Short extenders forcings – doing without preparations.

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Abstract

We introduce certain morass type structures and apply them to blowing up powers of singular cardinals. As a bonus, a forcing for adding clubs with finite conditions to higher cardinals is obtained.

1 Introduction.

We would like to present a way of doing short extenders forcings without forcing first with a preparation forcings of type $\mathcal{P}'$ of [6]. The main issue with short extenders forcings is to show that κ^{++} and cardinals above it are preserved in the final model. In [6] the preparation forcing (which added a structure with pistes) was used eventually to show κ^{++}-c.c. of the main forcing. A negative side of this preparation forcing is that it is only strategically closed which is not enough in order to preserve large cardinals like a supercompact. Actually it adds a version of the square principle which is incompatible with supercompacts [7].
Carmi Merimovich [13] used for the gap 3 a variation of Velleman's simplified morass [17] instead. κ^{++}-c.c. breaks down but he was able to show κ^{++}–properness instead. The forcing adding a simplified morass is directed closed enough in order to preserve supercompact cardinals. Unfortunately generalizations (at least those that we considered) of Merimovich's idea of first adding a simplified morass and then using a properness instead of a chain condition of the main forcing run into severe difficulties already for Gap 4.

Here we suggest an other way. The main forcing will be used directly over V without a preparation. Actually a simple version of the preparation forcing of [6] will be incorporated directly into the main forcing. Again as in [13] κ^{++}-c.c. will break down and we will show a properness instead.

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In this paper we will deal with a general situation - no bounds on a gap between a singular cardinal and its power. The situation where the gap between a singular cardinal κ and its power is bounded by $\kappa^{+\kappa^+} = \aleph_{\kappa^+}$ is considered in [9]. The arguments there are slightly easier, but not essentially.

The main instrument introduced here is called *structures with pistes*. It seems of an interest by its own. Beyond cardinal arithmetic applications stated above, it is applied in a further paper to certain generalizations of Forcing Axioms to higher cardinals, see [10]. Here we will apply it to adding clubs by finite conditions.

The paper is organized as follows.
In Section 2 we introduce a $\delta-$structure with pistes over η of the length θ. Basic properties of such structures like the intersection property, possible extensions etc. are studied here. A forcing with piste structures is introduced. Its properness is proved. An application to adding clubs is given at the end of this section.
Section 3 is devoted to a cardinal arithmetic application. We show how using structures of this type it is possible to blow up the power of a singular cardinal.

2 Structures with pistes–general setting.

Assume GCH.

The basic idea behind the structures defined below (Definition 2.3, $\delta-$structure with pistes over η of length θ) is to stay as close as possible to an elementary chain of models. It cannot be literally a chain since models of different sizes are involved and models of bigger cardinality can come before ones of a smaller. The first part (Definition 2.1) describes this "linear" part of conditions in the main forcing. It is called *a wide piste* and incorporates together elementary chains of models of different cardinalities. The main forcing, defined in Section 2.3, will be based on such wide pistes and involves an additional natural but non-linear component called splitting or reflection.

Definition 2.1 Let $\delta \leq \eta < \theta$ be regular cardinals.
A $(\theta, \eta, \delta)-$wide piste is a set $\langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle$ such that the following hold.[1]
Let us first specify sizes of models that are involved.

1. (Support) s is a set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:
   (a) $|s| < \delta$,
   (b) $\eta, \theta \in s$.
   Which means that the minimal and the maximal possible sizes are always present.
2. (Models) For every $\tau \in s$ and $A \in C^\tau$ the following holds:
   (a) $A \preccurlyeq \langle H(\theta^+), \in, \leq, \delta, \eta\rangle$, where $\leq$ is some fixed well ordering of $H(\theta^+)$,
   (b) $|A| = \tau$,
   (c) $A \supseteq \tau + 1$,
   (d) $A \cap \tau^+$ is an ordinal,
   (e) elements of C^τ form a closed $\in -$chain, of length $< \delta$, with a largest element,
   (f) if $X \in C^\tau \setminus C^{\tau lim}$ is a non-limit model (i.e. is not a union of elements of C^τ), then ${}^{\tau>}X \subseteq X$,
   (g) if $X, Y \in C^\tau$ then $X \in Y$ iff $X \subsetneq Y$.

[1] The main application will be to the case when $\eta = \kappa^+$ for a cardinal κ which is an $\omega-$limit of sufficiently strong cardinals, but still without extenders which overlap κ. An other application is to forcing axioms, and for it we use $\delta = \eta = \omega$.

3. (Potentially limit points) Let $\tau \in s$.
$C^{\tau lim} \subseteq C^{\tau}$. We refer to its elements as *potentially limit points.*
The intuition behind this is that it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.
Let $X \in C^{\tau lim}$. Require the following:

   (a) X is a successor point of C^{τ}.

   (b) (Increasing union) There is an increasing continuous $\in -$chain $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ [2] of elementary submodels of X such that

      i. $\bigcup_{i<\mathrm{cof}(\sup(X\cap\theta^+))} X_i = X$,

      ii. $|X_i| = \tau$,

      iii. $X_i \supseteq \tau$,

      iv. $X_i \in X$,

      v. ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

   If such sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ is definable from X (for example, the least one in the well ordering of $H(\theta^+)$), then we will call it *an $X-$sequence.*

   (c) (Degree of closure of potentially limit point)
   Either

      i. ${}^{\tau>}X \subseteq X$, and then we call X *a closed model*,
      or

      ii. $\mathrm{cof}(\sup(X \cap \theta^+)) = \xi$ for some $\xi \in s \cap \tau$ and then either

         - there are $X^* \in C^{\tau lim}$ and $E^* \in C^{\rho lim}$, for some $\rho \in s, \rho < \tau$ such that ${}^{\rho}X^* \subseteq X^*$, X is the Skolem Hull of $(X^* \cap E^*) \cup \tau + 1$ in the structure $\langle H(\theta^+), \in, \leq, \delta, \eta\rangle$.
         Denote such Skolem Hull further by cl.
         Or

         - there are $X_\xi \in C^{\xi lim}, X_\theta \in C^{\theta lim}$ such that

            A. $\sup(X_\xi \cap \theta^+) = \sup(X \cap \theta^+) = X_\theta \cap \theta^+$,

            B. $X_\xi \subseteq X \subseteq X_\theta$,

            C. $|X| = \tau \in X_\xi$,

[2] These models need not be in C^{τ}, but rather allow to add in future extensions models below X.

D. X is the least (in the well order $\leq$ of $H(\theta^+)$) elementary submodel of $H(\theta)$ elementary submodel of X_θ elementary submodel of $H(\theta)$ which includes X_ξ and $\tau+1$,

E. there is a sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ witnessing 3(b) for X_θ whose members belong to X_ξ.

F. For any $\mu \in s, \xi < \mu < \theta$, if $Y \in C^\mu$ is the least in C^μ such that $X_\xi \in Y$, then $X \in Y$ and Y is a potentially limit point.

Note that if $\tau > \omega$, $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ and $\langle X'_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ are two sequences which witness (3b) above, then the set $\{i < \mathrm{cof}(\sup(X \cap \theta^+)) \mid X_i = X'_i\}$ is closed and unbounded.

It is possible using the well ordering $\leq$ to define a canonical witnessing sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ for X.

Let us first do this for X such that $\mathrm{cof}(\sup(X \cap \theta^+)) = \tau$ (or for X_ξ of (3c(ii)(C)) above). Let $\langle x_\nu \mid \nu < \tau\rangle$ be an enumeration of X (defined using $\leq$). We proceed by induction. If $i < \tau$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least elementary submodel of X such that

- $x_i \in X_{i+1}$,
- $X_i \in X_{i+1}$,
- $|X_{i+1}| = \tau$,
- $X_{i+1} \supseteq \tau$,
- ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

By (3b), it is possible to find such X_{i+1}.

Clearly $\bigcup_{i<\tau} X_i = X$.

Suppose now that $\mathrm{cof}(\sup(X \cap \theta^+)) = \xi \in s \cap \tau$. Then let us use the canonical sequence $\langle X_{i\xi} \mid i < \xi = \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ for X_ξ in order to define the canonical sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ for X.

Proceed by induction. If $i < \xi$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least (in the well order $\leq$ of $H(\theta^+)$) elementary submodel of $H(\theta^+)$ such that

- $X_{i+1} \in X_\xi$,
- $X_{i\xi} \in X_{i+1}$,
- $X_i \in X_{i+1}$,
- $|X_{i+1}| = \tau$,

– $X_{i+1} \supseteq \tau + 1$,

– ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

By (3c(ii)B), it is possible to find such X_{i+1} inside X_ξ.

Note that the existence of such canonical sequences implies that X itself is definable from X_ξ and τ.

The next condition prevents unneeded appearances of small models between big ones.

4. If $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point and B_0 is its immediate predecessor, then there is no potentially limit point $A \in C^\tau$ with $\tau < \rho$ such that $B_0 \in A \in B_1$.

   The requirement that B_1 is not a potentially limit point is important here. Once dealing with potentially limit points, we would like to allow reflections which may add small intermediate models.
   However, small models which are non-potentially limit points are allowed.

