Short Extenders Forcings II

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March 7, 2017

Abstract

A model with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) = \omega_1 + 1$ is constructed, for countable set $\mathfrak{a}$ of regular cardinals.

1 Preliminary Settings

Let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an an increasing continuous sequence of singular cardinals of cofinality ω so that for each $\alpha < \omega_1$, if $\alpha = 0$ or α is a successor ordinal, then κ_α is a limit of an increasing sequence $\langle \kappa_{\alpha,n} \mid n < \omega \rangle$ of cardinals such that

(1) $\kappa_{\alpha,n}$ is strong up to a 2-Mahlo cardinal $< \kappa_{\alpha,n+1}$,

(2) $\kappa_{\alpha,0} > \kappa_{\alpha-1}$.

Fix a sequence $\langle g_\alpha \mid \alpha < \omega_1,\ \alpha = 0 \text{ or it is a successor ordinal} \rangle$ of functions from ω to ω such that for every $\alpha, \beta, \alpha < \beta$ which are zero or successor ordinals below ω_1 the following holds

(a) $\langle g_\alpha(n) \mid n < \omega \rangle$ is increasing

(b) there is $m(\alpha, \beta) < \omega$ such that for every $n \geq m(\alpha, \beta)$

$$g_\alpha(n) \geq \sum_{m=0}^{n} g_\beta(m)\ .$$

*The work was partially supported by ISF grants 234/08,58/14. The material was presented during 2014-2015 in a course at the Hebrew University of Jerusalem. We are grateful to all participants for their helpful comments, remarks and corrections. Our special thanks are due to Menachem Magidor for his enormous patience in listening the arguments, going through previous versions and his crucial comments and corrections.

The easiest way is probably to force such a sequence.

Conditions are of the form

$\langle n, \{h_\alpha | \alpha \in I\}\rangle$, where $n < \omega, I$ is a finite subset of ω_1 and $h_\alpha : n \to \omega$.

The order is defined as follows:

$\langle n, \{h_\alpha | \alpha \in I\}\rangle \leq \langle m, \{t_\beta | \beta \in J\}\rangle$ iff $n \leq m, I \subseteq J$, for every $\alpha \leq \beta, \alpha, \beta \in I$, we have $t_\alpha | n = h_\alpha$ and if $n \leq k < m$ then require that $t_\alpha(k) \geq \sum_{0 \leq s \leq k} t_\beta(s)$.

It is possible to construct such a sequence in ZFC. Pick first a sequence $\langle h_\alpha | \alpha < \omega_1\rangle$ of functions from ω to ω such that

(1) $\langle h_\alpha(n) \mid n < \omega\rangle$ is non-decreasing and converges to infinity;

(2) if $\alpha < \beta$ then $h_\alpha > h_\beta$ mod finite.

Replace now each h_α by h'_α such that $h'_\alpha(n) = h_\alpha(n) + n + 1$.

Define $g_\alpha(n)$ to be $2^{(2^{\cdots(2^{h'_\alpha(n)})\cdots})}$ where the number of powers is $h'_\alpha(n)$.

Let us argue that it is as required. Let $\alpha < \beta$. Pick $m(\alpha, \beta)$ to be such that for every $n \geq m(\alpha, \beta)$ we have $h'_\alpha(n) > h'_\beta(n)$.

Let $n \geq m(\alpha, \beta)$. Consider $\sum_{0 \leq s \leq n} g_\beta(s)$.

Then

$$\sum_{0 \leq s \leq n} g_\beta(s) \leq (n+1) \cdot g_\beta(n) \leq (g_\beta(n))^2 \leq 2^{g_\beta(n)} \leq g_\alpha(n).$$

2 A basic description of the pcf- structure with ω_1–many cardinals

We would like to blow up the powers and pseudo-powers (pp) of all $\kappa_\alpha, \alpha < \omega_1$ to $\kappa^+_{\omega_1}$.[1]
The first tusk will be to arrange an appropriate pcf–structure that will be realized further. It requires some work since we allow only finitely many blocks at each level. Note that in view of [9] one cannot allow infinitely many blocks at least not under the large cardinals assumptions used here (below a strong or a little bit more).

Organize the things as follows.

Let $n < \omega$ and $1 \leq \alpha < \omega_1$ be a successor ordinal or $\alpha = 0$. We reserve at level n a splitting into $g_\alpha(n)$–blocks one above another:

$$\langle \kappa_{\alpha,n,m,i} \mid m < g_\alpha(n), i \leq \omega_1\rangle,$$

so that

[1] $\mathrm{pp}(\lambda) = \sup\{\mathrm{cof}(\prod a/D \mid a \subseteq \lambda$ is a set of at most $\mathrm{cof}(\lambda)$ of regular cardinals, unbounded in λ and D an ultrafilter over a including all cobounded subsets of $a\}$.

1. $\kappa_{\alpha,n} < \kappa_{\alpha,n,0,0}$,

2. $\kappa_{\alpha,n,m,i'} < \kappa_{\alpha,n,m,i}$, for every $m < g_\alpha(n), i' < i \leq \omega_1$,

3. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n,m+1,0}$, for every $m, m+1 < g_\alpha(n)$,

4. for every successor ordinal $i < \omega_1$ or if $i = 0$ let $\kappa_{\alpha,n,m,i}$ be large enough (say a Mahlo or even measurable),

5. for every limit $i, 0 < i \leq \omega_1$ let $\kappa_{\alpha,n,m,i} = \sup(\{\kappa_{\alpha,n,m,i'} \mid i' < i\})$,

6. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n+1}$, for every $m < g_\alpha(n)$.

For each successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$, we will fix a $(\kappa_{\alpha,n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1})$–extender $E_{\alpha n}$, i.e. an extender with the critical point $\kappa_{\alpha,n}$ which ultrapower contains $V_{\kappa_{\alpha,n,g_\alpha(n)-1,\omega_1}+2}$.

Let $\alpha < \omega_1$ be a successor ordinal or 0.

We will refer further to $\kappa_{\alpha n}$'s (or, simplicity just to n's) as levels of κ_α (or, again, for simplicity just of α). In addition, if $n < \omega$ and $m < g_\alpha(n)$, then we refer to $\kappa_{\alpha,n,m,i}$'s ($i \leq \omega_1$) as members of the $m-$th block (of the level n of α).

Let us incorporate indiscernibles that will be generated by extender based forcings with $E_{\alpha n}$'s into the blocks as follows.[2] Denote as above the indiscernible for $\kappa_{\alpha,n,m,i}$ by $\rho_{\alpha,n,m,i}$.

$[\kappa^+_{\alpha-1}, \rho^+_{\alpha,0,0,\omega_1}]$ will be the first block of α of the level 0 (if $\alpha = 0$, then let it be $[\omega_1, \rho^+_{0,0,0,\omega_1}]$). Then for every $m < g_\alpha(0)$ let m–th block of α of the level 0 be $[\rho^{++}_{\alpha,0,m-1,\omega_1}, \rho^+_{\alpha,0,m,\omega_1}]$. The first block of the level 1 of α will be $[\rho^{++}_{\alpha,0,g_\alpha(0)-1,\omega_1}, \rho^+_{\alpha,1,0,\omega_1}]$. In general the first block of the level $n > 0$ of α will be $[\rho^{++}_{\alpha,n-1,g_\alpha(n-1)-1,\omega_1}, \rho^+_{\alpha,n,0,\omega_1}]$. The m-th block ($m > 0$) of the level $n > 0$ of α will be $[\rho^{++}_{\alpha,n,m-1,\omega_1}, \rho^+_{\alpha,n,m,\omega_1}]$.

Special attention will be devoted to the very last blocks of each level,
i.e. to $[\rho^{++}_{\alpha,n,g_\alpha(n)-2,\omega_1}, \rho^+_{\alpha,n,g_\alpha(n)-1,\omega_1}]$.

In the final (after the main forcing) model we will have the following structure:

1. every element of the set $\{\kappa^+_\beta \mid \alpha < \beta < \omega_1\}$ will be represented at each $\alpha' \leq \alpha$;

2. the set of indiscernibles

$$\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\}$$

   will be a countable set with uncountable pcf over α;

[2] By indiscernibles here we mean members of a generic (one element) Prikry sequences produced by one-element Extender Based Prikry forcings with $E_{\alpha,n}$'s.

3. for every successor ordinal β, $\alpha < \beta < \omega_1$, each indiscernible $\rho^+_{\beta,n,m,\omega_1}$ $(n < \omega, m < g_\beta(n))$ will be in the pcf of this set. Thus, we will have the following:

$$\mathrm{pcf}(\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\}) \supseteq$$

$$\{\rho^+_{\beta,n,m,\omega_1} \mid \alpha < \beta < \omega_1, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_{\omega_1}\};$$

4. for each limit ordinal $\gamma, \alpha < \gamma \le \omega_1$, the following will hold:

$$\mathrm{pcf}(\{\rho^+_{\alpha,n,m,\gamma} \mid n < \omega, m < g_\alpha(n)\}) \supseteq$$

$$\{\rho^+_{\beta,n,m,\gamma} \mid \alpha < \beta < \gamma, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_\gamma\}.$$

   Note that for $\gamma < \omega_1$ the set on the right side is countable.

3 Automatic connection

Let us establish the first connection between the levels and blocks by induction.
Start with a connection of levels and blocks of κ_1 to the levels and blocks of κ_0.
Consider $m(0,1)$, i.e. the least $m < \omega$ such that for every $n \ge m$ we have

$$g_0(n) \ge \sum_{k=0}^{n} g_1(k) \ .$$

This is a place from which blocks of the second level fit nicely inside those of the first level. Let us arrange the connection as follows. Connect all the blocks of the levels $n, n \le m(0,1)$ of κ_1 to the blocks of the level $m(0,1)$ of κ_0 (or in short - of 0) moving to the right as much as possible, i.e. if $r = g_0(m(0,1)) - \sum_{k=0}^{m(0,1)} g_1(k)$, then the first block of κ_1 (in short - 1) is connected to the r–th block of the level $m(0,1)$ of κ_0, the second block of κ_1 is connected to $r+1$-th block of the level $m(0,1)$ of κ_0 etc., the last block of the level $m(0,1)$ of κ_1 will be connected to the last block of the level $m(0,1)$ of κ_0.
For every $s, m(0,1) \le s < \omega$, we continue to connect blocks of all the levels $s' \le s$ of 1 to the s block of 0 in the same fashion, moving to the right as much as possible.

