Structures with pistes - different sizes.

Moti Gitik[*]

June 27, 2016

Abstract

We generalize [5] and introduce structures with pistes which may have different number of models in every cardinality.

1 Structures with pistes–general setting.

Assume GCH.
As in [5], the first part (1.1) describes this "linear" part of conditions in the main forcing. It is called *a wide piste* and incorporates together elementary chains of models of different cardinalities. The main forcing, defined in 1.2, will be based on such wide pistes and involves an additional natural but non-linear component called splitting or reflection.

Definition 1.1 Let $\eta < \theta$ be regular cardinals, $\mathfrak{S}$ be a function from the set $\{\tau \mid \eta \leq \tau \leq \theta$ and τ is a regular cardinal $\}$ to θ, such that for every $\tau \in \mathrm{dom}(\mathfrak{S})$, $\mathfrak{S}(\tau)$ is a cardinal $< \tau$[1].
Assume also that $\mathfrak{S}(\tau) \leq \mathfrak{S}(\theta)$, for every $\tau \in \mathrm{dom}(\mathfrak{S})$.
A $(\theta, \eta, \mathfrak{S})-$wide piste is a set $\langle \langle C^\tau, C^{\tau lim} \rangle \mid \tau \in s \rangle$ such that the following hold.
Let us first specify sizes of models that are involved.

1. (Support) s is a closed set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:

   (a) $|s| < \eta$,[2]

[*]The author was partially supported by Israel Science Foundation Grant No. 58/14.
[1]The requirement $\mathfrak{S}(\tau) < \tau$ is essential for properness arguments. Once $\mathfrak{S}(\tau) = \tau$, then we are basically in a situation of [1] and arguments around chain condition and strategic closure replace properness.
[2]We need the ability to cover s by models of the least possible size, so s of cardinality above η is not allowed. Also, models of the least allowed cardinality η are not more than $< \eta-$closed, so it puts an additional restriction on the size of the support.

(b) $\eta, \theta \in s$.

Which means that the minimal and the maximal possible sizes are always present.

2. (Models) For every $\tau \in s$ and $A \in C^\tau$ the following holds:

   (a) $A \preccurlyeq \langle H(\theta^+), \in, \leq, \mathfrak{S}, \eta \rangle$,

   (b) $|A| = \tau$,

   (c) $A \supseteq \tau$,

   (d) $A \cap \tau^+$ is an ordinal,

   (e) elements of C^τ form a closed $\in -$chain with a largest element of a length $< \mathfrak{S}(\tau)$,

   (f) if $X \in C^\tau \setminus C^{\tau lim}$ is a non-limit model (i.e. is not a union of elements of C^τ), then ${}^{\tau>}X \subseteq X$.

   (g) if $X, Y \in C^\tau$ then $X \in Y$ iff $X \subsetneq Y$,

3. (Potentially limit points) Let $\tau \in s$.
   $C^{\tau lim} \subseteq C^\tau$. We refer to its elements as *potentially limit points.*
   The intuition behind is that once extending it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.
   Let $X \in C^{\tau lim}$. Require the following:

   (a) X is a successor point of C^τ.

   (b) (Increasing union) There is an increasing continuous $\in -$chain $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+)) \rangle$ [3] of elementary submodels of X such that

      i. $\bigcup_{i<\mathrm{cof}(\sup(X\cap\theta^+))} X_i = X$,

      ii. $|X_i| = \tau$,

      iii. $X_i \supseteq \tau$,

      iv. $X_i \in X$,

      v. ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

   (c) (Degree of closure of potentially limit point)
   Either

[3] This models need not be in C^τ, but rather allow to add in future extensions models below X

i. ${}^{\tau>}X \subseteq X$

or

ii. $\mathrm{cof}(\sup(X \cap \theta^+)) = \xi$ for some $\xi \in s \cap \tau$ and then

A. ${}^{\xi>}X \subseteq X$,

B. there are $X_\theta \in C^{\theta lim}, X_\xi \in C^{\xi lim}$ such that $X \cap \theta^+ = \sup(X_\xi \cap \theta^+) = \sup(X \cap \theta^+)$ and there is a sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ witnessing 3(b) which members belong to X_ξ.

Further the condition (9(b)) will imply that $X' \supseteq X \supseteq X''$. Eventually (once extending) for every regular $\mu, \tau \leq \mu \leq \theta$ there will be $X''' \in C^{\mu lim}, X \subseteq X''' \subseteq X'$.

Note that if $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ and $\langle X'_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ are two sequences which witness (3b) above, then the set $\{i < \mathrm{cof}(\sup(X \cap \theta^+)) \mid X_i = X'_i\}$ is closed and unbounded.

It is possible using the well ordering $\leq$ to define a canonical witnessing sequence $\langle X_i \mid i < \mathrm{cof}(X \cap \theta^+)\rangle$ for X.

Let first do this for X such that $\mathrm{cof}(X \cap \theta^+) = \tau$ (or for X_ξ of (3c(ii)(B)) above). Fix the well ordering $\langle x_\nu \mid \nu < \tau\rangle$. We proceed by induction. Once $i < \tau$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least elementary submodel of X such that

- $x_i \in X_{i+1}$,
- $X_i \in X_{i+1}$,
- $|X_i| = \tau$,
- $X_i \supseteq \tau$,
- ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

By (3b), it is possible to find such X_{i+1}.

Clearly $\bigcup_{i<\tau} X_i = X$.

Suppose now that $\mathrm{cof}(X \cap \theta^+) = \xi \in s \cap \tau$. Then let us use the canonical sequence $\langle X_{i\xi} \mid i < \xi = \mathrm{cof}(X \cap \theta^+)\rangle$ for X_ξ in order to define the canonical sequence $\langle X_i \mid i < \mathrm{cof}(X \cap \theta^+)\rangle$ for X.