5. Let $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point, B_0 is its immediate predecessor and $A \in C^\tau \cap B_1$, with $\tau < \rho$. If $\sup(A \cap \theta^+) > \sup(B_0 \cap \theta^+)$, then $B_0 \in A$.

   The next condition is of a similar flavor, but deals with smallest models.

6. If $B \in C^\rho$, for some $\rho \in s$, is not a potentially limit point and it is the least element of C^ρ, then there is no potentially limit point $A \in C^\tau$ with $\tau > \rho$ such that $A \in B$[3].

   Both conditions 4 and 6 are designed to allow one to add new models below potentially limit points, which will be essential for properness of the forcing.

   The next condition deals with closure and is desired to prevent some pathological patterns.

7. Let $B \in C^\rho$, for some $\rho \in s$, be a non-limit point of C^ρ. If there are models $A \in \bigcup_{\xi \in s} C^\xi$ with $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then there is $A \in B \cap \bigcup_{\xi \in s} C^\xi$ such that

   (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$,

   (b) for every $A' \in \bigcup_{\xi \in s} C^\xi$ with $\sup(A' \cap \theta^+) < \sup(B \cap \theta^+)$, $\sup(A' \cap \theta^+) \leq \sup(A \cap \theta^+)$.

[3] If we drop the requirement $\tau > \rho$, then it may be impossible further to add models of sizes $> \eta$ once a potentially limit point of size η is around.

Such A is the "real" immediate predecessor of B. Further, in the definition of the order, we will require that if B is not a potentially limit point, then no models E such that $A \in E \in B$ can be added.

The purpose of the next four conditions is to allow to proceed down the pistes without interruptions at least before reaching a potentially limit point.

8. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^ρ. Then $B' \in A$.

9. Let $\tau, \rho, \rho^* \in s, \tau < \rho < \rho^*$, $A \in C^\tau, B \in C^{\rho^*}, D \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^{ρ^*}.
Then $B' \in D \in B$ implies $D \in A$.

10. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is a limit point in C^ρ. Let $\langle B_\nu \mid \nu < \nu^* < \delta\rangle$ be $C^\rho \cap B$. Then a closed unbounded subsequence of $\langle B_\nu \mid \nu < \nu^*\rangle$ is in A.

11. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is a limit point in C^ρ. Let $D \in \bigcup_{\mu \in s} C^\mu$ be such that $\sup(D \cap \theta^+) = \sup(B \cap \theta^+)$. Then $|D| \in A$ implies that $D \in A$.

12. (Linearity) If $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$, then

    (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ implies $A \in B$,

    (b) $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$ implies $A \subseteq B$.

The next condition will be used to ensure that the maximal models are linearly ordered by a combination of $\in, \subset -$relations.

13. (Linearity at the top) Let $\tau, \rho \in s, \tau < \rho$, B be the maximal model of C^ρ and A be the maximal model of C^τ. Then

    (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ implies $A \in B$,

    (b) $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$ implies $A \subset B$,

    (c) $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ implies $B \in A$.

This condition is unneeded if $\theta < \aleph_{\eta^+}$, as will be shown in Lemma 2.2.

14. Let $\tau, \rho \in s, \tau \leq \rho$, $A \in C^\tau, X \in C^{\rho lim}$,$X \in A$ and ${}^{\rho>}X \not\subseteq X$. Then there are $X' \in A \cap C^{\rho lim}$ and $E' \in A \cap C^{\mu lim}$, for some $\mu \in s, \mu \geq \rho$, such that $X = cl((X' \cap E') \cup \rho+1)$. Moreover, it is possible to find such X', E' so that $\mathrm{cof}(\sup(X' \cap E' \cap \theta^+)) = |X'|$.

15. Let $B \in C^\tau$, $D^* \in C^\rho, \rho > \tau$, $\mathrm{cof}(\sup(D^* \cap \theta^+)) = \xi$ for some $\xi \in s \cap \rho$, $\xi > |B| = \tau \geq \eta$. Suppose that $E \in C^{\xi lim}$ is such that ${}^{\xi>}E \subseteq E, \sup(E \cap \theta^+) = \sup(D^* \cap \theta^+)$. Then $E \in B$.

16. Let $\tau, \rho, \xi \in s, \tau < \rho < \xi$, $A \in C^\tau, B \in C^\rho, M \in C^\xi$, $B, M \in A$ and $\sup(B \cap \theta^+) > \sup(M \cap \theta^+)$. Suppose that there is a model $M^* \in B$ such that $M^* \cap B = M \cap B$. Then such model is in A.[4]

    Further conditions (called covering conditions) will insure existence of such M^*. So the above condition guarantees the closure of A under this type of covering.

17. (Immediate successor restriction) Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^{\rho lim}$ and $B \in A$. Suppose that there a model $B' \in B \cap C^\rho$ such that $\sup(B' \cap \theta^+) > \sup((A \cap B) \cap \theta^+)$.[5] Then the least such B' is a potentially limit model. I.e., if there is a model in C^ρ between $A \cap B$ and B, then the least such model is a potentially limit model.

    The condition 17 is designed to prevent the situation when there is $E \in A \cap C^\rho$ which has a non-potentially limit immediate successor E'' in B but not in A. Also it prevents a possibility that the least element Y of C^ρ is a non-potentially limit point which belongs to B and is above $A \cap B$.

    The condition above is needed further for $\tau-$properness argument[6].

18. Let $M \in C^\tau, E \in C^{\xi lim}, E \notin M$, for some $\tau < \xi$ in s. Suppose that $\sup(E \cap \theta^+) < \sup(M \cap \theta^+)$. Let E^* be the cover of E in M, i.e. $E^* \in M \cap C^{\xi lim}, E^* \supseteq E$ is such

[4]Note such M^* is unique, if exists. Thus suppose that $M', M'' \in B$, $M' \neq M''$ and $M' \cap B = M \cap B = M'' \cap B$. There is $x \in M' \setminus M''$ or $x \in M'' \setminus M'$. Suppose for example that there is $x \in M' \setminus M''$. By elementarity, then there is $x \in M' \setminus M'', x \in B$, but this is clearly impossible.
Also note that $M^* = M$, if $M \in B$.

[5]Note that then $\mathrm{cof}(\sup(B \cap \theta^+)) > \tau$. Otherwise, if $\mathrm{cof}(\sup(B \cap \theta^+)) \leq \tau$, then the least cofinal in $\sup(B \cap \theta^+)$ sequence $\langle \zeta_i \mid i < \mathrm{cof}(\sup(B \cap \theta^+)) \rangle$ which consists of elements of B is in A. We have $A \in C^\tau$, hence $\tau \subseteq A$, but then $\{\zeta_i \mid i < \mathrm{cof}(\sup(B \cap \theta^+))\} \subseteq A$. So, $\sup((A \cap B) \cap \theta^+) = \sup(B \cap \theta^+)$. This leaves no room for $B' \in B \cap C^\rho$ such that $\sup(B' \cap \theta^+) > \sup((A \cap B) \cap \theta^+)$.

[6]In an earlier version of the paper, we defined a model $B_A := \bigcup_{i \in A \cap \mathrm{cof}(\sup(B \cap \theta^+))} B_i$ (where $\langle B_i \mid i < \mathrm{cof}(\sup(B \cap \theta^+)) \rangle$ is a chain which witnesses (3(b)) and added it to $C^{\rho lim}$. Having such B_A in $C^{\rho lim}$ implies impossibility of the situations above. Here we do without B_A and this simplifies the major arguments like intersection properties and properness. However, getting a club that runs away from sets in V becomes a bit more complicated.

that $M \cap E^* \subseteq E$.
Let $\langle E_i^* \mid i < \mathrm{cof}(E^* \cap \theta^+)\rangle$ be an E^*-sequence. Set $i_M = \sup(M \cap \mathrm{cof}(E^* \cap \theta^+))$.
Then either $E^*_{i_M} = E$ or $E^*_{i_M} \in E$.

The next conditions deal with the possibility of covering a model $D \notin B$ such that $\sup(B \cap \theta^+) > \sup(D \cap \theta^+)$ by a model that belongs to B.
We deal separately with cases $\theta < \aleph_{\eta^+}$ and $\theta > \aleph_{\eta^+}$. It is not too hard to combine both cases together, but our imprecision is due to the fact that first considering a simpler case $\theta < \aleph_{\eta^+}$ will better explain the intuition behind the general case. This seems to be true not only in this definition, but rather through the paper.

19. (Covering under $\theta < \aleph_{\eta^+}$) If $\tau, \rho \in s, \tau < \rho$, $B \in C^\tau, D \in C^\rho$ and $\sup(B \cap \theta^+) > \sup(D \cap \theta^+)$, then there is $D^* \in B \cap C^\rho$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$.[7]
Note that the set $Z := \{\mu \leq \theta \mid \mu$ is a regular cardinal $\}$ belongs to B, by elementarity. Its cardinality is at most η, so $Z \subseteq B$.