Let us deal now with a level $\alpha > 1$. Fix an enumeration $\langle \alpha_i \mid i < \omega \rangle$ of α (if $\alpha < \omega$, then the construction is the same). Connect blocks of levels of α to those of levels of α_0 exactly as above (i.e. κ_1 and κ_0).
Let us deal now with α_1. We would like to have a tree order at least on the very last blocks of each level. Thus we would not allow a block of α to be connected to two unconnected

blocks of α_0 and α_1. Split into two cases.
Case 1. $\alpha_1 > \alpha_0$. Let $l(\alpha_0, \alpha_1)$ be the first level where the connection between α_0 and α_1 starts. Then, by induction, $l(\alpha_0, \alpha_1) \geq m(\alpha_0, \alpha_1)$. Let $l(\alpha_0, \alpha)$ be the first level where the connection between α_0 and α starts. By the definition we have $l(\alpha_0, \alpha) = m(\alpha_0, \alpha)$. Consider $m(\alpha_1, \alpha)$. It is tempting to start the connection between α and α_1 with the levels $m(\alpha_1, \alpha)$, but we would like to avoid a situation when the last block of a level n of α is connected to last blocks of levels n of both α_0 and α_1, which are disconnected, i.e. the connection order is not a tree order. So set $l(\alpha_1, \alpha) = \max(l(\alpha_0, \alpha_1), m(\alpha_1, \alpha))$. Note that $m(\alpha_0, \alpha) = l(\alpha_0, \alpha) \leq l(\alpha_1, \alpha)$, since $l(\alpha_0, \alpha_1) \geq m(\alpha_0, \alpha_1)$.
Also note that there is a commutativity here, and for each $n \geq l(\alpha_1, \alpha)$, blocks of α of levels $\leq n$ are connected to the level n of α_1 and the levels $\leq n$ of α_1 are connected to the level n of α_0.
Case 2. $\alpha_1 < \alpha_0$.
The treatment is similar only now α_0 is connected to α_1. Set

$$l(\alpha_1, \alpha) = \max(l(\alpha_1, \alpha_0), l(\alpha_0, \alpha), m(\alpha_1, \alpha)).$$

Continue in the same fashion by induction.

Let us called the established connection *automatic connection.* Last blocks of levels ordered by this connection form a tree order by the construction.
It is not hard to show that there is no ω_1–branches. The automatic connection is defined so that if $\alpha < \alpha' < \omega_1$ and for some $n < \omega$, blocks of n-th levels of α and α' are connected, then for every $n', n \leq n' < \omega$, blocks of n'-th levels of α and α' are automatically connected as well. However, the ability to connect between blocks of lower levels does not imply that they will be actually connected by the automatic connection.
For example, suppose that $g_\alpha(0) = g_{\alpha'}(0) = 1$, but $g_\alpha(1) < g_{\alpha'}(1)$. In this case we have $m(\alpha, \alpha') > 1$, so the automatic connection will not start at 0, nerveless, the number of blocks of the first levels of α and α' is the same.

Let $\alpha < \omega_1, n < \omega$ and $m < g_\alpha(n)$. Set

$$a_\alpha(n, m) = \{(\alpha', n', m') \mid \alpha' < \alpha, \text{ the block } m' \text{ of level } n' \text{ of } \alpha'$$

$$\text{is connected automatically to those of } m \text{ of level } n \text{ of } \alpha\}.$$

The next lemma is obvious.

Lemma 3.1 *Let $\alpha < \omega_1, n_1, n_2 < \omega$, $m_1 < g_\alpha(n_1), m_2 < g_\alpha(n_2)$ and ($n_1 \neq n_2$ or $n_1 = n_2$ but $m_1 \neq m_2$). Then $a_\alpha(n_1, m_1) \cap a_\alpha(n_2, m_2) = \emptyset$.*

4 Manual connection

Note that the automatic connection of the previous section leaves many blocks unconnected. If no further connection will be made, then the following will occur. Unconnected blocks of levels of an $\alpha < \omega_1$ will correspond to κ_α^+ in the sense of pcf generators, i.e. their regular cardinals will be in the pcf generator $\mathfrak{b}_{\kappa_\alpha^+}$. By [8], we will have here always $\max(\mathrm{pcf}(\kappa_\alpha^+ \mid \alpha < \beta)) = \kappa_\beta^+$, for every $\beta < \omega_1$, since the initial large cardinal assumptions are mild ones. So, eventually there may be $\beta < \omega_1$ such that all blocks of all levels of all $\alpha < \beta$ will correspond to κ_β^+. It is clearly bad for our purpose.

We would like to extend the automatic connection such that for every α, if ρ and η are the last members of different blocks of α (it does not matter if levels are the same or not), then $\mathfrak{b}_{\rho^+} \neq \mathfrak{b}_{\eta^+}$, where $\mathfrak{b}_\lambda$ denotes the pcf generator of λ. A problematic for us situation is once a connection was established in a way that for some $\alpha < \omega_1$ there are two different blocks of α that are connected to same blocks of $\alpha - 1$, for unboundedly many levels, if α is a successor ordinal, and to same blocks of α''s for unboundedly many α''s below α, if α is a limit ordinal. A problem will be then with $\kappa_\alpha^{++}-$chain condition of the forcing that will realize the pcf-structure.
Note that by Localization Property (see [12] or [1]) once pcf of a countable set is uncountable, there will be countable sets which correspond to cardinals much above their sup. Our construction uses only finitely many blocks at each level. If the connection is not built properly, then some countable set of blocks that should be connected with $\aleph_1$–many may turn to be connected with a single block of some $\alpha < \omega_1$ which will spoil everything.

In order to take care of above problems, let us force with a c.c.c. forcing a new connection based on the automatic connection.

Definition 4.1 Let Q be a set consisting of all pairs of finite functions q, ρ such that

1. $\rho : \{(\alpha, \beta) \mid \alpha < \beta < \omega_1, \alpha = 0$ or it is a successor ordinal, β is a successor ordinal $\} \to \omega \setminus \{0\}$ is a partial finite function,

2. $l(\alpha, \beta) \leq \rho(\alpha, \beta) < \omega$, for every $\alpha < \beta$ in the domain of ρ.
   Intuitively, $\rho(\alpha, \beta)$ will specify the place from which the automatic connection between α and β will step into the play.

3. $\mathrm{dom}(q) \subseteq \{\alpha < \omega_1 \mid \alpha = 0$ or it is a successor ordinal$\} \times (\omega \times \omega)$,

4. for every $(\alpha, n, m) \in \mathrm{dom}(q)$, $q(\alpha, n, m)$ is a non-empty finite subset of $\alpha \times \omega \times \omega$ such that

(a) $m < g_\alpha(n)$,

(b) if $\langle \beta, r, s\rangle \in q(\alpha, n, m)$, then $s < g_\beta(r)$.
This will mean that s-th block of the level r of β is connected to m-th block of the level n of α.

(c) $(\beta, \alpha) \in \mathrm{dom}(\rho)$ iff $\alpha > \beta$ and $\alpha \in \mathrm{dom}(\mathrm{dom}(q))$ and there are $n, m, r, s < \omega$ such that $(\beta, s, r) \in q(\alpha, n, m)$,

(d) if $(\beta, r, s) \in q(\alpha, n, m)$, $(\beta, r, s') \in q(\alpha, n', m')$, and if $s \neq s'$, then either $n \neq n'$ or, $n = n'$ and then $m' \neq m$, moreover $s' < s$ implies $m' < m$,

(e) if $(\beta, r, s) \in q(\alpha, n, m)$ and the connection is not automatic, then $n > r$,

(f) if $(\beta, r, s) \in q(\alpha, n, m)$ and $(\alpha, n, m) \in q(\alpha', n', m')$, then $(\beta, r, s) \in q(\alpha', n', m')$,

Let us define the order on Q.

Definition 4.2 Let $\langle q_1, \rho_1\rangle, \langle q_2, \rho_2\rangle \in Q$. Set $\langle q_1, \rho_1\rangle \geq \langle q_2, \rho_2\rangle$ iff

1. $\rho_1 \supseteq \rho_2$,
2. $q_1 \supset q_2$, i.e.

   (a) $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_2)$,

   (b) for every $\langle \alpha, n, m\rangle \in \mathrm{dom}(q_2)$ we have $q_2(\alpha, n, m) = q_1(\alpha, n, m)$.

Let us give a bit more intuition behind the definition of Q and explain the reason of adding ρ instead of just using the function l of the automatic connection.
The point is to prevent a situation like this: let $\gamma < \beta < \alpha$, $\alpha, \gamma \in \mathrm{dom}(\mathrm{dom}(q))$, $\beta \notin \mathrm{dom}(\mathrm{dom}(q))$ and we like to add it, for some $q \in Q$. Suppose that $l(\gamma, \alpha) = n < l(\beta, \alpha)$ and the level n of γ is connected automatically in q to all the blocks of α up to and including the level n. We need to add β. In order to do this the level n of γ should be connected to β. Then, due to the commutativity, the established connection is continued to α to the level n or below. One may try to use blocks of β of the level n and below for this purpose, but the total number of such blocks may be less than the number of blocks of the level n of γ, i.e. of $g_\gamma(n)$. So some non connected automatically to α blocks of higher levels of β should be used. There may be no such blocks at all or even if there are still this may conflict with automatic connections of bigger than α ordinals in the domain of q.
Once we have ρ, it is possible just to "fix" the automatic connection setting $\rho(\beta, \alpha)$ (i.e. the point from which the automatic connection starts actually to work) higher enough.

Lemma 4.3 *Q satisfies c.c.c.*

Proof. Suppose that $\langle\langle q_i, \rho_i\rangle \mid i < \omega_1\rangle$ is a sequence of ω_1 elements of Q. Let us concentrate on q_i's. Set $b_i = \mathrm{dom}(\mathrm{dom}(q_i)) \cup \mathrm{dom}(\mathrm{rng}(q_i)$ (i.e. the finite sequence of ordinals of $\mathrm{dom}(q_i)$ and of its range). Form a Δ-system. Suppose that $\langle b_i \mid i < \omega_1\rangle$ is already a Δ-system and let b^* be its kernel. By shrinking more if necessary, we can assume that the following holds:

- for every $i < j < \omega_1$, $\sup(b_i) < \min(b_j \setminus b^*)$,
- for every $i < j < \omega_1$, q_i and q_j are isomorphic over $\omega \cup \sup(b^*)$,
- $\rho_i \restriction b^* = \rho_j \restriction b^*$.

Now, let $i < j < \omega_1$. Then $\langle q_i \cup q_j, \rho_i \cup \rho_j\rangle$ will be a condition in Q stronger than both $\langle q_i, \rho_i\rangle$ and $\langle q_j, \rho_j\rangle$.
$\square$

Let G be a generic subset of Q. It naturally defines a connection between blocks. Namely we connect s-th block of a level r of β with m-th block of a level n of α iff for some $(q, \rho) \in G$, $\langle\beta, r, s\rangle \in q(\alpha, n, m)$. Let us call further the part of this connection that is not the automatic connection by *manual connection* .
Denote for $\alpha, n < \omega, m < g_\alpha(m), \alpha_1 < \alpha_2 < \omega_1$, $\alpha, \alpha_1, \alpha_2$ either 0 or successor ordinals

$$\mathit{connect}'(\alpha, n, m) = \{\langle\beta, n_1, m_1\rangle \mid \exists(q, \rho) \in G \quad \langle\beta, n_1, m_1\rangle \in q(\alpha, n, m)\}, \text{ or } \langle\beta, n_1, m_1\rangle$$

$$\text{is automatically connected to } \langle a, n, m\rangle \text{ and } \rho(\beta, \alpha) \leq n_1\},$$

$$\mathit{connect}'(\alpha_1, \alpha_2) = \{(n_1, m_1), (n_2, m_2)) \mid \langle\alpha_1, n_1, m_1\rangle \in \mathit{connect}'(\alpha_2, n_2, m_2)\},$$

$$\mathit{aconnect}'(\alpha_1, \alpha_2) = \{(n_1, m_1), (n_2, m_2)) \in \mathit{connect}'(\alpha_1, \alpha_2) \mid \langle\alpha_1, n_1, m_1\rangle, \langle\alpha_2, n_2, m_2\rangle$$

$$\text{are automatically connected and for some } (q, \rho) \in G \text{ we have } \rho(\alpha_1, \alpha_2) \leq n_1\}.$$

$$\mathit{mconnect}'(\alpha_1, \alpha_2) = \mathit{connect}'(\alpha_1, \alpha_2) \setminus \mathit{aconnect}'(\alpha_1, \alpha_2).$$

Let us refer further to elements of $\mathit{mconnect}'(\alpha_1, \alpha_2)$ connected by the *manual connection* .