Proceed by induction. Once $i < \tau$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least elementary submodel of $H(\theta)$ such that

- $X_{i+1} \in X_\xi$,
- $X_{i\xi} \in X_{i+1}$,

- $X_i \in X_{i+1}$,
- $|X_i| = \xi$,
- $X_i \supseteq \xi$,
- ${}^{\xi>}X_{i+1} \subseteq X_{i+1}$.

By (3c(ii)B), it is possible to find such X_{i+1} inside X_ξ.
Note that the existence of such canonical sequences implies that X itself is definable from X_ξ.

The next condition prevent unneeded appearances of small models between big ones.

4. If $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point and B_0 is its immediate predecessor, then there is no potentially limit point $A \in C^\tau$ with $\tau < \rho$ such that $B_0 \in A \in B_1$.

   It is possible to require that no A at all, i.e. potentially limit or not, appears between B_0 and B_1. The requirement that B_1 is not a potentially limit point is important here. Once dealing with potentially limit points, we would like to allow reflections which may add small intermediate models.

   Next condition is of a similar flavor, but deals with smallest models.

5. If $B \in C^\rho$, for some $\rho \in s$, is not a potentially limit point and it is the least element of C^ρ, then there is no potentially limit point $A \in C^\tau$ with $\tau > \rho$ such that $A \in B$[4].

   Both conditions 4 and 5 are desired to allow to add new models below potentially limit points which will be essential further for properness of the forcing.

   The next condition deals with with closure and is desired to prevent some pathological patterns.

6. Let $B \in C^\rho$, for some $\rho \in s$, be a non-limit point of C^ρ. If there are models $A \in \bigcup_{\xi\in s} C^\xi$ with $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then there is $A \in B \cap \bigcup_{\xi \in s} C^\xi$ such that
   (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$,
   (b) for every $A' \in \bigcup_{\xi\in s} C^\xi$ with $\sup(A' \cap \theta^+) < \sup(B \cap \theta^+)$, $\sup(A' \cap \theta^+) \leq \sup(A \cap \theta^+)$.

[4]If we drop the requirement $\tau > \rho$, then it may be impossible further to add models of sizes $> \eta$ once a potencially limit point of size η is around.

Such A is the "real" immediate predecessor of B. Further, in the definition of the order, we will require that once B is not a potentially limit point, then no models E such that $A \in E \in B$ can be added.

The purpose of the next two conditions is to allow to proceed down the pistes without interruptions at least before reaching a potentially limit point.

7. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^ρ, then $B' \in A$.

8. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is a limit point in C^ρ. Let $\langle B_\nu \mid \nu < \nu^* < \delta \rangle$ be $C^\rho \cap B$. Then a closed unbounded subsequence of $\langle B_\nu \mid \nu < \nu^* \rangle$ is in A.

9. (Linearity) If $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$, then

   (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ implies $A \in B$,

   (b) $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$ implies $A \subseteq B$.

10. If $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$, $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \in A$, then for every $X \in \bigcup_{\mu \in s} C^\mu$, $\sup(X \cap \theta^+) = \sup(B \cap \theta^+)$ and $|X| \in A$ implies $X \in A$.

11. (Immediate successor restriction) Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^{\rho lim}$,cof$(\sup(B' \cap \theta^+)) > \tau$ and $B \in A$. Suppose that there a model $B' \in B \cap C^\rho$ such that $\sup(B' \cap \theta^+) > \sup((A \cap B) \cap \theta^+)$, then the least such B' is a potentially limit model. I.e., if there is a model in C^ρ between $A \cap B$ and B, then the least such model is a potentially limit model.
    It is designed to prevent the situation when there is $E \in A \cap C^\rho$ which has a non-potentially limit immediate successor E'' in B but not in A. Also it prevents a possibility that the least element Y of C^ρ is a non-potentially limit point which belongs to B is above $A \cap B$.
    This condition is needed further for $\tau-$properness argument.

12. (Covering) If $\tau, \rho \in s, \tau < \rho$, $B \in C^\tau, D \in C^\rho$ and $\sup(B \cap \theta^+) > \sup(D \cap \theta^+)$, then there is $D^* \in B \cap C^{\rho^*}$ such that $D^* \supseteq D$[5], where $\rho^* = \min((B \setminus \rho) \cap Regular)$, i.e. the least regular cardinal in the interval $[\rho, \theta]$ which belongs to B. In particular, $\rho^* \in s$[6].

[5] Note that the least such D^* must be a potentially limit point by 7, 8 above.

[6] Note that the set $Z := \{\mu \leq \theta \mid \mu \text{ is a regular cardinal } \}$ belongs to B, by elementarity. If its cardinality is at most τ, then $Z \subseteq B$. So, in this case $\rho^* = \rho$.

The last condition describes a very particular way of covering and it is crucial for the properness arguments.

13. (Strong covering) Let $B \in C^{\tau}$, $D \in C^{\rho}, \rho > \tau$ and $\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Then either

    (a) $D \in B$,
    or

    (b) $D \notin B$ and the least $D^* \in C^{\rho^*} \cap B, D^* \supset D$ is closed under $< \rho^*-$ sequence of its elements, where $\rho^* = \min((B \setminus \rho) \cap Regular)$. Then $B \cap D^* \subseteq D$ and

$$\{D' \in D^* \mid ((|D'| = \rho^*) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$
$$((\forall k < n)(|Z_k| < \rho^*)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D^{7}.$$

