^{1\kappa^{+3}}(\mathfrak{X}')$, then for some $k, 2 < k < \omega$, $t \prec H(\chi^{+k})$, with χ big enough fixed in advance.
   Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image t of a model from X will be elementary submodel of $H(\chi^{+k})$.
   The way to compare such models $t_1 \prec H(\chi^{+k_1})$, $t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
   move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{nX'}$,

4. for every ordinals α, β, γ which code models in $\bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')$, we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \text{ implies}$$

$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho)),$$

   for every $\rho \in \pi''_{X'\alpha} A$.

Definition 2.12 Let $\langle a, A, f\rangle, \langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $\mathrm{dom}(a) \geq \mathrm{dom}(b)$,

2. $\mathrm{ran}(a) \geq \mathrm{ran}(b)$,

3. $a \supseteq b$,

4. $f \supseteq g$,

5. $\pi^{E_n}_{\max(\mathrm{ran}(a)),\max(\mathrm{ran}(b)}$ “$A \subseteq B$.

Definition 2.13 Q_{n1} consists of all partial functions $f : \kappa^{+3} \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

Definition 2.14 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq^*_n = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f \rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown\nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{ran}(a)),a(\alpha)}(\nu) \mid \alpha \in A^{1\kappa^{+3}}(\mathrm{dom}(a)) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $A^{1\theta}(\mathrm{dom}(a))$ are used to produce non direct extensions, the rest of models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq^*_n q$ or $p \in Q_{n1}, q = \langle b, B, g\rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown\nu$.

Definition 2.15 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega\rangle$ so that

1. for every $n < \omega$, $\quad p_n \in Q_n$,

2. there is $\ell(p) < \omega$ such that

   (a) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,

   (b) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n\rangle \in Q_{n0}$,

   (c) if $\ell(p) \leq n \leq m$, then $\mathrm{dom}(a_n) \leq \mathrm{dom}(a_m)$,

   (d) if $\ell(p) \leq n \leq m$, then $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$.

3. For every $n \geq m \geq \ell(p)$, $\quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,

4. for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).

5. For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.

6. There is a κ-structure with pistes $\mathfrak{p}$ over κ such that

   (a) $\mathfrak{p} \geq \mathrm{dom}(a_n)$, for every $n, \ell(p) \leq n < \omega$,

   (b) if a model A appears in $\mathfrak{p}$, then A appears in $\mathrm{dom}(a_n)$ for some $n, \ell(p) \leq n < \omega$ (and then in a final segment of them),

   (c) $\max(\mathrm{dom}(a_n)) = \max(\mathfrak{p})$ (actually this follows from the previous condition).

Note that $\mathfrak{p}$ of 2.15(6) is uniquely determined by p. Let us refer to it further as the κ-structure with pistes over κ of p.

Lemma 2.16 *$\langle Q_{n0}, \leq_{n0} \rangle$ is $< \kappa_n$-strategically closed.*

Lemma 2.17 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 2.18 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Lemma 2.19 *Let $p \in \mathcal{P}$ and $\alpha < \theta^+$, then there are $q \geq^* p$ and $\beta, \alpha < \beta < \theta^+$ such that $\beta = M \cap \theta^+$, for some M which appears in Q.*

Proof. Pick some $M \prec H(\theta^+)$ of size θ which is above the maximal model of $\mathfrak{p}$ (say $\mathfrak{p} \in M$) and such that $M \cap \theta^+ > \alpha$. Add it to p. Let q be the resulting condition. Then it is as desired.

□

The next lemma follows now:

Lemma 2.20 *Let G be a generic subset of $\langle \mathcal{P}, \leq \rangle$. Then in $V[G]$ there are $\mathrm{cof}((\theta^+)^V)$–many ω–sequences of ordinals below κ.*

Define $\rightarrow$ on $\mathcal{P}$ as in [1].

κ^{++}–c.c. and even θ^+–c.c. break down here for the forcing $\langle \mathcal{P}, \rightarrow \rangle$.
Following C. Merimovich [5] we replace them by properness.

2.4 Properness–arbitrary gaps.

The following basic definition is due to S. Shelah [6]:

Definition 2.21 Let $\eta > \omega$ be a regular cardinal and P a forcing notion. P is called η–proper iff for every $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \eta, {}^{\eta>}M \subseteq M$,

$P, p \in M$ there is $p' \geq_P p$ such that for every dense open $D \subseteq P, D \in M$, $p' \Vdash$ "$D \cap \underset{\sim}{G} \cap M \neq \emptyset$." Such p' is called (M, P)-generic.