Lemma 4.4 *Suppose that $\langle\beta, r, s\rangle$ is a block of β and $\alpha > \beta$. Then for some $n, m < \omega$ we have $\langle\beta, r, s\rangle \in \mathit{connect}'(\alpha, n, m)$.*

Proof. Let $\langle q, \rho\rangle \in Q$. We will construct a stronger condition $\langle q^*, \rho^*\rangle$ with $\langle \alpha, n, m\rangle \in \mathrm{dom}(q^*)$ and $\langle \beta, r, s\rangle \in q^*(\alpha, n, m)$, or $\rho^*(\beta, \alpha) \leq r$ and $\langle \beta, r, s\rangle$ is automatically connected with $\langle \alpha, n, m\rangle$, for some $n, m < \omega$.
If $\langle \beta, r, s\rangle$ is automatically connected with $\langle \alpha, n, m\rangle$, for some $n, m < \omega$ and $(\beta, \alpha) \notin \mathrm{dom}(\rho)$ or $(\beta, \alpha) \in \mathrm{dom}(\rho)$, $\rho(\beta, \alpha) \leq r$, then just set $\rho^*(\beta, \alpha) = r$ or $\rho^*(\beta, \alpha) = \rho(\beta, \alpha)$, if defined and we are done.
Suppose now that the above is not the case. So $r < \rho(\beta, \alpha)$ or $\rho(\beta, \alpha)$ is undefined. In the later case define it just to take any value above r. Pick $n < \omega$ to be big enough such that (α, n) does not appears in q. Extend q to q^* by adding $\langle \alpha, n, g_\alpha(n) - 1\rangle$ to its domain. Set $q^*(\alpha, n, g_\alpha(n) - 1) = \{(\beta, r, s)\}$.
□

Lemma 4.5 *For every $\langle \beta, r, s\rangle, \langle \beta, r', s'\rangle$ with $\beta < \omega_1, r, r' < \omega, r \neq r', s < g_\beta(r), s' < g_\beta(r')$ there are $\alpha < \omega_1$ and even $\alpha < \beta + \omega$,$n < \omega, m < g_\alpha(n)$ such that $\langle \beta, r, s\rangle, \langle \beta, r', s'\rangle \in$ connect$'(\alpha, n, m)$.*

Proof. This follows by the density argument.
Assume that $r < r'$.
Let $\langle q, \rho\rangle \in Q$. We will construct a stronger condition $\langle q^*, \rho^*\rangle$ as follows. First let us pick $\alpha > \beta$ which does not appear in q. If the automatic connection between β and α starts at a level $\leq r'$ and the block s' is connected by it, then just set $\rho^*(\beta, \alpha) = r'$. Let $n = r'$ and m be the block of the level n of α which corresponds to the block s' of the level r' of β. Extend q to q^* by adding $\langle \alpha, n, m\rangle$ to its domain. Set $q^*(\alpha, n, m) = \{(\beta, r, s)\}$.
Suppose now that the automatic connection between β and α starts at a level $> r'$ or it starts at a level $\leq r'$, but the block s' is too low and remains unconnected by it.
Set then $\rho^*(\beta, \alpha)$ to be the place where automatic connection between β and α starts. Pick a level n above it. Extend q to q^* by adding $\langle \alpha, n, g_\alpha(n) - 1\rangle$ to its domain. Set $q^*(\alpha, n, g_\alpha(n) - 1) = \{(\beta, r, s), (\beta, r', s')\}$.
□

The next lemma is similar.

Lemma 4.6 *For every $\langle \beta, r, s\rangle$ with $\beta < \omega_1, r < \omega, s < g_\beta(r)$ and every $\alpha', \beta + \omega \leq \alpha' < \omega_1$ the following hold:*

1. *$\langle \beta, r, s\rangle \in$ aconnect$'(\alpha', n', m')$, for some $n' < \omega, m' < g_{\alpha'}(n')$,*
   or

2. *there are $\alpha, \beta \leq \alpha < \beta+\omega, n < \omega, m < g_\alpha(n)$ such that $\langle \beta, r, s\rangle \in connect'(\alpha, n, m)$ and $\langle \alpha, n, m\rangle \in aconnect'(\alpha', n', m')$, for some $n' < \omega, m' < g_{\alpha'}(n')$.*

Proof. Use density argument.
Let $\langle q, \rho\rangle \in Q$. If for some $n' < \omega, m' < g_{\alpha'}(n')$, the triples $\langle \beta, r, s\rangle, \langle \alpha', n', m'\rangle$ are automatically connected and $\rho(\beta, \alpha') \leq r$, then the first possibility is forced by $\langle q, \rho\rangle$.
Suppose that this is not the case. Then for every $n' < \omega, m' < g_{\alpha'}(n')$, if the triples $\langle \beta, r, s\rangle, \langle \alpha', n', m'\rangle$ are automatically connected, then $\rho(\beta, \alpha') > r$.
We will construct a stronger condition $\langle q^*, \rho^*\rangle$ as follows.
Pick some $n' < \omega, m' < g_{\alpha'}(n')$ such that $(\alpha', n', m') \notin \mathrm{dom}(q)$. In addition pick $\alpha, \beta < \alpha < \beta + \omega$ which does not appear in q (possible since q is finite). Now, there are $n, r < n < \omega, m < g_\alpha(n)$ such that $\langle \alpha, n, m\rangle$ is automatically connected to $\langle \alpha', n', m'\rangle$.
Set $\rho^* = \rho \cup \{((\beta, \alpha), \max(\ell(\beta, \alpha), n)), ((\alpha, \alpha'), \ell(\alpha, \alpha')\}$ and let q^* be obtained from q by adding to its domain $(\alpha, n, m), (\alpha', n', m')$ and setting
$(\beta, r, s) \in q^*(\alpha, n, m)$, $(\beta, r, s), (\alpha, n, m) \in q^*(\alpha', n', m')$.
□

Lemma 4.7 *For every $\alpha < \omega_1$, $n, n' < \omega$ and $m < g_\alpha(n), m' < g_\alpha(n')$. $connect'(\alpha, n, m) \cap connect'(\alpha, n', m')$ is bounded in α, unless $n = n'$ and $m = m'$.*

Proof. Note that the automatic connection has this property (even we have disjoint sets by 3.1). The additions made (if at all) are finite.
□

In order to realize the defined above connection there is a need in dropping cofinalities technics. Thus, for example, for some α the very first block of α may be connected (by the manual connection) to the last block of a level $n > 0$ of $\alpha + 1$. So, in order to accommodate all the blocks of levels $\leq n$ of $\alpha + 1$ on and below the very first block of α there is a need to drop down below α. Note that on $\alpha - 1$ there is enough places to which such blocks are connected automatically, just starting with a higher enough level of $\alpha - 1$.

In this respect $\alpha = 0$ should be treated separately, since $\alpha - 1$ does not exist and so no place to drop. Let us just assume that all blocks of 0 are connected to blocks of 1 automatically. This can be achieved easily by changing g_0, g_1 a bit in order that numbers of blocks fit together nicely.
Under the same lines, we would like to simplify the connection defined generically above a bit more.

First we assume (arrange) that for every limit $\alpha > 0$,

$$g_{\alpha+1}(0) = g_{\alpha+2}(0) \text{ and } \forall n > 0 (g_{\alpha+1}(n) = \sum_{k \le n} g_{\alpha+2}(k)),$$

i.e. the blocks of $\alpha + 1$ and $\alpha + 2$ fit precisely one to another.
Do the same 0 and 1, i.e.

$$g_0(0) = g_1(0) \text{ and } \forall n > 0 (g_0(n) = \sum_{k \le n} g_1(k)).$$

We require that all connections to $\alpha+1$ or to 0 from above go via $\alpha+2$ or via 1, respectively, and the only connections between $\alpha + 2$ and $\alpha + 1$, 1 and 0, are automatic with all blocks of $\alpha + 1$ (or of 0) connected to blocks of $\alpha + 2$ (or, respectively, to 1) by $a-$connections.
This way $\alpha + 1$ (or 0) will be used for dropping from $\alpha + 2$ (or from 1).
Also, if $\alpha < \alpha'$ are both limit, then connections from $\alpha'+k', 0 < k' < \omega$ to $\alpha+k, 0 < k < \omega$ are only $a-$connections. I.e. manual connections applied only between $\alpha+s, \alpha+s', 0 < s, s' < \omega$ with a same limit $\alpha < \omega_1$.
Note that in view of Lemma 4.6, no harm is made by such a change, i.e. each block of a lower level will be still connected to blocks of arbitrary higher levels.
Let us define now explicitly the connections that will be used further in the main forcing and will eventually give the desired pcf-structure.

Set

$$connect(0, 1) = \{((n_1, m_1), (n_2, m_2)) \mid \langle 0, n_1, m_1 \rangle \text{ is automatically connected to } \langle 1, n_2, m_2 \rangle\}.$$

Since

$$g_0(0) = g_1(0) \text{ and } \forall n > 0 (g_0(n) = \sum_{k \le n} g_1(k)),$$

each block of each level of 0 will be connected automatically to those of 1.
Set $aconnect(0, 1) = connect(0, 1)$.

Suppose now that $\alpha < \omega_1$ is a limit non-zero ordinal. Set

$$connect(\alpha + 1, \alpha + 2) = \{((n_1, m_1), (n_2, m_2)) \mid \langle \alpha + 1, n_1, m_1 \rangle$$
$$\text{is automatically connected to } \langle \alpha + 2, n_2, m_2 \rangle\}.$$

Since

$$g_{\alpha+1}(0) = g_{\alpha+2}(0) \text{ and } \forall n > 0 (g_{\alpha+1}(n) = \sum_{k \le n} g_{\alpha+2}(k)),$$

each block of each level of $\alpha+1$ will be connected automatically to those of $\alpha+2$.
Set $aconnect(\alpha+1,\alpha+2) = connect(\alpha+1,\alpha+2)$.