    Or

    (c) $D \notin B$ and the least $D^* \in C^{\rho} \cap B, D^* \supset D$ is not closed under $< \rho-$ sequence of its elements.
    Let $\mathrm{cof}(\sup(D^* \cap \theta^+)) = \xi$ for some $\xi \in s \cap \rho$ and let $E \in C^{\xi lim}$ such that $\sup(E \cap \theta^+) = \sup(D^* \cap \theta^+)$ (such E exists by 3c(b) and $E \in B$ by 10, since $D^* \in B$).
    Then either

       i. $D \in E$, $B \cap D^* \subseteq D$ and

$$\{D' \in D^* \mid ((|D'| \leq \rho^*) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$
$$((\forall k < n)(|Z_k| < \xi)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D.$$

       ii. $D \notin E$, and then, let be the least $D^{**} \in C^{\rho^{**}} \cap E$ with $D^{**} \supset D$, where $\rho^{**} = \min((E \setminus \rho) \cap Regular)$. If D^{**} is closed under $< \rho^{**}-$ sequence of its elements, then $B \cap D^* \subseteq D, E \cap D^{**} \subseteq D$ and

$$\{D' \in D^{**} \mid ((|D'| \leq \rho^{**}) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$
$$((\forall k < n)(|Z_k| < \rho^{**})) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D$$

[7] Note that GCH is assumed, so the cardinality of this set is less than ρ. Then it is in D^*, once D^* is closed under $< \rho-$sequences of its elements.

If D^{**} is not closed under $< \rho^{**}-$ sequence of its elements, then the process repeats itself going down below D^{**}. After finitely many steps we will either reach D or D will be above everything related to B. Let us state this formally. So suppose that D^{**} is not closed under $< \rho-$ sequence of its elements.
Then are $n^* < \omega, \{\xi_n \mid n \leq n^*\} \subseteq s \setminus \eta + 1, \langle E_n \mid n \leq n^* \rangle, \langle D_n \mid n \leq n^* \rangle$ such that for every $n \leq n^*$ the following hold:

A. $D_0 = D^*$,

B. $E_0 = E$,

C. $\rho_0 = \rho^*$,

D. $D_n \in C^{\rho_n}$,

E. $D_n \supseteq D$,

F. $D_{n+1} \in D_n$,

G. $\mathrm{cof}(\sup(D_n \cap \theta^+)) = \xi_n$,

H. $E_n \in C^{\xi_n}$,

I. $\sup(D_n \cap \theta^+) = \sup(E_n \cap \theta^+)$,

J. $D_{n+1} \in E_n$ is the least in $C^{\rho_{n+1}} \cap E_n$ with $D_{n+1} \supset D$ and $\rho_{n+1} = \min((E_n \setminus \rho) \cap \mathit{Regular})$.

K. $B \cap D_0 \subseteq D$,

L. $E_n \cap D_{n+1} \subseteq D$,

M. $\{D' \in D_{n+1} \mid (|D'| = \rho_{n+1}) \wedge (\exists m < \omega)(\exists Z_{m-1} \in ... \in Z_0 \in B)$ $((\forall k < m)(|Z_k| < \xi_n)) \wedge D' \in B \cup \bigcup_{k<m} Z_k))\} \in D$,

N. $D_{n^*} = D$ or, we have, $D \in D_{n^*}, {}^{\rho_{n^*}} D_{n^*} \subseteq D_{n^*}$,

$$\{D' \in D_{n^*} \mid (|D'| = \rho_{n^*}) \wedge (\exists m < \omega)(\exists Z_{m-1} \in ... \in Z_0 \in B)$$

$$((\forall k < m)(|Z_k| < \rho_{n^*})) \wedge D' \in B \cup \bigcup_{k<m} Z_k))\} \in D.$$

14. (An addition to the strong covering condition) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$ and $\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Suppose that there is $X \in C^\theta$ with $\sup(B \cap \theta^+) = X \cap \theta^+$.
Then either

(a) $D \in B$,
or

(b) $D \notin B$ and (b),(c) of (13) hold with B replaced by any model $Y, B \subseteq Y \subseteq X$ of a regular cardinality $\mu \in s, \tau < \mu < \rho$ which is definable in $\langle H(\theta^+), \in, \leq, \delta, \eta \rangle$ with parameters from the set $B \cup (\mu + 1) \cup \{B\}$[8].

The next conditions are the versions of a strong covering used for chains of models. They are essential further for showing properness.

15. (Strong covering for chains of models) Let $B \in C^\tau$, $\rho > \tau, \rho \in B$, $\langle D_i \mid i \leq \alpha \rangle$ is an initial segment of C^ρ, for some $\alpha < \mathfrak{S}(\rho)$, and
$\sup(D_\alpha \cap \theta^+) < \sup(B \cap \theta^+)$. Then either

    (a) $\langle D_i \mid i \leq \alpha \rangle \in B$,
    or

    (b) there is $\alpha^* < \alpha$ such that $\langle D_i \mid i \leq \alpha^* \rangle \in B$ and the models of $\langle D_i \mid \alpha^* < i \leq \alpha \rangle$ satisfy 13(b,c), 14(b) with $D^* \in C^\rho \cap B$ the least above D_α.

16. (Strong covering for chains of models of size outside) Let $B \in C^\tau$, $\rho > \tau, \rho \notin B$, $\langle D_i \mid i \leq \alpha \rangle$ is an initial segment of C^ρ, for some $\alpha < \mathfrak{S}(\rho)$, and
$\sup(D_\alpha \cap \theta^+) < \sup(B \cap \theta^+)$. Let $D^* \in C^{\rho^*} \cap B$ the least above D_α.
Then either

    (a) the sequence $\langle D_i \mid i \leq \alpha \rangle$ satisfy 13(b,c), 14(b),
    or

    (b) there are $\alpha^* < \alpha$ and a closed chain $\langle D^*_i \mid i \leq \alpha^* \rangle$ of members of C^{ρ^*} such that

        i. $\langle D^*_i \mid i \leq \alpha^* \rangle \in B$,
        ii. $\langle D^*_i \mid \alpha^* < i \leq \alpha \rangle$ satisfy 13(b,c), 14(b),
        iii. for every $i \leq \alpha^*$, if $D^*_i \in B$, or equivalently $i \in B$, then D^*_i is the least member of B which covers D_i.