20. (Strong covering under $\theta < \aleph_{\eta^+}$) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$[8] and $\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Then either

    (a) $D \in B$,
    or

    (b) $D \notin B$ and the least $D^* \in C^\rho \cap B, D^* \supset D$ is closed under $< \rho-$ sequences of its elements. Then $B \cap D^* \subseteq D$.
    Or

    (c) $D \notin B$ and the least $D^* \in C^\rho \cap B, D^* \supset D$ is not closed under $< \rho-$sequences of its elements. Then $B \cap D^* \subseteq D$.
    Let $\mathrm{cof}(\sup(D^* \cap \theta^+)) = \xi$ for some $\xi \in s \cap \rho$. ***Then*** $\xi > |B| = \tau \geq \eta$.
    Suppose that there is $E \in C^{\xi lim}$ such that ${}^{\xi>}E \subseteq E, \sup(E \cap \theta^+) = \sup(D^* \cap \theta^+)$. This corresponds to the second bullet of Item 3c(ii). Note that $E \in B$ by

[7] Note that
(a) if $D \notin B$, then the least such D^* must be a potentially limit point by Items 8 and 10 above. Thus, it cannot be a successor non-potentially limit point, by Item 8, since its immediate predecessor $D^{*'}$ will be in B, and then, by minimality of D^*, $\sup(D^{*'} \cap \theta^+) < \sup(D \cap \theta^+)$, and so $D \supseteq D^{*'}$.
By Item 10, D^* cannot be a limit point of C^ρ.
(b) such D^* is the least model $D' \in B \cap C^\rho$ such that $D' \supseteq D$.
(c) if $D \notin B$, then $\mathrm{cof}(\sup(D^* \cap \theta^+)) > |B|$, since otherwise a cofinal in $\sup(D^* \cap \theta^+)$ consisting of elements of D^* will be in B. Which is impossible, since $D \in D^*$, and so, $\sup(D \cap \theta^+) \in D^*$, but $B \cap D^* \subseteq D$.
[8] Note that the assumption $\theta < \aleph_{\eta^+}$ implies that $\rho \in B$.

Item 15, since $D^* \in B$.

Then either

i. $D \in E$, $B \cap D^* \subseteq D$.

 Or

ii. $D \notin E$, $\sup(E \cap \theta^+) = \sup(D^* \cap \theta^+) > \sup(D \cap \theta^+)$,
 and then, let $D^{**} \in C^\rho \cap E$ be the least such that $D^{**} \supset D$.
 Let $\langle E_i \mid i < \xi \rangle$ be an $E-$sequence. Set $i^E_B = \sup(B \cap \xi)$. ***Then*** the least $i < \xi$, such that $\sup(D^{**} \cap \theta^+) \in E_i$, is above i^E_B.[9]
 If D^{**} is closed under $< \rho-$sequences of its elements, then $B \cap D^* \subseteq D, E \cap D^{**} \subseteq D$, and we are done.
 If D^{**} is not closed under $< \rho-$sequences of its elements, then the process repeats itself going down below D^{**}. After finitely many steps we will either reach D or D will be above everything related to B.
 We will state it formally below, but let us first deal with another possibility which corresponds to the first bullet of of Item 3c(ii).
 So, ***suppose that there are*** $\tilde{E} \in B \cap C^{\zeta lim}, \tilde{F} \in B \cap C^{\rho lim}$,[10] for some $\zeta, \tau < \zeta < \rho$, such that $D^* = cl((\tilde{E} \cap \tilde{F}) \cup \rho)$. In addition, $\operatorname{cof}(\sup(\tilde{F} \cap \theta^+)) > |\tilde{E}|$.
 Then either

iii. $D \in \tilde{E}$, $B \cap D^* \subseteq D$.

 Or

iv. $D \notin \tilde{E}$, $\sup(\tilde{E} \cap \theta^+) > \sup(D^* \cap \theta^+) > \sup(D \cap \theta^+)$, and then, let $D^{**} \in C^\rho \cap \tilde{E}$ be the least such that $D^{**} \supset D$.
 Unlike the previous case (ii), the model $\tilde{E}$ may be not closed, i.e. ${}^{|\tilde{E}|>}\tilde{E} \not\subseteq \tilde{E}$.
 If this is the case then we require existence of a closed model $E_0 \in B \cap C^{\operatorname{cof}(\sup(\tilde{E} \cap \theta^+)) lim}$ which is responsible for non-closure of $\tilde{E}$.
 Namely, there are finite sequences $\langle \tilde{E}_k \mid k \leq \ell \rangle, \langle E_k \mid k \leq \ell \rangle, \langle \tilde{F}_k \mid k < \ell \rangle \in B$ such that

 - $\tilde{E}_0 = \tilde{E}, \tilde{F}_0 = \tilde{F}$,
 - $E_k \in C^{|E_k| lim}$ is closed, for every $k < \ell$,
 - $E_{k+1}, \tilde{E}_{k+1} \in E_k$, for every $k < \ell$,

[9] Clearly, $D^{**} \cap \theta^+ \in E_i$ implies that $\sup(D^{**} \cap \theta^+) \in E_i$, as well.

[10] We may assume that $\tilde{F} \in \tilde{E}$, since otherwise the covering Item 19 can be applied to $\tilde{E}$, $\tilde{F}$, and $\tilde{F}$ can be replaced by its cover.

- $|E_k| = \mathrm{cof}(\sup(\tilde{E}_k \cap \theta^+))$, for every $k < \ell$,
- $\tilde{E}_k = cl((\tilde{E}_{k+1} \cap \tilde{F}_{k+1}) \cup |\tilde{F}_{k+1}|)$, for every $k < \ell$,
- $\tilde{E}_k \in C^{|\tilde{E}_k| lim}$, for every $k \leq \ell$,
- $\tilde{F}_k \in C^{|\tilde{F}_k| lim} \cap E_k$ and $|\tilde{E}_k| \leq |\tilde{F}_k|$, moreover if $|\tilde{E}_k| = |\tilde{F}_k|$, then $\tilde{E}_k = \tilde{F}_k$, i.e we can omit it from the intersection, for every $k < \ell$,
- $\tilde{E}_\ell$ is closed under $< |\tilde{E}_\ell| -$sequences of its elements.

Further let us denote such final model $\tilde{E}_\ell$ by $c(\tilde{E})$ and call it *the core* of $\tilde{E}$.
Let $E = c(\tilde{E})$.
Let $\langle E_i \mid i < |E| \rangle$ be an $E-$sequence. Set $i^E_B = \sup(B \cap |E|)$. ***Then*** $i^E_B < |E|$ and the least $i < |E|$, such that $\sup(D^{**} \cap \theta^+) \in E_i$, is above i^E_B.
If D^{**} is closed under $< \rho-$sequences of its elements, then $B \cap D^* \subseteq D, E \cap D^{**} \subseteq D$, and we are done.
Suppose now that D^{**} of (ii) or (iv) is not closed under $< \rho-$sequences of its elements.
Then there are $n^* < \omega, \langle E^n \mid n < n^* \rangle, \langle D^n \mid n \leq n^* \rangle$ such that for every $n < n^*$ the following hold:

A. $D^0 = D^*$,

B. $E^0 = E$,

C. $D^n \in C^\rho$,

D. $D^n \supseteq D$,

E. $D^{n+1} \in D^n$,

F. ${}^{|E^n|>}E^n \subseteq E^n$ and *either* $D^n = cl(E^n \cup \rho)$ *or* E^n is a core of a model $\tilde{E}^n$ such that
$D^n = cl((\tilde{E}^n \cap \tilde{F}^n) \cup \rho))$, for some $\tilde{F}^n \in \tilde{E}^n \cap C^\rho$, such that $|\tilde{F}^n| > |\tilde{E}^n|$,

G. $E^n \in C^{|E^n|}$,

H. $E^{n+1} \in E^n$, $|E^{n+1}| > |E^n|$ and if $\tilde{E}^{n+1}, \tilde{F}^{n+1}$ are defined, then they are in E_n, as well,

I. $D^{n+1} \in E^n$ is the least in $C^\rho \cap E^n$ such that $D^{n+1} \supset D$, i.e. $D^{**} = D^1$,

J. $B \cap D^0 \subseteq D$,

K. $E^n \cap D^{n+1} \subseteq D$,

L. Let $\langle E^n_i \mid i < |E^n| \rangle$ be an E^n-sequence. Set $i^n_B = \sup(B \cap |E^n|)$.
Then $i^n_B < |E^n|$ and the least $i < |E^n|$, such that $\sup(D^{n+1} \cap \theta^+) \in E^n_i$, is above i^n_B.

M. $D^{n^*} = D$ or, we have, $D \in D^{n^*}, {}^{\rho>} D^{n^*} \subseteq D^{n^*}$.

Let us turn to covering properties under the assumption $\theta \geq \aleph_{\eta^+}$. The new possibility here is that a model on the wide piste may be too small to include even cardinalities of other models there. In order to cover models in such situation, we may need models of singular cardinalities that we prefer not to include into C^ρ's for $\rho \in s$.

21. (Covering under $\theta \geq \aleph_{\eta^+}$.) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$ and $\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Then either

    (a) there is $D^* \in C^\rho \cap B$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$ (and then, clearly, $B \cap D^* = B \cap D$),

    or

    (b) there is no $D^* \in (\bigcup_{\mu \in s \setminus \rho+1} C^\mu) \cap B$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$. Then the following hold:

        i. $\rho \notin B, \min((B \cap \theta^+) \setminus \rho)$ is a singular cardinal and there is $D^* \in B, |D^*| = \min((B \cap \theta^+) \setminus \rho)$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$.

        or

        ii. $\rho \notin B, \min((B \cap \theta^+) \setminus \rho)$ is an inaccessible cardinal and there is $D^* \in B, |D^*| = \min((B \cap \theta^+) \setminus \rho)$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$. Moreover, $|D^*| \notin D^*$.