Let now $1 \le t_1 < t_2 < \omega$. Set $connect(t_1,t_2) = connect'(t_1,t_2), aconnect(t_1,t_2) = aconnect'(t_1,t_2)$ and $mconnect(t_1,t_2) = mconnect'(t_1,t_2)$.
Let $t, 1 < t < \omega$ connect 0 to t via the connections of 0 to 1 and of 1 to t:

$$connect(0,t) = \{((n_0,m_0),(n_2,m_2)) \mid \exists (n_1,m_1)(\langle 0,n_0,m_0\rangle \text{ is automatically connected to}$$
$$\langle 1,n_1,m_1\rangle \text{ and } ((n_1,m_1),(n_2,m_2)) \in connect(1,t))\}.$$

$$aconnect(0,t) = \{((n_0,m_0),(n_2,m_2)) \mid \exists (n_1,m_1)(\langle 0,n_0,m_0\rangle \text{ is automatically connected to}$$
$$\langle 1,n_1,m_1\rangle \text{ and } ((n_1,m_1),(n_2,m_2)) \in aconnect(1,t))\}.$$

$$mconnect(0,t) = \{((n_0,m_0),(n_2,m_2)) \mid \exists (n_1,m_1)(\langle 0,n_0,m_0\rangle \text{ is automatically connected to}$$
$$\langle 1,n_1,m_1\rangle \text{ and } ((n_1,m_1),(n_2,m_2)) \in mconnect(1,t))\}.$$

We deal similar with $\alpha < \omega_1$ which is a limit non-zero ordinal. Thus, let now $\alpha+2 \le \alpha_1 < \alpha_2 < \alpha+\omega$. Set $connect(\alpha_1,\alpha_2) = connect'(\alpha_1,\alpha_2), aconnect(\alpha_1,\alpha_2) = aconnect'(\alpha_1,\alpha_2)$ and $mconnect(\alpha_1,\alpha_2) = mconnect'(\alpha_1,\alpha_2)$.
Let $\gamma, \alpha+2 < \gamma < \alpha+\omega$ connect $\alpha+1$ to γ via the connections of $\alpha+1$ to $\alpha+2$ and of $\alpha+2$ to γ:

$$connect(\alpha+1,t) = \{((n_0,m_0),(n_2,m_2)) \mid \exists (n_1,m_1)(\langle \alpha+1,n_0,m_0\rangle$$
$$\text{is automatically connected to } \langle \alpha+2,n_1,m_1\rangle$$
$$\text{and } ((n_1,m_1),(n_2,m_2)) \in connect(\alpha+2,\gamma))\}.$$

$$aconnect(\alpha+1,t) = \{((n_0,m_0),(n_2,m_2)) \mid \exists (n_1,m_1)(\langle \alpha+1,n_0,m_0\rangle$$
$$\text{is automatically connected to } \langle \alpha+2,n_1,m_1\rangle$$
$$\text{and } ((n_1,m_1),(n_2,m_2)) \in aconnect(\alpha+2,\gamma))\}.$$

$$mconnect(\alpha+1,t) = \{((n_0,m_0),(n_2,m_2)) \mid \exists(n_1,m_1)(\langle\alpha+1,n_0,m_0\rangle$$
$$\text{is automatically connected to } \langle\alpha+2,n_1,m_1\rangle$$
$$\text{and } ((n_1,m_1),(n_2,m_2)) \in mconnect(\alpha+2,\gamma))\}.$$

Let now $\beta, \omega < \beta < \omega_1$ be a successor ordinal and $0 < t < \omega$. Define $aconnect(t,\beta) = aconnect'(t,\beta)$. Set

$$connect(t,\beta) = aconnect(t,\beta) \cup \{((n_0,m_0),(n_2,m_2)) \mid \exists t_1 < \omega \exists(n_1,m_1)$$
$$(((n_0,m_0),(n_1,m_1)) \in connect(t,t_1) \text{ and } ((n_1,m_1),(n_2,m_2)) \in aconnect(t_1,\beta))\}.$$

Connect 0 to β via 1. Namely, we set

$$connect(0,\beta) = \{((n_0,m_0),(n_2,m_2)) \mid \exists(n_1,m_1)(\langle 0,n_0,m_0\rangle \text{ is automatically connected to}$$
$$\langle 1,n_1,m_1\rangle \text{ and } ((n_1,m_1),(n_2,m_2)) \in connect(1,\beta))\}$$

and

$$aconnect(0,\beta) = \{((n_0,m_0),(n_2,m_2)) \mid \exists(n_1,m_1)(\langle 0,n_0,m_0\rangle \text{ is automatically connected to}$$
$$\langle 1,n_1,m_1\rangle \text{ and } ((n_1,m_1),(n_2,m_2)) \in aconnect(1,\beta))\}.$$

Deal in a similar fashion with $\alpha < \omega_1$ which is a limit non-zero ordinal.
Let $\beta, \alpha+\omega < \beta < \omega_1$ be a successor ordinal and $\alpha+2 \leq \gamma < \alpha+\omega$. Define $aconnect(\gamma,\beta) = aconnect'(\gamma,\beta)$.
Set

$$connect(\gamma,\beta) = aconnect(\gamma,\beta) \cup \{((n_0,m_0),(n_2,m_2)) \mid \exists\gamma_1 < \alpha+\omega\exists(n_1,m_1)$$
$$(((n_0,m_0),(n_1,m_1)) \in connect(\gamma,\gamma_1) \text{ and } ((n_1,m_1),(n_2,m_2)) \in aconnect(\gamma_1,\beta))\}.$$

Connect $\alpha+1$ to β via $\alpha+2$. Namely, we set

$$connect(\alpha+1,\beta) = \{((n_0,m_0),(n_2,m_2)) \mid \exists(n_1,m_1)(\langle\alpha+1,n_0,m_0\rangle$$
$$\text{is automatically connected to } \langle\alpha+2,n_1,m_1\rangle \text{ and } ((n_1,m_1),(n_2,m_2)) \in connect(\alpha+2,\beta))\}$$

and

$$aconnect(\alpha+1,\beta) = \{((n_0,m_0),(n_2,m_2)) \mid \exists(n_1,m_1)(\langle\alpha+1,n_0,m_0\rangle \text{ is automatically}$$
$$\text{connected to } \langle\alpha+2,n_1,m_1\rangle \text{ and } ((n_1,m_1),(n_2,m_2)) \in aconnect(\alpha+2,\beta))\}.$$

5 The preparation forcing.

We would like to use a generic set for the forcing $\mathcal{P}'$ of Chapter 3 (Preserving Strong Cardinals) of [6] in order to supply models for the main forcing defined further. Some degree of strongness of $\kappa_{\alpha,n}$ will be needed as well, for every successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$.

Two ways were described in Chapter 3 of [6]. Either can be applied for our purpose.

The first one is as follows.

Assume that for some regular cardinal θ the following set is stationary:

$$S = \{\nu < \theta \mid \nu \text{ is a superstrong with the target } \theta (\text{i.e. there is } i : V \to M, crit(i) = \nu$$

$$i(\nu) = \theta \text{ and } M \supseteq V_\theta)\}.$$

Return to the definition of κ_γ's and $\kappa_{\gamma,k}$'s. Let us choose them by induction such that all $\kappa_{\gamma,k}$'s are from S. Suppose that $\langle \kappa_{\gamma,k} \mid k < \omega \rangle$ is defined. Then $\kappa_\gamma = \bigcup_{k<\omega} \kappa_{\gamma,k}$. Let $\tilde{\kappa}_\gamma$ be the next element of S. Pick $\kappa_{\gamma+1,0}$ to be an element of S above $\tilde{\kappa}_\gamma$.

Force with $\mathcal{P}'(\theta)$ with a smallest size of models say $\aleph_8$. Then, by Lemma 3.0.23 of Chapter 3 (Preserving Strong Cardinals) of [6], each $\kappa_{\alpha,n}$ will remain $\tilde{\kappa}_\alpha$–strong (and even $\kappa^+_{\omega_1}$–strong). Moreover, $\mathcal{P}'(\tilde{\kappa}_\alpha)$ is a nice subforcing of $\mathcal{P}'(\theta)$ by Lemma 3.0.18 of Chapter 3 (Preserving Strong Cardinals) of [6], since $V_{\tilde{\kappa}_\alpha} \preceq V_\theta$ due to the choice of $\tilde{\kappa}_\alpha$ in S.

An other way, which uses initial assumptions below $\mathbf{0}^\P$, is as follows.

Let θ be a 2-Mahlo cardinal and $\kappa < \theta$ be a strong up to θ cardinal. Pick $\delta, \kappa < \delta < \theta$ a Mahlo cardinal such that $V_\delta \prec_{\Sigma_1} V_\theta$. By Lemma 3.0.15 of Chapter 3 (Preserving Strong Cardinals) of [6] or just directly, there will unboundedly many cardinals $\eta < \kappa$ with $\delta_\eta < \kappa$ such that the function $\eta \mapsto \delta_\eta$ represents δ and $V_{\delta_\eta} \prec_{\Sigma_1} V_\theta$. Then, by Lemma 3.0.18 of Chapter 3 of [6], $\mathcal{P}'(\delta_\eta)$ is a nice subforcing of $\mathcal{P}'(\theta)$.

Denote by S the set of all such η's.

Force now with $\mathcal{P}'(\theta)$. Let G' be a generic. By Lemma 3.0.24 of Chapter 3 of [6], embeddings wich witness δ-strongness of κ for large enough δ's below θ extend in $V[G']$. Then, below κ in $V[G']$, we will have unboundedly many η's which are strong up to δ_η for which $V_{\delta_\eta}[G' \cap V_{\delta_\eta}] \prec_{\Sigma_1} V_\theta[G']$, since every $\eta \in S$ is like this.

We define now by induction $\kappa_{\gamma,k}$'s, κ_γ's and $\tilde{\kappa}_\gamma$'s using such η's and δ_η's.

Let η_0 be the first element of S. Define $\tilde{\kappa}_0$ be δ_{η_0}. Set κ_{00} to be the least element of S above $\tilde{\kappa}_0$. Let κ_{01} to be the least element of S above $\delta_{\kappa_{00}}$. Continue by induction. Suppose that $n < \omega$ and $\kappa_{0n} \in S$ is defined. Let then κ_{0n+1} to be the least element of S above $\delta_{\kappa_{0n}}$.

Set $\kappa_0 = \bigcup_{k<\omega} \kappa_{0k}$.
Continue to $\gamma > 0$ in a similar fashion.
Thus, if $\gamma, 0 < \gamma < \omega_1$ is a limit ordinal and $\langle \kappa_{\gamma'} \mid \gamma' < \gamma \rangle$ is defined, then set $\kappa_\gamma = \bigcup_{\gamma'<\gamma} \kappa_{\gamma'}$.
Suppose now that κ_γ is defined. Define $\langle \kappa_{\gamma+1,k} \mid k < \omega \rangle$ and $\kappa_{\gamma+1}$.
Let $\tilde{\kappa}_\gamma = \delta_\eta$ for the least $\eta > \kappa_\gamma, \eta \in S$. Pick $\kappa_{\gamma+1,0}$ to be the first $\eta \in S$ above $\tilde{\kappa}_\gamma$ and $\kappa_{\gamma+1,1}$ to be the first $\eta \in S$ above $\delta_{\kappa_{\gamma+1,0}}$, etc. Finally, set $\kappa_{\gamma+1} = \bigcup_{k<\omega} \kappa_{\gamma+1,k}$.

6 Suitable and suitable generic structures.

Suitable structures and suitable generic structures are defined similar to those in Sections 1.2 or 2.4 of [6].

Let us briefly address main components of the preparation forcing $\mathcal{P}'$ ($\mathcal{P}'(\theta)$) used here. A typical member of $\mathcal{P}'$ is of the form $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$.

- s is a closed set of cardinals from the interval $[\aleph_8, \theta]$ having the Easton support. For every $\tau \in s$ the following holds:

- $A^{1\tau}$ is a set of cardinality at most τ consisting of elementary submodels of $H(\theta)$ of size τ, and $A^{0\tau}$ is its largest element under both $\in, \subseteq$.