Now we are ready to give the main definition.

Definition 1.2 Let $\eta < \theta$ be regular cardinals, $\mathfrak{S}$ be a function from the set $\{\tau \mid \eta \leq \tau \leq \theta$ and τ is a regular cardinal $\}$ to θ, such that for every $\tau \in \mathrm{dom}(\mathfrak{S}), \mathfrak{S}(\tau)$ is a cardinal $< \tau$.

[8]Note that the total number of such Y's for a fixed regular $\mu \in s, \tau < \mu < \rho$ is $|B| = \tau$. Hence, there are less than ρ possibilities for Y's. Also, note that the model X is definable from B, as it was observed above in (3)

A $(\theta, \eta, \mathfrak{S})-$structure with pistes is a set $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ such that the following hold.[9]
Let us first specify sizes of models that are involved.

1. (Support) s is a closed set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:

   (a) $|s| < \delta$,

   (b) $\eta, \theta \in s$.
   Which means that the minimal and the maximal possible sizes are always present.

2. (Models) For every $\tau \in s$ the following holds:

   (a) $A^{0\tau} \preccurlyeq \langle H(\theta^+), \in, \leq, \delta, \eta\rangle$,

   (b) $|A^{0\tau}| = \tau$,

   (c) $A^{0\tau} \in A^{1\tau}$,

   (d) $A^{1\tau}$ is a set of less than δ elementary submodels of $A^{0\tau}$,

   (e) each element A of $A^{1\tau}$ has cardinality τ, $A \supseteq \tau$ and $A \cap \tau^+$ is an ordinal and it is above the number of cardinals in the interval $[\eta, \theta]$.

3. (Potentially limit points) Let $\tau \in s$.
   $A^{1\tau lim} \subseteq A^{1\tau}$. We refer to its elements as *potentially limit points*.
   The intuition behind is that once extending it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.

4. (Piste function) The idea behind is to provide a canonical way to move from a model in the structure to one below.
   Let $\tau \in s$.
   Then, $\mathrm{dom}(C^\tau) = A^{1\tau}$ and
   for every $B \in \mathrm{dom}(C^\tau)$, $C^\tau(B)$ is a closed chain of models in $A^{1\tau} \cap (B \cup \{B\})$ such that the following holds:

   (a) $B \in C^\tau(B)$,

[9] If for some regular $\delta \leq \eta$, $\mathfrak{S}(\tau) = \delta$, for every τ, then it is just a $\delta-$structure with pistes over η of the length θ.

(b) if $X \in C^\tau(B)$, then $C^\tau(X) = \{Y \in C^\tau(B) \mid Y \in X \cup \{X\}\}$,

(c) if B has immediate predecessors in $A^{1\tau}$, then one (and only one) of them is in $C^\tau(B)$,

5. (Wide piste) The set

$$\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s\rangle$$

is a $(\theta, \eta, \mathfrak{S})-$wide piste.

Next two condition describe the ways of splittings from wide pistes. This describes the structure of $A^{1\tau}$ and the way pistes allow to move from one of its models to an other.

6. (Splitting points) Let $\tau \in s$. Let $X \in A^{1\tau}$ be a non-limit model (but possibly a potentially limit), then either

(a) X is a minimal under $\in$ or equivalently under $\supsetneq$,
or

(b) X has a unique immediate predecessor in $A^{1\tau}$,
or

(c) X has exactly two immediate predecessors X_0, X_1 in $A^{1\tau}$, non of X, X_0, X_1 is a limit or potentially limit points and X, X_0, X_1 form a $\Delta-$*system triple* relatively to some $F_0, F_1 \in A^{1\tau^* lim}$, for some $\tau^* \in s \setminus \tau + 1$[10], which means the following:

i. $F_0 \subsetneq F_1$ and then $F_0 \in C^{\tau^*}(F_1)$, or $F_1 \subsetneq F_0$ and then $F_1 \in C^{\tau^*}(F_0)$,

ii. ${}^{\tau^*>}F_0 \subseteq F_0$ and ${}^{\tau^*>}F_1 \subseteq F_1$,

iii. $X_0 \in F_1$ (or $X_1 \in F_0$),

iv. $F_0 \in X_0$ and $F_1 \in X_1$,

v. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

vi. ${}^{\tau>}X_0 \subseteq X_0$ and ${}^{\tau>}X_1 \subseteq X_1$,

vii. the structures

$$\langle X_0, \in, \langle X_0 \cap A^{1\rho}, X_0 \cap A^{1\rho lim}, (C^\rho \restriction X_0 \cap A^{1\rho}) \cap X_0 \mid \rho \in s \cap X_0\rangle\rangle$$

and

$$\langle X_1, \in, \langle X_1 \cap A^{1\rho}, X_1 \cap A^{1\rho lim}, (C^\rho \restriction X_1 \cap A^{1\rho}) \cap X_1 \mid \rho \in s \cap X_1\rangle\rangle$$

[10] If there are only finitely many cardinals between η and θ, then we can take τ^* to be just τ^+.

are isomorphic over $X_0 \cap X_1$. Denote by π_{X_0,X_1} the corresponding isomorphism.

viii. $X \in A^{0\tau *}$.

Further we will refer to such X as *a splitting point.*

Or

(d) (Splitting points of higher order) There are $G, G_0, G_1 \in X \cap A^{1\mu}$, for some $\mu \in s \setminus \min(s \setminus \tau+1)$, which form a Δ-system triple with witnessing models in X such that

i. $X_0 \in G_0$,

ii. $X_1 \in G_1$,

iii. $X_1 = \pi_{G_0 G_1}[X_0]$.

iv. X is not a limit or potentially limit point,

v. $X \in A^{0\mu}$,

vi. (Pistes go in the same direction) $G_i \in C^\mu(G) \Leftrightarrow X_i \in C^\tau(X), i < 2$.