        Or

        iii. there are $F \in C^\zeta$ and $H \in C^\theta$ such that $\tau < \zeta < \rho < \theta$, $\sup(D \cap \theta^+) = \sup(F \cap \theta^+) = H \cap \theta^+$ and there is no $F^* \in (\bigcup_{\mu \in s \setminus \zeta} C^\mu) \cap B$ such that $F^* \supseteq F$ and $B \cap F^* \subseteq F$. Then there is $D^* \in B, |D^*| = \min((B \cap \theta^+) \setminus \rho)$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$.

    We will elaborate this last possibility (b) below.[11]

22. (Strong covering under $\theta \geq \aleph_{\eta^+}$) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$ and $\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Then either

    (a) $D \in B$,

    or

[11]Such possibility may occur already if $\eta = \delta = \omega$ and $\theta = \aleph_{\omega_1+1}$. Let B be a countable model. Then $B \cap \omega_1 < \omega_1$. Let D be a model of cardinality $\aleph_{\alpha+1}$ for some $\alpha, B \cap \omega_1 < \alpha < \omega_1$. So, a model $D^* \in B$ of a singular cardinality $\aleph_{\omega_1}$ may be needed in order to cover such D.

(b) $D \notin B$ and the least $D^* \in B, D^* \supset D$ is in C^ρ and is closed under $< \rho-$ sequences of its elements. Then $B \cap D^* \subseteq D$.
Or

(c) $D \notin B$ and the least $D^* \in B, D^* \supset D$ is not in C^ρ or it is, but is not closed under $< \rho-$sequences of its elements. Then $B \cap D^* \subseteq D$.
If $D^* \in C^\rho$, then we repeat the requirements of (20).
Suppose that $D^* \notin C^\rho$. Then $|D^*| = \rho^* := \min(B \cap \theta^+ \setminus \rho)$. We require that there are $\tilde{E}, \tilde{F} \in B$, such that $D^* = cl((\tilde{E} \cap \tilde{F}) \cup \rho^*)$. In addition, if $\tilde{F} \neq \tilde{E}$, then $\mathrm{cof}(\sup(\tilde{F} \cap \theta^+)) > |\tilde{E}|$.
Deal first with the simplest case here.
Suppose $D^* = cl((\tilde{E} \cap \tilde{F}) \cup \rho^*)$, $|\tilde{E}| = \xi$, for some $\xi < \rho$, and E is closed under $< \xi-$sequences of its elements.[12]
Then require then that $\xi \in s$ and $\tilde{E} \in C^\xi$. In addition,

either

i. $D \in \tilde{E}$, $B \cap D^* \subseteq D$.
Or

ii. $D \notin \tilde{E}$, $\sup(\tilde{E} \cap \theta^+) = \sup(D^* \cap \theta^+) > \sup(D \cap \theta^+)$,
and then, let $D^{**} \in \tilde{E}$ be the least such that $D^{**} \supset D$.
Let $\langle \tilde{E}_i \mid i < \xi \rangle$ be an $\tilde{E}-$sequence. Set $i^{\tilde{E}}_B = \sup(B \cap \xi)$. ***Then*** the least $i < \xi$, such that $\sup(D^{**} \cap \theta^+) \in \tilde{E}_i$, is above $i^{\tilde{E}}_B$.
If D^{**} in C^ρ and is closed under $< \rho-$sequences of its elements, then $B \cap D^* \subseteq D, \tilde{E} \cap D^{**} \subseteq D$, and we are done.
If D^{**} is not in C^ρ or it is in C^ρ, but is not closed under $< \rho-$sequences of its elements, then the process repeats itself going down below D^{**}. After finitely many steps we will either reach D or D will be above everything related to B.
We will state it formally below, but let us first deal with the general case.
We require that either

iii. $D \in \tilde{E}$, $B \cap D^* \subseteq D$.
Or

iv. $D \notin \tilde{E}$, $\sup(\tilde{E} \cap \theta^+) > \sup(D^* \cap \theta^+) > \sup(D \cap \theta^+)$, and then, let $D^{**} \in C^\rho \cap \tilde{E}$ be the least such that $D^{**} \supset D$.

[12]It may be that already $D^* = cl(\tilde{E} \cap \rho^*)$, for example, if $\tilde{E} = \tilde{F}$.

Unlike the previous case, the model $\tilde{E}$ may be not closed, i.e. ${}^{|\tilde{E}|>}\tilde{E} \not\subseteq \tilde{E}$.
If this is the case then we require existence of a closed model $E_0 \in B \cap C^{\mathrm{cof}(\sup(\tilde{E}\cap\theta^+))lim}$ which is responsible for non-closure of $\tilde{E}$.
Namely, there are finite sequences $\langle \tilde{E}_k \mid k \leq \ell \rangle, \langle \tilde{F}_k \mid k \leq \ell \rangle \in B$ such that

- $\tilde{E}_0 = \tilde{E}, \tilde{F}_0 = \tilde{F}$,
- $\tilde{E}_k = cl((\tilde{E}_{k+1} \cap \tilde{F}_{k+1}) \cup |\tilde{F}_{k+1}|)$, for every $k < \ell$,
- for every $k < \ell$, either $|\tilde{E}_k| < |\tilde{F}_k|$, and then $\tilde{F}_k \in \tilde{E}_k$;
  or $|\tilde{E}_k| = |\tilde{F}_k|$, and then $\tilde{E}_k = \tilde{F}_k$, so, we can omit $\tilde{F}_k$ from the intersection in this case.
- $\tilde{E}_k$ is not closed under $< |\tilde{E}_k|-$sequences of its elements, for every $k < \ell$,
- $\tilde{E}_\ell$ is closed under $< |\tilde{E}_\ell|-$sequences of its elements and $\tilde{E}_\ell \in C^{|\tilde{E}_\ell|lim}$.

Further let us denote such final model $\tilde{E}_\ell$ by $c(\tilde{E})$ and call it *the core* of $\tilde{E}$.
Let $E = c(\tilde{E})$.
Let $\langle E_i \mid i < |E| \rangle$ be an $E-$sequence. Set $i^E_B = \sup(B \cap |E|)$.
We require that $i^E_B < |E|$ and the least $i < |E|$, such that $\sup(D^{**} \cap \theta^+) \in E_i$, is above i^E_B.
If D^{**} is closed under $< \rho-$sequences of its elements, then $B \cap D^* \subseteq D, E \cap D^{**} \subseteq D$, and we are done.
Suppose now that D^{**} of (ii) or (iv) is not closed under $< \rho-$sequences of its elements.
Then there are $n^* < \omega, \langle E^n \mid n < n^* \rangle, \langle D^n \mid n \leq n^* \rangle$ such that for every $n < n^*$ the following hold:

A. $D^0 = D^*$,

B. $E^0 = E$,

C. $D^n \in C^\rho$,

D. $D^n \supseteq D$,

E. $D^{n+1} \in D^n$,

F. ${}^{|E^n|>}E^n \subseteq E^n$ and *either* $D^n = cl(E^n \cup \rho)$ *or* E^n is a core of a model $\tilde{E}^n$ such that
$D^n = cl((\tilde{E}^n \cap \tilde{F}^n) \cup \rho))$, for some $\tilde{F}^n \in \tilde{E}^n \cap C^\rho$, such that $|\tilde{F}^n| > |\tilde{E}^n|$,

G. $E^n \in C^{|E^n|}$,

H. $E^{n+1} \in E^n$, $|E^{n+1}| > |E^n|$ and if $\tilde{E}^{n+1}, \tilde{F}^{n+1}$ are defined, then they are in E_n, as well,

I. $D^{n+1} \in E^n$ is the least in $C^\rho \cap E^n$ such that $D^{n+1} \supset D$, i.e. $D^{**} = D^1$,

J. $B \cap D^0 \subseteq D$,

K. $E^n \cap D^{n+1} \subseteq D$,

L. Let $\langle E^n_i \mid i < |E^n| \rangle$ be an E^n-sequence. Set $i^n_B = \sup(B \cap |E^n|)$.
Then $i^n_B < |E^n|$ and the least $i < |E^n|$, such that $\sup(D^{n+1} \cap \theta^+) \in E^n_i$, is above i^n_B.

M. $D^{n^*} = D$ or, we have, $D \in D^{n^*},^{\rho>} D^{n^*} \subseteq D^{n^*}$.

23. Let $A \in C^\tau$, for some $\tau \in s$. Let $E \in \bigcup_{\xi \in s} C^\xi$, or $E \notin \bigcup_{\xi \in s} C^\xi$, but E covers some model, as in 21(b). Suppose that $\sup(E \cap \theta^+) \in A$ and $|E| \in A$. Then $E \in A$.

□ of the definition.