The following is obvious:

Lemma 2.22 *If P is η-proper, then it preserves η^+.*

Our tusk will be to prove the following two lemmas:

Lemma 2.23 *$\langle \mathcal{P}, \to \rangle$ is κ^+-proper.*

Lemma 2.24 *$\langle \mathcal{P}, \to \rangle$ is η-proper, for every regular $\eta, \kappa^+ \leq \eta \leq \theta$.*

The proofs are similar to those of Section 1.

Proof of 2.23. Let $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \kappa^+, {}^\kappa M \subseteq M$, $P, p \in M$.

Set $M' := M \cap H(\kappa^{+4})$. Extend p by adding M' as the largest model, make it potentially limit point. We use 2.1(2p(i),2q(iii)) to insure that there are can be no overlapping of M' with non-potentially limit models of bigger cardinalities. This is needed at the final stage of the argument in order to show compatibility.
Let p' be the resulting condition. We claim that p' is (M, P)–generic.
Let $q \geq p'$ and $D \in M$ be a dense open. Let us show that there is an element of $D \cap M$ which is compatible with q. Consider $\mathfrak{q}$ the κ–structure with pistes over κ of q. Now, $\mathfrak{q} \restriction M'$ is κ–structure with pistes over κ of the length θ, by 2.8, since $(M')_\tau$'s exist by 2.5.
Pick some $M'' \prec H(\kappa^{+4})$ of size κ^+, $M'' \in M'$ and such that $\mathfrak{q} \restriction M'$ with M' removed is in M''. Add M'' to $\mathfrak{q} \restriction M'$. It is possible by 2.1(2p), since M' is a potentially limit model. Denote the result by $\mathfrak{q}'$ and a corresponding condition by q' (i.e. we extend q in order to incorporate M'').
Set $\mathfrak{q}'' = \mathfrak{q}' \restriction M''$. Then, as above it is a κ–structure with pistes over κ. Let $q'' \in M$ be a corresponding condition. Pick $r \in M \cap D$ above q''. Combine r with q passing to an equivalent condition if necessary. The result will be as desired.
$\square$
Proof of 2.24.
Let η be a regular cardinal such that $\kappa^+ < \eta \leq \theta$. Suppose that $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \eta, {}^{\eta>}M \subseteq M$, $P, p \in M$.

Set $M' := M \cap H(\theta^+)$. Extend p by adding M' as the largest model, make it potentially limit point We use 2.1(2p(i),2q(iii)) to insure that there are can be no overlapping of M'

with non-potentially limit models of bigger cardinalities. This is needed at the final stage of the argument in order to show compatibility.
Let p' be the resulting condition. We claim that p' is (M, P)–generic.
Let $q \geq p'$ and $D \in M$ be a dense open. Extending if necessary, we can assume that $q \in D$. Let us show that some condition in $D \cap M$ which is compatible with q.
Consider $\mathfrak{q}$ the κ–structure with pistes over κ of q. Extending if necessary, we can assume that $A^{0\kappa^+}(\mathfrak{q})$ is the maximal model of $\mathfrak{q}$. Consider also $\mathfrak{q} \restriction M'$. Note that it need not be κ–structure with pistes over κ, since there may be no single maximal model of size κ^+ inside. Let us reflect $A^{0\kappa^+}(\mathfrak{q})$ and q down to M over $A^{0\kappa^+}(\mathfrak{q}) \cap M$, i.e. we pick some $A' \in M$ and q' which realizes the same k-type (for some $k < \omega$ sufficiently big) over $A^{0\kappa^+}(\mathfrak{q}) \cap M$ as $A^{0\kappa^+}(\mathfrak{q})$ and q do in a rich enough language which includes D as well. [3] In particular $q' \in D \cap M$.
Now q' is compatible with q. Just pick some model A of cardinality κ^+ which includes all relevant information, i.e. $A^{0\kappa^+}(\mathfrak{q}), A', q, q', M'$ etc. The triple $A, A^{0\kappa^+}(\mathfrak{q}), A'$ will form a Δ–system triple relatively to M' and the model which corresponds to M' in A'. Combine q, q' together adding A as the maximal model and replacing models in the range of q by equivalent ones in order to fit with the range of q'.
$\square$

Finally, combining together Lemmas 2.17, 2.18, 2.20, 2.23, 2.24, we obtain the following:

Theorem 2.25 *Let G be a generic subset of $\langle \mathcal{P}, \rightarrow \rangle$. Then $V[G]$ is cofinalities preserving extension of V in which $2^\kappa = \kappa^\omega = \theta^+$.*

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