Further we will refer to such X as *a splitting point of higher order.*

7. Let $\tau, \rho \in s$, $X \in A^{1\tau}, Y \in A^{1\rho}$. Suppose that X is a successor point, but not potentially limit point and $X \in Y$. Then all immediate predecessors of X are in Y, as well as the witnesses, i.e. F_0, F_1 if (6c) holds and G_0, G_1, G if (6d) holds.

8. Let $\tau \in s$. If $X \in A^{1\tau}$, $Y \in \bigcup_{\rho \in s} A^{1\rho}$ and $Y \in X$, then Y is *a piste reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n \rangle$ of elements of $A^{1\tau}$ which we call *a piste leading to* Y such that

   (a) $X = X(0)$,

   (b) for every $i, 0 < i \leq n$, $X(i) \in C^\tau(X(i-1))$ or $X(i-1)$ has two immediate successors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^\tau(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$ or $Y = X(i-1)_1$,

   (c) $Y = X(n)$, if $Y \in A^{1\tau}$ and if $Y \in A^{1\rho}$, for some $\rho \neq \tau$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\tau}$.

In particular, every $Y \in A^{1\tau}$ is piste reachable from $A^{0\tau}$.

In order formulate further requirement, we will need to describe a simple process of changing the wide pistes. This leads to equivalent forcing conditions once the order will be defined.

Let $X \in A^{1\tau}$. We will define $X-$wide piste. The definition will be by induction on number of turns (splits) needed in order to reach X by the piste from $A^{0\tau}$.
First, if $X \in C^\tau(A^{0\tau})$, then $X-$wide piste is just $\langle C^\xi(A^{0\xi}), C^\xi(A^{0\xi}) \cap A^{1\xi lim} \mid \xi \in s\rangle$, i.e. the wide piste of the structure.
Second, if $X \notin C^\tau(A^{0\tau})$, but it is not a splitting point, then pick the least splitting point Y above X. Let Y_0, Y_1 be its immediate predecessors with $Y_0 \in C^\tau(Y)$. Then $X \in Y_i \cup \{Y_i\}$ for some $i < 2$. Set $X-$wide piste to be the Y_i-wide piste.
So, in order to complete the definition, it remain to deal with the following principle case:
$X \in A^{1\tau}$ a splitting point with witnesses $F_0, F_1 \in C^{\tau^*}(A^{0\tau^*})$. Let X_0, X_1 be its immediate predecessors with $X_0 \in C^\tau(X)$. Assume that $X-$wide piste $\langle C_X^\xi, C_X^{\xi lim} \mid \xi \in s\rangle$ for X is defined and assume that $C^\tau(X)$ is an initial segment of C_X^τ.
Let the X_0-wide piste be $\langle C_X^\xi, C_X^{\xi lim} \mid \xi \in s\rangle$.
Define X_1-wide piste $\langle C_{X_1}^\xi, C_{X_1}^{\xi lim} \mid \xi \in s\rangle$ as follows:

- $C_{X_1}^\xi = C_X^\xi$, for every $\xi \geq \tau^*$.
  I.e. no changes for models of cardinality $\geq \tau^*$.
- $C_{X_1}^{\xi lim} = C_{X_1}^\xi \cap A^{1\xi lim}$, for every $\xi \in s$.
  Models that were potentially limit remain such and no new are added.
- $C_{X_1}^\tau = (C_X^\tau \setminus X) \cup C^\tau(X_1)$.
  Here we switched the piste from X_0 to X_1.
- $C_{X_1}^\xi = \{Z \in C_X^\xi \mid \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))\} \cup \{\pi_{X_0,X_1}(Z) \mid Z \in C_X^\xi \cap X_0\}$, for every $\xi \in s \cap \tau^*$[11].

Now we require the following:

9. Let $\tau \in s$ and $X \in A^{1\tau}$. Then $X-$wide piste is a wide piste, i.e. it satisfies 1.1.
   The problem is with (3c) of 5 which, in general, is not preserved while splitting.

   Final conditions deal with largest models.

[11] In particular, due to this, the next condition implies that for $\xi \in s \cap \tau^*$, if $Z \in C_X^\xi, \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))$, then $\{\pi_{X_0 X_1}(Z') \mid Z' \in C_X^\xi \cap X_0\} \subseteq Z$.

10. (Maximal models are above all the rest) For every $\tau \in s$ and $Z \in \bigcup_{\rho \in s} A^{1\rho}$, if $Z \notin A^{0\tau}$, then there is $\mu \in s$ such that $Z = A^{0\mu}$.

    Recall that by 5, maximal models $A^{0\tau}, \tau \in s$ are linearly ordered as top parts of the wide piste $\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s \rangle$.

This completes the definition of $(\theta, \eta, \mathfrak{S})-$structure with pistes.

1.1 The intersection property.

Recall two definitions from [5].

Definition 1.3 (Models of different sizes). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $(\theta, \eta, \mathfrak{S})-$structure with pistes.

Let $A \in A^{1\tau}, B \in A^{1\rho}$ and $\tau < \rho$.

By $ip(A, B)$ we mean the following:

1. $B \in A$,
   or

2. $A \subset B$,
   or

3. $B \notin A$, $A \not\subset B$ and then

   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \rho) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

Definition 1.4 (Models of a same size). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $(\theta, \eta, \mathfrak{S})-$structure with pistes.

Let $A, B \in A^{1\tau}$. By $ip(A, B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then

   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \tau) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

If both $ip(A, B)$ and $ip(B, A)$ hold, then we denote this by $ipb(A, B)$.