Let us make now the following observation:

Lemma 2.2 *Let $\langle \langle C^\tau, C^{\tau lim} \rangle \mid \tau \in s \rangle$ be a $(\theta, \eta, \delta)-$wide piste. Let $\tau, \rho \in s, \tau < \rho$ and A and B be the maximal models in C^τ and C^ρ respectively. Suppose that if* $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$*, then $\rho \in A$. Then, without assuming 2.1(13), either $B \in A$ or $A \subset B$. Moreover,* $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ *implies $A \in B$ and* $\sup(B \cap \theta^+) < \sup(A \cap \theta^+)$ *implies $B \in A$.*
So, the maximal models are linearly ordered by a combination of $\in$ and $\subset -$relations.

Proof. If $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then by Definition 2.1(12) $A \in B$.
If $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$, then by Definition 2.1(12) $A \subseteq B$.
Suppose that $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$.
Then $\rho \in A$, by the assumption. Apply Definition 2.1(19,21). Then there is $D^* \in A \cap C^\rho$ such that $D^* \supseteq B$. The maximality of B implies then that $B = D^*$. So, $B \in A$.
□

Now we are ready to give the main definition.

Definition 2.3 Let $\delta \leq \eta < \theta$ be regular cardinals.
A $\delta-$structure with pistes over η of length θ is a set $\langle \langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau \rangle \mid \tau \in s \rangle$ such that the following hold.[13]
Let us first specify sizes of models that are involved.

1. (Support) s is a set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:

[13]If $\delta = \omega$, then we call $\delta-$structure with pistes over η of length θ just a finite structure with pistes over η of the length θ.

(a) $|s| < \delta$,

(b) $\eta, \theta \in s$.

Which means that the minimal and the maximal possible sizes are always present.

2. (Models) For every $\tau \in s$ the following holds:

   (a) $A^{0\tau} \preccurlyeq \langle H(\theta^+), \in, \leq, \delta, \eta \rangle$,

   (b) $|A^{0\tau}| = \tau$,

   (c) $A^{0\tau} \in A^{1\tau}$,

   (d) $A^{1\tau}$ is a set of less than δ elementary submodels of $A^{0\tau}$,

   (e) each element A of $A^{1\tau}$ has cardinality τ, $A \supseteq \tau + 1$ and $A \cap \tau^+$ is an ordinal.

3. (Potentially limit points) Let $\tau \in s$.
   $A^{1\tau lim} \subseteq A^{1\tau}$. We refer to its elements as *potentially limit points*.
   The intuition behind this is that once extending it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.

4. (Piste function) The idea behind this is to provide a canonical way to move from a model in the structure to one below.
   Let $\tau \in s$.
   Then, $\mathrm{dom}(C^\tau) = A^{1\tau}$ and
   for every $B \in \mathrm{dom}(C^\tau)$, $C^\tau(B)$ is a closed chain of models in $A^{1\tau} \cap (B \cup \{B\})$ such that the following holds:

   (a) $B \in C^\tau(B)$,

   (b) if $X \in C^\tau(B)$, then $C^\tau(X) = \{Y \in C^\tau(B) \mid Y \in X \cup \{X\}\}$,

   (c) if B has immediate predecessors in $A^{1\tau}$, then one (and only one) of them is in $C^\tau(B)$,

5. (Wide piste) The set

   $$\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s \rangle$$

   is a $(\theta, \eta, \delta)-$wide piste.

   The next two condition describe the ways of splittings from wide pistes. This describes the structure of $A^{1\tau}$ and the way pistes allow one to move from one of its models to an other.

6. (Splitting points) Let $\tau \in s$. Let $X \in A^{1\tau}$ be a non-limit model (but possibly a potentially limit). Then either

   (a) X is minimal under $\in$ or equivalently under $\subsetneq$,
   or

   (b) X has a unique immediate predecessor in $A^{1\tau}$,
   or

   (c) $\tau < \theta$, X has exactly two immediate predecessors X_0, X_1 in $A^{1\tau}$, and then the following hold:

      i. (Splitting points of type 1) None of X, X_0, X_1 is a limit or potentially limit point and X, X_0, X_1 form a Δ–*system triple* relative to some $F_0, F_1 \in A^{1\tau^* lim}$, where $\tau^* \in s \setminus \tau + 1$[14], which means the following:

         A. $F_0 \subsetneq F_1$ and then $F_0 \in C^{\tau^*}(F_1)$, or $F_1 \subsetneq F_0$ and then $F_1 \in C^{\tau^*}(F_0)$,

         B. ${}^{\tau^*>}F_0 \subseteq F_0$ and ${}^{\tau^*>}F_1 \subseteq F_1$,

         C. $X_0 \in F_1$, if $F_0 \subsetneq F_1$ and $X_1 \in F_0$, if $F_1 \subsetneq F_0$,

         D. $F_0 \in X_0$ and $F_1 \in X_1$,

         E. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

         F. ${}^{\tau>}X_0 \subseteq X_0$ and ${}^{\tau>}X_1 \subseteq X_1$,

         G. the structures

$$\langle X_0, \in, \langle X_0 \cap A^{1\rho}, X_0 \cap A^{1\rho lim}, (C^\rho \restriction X_0 \cap A^{1\rho}) \cap X_0 \mid \rho \in s \cap X_0 \rangle\rangle$$

         and

$$\langle X_1, \in, \langle X_1 \cap A^{1\rho}, X_1 \cap A^{1\rho lim}, (C^\rho \restriction X_1 \cap A^{1\rho}) \cap X_1 \mid \rho \in s \cap X_1 \rangle\rangle$$

         are isomorphic over $X_0 \cap X_1$. Denote by π_{X_0,X_1} the corresponding isomorphism.

         H. $X \in A^{0\tau*}$.

      Or

      ii. (Splitting points of type 2)[15] There is a singular cardinal $\lambda > \tau$ of cofinality $< \delta$[16] with $\lambda^+ \in s$, there are $\in$ –increasing sequences $\langle G_{0\xi} \mid \xi \in s \cap \lambda \rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda \rangle \in X$, $F_0, F_1 \in X \cap A^{1\lambda^+ lim}$ such that

[14] If there are only finitely many cardinals between η and θ, then we can take τ^* to be just τ^+.

[15] In previous versions of the paper models of singular cardinalities were allowed. This condition corresponds to splitting for such models.

[16] So this type of splitting does not occur if $\delta = \omega$.

A. $F_0 \subsetneq F_1$ and then $F_0 \in C^{\lambda^+}(F_1)$, or $F_1 \subsetneq F_0$ and then $F_1 \in C^{\lambda^+}(F_0)$,

B. ${}^{\lambda}F_0 \subseteq F_0$ and ${}^{\lambda}F_1 \subseteq F_1$,

C. for every $\xi \in s \cap \lambda$, ${}^{\xi>}G_{0\xi} \subseteq G_{0\xi}$ and ${}^{\xi>}G_{1\xi} \subseteq G_{1\xi}$

D. $G_{0\lambda} = \bigcup_{\xi \in s \cap \lambda} G_{0\xi}$ and $G_{1\lambda} = \bigcup_{\xi \in s \cap \lambda} G_{1\xi}$ are in X,

E. $G_{0\lambda} \in F_1$, if $F_0 \subsetneq F_1$ and $G_{1\lambda} \in F_0$, if $F_1 \subsetneq F_0$,

F. $F_0 \in G_{0\lambda}$ and $F_1 \in G_{1\lambda}$,

G. $G_{0\lambda} \cap G_{1\lambda} = G_{0\lambda} \cap F_0 = G_{1\lambda} \cap F_1$,

H. the structures

$$\langle G_{0\lambda}, \in, \langle G_{0\lambda} \cap A^{1\rho}, G_{0\lambda} \cap A^{1\rho lim}, (C^{\rho} \restriction G_{0\lambda} \cap A^{1\rho}) \cap G_{0\lambda} \mid \rho \in s \cap G_{0\lambda} \rangle \rangle$$

and

$$\langle G_{1\lambda}, \in, \langle G_{1\lambda} \cap A^{1\rho}, G_{1\lambda} \cap A^{1\rho lim}, (C^{\rho} \restriction G_{1\lambda} \cap A^{1\rho}) \cap G_{1\lambda} \mid \rho \in s \cap G_{1\lambda} \rangle \rangle$$

are isomorphic over $G_{0\lambda} \cap G_{1\lambda}$. Denote by $\pi_{G_{0\lambda}G_{1\lambda}}$ the corresponding isomorphism.

I. For every $\xi \in s \cap \lambda$, $\pi_{G_{0\lambda}G_{1\lambda}}(G_{0\xi}) = G_{1\xi}$.

J. $X_0 = G_{0\tau}$ and $X_1 = G_{1\tau}$.

K. (Pistes go in the same direction) If $\xi \in s \cap \lambda, \xi \neq \tau$, $G_{0\xi}, G_{1\xi} \in A^{1\xi}$ and there is $G_{\xi} \in A^{1\xi}$ which is the immediate successor of $G_{0\xi}, G_{1\xi}$ in $A^{1\xi}$, then $G_{i\xi} \in C^{\mu}(G_{\xi}) \Leftrightarrow X_i \in C^{\tau}(X), i < 2$.