- C^τ (pistes) is function with domain $A^{1\tau}$ which attach to every $X \in A^{1\tau}$ an increasing continuous sequence of models in $(X \cap A^{1\tau}) \cup \{X\}$ with X being the maximal element.

- The basic property here is that every $B \in A^{1\tau}$ can be reached in finitely many steps from the top model $A^{0\tau}$ going down by pistes of C^τ.

- For every $\tau \in s$, $C^\tau(A^{0\tau})$ is called $\tau-$*central line* and $\langle C^\tau(A^{0\tau} \mid \tau \in s \rangle$ is called *central line (or the main piste)* of the condition $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$.

It is allowed to change directions of pistes (this is called switching) of elements of $\mathcal{P}'$ in the obvious sense, i.e. at splitting points we can choose a direction which is different from one given by the central line. Such process will create a new central line. This way equivalent conditions (in the forcing sense) are obtained.
The order (pre-order) on $\mathcal{P}'$ is defined by combining switchings with end-extensions.

The notions of *a suitable and suitable generic structures* (from SEF I) are used in the main forcing.
The idea is to code elements of $\mathcal{P}'$ as a single structure (i.e. not three sorted as appears in

the definition of $\mathcal{P}'$) and then to deal with isomorphisms of such structures over different cardinals. Suitable structures are such codes. Let us recall the definition.

Definition 6.1 A structure $\mathfrak{X} = \langle X \cup \{X\}, E, C \ \in, \subseteq \ \rangle$, where $E \subseteq [X \cup \{X\}]^2$ and $C \subseteq [X \cup \{X\}]^3$ is called *suitable structure* iff there is
$p(\mathfrak{X}) = \langle \langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), C^{\tau}(\mathfrak{X}) \rangle \mid \tau \in s(\mathfrak{X}) \rangle \in \mathcal{P}'$ such that

1. $X = A^{0\kappa^+}(\mathfrak{X})$,
2. $s(\mathfrak{X}) \in X$,
3. $s(\mathfrak{X}) \subseteq X$,
4. $\langle a, b \rangle \in E$ iff $a \in s(\mathfrak{X})$ and $b \in A^{1a}(\mathfrak{X})$,
5. $\langle a, b, d \rangle \in C$ iff $a \in s(\mathfrak{X}), b \in A^{1a}(\mathfrak{X})$ and $d \in C^a(\mathfrak{X})(b)$.

We will use further *suitable structures over β of level n*, where $\beta = 0$ or is a successor ordinal $< \omega_1$ and $n < \omega$. The definition is the same only $\mathcal{P}'$ is replaced by $\mathcal{P}' \cap V_{\delta_{\kappa_{\beta,n}}}$.

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.
A suitable generic structure is basically a substructure (not necessarily elementary) of the suitable structure generated by an element of $G(\mathcal{P}')$. It can and, typically, would have a smaller cardinality, which is archived by omitting some models from the pistes.

Let us state the main properties.

A suitable structure $\mathfrak{X} = \langle X, E, C \in, \subseteq \rangle$ is called *suitable generic structure* iff there is $\langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \rangle \in G(\mathcal{P}')$ such that

- $\langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \setminus \{\mu\} \rangle \in A^{0\mu}$, where $\mu \in s$ is a regular cardinal and it is the least size of models of our particular interest. Typically, μ is a successor of a singular cardinal which power blows up and the forcing used for this purpose satisfies μ^+−c.c. For example, in our particular setting, once we would like to show that κ_α^{++} is preserved, for some $\alpha < \omega_1$, $\mu = \kappa_\alpha^+$ will be taken.

- $\mathfrak{X}$ is a substructure (not necessarily elementary) of the suitable structure generated by $\langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \rangle$, i.e.
$\langle A^{0\mu} \cup \{A^{0\mu}\}, \{\langle \tau, B \rangle \mid \tau \in s, B \in A^{1\tau}\}, \{\langle \tau, B, D \rangle \mid \tau \in s, B \in A^{1\tau}, D \in C^{\tau}(B)\}$,

- $X \in C^{\mu}(A^{0\mu})$,

- $p(\mathfrak{X})$ (the decoding of $\mathfrak{X}$) and $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ agree about the pistes to members of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$. In other words we require that all the elements of pistes in $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ to elements of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$ are in X.
  The idea here is to reduce the cardinality of the structure still keeping all the essential information.

7 Types of Models

Force with $\mathcal{P}'$. Let $G' \subseteq \mathcal{P}'$ be a generic subset. Work in $V[G']$. For each successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$ let us fix a $(\kappa_{\alpha,n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1})$– extender $E_{\alpha n}$, i.e. an extender with the critical point $\kappa_{\alpha,n}$ which ultrapower contains $V_{\kappa_{\alpha,n,g_\alpha(n)-1,\omega_1}+2}$.
Also, using GCH (we assume GCH in V and then it will holds in $V[G']$ as well), fix an enumeration $\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle$ of $[\kappa_{\alpha n}]^{<\kappa_{\alpha n}}$ so that for every successor cardinal $\delta < \kappa_{\alpha n}$ the initial segment $\langle x_\gamma \mid \gamma < \delta\rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_{\alpha n}(\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle) = \langle x_\gamma \mid \gamma < j_{\alpha n}(\kappa_{\alpha n})\rangle$, where $j_{\alpha n}$ is a canonical embedding of $E_{\alpha n}$. Then $\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle$ will enumerate $[\kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}]^{\leq\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}}$.

For every $k < \omega$, we consider a structure

$$\mathfrak{A}_{\alpha,n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, \chi, E_{\alpha n}, \langle \kappa_\beta \mid \beta < \omega_1\rangle,$$

$$\langle \kappa_{\beta s} \mid \beta < \omega_1 \text{ is a successor ordinal or zero }, s < \omega\rangle,$$

$$\langle \kappa_{\beta,s,r,i} \mid \beta < \omega_1 \text{ is a successor ordinal or zero }, s < \omega, r < g_\beta(s), i \leq \omega_1\rangle,$$

$$\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle, G', \theta, 0, 1, \ldots, \xi, \ldots \mid \xi < \kappa^{+k}_{\alpha n}\rangle$$

in an appropriate language which we denote $\mathcal{L}_{\alpha,n,k}$, with a large enough regular cardinal χ. Note that we have G' inside, so suitable structures may be chosen inside G' or $G' \cap \mathcal{P}'(\kappa_{\alpha,n})$.

Let $\mathcal{L}'_{\alpha,n,k}$ be the expansion of $\mathcal{L}_{\alpha,n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$ let $\mathfrak{A}_{\alpha,n,k,a}$ be the expansion of $\mathfrak{A}_{\alpha,n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. Denote by $tp_{\alpha,n,k}(b)$ the $\mathcal{L}_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k}$. Further we identify it with the ordinal coding it and refer to it as the k-type of b. Let $tp_{\alpha,n,k}(a, b)$ be a the $\mathcal{L}'_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [2].

Now, repeating the usual arguments we obtain the following:

Lemma 7.1 *(a)* $|\{tp_{\alpha,n,k}(b) \mid b \in H(\chi^{+k})\}| = \kappa_{\alpha n}^{+k+1}$

(b) $|\{tp_{\alpha,n,\kappa}(a,b) \mid a,b \in H(\chi^{+k})\}| = \kappa_{\alpha n}^{+k+1}$

Lemma 7.2 *Let* $A \prec \mathfrak{A}_{\alpha,n,k+1}$ *and* $|A| \geq \kappa_{\alpha n}^{+k+1}$. *Then the following holds:*

(a) for every $a,b \in H(\chi^{+k})$ *there* $c,d \in A \cap H(\chi^{+k})$ *with* $tp_{\alpha,n,k}(a,b) = tp_{\alpha,n,k}(c,d)$

(b) for every $a \in A$ *and* $b \in H(\chi^{+k})$ *there is* $d \in A \cap H(\chi^{+k})$ *so that* $tp_{\alpha,n,k}(a \cap H(\chi^{+k}),\ b)$=$tp_{\alpha,n,k}(a \cap H(\chi^{+k}),\ d)$.

Lemma 7.3 *Suppose that* $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_{\alpha n}^{+k+1}$. *Let* τ *be an ordinal less than* $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$ *those* $k+1$*-type is realized unboundedly often below* $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. *Then there are* τ' *and* $A' \prec A \cap H(\chi^{+k})$ *such that* $\tau', A' \in A$ *and* $\langle \tau', A' \rangle$ *and* $\langle \tau, A \cap H(\chi^{+k}) \rangle$ *realize the same* $tp_{\alpha,n,k}$. *Moreover, if* $|A| \in A$*, then we can find such* A' *of cardinality* $|A|$.

Lemma 7.4 *Suppose that* $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_{\alpha n}^{+k+1}$, $B \prec \mathfrak{A}_{\alpha,n,k}$, *and* $C \in \mathcal{P}(B) \cap A \cap H(\chi^{+k})$. *Then there is* D *so that*

(a) $D \in A$

(b) $C \subseteq D$

(c) $D \prec H(\chi^{+k})$.

(d) $tp_{\alpha,n,k}(C,B) = tp_{\alpha,n,k}(C,D)$.

8 The Main Forcing.

Generic connections (*connect*(α_1,α_2)'s, *aconnect*(α_1,α_2)'s,*mconnect*(α_1,α_2)'s) were defined in Section 4. Our aim here will be to define a forcing that turns them into pcf-structure.

In order to do this, we realize the connection as isomorphism functions between suitable structures.

It is rather natural to define (see 8.1.1 below) such isomorphisms for automatically connected blocks, i.e. those in *aconnect*(α_1,α_2)'s.

A slight complication here is that rather than connecting (i.e. making correspond by isomorphisms) blocks of cardinals provided by the automatic connection, we connect the one element Prikry sequences (i.e. indiscernibles) of higher blocks to blocks below. The reason for doing this is that the cardinals outside of blocks of indiscernibles naturally belong to pcf-generators for some κ_α^+'s, since GCH is assumed in the ground model. This will not allow us to proceed all the way up to $\kappa_{\omega_1}^+$.

Turn now to the manual connection.

Let us explain the reason for using it at all.

The point is that without it there will be plenty blocks that left unconnected to higher ones. Just note that the functions $g_\alpha, \alpha < \omega_1, \alpha = 0$ or α is a successor ordinal, which were used for the connections, satisfy

$$g_\alpha(n) \geq \sum_{m=0}^{n} g_\beta(m) \ ,$$

for $\beta > \alpha$ and actually,

$$g_\alpha(n) > \sum_{m=0}^{n} g_\beta(m) \ ,$$

must hold at many places. This strong inequality generates unconnected (automatically) blocks from the level α to the level β.

Now this unconnected blocks (or more precisely the cardinals inside them) will be then in pcf-generators of some κ_ξ^+, for limit ξ's below ω_1. Then the Localization Property ([12]) will not allow to climb all the way up to $\kappa_{\omega_1}^+$.

A complication with manual connections is that in contrast with the automatic ones, the number of blocks does not fit together nicely.

For example - for some $\alpha < \beta < \omega_1$, α limit non-zero ordinal and $\alpha + 2 < \beta < \alpha + \omega$, the very first block of the first level $[\kappa_{\alpha+2}^+, \kappa_{\alpha+2,0,0,\omega_1}^+]$ of $\alpha + 2$ may correspond (by a manual connection) to say 10-th block of the second level of β, i.e. to $[\kappa_{\beta,1,9,\omega_1}^{++}, \kappa_{\beta,1,10,\omega_1}^+]$.