Lemma 1.5 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $(\theta, \eta, \mathfrak{S})-$structure with pistes. Assume $A \in A^{1\tau}, B \in A^{1\rho}$, for some $\tau \leq \rho, \tau, \rho \in s$. Then $ip(A, B)$ and if $\tau = \rho$, then also $ipb(A, B)$.*

The proof repeats those of the corresponding lemma of [5].

1.2 Forcing with structures with pistes of different sizes.

Definition 1.6 Define $\mathcal{P}_{\theta\eta\mathfrak{S}}$ to be the set of all $(\theta,\eta,\mathfrak{S})-$structures with pistes.
Let $p=\langle\langle A^{0\tau},A^{1\tau},A^{1\tau lim},C^{\tau}\rangle\mid\tau\in s\rangle\in\mathcal{P}_{\theta\eta\mathfrak{S}}$.
Denote further $A^{0\tau}$ by $A^{0\tau}(p)$, $A^{1\tau}$ by $A^{0\tau}(p)$, $A^{1\tau lim}$ by $A^{1\tau lim}(p)$, C^{τ} by $C^{\tau}(p)$ and s by $s(p)$.
Call s the support of p.

Let us define a partial order on $\mathcal{P}_{\theta\eta\mathfrak{S}}$ as follows.

Definition 1.7 Let
$p_0=\langle\langle A_0^{0\tau},A_0^{1\tau},A_0^{1\tau lim},C_0^{\tau}\rangle\mid\tau\in s_0\rangle$, $p_1=\langle\langle A_1^{0\tau},A_1^{1\tau},A_1^{1\tau lim},C_1^{\tau}\rangle\mid\tau\in s_1\rangle$ be two elements of $\mathcal{P}_{\theta\eta\mathfrak{S}}$.
Set $p_0\leq p_1$ (p_1 extends p_0) iff

1. $s_0\subseteq s_1$,

2. $A_0^{1\tau}\subseteq A_1^{1\tau}$, for every $\tau\in s_0$,

3. let $A\in A_0^{1\tau}$, then $A\in A_0^{1\tau lim}$ iff $A\in A_1^{1\tau lim}$.
   The next item deals with a property called switching in [1]. It allows to change piste directions.

4. For every $A\in A_0^{1\tau}$, $C_0^{\tau}(A)\subseteq C_1^{\tau}(A)$,
   or
   there are finitely many splitting (or generalized splitting) points $B(0),...,B(k)\in A_0^{1\tau}$ with $B(j)',B(j)''$ the immediate predecessors of $B(j)$ $(j\leq k)$ such that

   (a) $B(j)'\in C_0^{\tau}(B(j))$,

   (b) $B(j)''\in C_1^{\tau}(B(j))$.

5. If $A\in A_0^{1\tau}$ is a splitting point or a splitting point of higher order in p_0, then it remains such in p_1 with the same immediate predecessors.

6. Let $B\in A_0^{1\tau}$ be a successor point, not in $A_0^{1\tau\,\mathrm{lim}}$ and with a unique immediate predecessor. Consider the wide piste that runs via B (in p_0). Let A be as in 1.1(6). Then there is no model E in p_1 such that $A\in E\in B$.

   This requirement guaranties intervals without models, even after extending a condition.

By 1.7(6), potentially limit points are the only places where not end-extensions can be made.

Next two lemmas will insure that generic clubs produced by $\mathcal{P}_{\theta\eta\mathfrak{S}}$ run away from old sets of corresponding sizes. Their proofs repeat those of [5].

Lemma 1.8 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ *be an element of* $\mathcal{P}_{\theta\eta\mathfrak{S}}$. *Let* $X \in A^{1\rho lim}$, *for some* $\rho \in s$.
Assume that if $\mathrm{cof}(\sup(X \cap \theta^+)) < \rho$, *then* $\rho \in B$, *where* $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ *is the model with* $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$ *(exists by 1.1(3)(c)B))*[12].
Suppose that for every $t \in X$ *there is* $D \preceq X$ *such that*

1. $D \in X$,
2. $t \in D$,
3. $|D| = \rho$,
4. $D \supseteq \rho$
5. ${}^{\rho>}D \subseteq D$,
6. *D is a union of a chain of its elementary submodels which satisfy items 1-5* [13].

Then for every $\beta < \sup(X \cap \theta^+)$ *there is* T *of size* ρ *with* $\sup(T \cap \theta^+) > \beta, T \in X$ *such that adding* T *as a potentially limit point and reflecting it through* $\Delta-$*system type triples gives an extension of* p.

Lemma 1.9 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ *be an element of* $\mathcal{P}_{\theta\eta\delta}$. *Let* $X \in A^{1\rho lim}$, *for some* $\rho \in s$.
Assume that if $\mathrm{cof}(\sup(X \cap \theta^+)) < \rho$, *then* $\rho \in B$, *where* $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ *is the model with* $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$ *(exists by 1.1(3)(c)B)).*
Suppose that for every $t \in X$ *there is* $D \preceq X$ *such that*

1. $D \in X$,
2. $t \in D$,
3. $|D| = \rho$,

[12] We will see further that it is possible to remove this assumption at least in interesting cases.
[13] The issue here is to satisfy 1.1(3(b)).