L. X is not a limit or potentially limit point,

M. $X \in A^{0\lambda^+}$.

Or

iii. (Splitting points of type 3) There are $G, G_0, G_1 \in X \cap A^{1\mu}$, G is a splitting point of types 1 or 2 and G_0, G_1 are its immediate predecessors, for some $\mu \in s \setminus (\min(s \setminus \tau + 1) + 1)$, with witnessing models in X such that

A. $X_0 \in G_0$,

B. $X_1 \in G_1$,

C. $X_1 = \pi_{G_0G_1}[X_0]$.

D. X is not a limit or potentially limit point,

E. $X \in A^{0\mu}$,

F. (Pistes go in the same direction) $G_i \in C^{\mu}(G) \Leftrightarrow X_i \in C^{\tau}(X), i < 2$.

Further we will refer to such X, i.e. of types 1,2 or 3, as *splitting points.*

7. Let $\tau, \rho \in s$, $X \in A^{1\tau}, Y \in A^{1\rho}$. Suppose that X is a successor point, but not potentially limit point and $X \in Y$. Then all immediate predecessors of X are in Y, as well as the witnesses, i.e. F_0, F_1 if (6(c)i) holds, $\langle G_{0\xi} \mid \xi \in s \cap \lambda\rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda\rangle, G_{0\lambda}, G_{1\lambda}$, F_0, F_1 if (6(c)ii) holds and G_0, G_1, G if (6(c)iii) holds.

8. Let $\tau \in s$. If $X \in A^{1\tau}$, $Y \in \bigcup_{\rho \in s} A^{1\rho}$ and $Y \in X$, then Y is *a piste-reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n\rangle$ of elements of $A^{1\tau}$ which we call *the piste leading to* Y *from* X such that
   (a) $X = X(0)$,
   (b) for every $i, 0 < i < n$, either
      i. $X(i-1)$ has two immediate predecessors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^\tau(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$,
      or
      ii. $X(i) \in C^\tau(X(i-1))$, and then either $i = n$ or
      $i < n$, $X(i)$ has two immediate predecessors $X(i)_0, X(i)_1$ with $X(i)_0 \in C^\tau(X(i))$, $X(i+1) = X(i)_1$ and $Y \in X(i)_1 \setminus X(i)_0$
   (c) $Y = X(n)$, if $Y \in A^{1\tau}$ and if $Y \in A^{1\rho}$, for some $\rho \neq \tau$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\tau}$.

Let us give two examples.

Example 1. *Suppose that $A^{1\tau}$ consists of three models, $Y \in Z \in X$.*
Then the piste from X to Y will be $\langle X, Y\rangle$.

Example 2. *Suppose that $A^{1\tau}$ consists of models $X, Z, T, T_0, T_1, Y_0, Y_1$ such that $Y_0 \in T_0 \in T \in Z \in X$ is $C^\tau(X)$, T is a splitting point with T_0, T_1 its immediate predecessors, $Y_0 \in T_0, Y_1 \in T_1$.*
Then the piste from X to Y_1 goes like this: From X we go down to T, then at T we turn to T_1 and from T_1 we continue to the final destination Y_1.
So the piste from X to Y_1 is $\langle X, T, T_1, Y_1\rangle$.

The sequence $\langle X(i) \mid i \leq n\rangle$ is defined uniquely from X and Y.
In particular, every $Y \in A^{1\tau}$ is piste reachable from $A^{0\tau}$.
In order to formulate further requirements, we will need to describe a simple process

of changing the wide pistes. This leads to equivalent forcing conditions once the order will be defined.

Let $X \in A^{1\tau}$. We will define the $X-$wide piste. The definition will be by induction on number of turns (splits) needed in order to reach X by the piste from $A^{0\tau}$.
First, if $X \in C^\tau(A^{0\tau})$, then the $X-$wide piste is just $\langle C^\xi(A^{0\xi}), C^\xi(A^{0\xi}) \cap A^{1\xi lim} \mid \xi \in s\rangle$, i.e. the wide piste of the structure.
Second, if $X \notin C^\tau(A^{0\tau})$, but it is not an immediate predecessor of a splitting point, then pick the least splitting point Y above X. Let Y_0, Y_1 be its immediate predecessors with $Y_0 \in C^\tau(Y)$. Then $X \in Y_i$ for some $i < 2$. Set the $X-$wide piste to be the Y_i-wide piste.
So, in order to complete the definition, it remain to deal with the following principal case:
$X \in A^{1\tau}$ a splitting point of one of the types 1,2 or 3.

Let X_0, X_1 be its immediate predecessors with $X_0 \in C^\tau(X)$. Assume that the $X-$wide piste $\langle C^\xi_X, C^{\xi lim}_X \mid \xi \in s\rangle$ is defined and assume that $C^\tau(X)$ is an initial segment of C^τ_X. Let the X_0-wide piste be $\langle C^\xi_X, C^{\xi lim}_X \mid \xi \in s\rangle$.

Let us deal with type of splitting separately.
Case 1. X is a splitting point of type 1.
Define the X_1-wide piste $\langle C^\xi_{X_1}, C^{\xi lim}_{X_1} \mid \xi \in s\rangle$ as follows:

- $C^\xi_{X_1} = C^\xi_X$, for every $\xi > \tau$.
  I.e. no changes for models of cardinality $> \tau$.
- $C^{\xi lim}_{X_1} = C^\xi_{X_1} \cap A^{1\xi lim}$, for every $\xi \in s$.
  Models that were potentially limit remain such and no new are added.
- $C^\tau_{X_1} = (C^\tau_X \setminus X) \cup C^\tau(X_1)$.
  Here we switched the piste from X_0 to X_1.
- $C^\xi_{X_1} = \{Z \in C^\xi_X \mid \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))\} \cup \{\pi_{X_0,X_1}(Z) \mid Z \in C^\xi_X \cap X_0\}$, for every $\xi \in s \cap \tau$.[17]

Note that such defined switch from X_0 to X_1 does not affect at all models of sizes above τ. Models of sizes $\leq \tau$ are effected only if they are contained in X_0 or in X_1.

[17]In particular, due to this, the next condition implies that for $\xi \in s \cap \tau$, if $Z \in C^\xi_X, \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))$, then $\{\pi_{X_0X_1}(Z') \mid Z' \in C^\xi_X \cap X_0\} \subseteq Z$.

If X is a splitting point of types 2 or 3, then we may need to turn some piste for models of cardinalities $> \tau$ into other directions, in order to satisfy the items 6(c)iiK,6(c)iiiF above.

Proceed as follows.

Case 2. X **is a splitting point of type 2**.

Let $\lambda > \tau$, $\in$ $-$increasing sequences $\langle G_{0\xi} \mid \xi \in s \cap \lambda\rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda\rangle \in X$, $F_0, F_1 \in X \cap A^{1\lambda^+ lim}$, $G_{0\lambda} = \bigcup_{\xi \in s \cap \lambda} G_{0\xi}, G_{1\lambda} = \bigcup_{\xi \in s \cap \lambda} G_{1\xi}$ be as in the item 6(c)ii.

Define the X_1-wide piste $\langle C^{\xi}_{X_1}, C^{\xi lim}_{X_1} \mid \xi \in s\rangle$ as follows:

- $C^{\xi}_{X_1} = C^{\xi}_X$, for every $\xi > \lambda$.
  I.e. no changes for models of cardinality $> \lambda$.
- $C^{\xi lim}_{X_1} = C^{\xi}_{X_1} \cap A^{1\xi lim}$, for every $\xi \in s$.
  Models that were potentially limit remain such and no new are added.
- $C^{\tau}_{X_1} = (C^{\tau}_X \setminus X) \cup C^{\tau}(X_1)$.
  Here we switched the piste from X_0 to X_1.
  Now, simultaneously, for every $\xi \in s \cap \lambda$, if $G_{0\xi}, G_{1\xi} \in A^{1\xi}$ and there is $G_{\xi} \in A^{1\xi}$ which is the immediate successor of $G_{0\xi}, G_{1\xi}$ in $A^{1\xi}$, then we switch pistes from $G_{0\xi}$ to $G_{1\xi}$ in $C^{\mu}(G_{\xi})$, exactly as it is done above with X_0, X_1, X.

Case 3. X **is a splitting point of type 3**.

Let $G, G_0, G_1 \in X \cap A^{1\mu}$ be models which witness that X is a splitting point of type 3 and X_0, X_1 are its immediate predecessors. Now using the induction[18] we can assume that the G_1-wide piste is already defined.

Define the X_1-wide piste to be the G_1-wide piste.

Now we require the following:

9. Let $\tau \in s$ and $X \in A^{1\tau}$. Then the $X-$wide piste is a wide piste, i.e. it satisfies 2.1.
   The problem is with the item (3c) of Definition 2.1 which, in general, is not preserved while splitting.

Final conditions deal with largest models.

[18]The induction is on pairs (n, ζ) ordered lexicographically, where n is the number of turns from the wide piste and ζ is the rank (the usual one as sets) of the model.
We have $G, G_0, G_1 \in X$, so the rank of G, G_0, G_1 is smaller than the rank of X. The number of turns needed to get to G and to X from the top is the same.

10. (Maximal models are above all the rest) For every $\tau \in s$ and $Z \in \bigcup_{\rho \in s} A^{1\rho}$, if $Z \not\in A^{0\tau}$, then there is $\mu \in s$ such that $Z = A^{0\mu}$.