Now, by No Hole Principle ([12]), the blocks of β starting from 9 and below (or their regular cardinals) should be connected to those below the first of $\alpha + 2$. However, $\alpha + 2$ has no blocks below its first one.

The solution is to drop down to $\alpha + 1$. In order to so, we need a variation of *a drop in cofinality* which was used in Section 4 of [6].

The situation (a bit simplified one) is as follows. Suppose that we would like to have a nice scale of functions $\langle f_\xi \mid \xi < \mu^{+3} \rangle$ in $\prod_{n<\omega} \mu^{+n+2}$, for some regular cardinals μ_n's unbounded in μ such that $\{\mu_n^{+r_n} \mid n < \omega, r_n < n+2\}$ correspond to μ^+. We must have that $\{\mu^{+n+2} \mid n < \omega\}$ corresponds to μ^{+3}. But what about μ^{++}? Usually, the set which correspond to it is obtained from the one for μ^{+3} by reducing each member by one, i.e. $\{\mu^{+n+1} \mid n < \omega\}$. But here we have that $\{\mu^{+n+1} \mid n < \omega\}$ corresponds to μ^+. So, going down is needed in order to realize such configuration. Namely, for ξ's less than μ^{+3} of cofinality μ^{++}, the cofinality of $f_\xi(n)$'s should drop down below μ_n.

We turn now to the definition of the main forcing $\mathcal{P}$. Let us split the definition into ω–many steps. First we define pure conditions $\mathcal{P}_0$, at the next step $\mathcal{P}_1$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_0$, then $\mathcal{P}_2$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_1$, etc. Finally $\mathcal{P}$ will be $\bigcup_{n<\omega}\mathcal{P}_n$.

Definition 8.1 The set $\mathcal{P}_0$ consists of all sequences

$$\langle p_\alpha \mid \alpha<\omega_1 \text{ and } (\alpha=0 \text{ or } \alpha \text{ is a successor ordinal })\rangle$$

such that $p_\alpha = \langle p_{\alpha\beta} \mid \alpha<\beta<\omega_1 \text{ is a successor ordinal } \rangle$, and for all $n<\omega, \alpha<\beta<\omega_1$ is a successor ordinal ,
$p_{\alpha\beta} = \langle p_{\alpha\beta x} \mid x \in \mathit{connect}(\alpha,\beta)\rangle$, where for every $x\in \mathit{connect}(\alpha,\beta)$,
$p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha\beta x}, f_{\alpha\beta x}\rangle$ is such that:

1. (Automatic connection)
   If $x\in \mathit{aconnect}(\alpha,\beta)$, $x=((n_1,k_1),(n_2,k_2))$, for some $k_1,k_2,n_1,n_2<\omega$, [3] then

   (a) $A_{\alpha\beta x}=A_{\alpha n_1}$, i.e. it does not depend on β, x, but rather on on level n_1 of α (and α itself).
   It is a set of measure one for some measure of the extender $E_{\kappa_{\alpha n_1}}$. Denote the corresponding coordinate in $E_{\kappa_{\alpha n_1}}$ by $\mathit{coor}(A_{\alpha n_1})$.

   (b) $a_{\alpha\beta x}=a_{\alpha\beta n_1}$, i.e. it depends on α,β and n_1 only.

   (c) $a_{\alpha\beta n_1}$ is an isomorphism between a $(\prod_{k\le n_1} A_{\beta k})-$name of a generic suitable structure $\underset{\sim}{\mathfrak{X}}^{\beta}_{\alpha\beta n_1}$ of size $<\kappa_{\alpha n_1}$ over β of the level n_1 and a suitable structure $\mathfrak{X}^{\alpha}_{\alpha\beta n_1}$ of α of the level n_1.

   (d) For each $k\le n_1$ and $\eta\in A_{\beta k}$ let us denote by $\rho_{\beta k}$ the projection of η to the normal measure of the extender $E_{\beta k}$.
   For each $m<g_\beta(k)$ and $\gamma\le\omega_1$ let $\rho_{\beta k m\gamma}$ be $\pi_{\mathit{coor}(A_{\beta k})\kappa_{\beta k m\gamma}}(\eta)$, i.e. the indiscernible which corresponds to $\kappa_{\beta k m\gamma}$, where $\mathit{coor}(A_{\beta k})$ is the coordinate of $E_{\kappa_\beta k}$ to which $A_{\beta k}$ belongs.[4]
   We require that for each

[3] Note that then $\rho(\alpha,\beta)\le n_2\le n_1$ and the level n_1 of β is also automatically connected to the level n_1 of α.

[4] It is not hard to arrange that $\rho_{\beta k}$ already determines all $\rho_{\beta k m\gamma}$'s, for every $k\le n_1, m<g_\beta(k)$ and $\beta<\gamma\le\omega_1$.

$\langle \eta_0, ..., \eta_{n_1} \rangle \in \prod_{k \le n_1} A_{\beta k}$, for every $k \le n_1, m < g_\beta(k)$ and $\beta < \gamma \le \omega_1$,
$a_{\alpha\beta n_1}[\langle \eta_0, ..., \eta_{n_2} \rangle]$ (i.e. the interpretation of $a_{\alpha\beta n_2}$ according to $\langle \eta_0, ..., \eta_{n_2} \rangle$) is the isomorphism between $\underset{\sim}{\mathfrak{X}}^\beta_{\alpha\beta n_1}[\langle \eta_0, ..., \eta_{n_2} \rangle]$ and $\mathfrak{X}^\alpha_{\alpha\beta n_1}$ which maps models of sizes $\rho_{\beta k m \gamma}$ and $(\rho_{\beta k m \gamma})^+$ to models over the level n_1 of α of cardinalities $\kappa_{\alpha n_1 m^* \gamma}$ and $(\kappa_{\alpha n_1 m^* \gamma})^+$ respectively, where $m^* = (g_\alpha(n_1) - \sum_{s=k}^{n_1} g_\beta(s)) + m$ (i.e. we start as far right as possible).
This means, in particular, that once a non-direct extension was made at the level n_1 of α, then $\rho_{\beta k m \gamma}$ and $(\rho_{\beta k m \gamma})^+$ will correspond to $\rho_{\alpha n_1 m^* \gamma}$ and $(\rho_{\alpha n_1 m^* \gamma})^+$ respectively.
Models of sizes from the interval $((\rho_{\beta k m-1 \omega_1})^+, \rho_{\beta k m \beta+1})$ will be connected with models of sizes in the interval $(\kappa^+_{\alpha n_1 m^* 0}, \kappa_{\alpha n_1 m^* \beta+1})$, if $m > 0$.
If $m = 0$ and $k > 0$, then models of sizes from $((\rho_{\beta k-1 g_\beta(k-1)-1 \omega_1})^+, \kappa_{\beta k-1}) \cup [\kappa_{\beta k-1}, \rho_{\beta k 0 \beta+1})$ will be connected with $(\kappa^+_{\alpha n_1 m^* 0}, \kappa_{\alpha n_1 m^* \beta+1})$.
If $m = 0$ and $k = 0$, then $(\kappa_{\beta-1}, \rho_{\beta 0 0 \beta+1})$ will be connected with $(\kappa_{\alpha n_1 m^* \beta}, \kappa_{\alpha n_1 m^* \beta+1})$.

(e) $f_{\alpha\beta x} = f_{\alpha\beta n_1}$ is a $(\prod_{k \le n_1} A_{\beta k})-$name of a partial function from κ_{β, n_1} to κ_{α, n_1} of cardinality at most $\kappa_{\beta-1}$.

2. (Manual connection)
$x \in mconnect(\alpha, \beta)$, $x = ((n_1, k_1), (n_2, k_2))$, for some $k_1, k_2, n_1, n_2 < \omega$.
The cardinals corresponding is similar to the case of the automatic connection. Note that $m-$connection connects to a single level and the rest drops down. Describe such droppings.

Describe now manual connections droppings and state commutativity requirements.

Suppose that two blocks of some levels of β are connected to the same block of some level of α. In particular, one (at least one) must be then $m-$connected.
Let (α, n, m) be connected with (β, r, s) and (β, r', s'), where $\beta > \alpha, r' > r$.
It may be the case that $r' \le n$, and then necessary (β, r', s') is a part of $a-$connection of the level n of β to the level n of α.
By genericity of connections, there will be $\gamma, \beta < \gamma < \beta + \omega, t, u < \omega$ such that (γ, u, t) is $a-$connected to (β, r', s') and (β, r, s) is $m-$connected to it.
We will require the obvious commutativity here in the further item (7).

Similar we treat the situation in which $\beta > \alpha + \omega$ is connected ($a-$connection, since $\beta > \alpha + \omega$) to (α, n), but its connection to $\alpha+1$ starts above level n, i.e. $\rho(\alpha+1, \beta) > n$.

Thus, by genericity of connections, there will be some (r, s)(actually, there will be infinitely many such r's) so that $(\alpha, n, g_\alpha(n) - 1)$ is connected manually to $(\alpha + 1, r, s)$ and also, (β, n) is $a-$connected to it. We will require commutativity here as well.

This way, in particular, we will not loose information on connection of $(\beta, n), (\alpha, n)$ once a non-direct extension made over $\alpha + 1$.

Let us describe dropping in cofinality that occurs here.

Suppose that α is a non-limit ordinal, $\beta > \alpha + 1$ and we have the following:

(β, n') connected to levels n and n' of $\alpha + 1$ (say $n' > n$), with $a-$connection to $(\alpha + 1, n')$.

Both $(\alpha + 1, n), (\alpha + 1, n')$ are connected to a same level at α.

It may be the case that level n of α is $a-$connected with the level n of $\alpha + 1$, but it is possible that this does not occurs and then a block (α, k, l) is $m-$connected to blocks $(\alpha + 1, n, m)$ and $(\alpha + 1, n', m')$. Consider the least block $s' < g_{\alpha+1}(n')$ which is connected to α (i.e. to $k-$th level of α). Then below it the drop to $\alpha - 1$ occurs.

If the connection from α to the $n-$th level of $\alpha + 1$ is not automatic, then there will be $s < g_{\alpha+1}(n)$ the last with connection to α (i.e. to $k-$th level of α). So, again, drop will occur here to $\alpha - 1$. We have n' big enough so the connections to $(\alpha + 1, n')$, $(\alpha + 1, n)$ which are from $n'-$level of β cover all blocks including s, s' (counting down from above). So, commutativity requirements apply to all relevant blocks before those that drop to $\alpha - 1$.

Let n' drops to some $\tilde{n}' \geq n'$ (over $\alpha - 1$) and n to $\tilde{n} \geq n$, where $\tilde{n}', \tilde{n}$ depend on places where the $a-$connection between β and $\alpha - 1$ starts to work. Also $\tilde{n}' > \tilde{n}$. This implies that the corresponding assignment functions ($\underset{\sim}{b}$'s) will have domains of different cardinalities ($< \kappa_{\alpha-1\tilde{n}'}$ and $< \kappa_{\alpha-1\tilde{n}}$). Repeat Section 4 of [6] and split into intervals over central pistes of suitable structures.