4. $D \supseteq \rho$

5. ${}^{\rho>}D \subseteq D$*,*

6. D *is a union of a chain of its elementary submodels which satisfy items 1-5.*

Let $\beta < \sup(X \cap \theta^+)$ *and* T *be a potentially limit point of size* ρ *with* $\sup(T \cap \theta^+) > \beta, T \in X$ *added by the previous lemma 1.8. Then for every* $\gamma, \sup(T \cap \theta^+) < \gamma < \sup(X \cap \theta^+)$ *there is* T' *of size* ρ *with* $\sup(T' \cap \theta^+) > \gamma, T' \in X$ *such that adding* T' *as a non-potentially limit point and reflecting it through* $\Delta-$*system type triples gives an extension of the previous condition.*

Lemma 1.10 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ *be an element of* $\mathcal{P}_{\theta\eta\delta}$*. Let* $X \in A^{1\rho lim}$*, for some* $\rho \in s$*.*
Assume that $\mathrm{cof}(\sup(X \cap \theta^+)) = \tau < \rho$*,* $\rho \notin B$*, for* $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ *such that* $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$*. Let* $Y \in A^{1\theta lim}$ *with* $Y \cap \theta^+ = \sup(X \cap \theta^+)$ *(it exists by 1.1(3(c)B)). Suppose that for every* $t \in Y$ *there is* $D \preceq Y$ *such that*

1. $D \in Y$*,*

2. $t \in D$*,*

3. $|D| = \theta$*,*

4. $D \supseteq \theta$

5. ${}^{\theta>}D \subseteq D$*,*

6. D *is a union of a chain of its elementary submodels which satisfy items 1-5.*

Then for every $\beta < \sup(X \cap \theta^+)$ *there is* T *of size* ρ *with* $\sup(T \cap \theta^+) > \beta, T \in X$ *such that adding* T *as a potentially limit point and reflecting it through* $\Delta-$*system type triples gives an extension of* p*.*

We turn now to properness of $\mathcal{P}_{\theta\eta\mathfrak{S}}$. The arguments are similar to those of [5], but require a certain addition which allow to deal with unbounded chains.

Lemma 1.11 *The forcing notion* $\langle \mathcal{P}_{\theta\eta\mathfrak{S}}, \leq\rangle$ *is* $\eta-$*proper*

Proof. If $\mathfrak{S}(\tau) \leq \eta$, for every $\tau \in \mathrm{dom}(\mathfrak{S})$ then the proof completely repeats those of [5]. Suppose that it is not the case and suppose that $\mathfrak{S}(\theta) > \eta$.

Let $p \in \mathcal{P}_{\theta\eta\mathfrak{S}}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ regular large enough such that such that

1. $|\mathfrak{M}| = \eta$,

2. $\mathfrak{M} \supseteq \eta$,

3. $\mathcal{P}_{\theta\eta\mathfrak{S}}, p \in \mathfrak{M}$,

4. ${}^{\eta>}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\theta^+)$.
Clearly, M satisfies 1.1(3(b)). Moreover, using the elementarity of $\mathfrak{M}$, for every $x \in M$ there will be $Z \in M$ such that

- $Z \preceq H(\theta^+)$,

- $|Z| = \theta$,

- $Z \supseteq \theta$,

- ${}^{\theta>}Z \subseteq Z$,

- $x \in Z$.

This allows to find a chain of models $\langle N_i \mid i < \eta\rangle$ of size θ which members are in M, witnesses 1.1(3(b)) for $N := \bigcup_{i<\eta} N_i$ and $N \supseteq M$.

Extend p by adding M as a new $A^{0\eta}$, N as a new $A^{0\theta}$ and, in addition we add now the sequence $\langle N_i \mid i < \eta\rangle$. Require $M, N, N_{i+1}, i < \eta$ to be potentially limit points. Denote the result by $p^\frown\{M, N, \langle N_i \mid i < \eta\rangle\}$.
We claim that $p^\frown\{M, N, \langle N_i \mid i < \eta\rangle\}$ is $(\mathcal{P}_{\theta\eta\mathfrak{S}}, \mathfrak{M})-$generic. So, let $p' \geq p^\frown\{M, N\}$ and $D \in M$ be a dense open subset of $\mathcal{P}_{\theta\eta\mathfrak{S}}$. It is enough to find $q \in \mathfrak{M} \cap D$ which is compatible with p'.

Pick $i < \eta$ big enough such that $D \in N_{i+1}$. Now we pick $M' \preceq N_{i+1}$ of size η, inside M such that

1. ${}^{\eta>}M' \subseteq M'$

2. $D \in M'$,

3. all components of p' which belong to $N_{i+1} \cap M$ are in M'.

Note that $\mathfrak{S}(\tau) \leq \tau$, and, so $\mathfrak{S}(\tau) \leq \theta$, for every $\tau \in \mathrm{dom}(\mathfrak{S})$. Hence, models of p' which belong to N_{i+1} are bounded there. Remember that ${}^{\theta>}N_{i+1} \subseteq N_{i+1}$. So, the set of models of p' which belong to N_{i+1} is a member of N_{i+1}. Using (15, 16) of 1.1, ${}^{\eta>}M \subseteq M$ and since the support of p', $s(p')$, has cardinality $< \eta$, by 1.1(1(a)), it is possible to satisfy the requirement 3 above.
Now we continue as in [5], only replacing M there by M'.
$\square$

Our next tusk will be to show that the forcing notion $\langle \mathcal{P}_{\theta\eta\mathfrak{S}}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$. The proof follows closely those of [5]. Let us address only a new point which appears in the present context.

Lemma 1.12 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$.*

Proof. Let τ be a regular cardinal in the interval $[\eta, \theta]$. We would like to show that $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper. If $\tau = \eta$, then this follows by the previous lemma (1.11). Suppose that $\tau > \eta$. Let $p \in \mathcal{P}_{\theta\eta\delta}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ regular large enough such that such that

1. $|\mathfrak{M}| = \tau$,

2. $\mathfrak{M} \supseteq \tau$,

3. $\mathcal{P}_{\theta\eta\mathfrak{S}}, p \in \mathfrak{M}$,

4. ${}^{\tau>}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\theta^+)$.
Clearly, M satisfies 1.1(3(b)). Moreover, using the elementarity of $\mathfrak{M}$, for every $x \in M$ there will be $Z \in M$ such that

- $Z \preceq H(\theta^+)$,

- $|Z| = \theta$,

- $Z \supseteq \theta$,

- ${}^{\theta>}Z \subseteq Z$,

- $x \in Z$.