    Recall that by Lemma 2.2, maximal models $A^{0\tau}, \tau \in s$ are linearly ordered as top parts of the wide piste $\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s \rangle$.

This completes the definition of $\delta-$structure with pistes over η of length θ.

2.1 Some properties of structures with pistes.

Let us turn now to the intersection property.
The intuition behind this is to replace an arbitrary intersection of models by an internal one. We split the definitions below according to $\theta > \aleph_{\eta^+}$ and $\theta < \aleph_{\eta^+}$.

Definition 2.4 (Models of different sizes, $\theta < \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ.

Let $A \in A^{1\tau}, B \in A^{1\rho}$ and $\tau < \rho$.
By $ip(A, B)$ we mean the following:

1. $B \in A$,
   or

2. $A \subset B$,
   or

3. $B \notin A$, $A \not\subset B$ and then

   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \rho) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

Definition 2.5 (Models of the same size, $\theta < \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ.

Let $A, B \in A^{1\tau}$. By $ip(A, B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then

   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \tau) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

If both $ip(A, B)$ and $ip(B, A)$ hold, then we denote this by $ipb(A, B)$.

Definition 2.6 (Models of different sizes, $\theta > \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ.

Let $A \in A^{1\tau}, B \in A^{1\rho}$ and $\tau < \rho$.

By $ip(A, B)$ we mean the following:

1. $B \in A$,
   or

2. $A \subset B$,
   or

3. $B \notin A$, $A \not\subset B$ and then

   - there are $\eta_1 < ... < \eta_m$ in $A \setminus \rho$ and $X_1, ..., X_m \in A$ such that
     - $A \cap B = A \cap X_1 \cap ... \cap X_m$,
     - $|X_i| = \eta_i$, for every $i, 1 \leq i \leq m$,
     - if $\eta_i \in s$, then $X_i \in A^{1\eta_i}$, for every $i, 1 \leq i \leq m$,
     - if $\eta_i \notin s$, then there are $G_i \in A \cap A^{1|G_i|}, H_i \in A \cap A^{1\theta}$, $\sup(G_i \cap \theta^+) = H_i \cap \theta^+$ such that X_i is the smallest elementary submodel of H_i which includes $G_i \cup \eta_i$, for every $i, 1 \leq i \leq m$.

Definition 2.7 (Models of the same size, $\theta > \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ.

Let $A, B \in A^{1\tau}$. By $ip(A, B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then

   - there are $\eta_1 < ... < \eta_m$ in $A \setminus \rho$ and $X_1, ..., X_m \in A$ such that
     - $A \cap B = A \cap X_1 \cap ... \cap X_m$,
     - $|X_i| = \eta_i$, for every $i, 1 \leq i \leq m$,
     - if $\eta_i \in s$, then $X_i \in A^{1\eta_i}$, for every $i, 1 \leq i \leq m$,

– if $\eta_i \notin s$, then there are $G_i \in A \cap A^{1|G_i|}, H_i \in A \cap A^{1\theta}$, $\sup(G_i \cap \theta^+) = H_i \cap \theta^+$ such that X_i is the smallest elementary submodel of H_i which includes $G_i \cup \eta_i$, for every $i, 1 \leq i \leq m$.

If both $ip(A, B)$ and $ip(B, A)$ hold, then we denote this by $ipb(A, B)$.

Lemma 2.8 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ. Assume $A \in A^{1\tau}, B \in A^{1\rho}$, for some $\tau \leq \rho, \tau, \rho \in s$. Then $ip(A, B)$ and if $\tau = \rho$, then also $ipb(A, B)$.*

Proof. We will basically split the proof into two main cases: $\rho = \tau$ and $\rho \neq \tau$. However, the inductive assumption[19] will be used simultaneously for both.

Case A. $\rho = \tau$**.**

So, $A, B \in A^{1\tau}$. Assume that $A \not\subseteq B$ and $B \not\subseteq A$. Consider the pistes leading from $A^{0\tau}$ to A and to B. Let X be their last common point. Then, by 2.3(8), X is a successor model.

Subcase A1. *X has a unique immediate predecessor.* Let X_0 be this immediate predecessor. Then, one of A or B is in X_0 and the other one is not. But, then it must be equal to X_0, which is impossible by our assumptions that $A \not\subseteq B$ and $B \not\subseteq A$.

Subcase A2. *X is a splitting point of type 1.*

Let X_0 and X_1 be the immediate predecessors of X. Let $F_0 \in X_0$ and $F_1 \in X_1$ witness that X, X_0, X_1 form a $\Delta-$system triple. Then $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$.
Assume that $A \in X_0 \cup \{X_0\}$ and $B \in X_1 \cup \{X_1\}$.
If $A = X_0$ and $B = X_1$, then $ipb(A, B)$ follows.
Suppose that $A \neq X_0$ or $B \neq X_1$. Say, $B \neq X_1$. Set $B' = \pi_{X_1,X_0}[B]$. Then $B' \in X_0$ and $B \cap X_0 = B' \cap F_0$. Hence,

$$A \cap B = A \cap B \cap X_0 = A \cap B' \cap F_0 = (A \cap B') \cap (A \cap F_0).$$

Now we apply induction to get $ip(A, B')$ and $ip(A, F_0)$.

Subcase A3. *X is a splitting point of type 2.*

Let X_0 and X_1 be the immediate predecessors of X.
Let $\langle G_{0\xi} \mid \xi \in s \cap \lambda\rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda\rangle, G_{0\lambda}, G_{1\lambda},\ F_0, F_1 \in X \cap A^{1\lambda^+ lim}$ be as in Definition 2.3(6(c)ii). Again, we have $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$, since $G_{0\lambda} \cap G_{1\lambda} = G_{0\lambda} \cap F_0 = G_{1\lambda} \cap F_1$ and the isomorphism $\pi_{G_{0\lambda}G_{1\lambda}}$ moves $X_0 \subseteq G_{0\lambda}$ to $X_1 \subseteq G_{1\lambda}$.
Hence, it is possible to proceed as in the previous case.

[19] As before, the induction is on pairs (n, ζ) ordered lexicographically, where n is the number of turns from the wide piste and ζ is the rank (the usual one as sets) of the model.

Subcase A4. *X is a splitting point of types 3.*

The proof essentially the same as in **Subcase A3**.

Case B. $\rho > \tau$.

So, $A \in A^{1\tau}, B \in A^{1\rho}$. Assume that $A \not\subseteq B$ and $B \notin A$.

Suppose first that $A \notin A^{0\rho}$. Then Definition 2.3(10), $A = A^{0\tau}$ and if $B \notin A^{0\tau}$, then, again by Definition 2.3(10), $B = A^{0\rho}$. But any two maximal models on the wide piste of the structure are compatible by Lemma 2.2. Namely, if $\sup(A^{0\tau} \cap \theta^+) \leq \sup(A^{0\rho} \cap \theta^+)$, then $A^{0\tau} \subseteq A^{0\rho}$ by Definition 2.1(12). If $\sup(A^{0\tau} \cap \theta^+) > \sup(A^{0\rho} \cap \theta^+)$, then $A^{0\rho} \in A^{0\tau}$, by Lemma 2.2.

Suppose that $A \in A^{0\rho}$. Then $B \neq A^{0\rho}$, as $A \not\subseteq B$, and hence $A, B \in A^{0\rho}$.

By Definition 2.3(9) we can assume that A is on the wide piste of the structure. Consider the pistes leading from $A^{0\rho}$ to A and to B. Let $X \in C^{\rho}(A^{0\rho})$ be their last common point. The proof proceeds by induction on rank(X). Then, by Definition 2.3(8), X is a successor model.

Subcase B1. *X is a splitting point of type 1.*

The proof is essentially as in **Subcase A2** above.

Subcase B2. *X is a splitting point of type 3.*

Let X_0 and X_1 be the immediate predecessors of X. Let $G, G_0, G_1 \in X \cap A^{1\mu}$ be a corresponding $\Delta-$system triple, for some $\mu \in s \setminus \rho+1$. Also let $F_0 \in G_0 \cap X$ and $F_1 \in G_1 \cap X$ witness this, i.e. $G_0 \cap G_1 = G_0 \cap F_0 = G_1 \cap F_1$.

Assume that $A \in X_0$ and $B \in X_1 \cup \{X_1\}$.

Set $B' = \pi_{G_1,G_0}[B]$. Then $A \cap B = A \cap B' \cap F_0$. The induction applies to A and B', since $B' \subseteq X_0 \in X$. Also it applies to A and F_0, since $F_0 \in X$. Hence, $ip(A, B)$.

Subcase B3. *X is a splitting point of type 2.*

It is similar to the previous case.

Subcase B4. *X has a unique immediate predecessor.*

Let X_0 be this predecessor. Then either $B = X_0$ or $B \in X_0$.

Split into three cases according to the relation between A and X_0.

Subsubcase B4.1. $\sup(A \cap \theta^+) < \sup(X_0 \cap \theta^+)$.

Then $A \in X_0$, by Definition 2.1(12). But $B \in X_0$ as well, and we get a contradiction to the choice of X.

Subsubcase B4.2. $\sup(A \cap \theta^+) > \sup(X_0 \cap \theta^+)$.