3. Let β be a successor ordinal. Assume that for some successor or zero ordinal $\alpha < \beta$ and $x \in$ *connect*(α, β), $a_{\alpha,\beta,x}$ is defined. Then for every $Z \in \mathrm{dom}(a_{\alpha,\beta,x})$, for every $k < \omega$

   the set

   $\{(\gamma, y) \mid \gamma < \beta, x \in \mathit{connect}(\gamma, \beta), a_{\gamma,\beta,y}$ is defined, $Z \in \mathrm{dom}(a_{\gamma,\beta,y})$ and

   $\neg(a_{\gamma,\beta,y}(Z) \cap H(\chi^{+k}_{\gamma,n_1}) \preceq H(\chi^{+k}_{\gamma,n_1}))\}$ is finite,

   where $y = ((n_1, m_1), (n_2, m_2))$ and χ_{γ,n_1} is a regular cardinal large enough in the interval $(\kappa^{++}_{\gamma,n_1,g_\gamma(n_1)-1,\omega_1}, \kappa_{\gamma,n_1+1})$

4. Let α be zero or a successor ordinal and $\beta, \alpha < \beta < \omega_1$, be a successor ordinal. Suppose that $((n_1, m_1), (n_2, m_2)), ((n_1', m_1'), (n_2', m_2')) \in \mathit{aconnect}(\alpha, \beta)$ and $n_1 < n_1'$. Then $\mathrm{dom}(a_{\alpha\beta((n_1,m_1),(n_2,m_2))}) \subseteq \mathrm{dom}(a_{\alpha\beta((n_1',m_1'),(n_2',m_2'))})$.

5. Let α be zero or a successor ordinal and $\beta, \alpha < \beta < \omega_1$, be a successor ordinal. Suppose that $((n_1, m_1), (n_2, m_2))) \in \mathit{mconnect}(\alpha, \beta)$ and $((n_1', m_1'), (n_2, m_2))) \in \mathit{aconnect}(\alpha, \beta)$. Then $\mathrm{dom}(a_{\alpha\beta((n_1,m_1),(n_2,m_2))}) \subseteq \mathrm{dom}(a_{\alpha\beta((n_1',m_1'),(n_2,m_2))})$.

6. Let α be zero or a successor ordinal and $\beta, \alpha < \beta < \omega_1$, be a successor ordinal. Suppose that $x \in \mathit{connect}(\alpha, \beta)$. Then $\mathrm{dom}(f_{\alpha\beta x}) \cap \mathrm{dom}(a_{\alpha\beta x}) = \emptyset$.

7. (Commutativity of connections) Let α be zero or a successor ordinal and β, γ be successor ordinals, $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$. Assume that $k_\alpha-$th block of $n_\alpha-$th level of α is connected to $k_\beta-$th block of a level n_β of β and to $k_\gamma-$th block of a level n_γ of γ. Suppose that in addition that $k_\beta-$th block of a level n_β of β and $k_\gamma-$th block of a level n_γ of γ are connected.
Then for each $Z \in \mathrm{dom}(a_{\alpha\gamma((n_\alpha,k_\alpha),(n_\gamma,k_\gamma))})$ we have $Z \in \mathrm{dom}(a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,kn_\gamma))})$ and
$$a_{\alpha\gamma((n_\alpha,k_\alpha),(n_\gamma,k_\gamma))}(Z) = a_{\alpha\beta((n_\alpha,k_\alpha),(n_\beta,k_\beta))}(\underset{\sim}{a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}}(Z)),$$
where $\underset{\sim}{a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}}(Z)$ is a name of the indiscernible which corresponds to $a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}(Z)$.

Definition 8.2 (One element extension.)
Suppose $p = \langle p_\alpha \mid \alpha < \omega_1$ and ($\alpha = 0$ or α is a successor ordinal)$\rangle \in \mathcal{P}_0$, $\alpha < \omega_1$ be zero or a successor ordinal, $\beta, \alpha < \beta < \omega_1$ a successor ordinal and $x = ((n_\alpha, m_\alpha), (n_\beta, m_\beta)) \in \mathit{connect}(\alpha, \beta)$. Let $p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha,x}, f_{\alpha\beta x}\rangle$ and $\eta \in A_{\alpha,x}$.
Assume that $n_\alpha = 0$. In general, if $n_\alpha \neq 0$, then taking a non-direct extension over the level n_α we would like simultaneously to make a non-direct extension at each level $n < n_\alpha$ over α. Define $p^\frown\eta$, the one element non direct extension of p by η, to be $q = \langle q_\xi \mid \xi < \omega_1$ and ($\xi = 0$ or ξ is a successor ordinal)$\rangle$ so that

1. for every $\xi, \zeta, \alpha < \xi < \zeta < \omega_1, y \in \mathit{connect}(\xi, \zeta)$, $\quad p_{\xi\zeta y} = q_{\xi\zeta y}$,

2. for every $y \in \mathit{connect}(\alpha, \gamma)$ with the level on α bigger than n_α we have $p_{\alpha\beta y} = q_{\alpha\beta y}$.

3. for every successor ordinal $\gamma, \alpha < \gamma < \omega_1$,
$$q_{\alpha\gamma y} = f_{\alpha\gamma y} \cup \{\langle \tau, \pi^{E_{\kappa_\alpha,n_\alpha}}_{mc(\alpha,n),a_{\alpha\gamma y}(\tau)}(\eta)\rangle \mid \tau \in \mathrm{dom}(a_{\alpha\gamma y})\},$$
where $y \in \mathit{connect}(\alpha, \gamma)$ and the level of y over α is n_α as those of x.

4. Let $\alpha', \tau, \alpha' > \alpha > \tau$, be successor ordinals or zero. Then connections $a_{\tau\alpha' y}$ of p will split now in q into connections from α' to α followed by a connection from α to τ. Namely, let $\langle \tau, r, s\rangle$ be connected with $\langle \alpha', n', m'\rangle$. For each (n, m) such that $((n, m), (n', m')) \in \mathit{aconnect}(\alpha', \alpha)$ and $\langle \tau, r, s\rangle \in \mathit{connect}(\alpha, n, m)$ (there are such n, m by Lemma 4.5) split $a_{(\alpha',n',m'),(\gamma,r,s)}$ into $a_{(\alpha',n',m'),(\alpha,n,m)}$ followed by $a_{(\gamma,r,s),(\alpha,n,m)}$.

5. For each level $n' < n_\alpha$ of α, the same things occur, i.e. 2-4 above hold with (n_α, m_α) replaced by (n', k'), where k' is any block of the level n'.

6. For every connection which drops in cofinality below the block of η, i.e. below the level n_α of α, we freeze such drops and deal only with drops to cofinalities above η in a fashion used in Section 6 of [6] for same purpose.

Definition 8.3 Set $\mathcal{P}_1$ to be the set all $p^\frown\eta$ as in Definition 8.2. Proceed by induction. For each $n < \omega$, once $\mathcal{P}_n$ is defined, define $\mathcal{P}_{n+1}$ to be the set of all $p^\frown\eta$, where $p \in \mathcal{P}_n$. Finally set $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}_n$.

Definition 8.4 Let $p, q \in \mathcal{P}$.

1. We say that p is a direct extension of q and denote this by $p \geq^* q$ iff p is obtained from q by extending $a_{\alpha\beta x}, f_{\alpha\beta x}$'s and by shrinking the sets of measures one probably by passing to bigger measure first.

2. The forcing order $\geq$ is defined as follows:
   $p \geq q$ iff there are $q_1, ..., q_n \in \mathcal{P}, \eta_1, ..., \eta_n$ such that

   (a) $q \leq^* q_1$,

   (b) for every $k, 1 \leq k \leq n, \quad q_k{}^\frown\eta_k \in \mathcal{P}$,

   (c) for every $k, 1 \leq k < n, \quad q_k{}^\frown\eta_k \leq^* q_{k+1}$,

   (d) $q_n{}^\frown\eta_n \leq^* p$.

For each $\alpha < \omega_1$. $\mathcal{P}$ splits into $(\mathcal{P}\backslash\kappa_\alpha) * \mathcal{P} \restriction \kappa_{\alpha+1}$, where $\mathcal{P}\backslash\kappa_\alpha$ is the part of $\mathcal{P}$ is defined as $\mathcal{P}$ but with $\kappa_{\alpha+1}$ replacing κ_0, i.e. everything is above κ_α and the first cardinal we deal with is $\kappa_{\alpha+1,0}$. $\mathcal{P} \restriction \kappa_{\alpha+1}$ is defined in $V[G']^{\mathcal{P}\backslash\kappa_\alpha}$ as $\mathcal{P}$ was defined in $V[G']$, but cutting everything at $\kappa_{\alpha+1}$, where $G' = G(\mathcal{P}')$ is a generic subset of the preparation forcing $\mathcal{P}'$.

Let us prove now the Prikry condition.

Lemma 8.5 $\langle \mathcal{P}, \leq, \leq^* \rangle$ *is a Prikry type forcing notion.*

Proof. Work in $V[G(\mathcal{P}')]$. Let σ be a statement of the forcing language and $p \in \mathcal{P}$. Suppose for simplicity that $p \in \mathcal{P}_0$.
We peak an elementary chain of elementary submodels of H_χ (for χ big enough)

$$\langle M(\kappa_{\alpha n}, \xi) \mid \alpha < \omega_1,\ 0 \text{ or non-limit ordinal },n < \omega, \xi \leq \kappa_{\alpha n} \rangle$$

such that

1. $p, \sigma \in M(\kappa_{00}, 0)$,
2. $|M(\kappa_{\alpha n}, \xi)| = \kappa_{\alpha n}$,
3. if ξ is a limit ordinal then $M(\kappa_{\alpha n}, \xi) = \bigcup_{\xi' < \xi} M(\kappa_{\alpha n}, \xi')$,
4. $\langle M(\kappa_{\alpha n}, \xi') \mid \xi' < \xi \rangle \in M(\kappa_{\alpha n}, \xi)$, for every successor ξ,
5. $\langle M(\kappa_{\alpha n}, \xi) \mid \xi \leq \kappa_{\alpha n} \rangle \in M(\kappa_{\alpha n+1}, 0)$,
6. $\langle M(\kappa_{\alpha n}, \xi) \in G(\mathcal{P}')$.
7. Let $M(\kappa_{\omega_1}) = \bigcup_{\alpha < \omega_1, n < \omega} M(\kappa_{\alpha n}, \kappa_{\alpha n})$.
   Then
   (a) $M(\kappa_{\omega_1}) \in G(\mathcal{P}')$,
   (b) each model $M(\kappa_{\alpha n}, \xi)$ is on the main piste of $M(\kappa_{\omega_1})$.