This allows to find a chain of models $\langle N_i \mid i < \tau \rangle$ of size θ which members are in M, witnesses 1.1(3(b)) for $N := \bigcup_{i<\tau} N_i$ and $N \supseteq M$.

Extend p by adding M as a new $A^{0\eta}$, N as a new $A^{0\theta}$ and, in addition we add now the sequence $\langle N_i \mid i < \tau \rangle$. Require $M, N, N_{i+1}, i < \eta$ to be potentially limit points. Denote the result by $p^\frown\{M, N, \langle N_i \mid i < \eta \rangle\}$.
We claim that $p^\frown\{M, N, \langle N_i \mid i < \eta \rangle\}$ is $(\mathcal{P}_{\theta\eta\mathfrak{S}}, \mathfrak{M})-$generic. So, let $D \in M$ be a dense open subset of $\mathcal{P}_{\theta\eta\mathfrak{S}}$ and $p' \geq p^\frown\{M, N\}$ be in D.
Extend p' further in order to achieve the following:

- for every $\xi \in s(p')$, there is a model B on the wide piste of p' of cardinality ξ such that $M \subseteq B \subseteq N$.

In particular, $\sup(M \cap \theta^+) = \sup(B \cap \theta^+) = N \cap \theta^+$. Denote such B by M_ξ.
Let us denote such extension of p' still by p'.

Pick now $A \preceq H(\theta^+)$ which satisfies the following:

- $|A| = \eta$,

- $A \supseteq \eta$,

- $A \cap \eta^+$ is an ordinal,

- ${}^{\eta>}A \subseteq A$,

- $p' \in A$.
  In particular every model of p' belongs to A.

Extend p' to p'' by adding A as new largest model of cardinality η, i.e. $p'' = p'^\frown A$.

Pick $i < \eta$ big enough such that $D \in N_{i+1}$. Now we pick $M' \preceq N_{i+1}$ of size τ, inside M such that

1. ${}^{\eta>}M' \subseteq M'$

2. $D \in M'$,

3. all components of p'' which belong to $N_{i+1} \cap M$ are in M'.

Note that $\mathfrak{S}(\tau) \leq \tau$, and, so $\mathfrak{S}(\tau) \leq \theta$, for every $\tau \in \mathrm{dom}(\mathfrak{S})$. Hence, models of p' which belong to N_{i+1} are bounded there. Remember that ${}^{\theta>}N_{i+1} \subseteq N_{i+1}$. So, the set of models of p' which belong to N_{i+1} is a member of N_{i+1}. Using (15, 16) of 1.1, ${}^{\eta>}M \subseteq M$ and since the support of p', $s(p')$, has cardinality $< \eta$, by 1.1(1(a)), it is possible to satisfy the requirement 3 above.

We reflect $A = A^{0\eta}(p'')$ down to M' over over $A^{0\eta}(p'') \cap M'$, i.e. we pick some $A' \in M'$ and q which realizes the same k-type (for some $k < \omega$ sufficiently big) over $A^{0\eta}(p'') \cap M'$ as $A^{0\eta}(p'')$ and p''. Do this in a rich enough language which includes D as well.

Now we continue as in [5], only replacing M there by M'.

$\square$

The next lemma is straightforward.

Lemma 1.13 *The forcing notion* $\langle \mathcal{P}_{\theta\eta\mathfrak{S}}, \leq \rangle$ *is* $< \min(\{\mathfrak{S}(\tau) \mid \tau \in \mathrm{dom}(\mathfrak{S}\})-$*strategically closed.*

Combining together Lemmas 1.11,1.12, 1.13, we obtain the following:

Theorem 1.14 *The forcing notion* $\langle \mathcal{P}_{\theta\eta\mathfrak{S}}, \leq \rangle$ *preserves all cardinals* $\leq \min(\{\mathfrak{S}(\tau) \mid \tau \in \mathrm{dom}(\mathfrak{S}\})$ *and all cardinals* $> \eta$.
In particular, if $\delta = \min(\{\mathfrak{S}(\tau) \mid \tau \in \mathrm{dom}(\mathfrak{S}\})$, *then all cardinals are preserved.*

As in [5], it is possible to use $\mathcal{P}_{\theta\eta\mathfrak{S}}$ for adding clubs and for blowing up the power of a singular cardinal.

References

[1] M. Gitik, Short extenders forcings I,
http://www.math.tau.ac.il/~gitik/short%20extenders%20forcings%201.pdf

[2] M. Gitik, Remarks on non-closure of the preparation forcing and off-piste versions of it,
http://www.math.tau.ac.il/~gitik/closed.pdf

[3] M. Gitik and S. Unger, Adding many ω–sequences to a singular cardinal, Notes from Appalachian Set Theory workshop April 3, 2010,
http://www.math.tau.ac.il/~gitik/GitikAST.pdf

[4] M. Gitik, Short extenders forcings – doing without preparations.Bounded gaps. http://www.math.tau.ac.il/∼gitik

[5] M. Gitik, Short extenders forcings – doing without preparations. http://www.math.tau.ac.il/∼gitik

[6] M. Magidor, Changing co

nality of cardinals, Fundamenta Mathematicae, 99(1):61-71, 1978.

[7] C. Merimovich, The short extenders gap two forcing is of Prikry type, Arch. Math. Logic, 48(2009),737-747.

[8] C. Merimovich, The short extenders gap three forcing using a morass, Arch. Math. Logic,50(2011),115-135.

[9] W. Mitchell,

[10] L. Radin, Adding closed co

nal sequences to large cardinals, Annals of Mathematical Logic, 22(3):243-261, 1982.

[11] S. Shelah, Proper and Improper forcing, Springer 1998.

[12] D. Velleman, Simplified morasses, JSL 49(1),257-271(1984)