Proceed by induction. Suppose we got to level n of some α. Denote by X the corresponding set of measure one of the condition q built (i.e. $A_{\alpha n}$ of it). Continue by induction on members of X. We use here models $\langle M(\kappa_{\alpha n}, \xi) \mid \xi \leq \kappa_{\alpha n} \rangle$. Thus, if $\nu \in X$, then work inside $M(\kappa_{\alpha n}, \nu + 1)$. We ask if there is an extension of $q(\nu)^\frown \nu$ (where $q(\nu)$ was formed on the previous stage) which decides σ and is a direct extension above α, n. If so, then pick such extension and add $M(\kappa_{\alpha n}, \nu + 1)$ to be the largest model. Otherwise, we make no change.
Non-direct parts below α, n will be stabilized once all ν's in X are considered. More precisely, we stabilize each $\langle \tau_1^\nu, ..., \tau_s^\nu \rangle$ that is below ν and a direct extension of $q(\nu)^\frown \langle \tau_1^\nu, ..., \tau_s^\nu \rangle^\frown \nu$ decides σ. Isomorphisms between structures (a's) and Cohen functions (f's) below are dealt as names depending on ν's.
Being of models $M(\kappa_{\beta m})$'s on the central line of $M(\kappa_{\omega_1})$ allows freely to take unions.
□

Lemma 8.6 *$\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets to κ_0.*

Proof. Let $p \in \mathcal{P}$, $\underset{\sim}{z}$ be a $\mathcal{P}-$name and $p \Vdash \underset{\sim}{z}$ is a bounded subset of κ_0. Extending p if necessary we can assume that $p \Vdash \underset{\sim}{z} \subseteq \kappa_{0m}$, for some $m < \omega$. Extend p further, if necessary, and assume that non-direct extensions were made in it at every level $n \leq m$ of 0.
Pick an elementary submodel $M \preceq H\chi$ of cardinality κ_0^+ such that

1. $p, \underset{\sim}{z} \in M$,
2. $M \in G(\mathcal{P}')$,
3. there is an increasing continuous sequence $\langle M_\xi \mid \xi < \kappa_{0m} \rangle$ of elementary submodels of M such that
   (a) $p, \underset{\sim}{z} \in M_0$,
   (b) $\langle M_\xi \mid \xi < \kappa_{0m} \rangle$ on the piste of M of models of size κ_0^+,
   (c) $M = \bigcup_{\xi < \kappa_{0m}} M_\xi$.

Now, we use the previous lemma 8.5 and build by induction a $\leq^*$ $-$increasing sequence $\langle p(\xi) \mid \xi < \kappa_{0m} \rangle$ of extensions of p such that $p(\xi) \in M_{\xi+1}$ and $p(\xi) \| \xi \in \underset{\sim}{z}$.
We have enough closure to run the process and eventually the upper bound of $\langle p(\xi) \mid \xi < \kappa_{0m} \rangle$ will decide $\underset{\sim}{z}$ completely.
$\square$

Similar argument gives the following:

Lemma 8.7 *For every $\alpha < \omega_1$, $\langle \mathcal{P} \setminus \kappa_\alpha, \leq \rangle$ does not add new bounded subsets to $\kappa_{\alpha+1}$.*

Define now $\longleftrightarrow$ and $\longrightarrow$.

Definition 8.8 Let $p, q \in \mathcal{P}$. Set $p \longleftrightarrow q$ iff there is $\alpha < \omega_1$ such that

1. $p \setminus \kappa_\alpha = q \setminus \kappa_\alpha$,
2. for every $k < \omega$, for all but finitely many $\beta \leq \alpha$, for all but finitely many $n < \omega$ the following hold:
   (a) if no non-direct extension was made at the level n of β in p_β and q_β, then $0_{\mathcal{P} \setminus \kappa_\beta} \Vdash_{\mathcal{P} \setminus \kappa_\beta}$ over the level n of β the following hold in p_β and q_β:
      i. f's, A's and dom(a)'s are the same,

ii. $\mathrm{rng}(a)$'s realize the same k-type;

(b) if a non-direct extension was made at the level n of β in one of p_β or q_β, then it was made in another as well, and they are equal.

This means basically that $p \restriction \kappa_{\alpha+1} \longleftrightarrow_{\mathcal{P}\restriction\kappa_{\alpha+1}} q \restriction \kappa_{\alpha+1}$,
where $\longleftrightarrow_{\mathcal{P}\restriction\kappa_{\alpha+1}}$ states that for each $k < \omega$ all but finitely many coordinates realize the same k-type.

Now we define $\longrightarrow$ in the usual fashion.

Definition 8.9 Let $p, q \in \mathcal{P}$. Set $p \longrightarrow q$ iff there is a sequence of conditions $\langle r_k \mid k < m < \omega\rangle$ so that

(1) $r_0 = p$

(2) $r_{m-1} = q$

(3) for every $k < m - 1$,
$r_k \leq r_{k+1}$ or $r_k \longleftrightarrow r_{k+1}$.

Lemma 8.10 *Let $\alpha < \omega_1$. Then, in $V^{\mathcal{P}' * \mathcal{P}\setminus\kappa_\alpha}$, the forcing $\langle \mathcal{P} \restriction \kappa_{\alpha+1}, \longrightarrow\rangle$ satisfies κ_α^{++}-c.c.*

Proof. Suppose otherwise. Assume that

$$0_{\mathcal{P}\setminus\kappa_\alpha} \Vdash_{\mathcal{P}\setminus\kappa_\alpha} \{\underset{\sim}{p}_\xi \mid \xi < \kappa_\alpha^{++}\} \subseteq \mathcal{P} \restriction \kappa_{\alpha+1} \text{ is an antichain} .$$

Force over $V[G(\mathcal{P}')]$ (not over $V[G(\mathcal{P}')][G(\mathcal{P} \setminus \kappa_\alpha)]$!) with the obvious forcing (i.e. initial segments) which produces a κ_α^{++}-chain of members of $G(\mathcal{P}')$ of size κ_α^+. This forcing does not add new sequences of length $\leq \kappa_\alpha^+$.
Pick an elementary submodel $M \preceq H_\chi$ of such generic extension which is a union of an elementary chain $\langle M_\xi \mid \xi < \kappa_\alpha^{++}\rangle$ of its elementary submodels of size κ_α^+ which are in $G(\mathcal{P}')$, and such that for every $\xi < \kappa_\alpha^{++}$,
$\langle M_{\xi'} \mid \xi' < \xi\rangle \in M_\xi$ and is on the central piste of M_ξ.
Now we proceed by induction. On stage ξ decide $\underset{\sim}{p}_\xi$ inside M_ξ and add M_ξ as a largest model. The rest of the proof follows completely the lines of the analogues arguments for short extenders forcings (see, for example, Sec 1 of [6]). Eventually, we will have $\xi < \rho < \kappa_\alpha^{++}$ and a condition in $\mathcal{P} \setminus \kappa_\alpha$ which forces compatibility of $\underset{\sim}{p}_\xi$ and $\underset{\sim}{p}_\rho$.
□

Lemma 8.11 *The forcing $\langle \mathcal{P}, \longrightarrow \rangle$ over $V[G']$ preserves all the cardinals (and every cofinality).*

Proof. Let η be a cardinal in $V[G']$. We show by induction on $\alpha < \omega_1$ that if $\eta \leq \kappa_\alpha$ then it is preserved in the generic extension. Clearly, it is enough to deal only with regular η's. Hence, we need to consider only the following situation:

$$\kappa_\alpha < \eta < \kappa_{\alpha+1},$$

for some $\alpha < \omega_1$. Split the forcing $\mathcal{P}$ into $\mathcal{P} \backslash \kappa_\alpha$ followed by $\mathcal{P} \restriction \kappa_{\alpha+1}$. By Lemma 8.7, $\mathcal{P} \backslash \kappa_\alpha$ does not add new bounded subsets to $\kappa_{\alpha+1}$ (namely, this lemma together with the Prikry condition imply that no new subsets are added to $\kappa_{\alpha+1,0}$, but taking non-direct extensions over $\kappa_{\alpha+1,n}$'s it is easy to push this up to $\kappa_{\alpha+1}$). By Lemma 8.10 the forcing $\mathcal{P} \restriction \kappa_{\alpha+1}$ preserves all the cardinals above κ_α^+. So, the only case that remains is $\eta = \kappa_\alpha^+$. But it is not problematic, since we have here the successor of the singular cardinal and the usual arguments apply.

□

Lemma 8.12 *For every $\alpha < \omega_1$, α non-accumulation point (i.e. $\alpha = 0$ or α non-limit ordinal) the following hold in $V^{\mathcal{P}' * \langle \mathcal{P}, \longrightarrow \rangle}$:*

$$\operatorname{pcf}(\{(\rho_{\alpha n m \omega_1})^+ \mid n < \omega, m < g_\alpha(n)\}) \setminus \kappa_\alpha =$$

$$\{(\rho_{\beta r s \omega_1})^+ \mid \alpha < \beta < \omega_1 \textit{ is a successor ordinal}, r < \omega, s < g_\beta(r)\} \cup \{\kappa_{\omega_1}^+\},$$

moreover, for every limit $\gamma, \alpha < \gamma < \omega_1$,

$$\operatorname{pcf}(\{(\rho_{\alpha n m \gamma})^+ \mid n < \omega, m < g_\alpha(n)\}) \setminus \kappa_\alpha =$$

$$\{(\rho_{\beta r s \gamma})^+ \mid \alpha < \beta < \gamma \textit{ is a successor ordinal}, r < \omega, s < g_\beta(r)\} \cup \{\kappa_\gamma^+\},$$

where $\rho_{\delta t u \xi}$ denotes the indiscernible for $\kappa_{\delta t u \xi}$.

Proof. The proof is by induction on β using the assignment functions (a's) of the conditions and that $\operatorname{pcf}(\operatorname{pcf}(A)) = \operatorname{pcf}(A)$.

□

9 Concluding remarks.

The construction of the previous section gives a countable set of regular cardinals $\mathfrak{a}$ with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) = \omega_1 + 1$. It is natural to try to get a bigger order type. The present methods allow to obtain $\omega_1 \cdot \alpha + 1$, for every $\alpha < \omega_1$. Just repeat the construction α– many times (one above another). However it is unclear how to get to $\omega_1 \cdot \omega_1 + 1$ and beyond. In addition the resulting countable set $\mathfrak{a}$ will have the order type $\omega \cdot \alpha$, and it is unclear whether it is possible to have a set of regular cardinals $\mathfrak{a}$ of order type ω with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) > \omega_1 + 1$.

Question 1. *Is it possible to increase* $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a}))$ *beyond* $\omega_1 \cdot \omega_1$*, for a countable set of regular cardinals* $\mathfrak{a}$*?*

We think that it may be possible under same lines, but using more elaborated techniques, to get any successor order type $< \omega_2$.

Shelah Weak Hypothesis (SWH) states that the set

$$\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$$

is at most countable.
The construction of the previous section provides a counterexample, but very restricted one. The cardinality and even the order type there is ω_1. So the following question is natural:

Question 2. *Is it possible to increase the cardinality of the set*

$$\{\eta \mid \eta < \kappa, \eta \textit{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$$

beyond ω_1*, for a cardinal* κ*?*
Note that no upper bound on cardinality of

$$\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$$

is known.
Going further beyond ω_1, in view of results of [7] and [9] will require some completely new ideas. The same once one likes to have a set $\{\eta \mid \eta < \kappa, \mathrm{cof}(\eta) > \omega, \mathrm{pp}(\eta) > \kappa\}$ infinite, for some κ.

Question 3. *How to move everything down, in particular is it possible to get down to* $\aleph_\omega$*?*

It is possible to add collapses to the present construction, but only very inessential ones. By [8], the supercompact Prikry forcing looks be needed in order to collapse successors of singular cardinals, but this complicates the matters largely. It is unclear how to combine this forcing with short extenders forcings in a productive way.

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