Using Definition 2.3(9), we may assume that both A and B are on the wide piste of the structure. Just X_0 is there as the unique immediate predecessor of X. Switching pistes below X_0 would not effect the piste leading to A, since $\sup(A \cap \theta^+) > \sup(X_0 \cap \theta^+)$.

Apply Definition 2.1(20) or (22) to A and B. Let $Z \in A, Z \supseteq B$, $A \cap Z \subseteq B$ be the result. Then $A \cap B = A \cap Z$, and we are done.

Subsubcase B4.3. $\sup(A \cap \theta^+) = \sup(X_0 \cap \theta^+)$.

Then $A \subset X_0$. If $B = X_0$, then $A \cap B = A$ and we are done. So, $B \in X_0$. Then, $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$. Apply Definition 2.1(20) or (22) to A and B and continue as in **Subsubcase B4.2**.

□

Lemma 2.9 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ. Suppose that $\tau \in s$, $A \in A^{1\tau}$ is a potentially limit point and $A \cap A^{1\theta} \neq \emptyset$. Then there is $X \in A \cap A^{1\theta}$ which includes every element of $A \cap A^{1\theta}$.*

Proof. First note that $A^{1\theta}$ has no splitting points, since θ is the maximal cardinal involved. So, $A^{1\theta} = C^\theta(A^{0\theta})$ is a closed chain. A is a potentially limit point in $A^{1\tau}, \tau < \theta$, hence $\mathrm{cof}(A \cap \theta^+) \geq \eta \geq \delta$. Hence $A \cap A^{1\theta}$ is bounded in A. Set X to be the maximal element of $A \cap A^{1\theta}$.

□

Lemma 2.10 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ. Suppose that $\tau, \rho \in s, \sup(A^{0\tau} \cap \theta^+) \neq \sup(A^{0\rho} \cap \theta^+), \tau < \rho$, $A \in A^{1\tau}$ is a potentially limit point and $A \cap A^{1\rho} \neq \emptyset$. Then there is $X \in A \cap A^{1\rho}$ which includes every element of $A \cap A^{1\rho}$.*

Proof. If $A \notin A^{0\rho}$, then $A = A^{0\tau}$, by Definition 2.3(10). By Lemma 2.2, then $A^{0\rho} \in A = A^{0\tau}$. So $A^{0\rho}$ will be as required.

Assume that $A \in A^{0\rho}$. By Definition 2.3(9), we may assume that A is on the wide piste of the structure.

Let $Z \in C^\rho(A^{0\rho})$ be the least model which includes A. Consider its immediate predecessor Z' on the piste. It exists since, by the assumption of the lemma A is not a limit model, so Z cannot be a limit model.

Now, both A and Z' are on the wide piste, $\tau < \rho$ and $A \not\subseteq Z'$. Hence, by Definition 2.1(12), $\sup(A \cap \theta^+) > \sup(Z' \cap \theta^+)$. Apply now Definition 2.1(19) or (21) to A and Z' (note that $|Z'| = \rho \in A$). So, there will be $X \in A \cap C^\rho(A^{0\rho})$ such that $X \supseteq Z'$. But then $Z' = X$ and we are done, since if $Y \in A \cap C^\rho(A^{0\rho})$, then $Y \in Z \supset A$. Hence $Y \in Z' \cup \{Z'\}$.

□

Lemma 2.11 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of length θ. Suppose that $\tau \in s$, $A \in A^{1\tau}$ and $A \cap A^{1\tau} \neq \emptyset$. If A is a potentially limit point then there is $X \in A \cap A^{1\tau}$ which includes every element of $A \cap A^{1\tau}$.*

Proof. Just by Definition 2.3(6), A has a unique immediate predecessor. It will be as desired. $\square$

Note that if A is a splitting point, then the lemma is not true anymore.
Also, if one likes to find the largest model of a small cardinality inside a larger one, then it should not be true in general (however any $\delta-$structure with pistes over η of the length θ can be extended to one that satisfies this). Thus, for example reflect in an increasing order $\omega-$many models of size η into a fixed potentially limit model A of size η^+. There will be no maximal model of cardinality η inside A. But an additional reflection will produce such.

2.2 Forcing with structures with pistes.

Definition 2.12 Define $\mathcal{P}_{\theta\eta\delta}$ to be the set of all
$\delta-$structures with pistes over η of length θ.
Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}_{\theta\eta\delta}$.
Denote further $A^{0\tau}$ by $A^{0\tau}(p)$, $A^{1\tau}$ by $A^{1\tau}(p)$, $A^{1\tau lim}$ by $A^{1\tau lim}(p)$, C^τ by $C^\tau(p)$ and s by $s(p)$.
Call s the support of p.

Let us define a partial order on $\mathcal{P}_{\theta\eta\delta}$ as follows.

Definition 2.13 Let
$p_0 = \langle\langle A_0^{0\tau}, A_0^{1\tau}, A_0^{1\tau lim}, C_0^\tau\rangle \mid \tau \in s_0\rangle$, $p_1 = \langle\langle A_1^{0\tau}, A_1^{1\tau}, A_1^{1\tau lim}, C_1^\tau\rangle \mid \tau \in s_1\rangle$ be two elements of $\mathcal{P}_{\theta\eta\delta}$.
Set $p_0 \leq p_1$ (p_1 extends p_0) iff

1. $s_0 \subseteq s_1$,
2. $A_0^{1\tau} \subseteq A_1^{1\tau}$, for every $\tau \in s_0$,
3. let $A \in A_0^{1\tau}$, for some $\tau \in s_0$, then $A \in A_0^{1\tau lim}$ iff $A \in A_1^{1\tau lim}$.
   The next item deals with a property called switching in [6]. It allows to change piste directions.
4. Let $\tau \in s_0$.
   For every $A \in A_0^{1\tau}$, $C_0^\tau(A) \subseteq C_1^\tau(A)$,

or

there are finitely many places below A where pistes change their directions, i.e. there are splitting points $B(0), ..., B(k) \in A_0^{1\tau} \cap (A \cup \{A\})$ with $B(j)', B(j)''$ the immediate predecessors of $B(j)$ $(j \leq k)$ such that

(a) $B(j)' \in C_0^\tau(B(j))$,

(b) $B(j)'' \in C_1^\tau(B(j))$.

If $B \in A_0^{1\tau} \cap (A \cup \{A\})$ is a splitting point different from $B(0), ..., B(k)$ and B', B'' are its immediate predecessors, then
$B' \in C_0^\tau(B)$ iff $B' \in C_1^\tau(B)$.

5. Let $\tau \in s_0$.
If $A \in A_0^{1\tau}$ is a splitting point in p_0, then it remains such in p_1 with the same immediate predecessors.

6. Let $\tau \in s_0$.
Let $B \in A_0^{1\tau}$ be a successor point, not in $A_0^{1\tau\,\lim}$ and with a unique immediate predecessor. Consider the wide piste that runs via B (in p_0). Let A be as in 2.1(7). Then there is no model E in p_1 such that $A \in E \in B$.

This requirement guarantees intervals without models, even after extending a condition.

By 2.13(6), potentially limit points are the only places where non-end-extensions can be made.

Lemma 2.14 *Suppose that $E, H \preceq H(\theta^+), M \preceq H(\theta^{++})$, $H \in E \in M$, $|H| \in M$,$M \cap E \subseteq H$, $|H|$ is a regular cardinal and H is closed, i.e. ${}^{|H|>}H \subseteq H$. Then for every $x \in E$ there is a closed model $H_x \in E$ such that $|H_x| = |H|$ and $x \in H_x$.*

Proof. Suppose otherwise. Then

$$H(\theta^{++}) \models \exists x \in E \forall F \in E(x \in F \wedge F \preceq H(\theta^+) \wedge |F| = |H| \Rightarrow F \text{ is not closed }).$$

By elementarity of M and since $E, |H| \in M$, there is $x \in M \cap E$ which witnesses the above. But $M \cap E \subseteq H$, so $x \in H$. This is impossible. Contradiction.
□

Lemma 2.15 *Suppose that $E, H \preceq H(\theta^+), M \preceq H(\theta^{++})$, $H \in E \in M$, $|H| = \theta$,$\theta \in M$,$M \cap E \subseteq H$ and H is closed, i.e. ${}^{|H|>}H \subseteq H$. Then for every $x \in E$ and every regular $\tau \in E \cap \theta + 1$ there is a closed model $K_x \in E$ such that $|K_x| = \tau$ and $x \in K_x$.*

Proof. Follows from the previous lemma.
□

Lemma 2.16 *Suppose that $E, H, H', M \preceq H(\theta^+)$, $H \in E \in M$, $H' \in M$,$M \cap E \subseteq H$,$H' \supseteq E \cup H$, $|H| = |H'|$ is a regular cardinal and H is closed. Then for every $x \in E$ there is a closed model $H_x \in E$ such that $|H_x| = |H|$ and $x \in H_x$.*

Proof. Suppose otherwise. Then

$$H(\theta^+) \models \exists x \in E \forall F \in E(x \in F \wedge F \preceq H' \wedge |F| = |H| \Rightarrow F \text{ is not closed }).$$

By elementarity of M and since $E, H' \in M$, there is $x \in M \cap E$ which witnesses the above. But $M \cap E \subseteq H$, so $x \in H$. In addition, $H \subseteq H'$, and hence $H \preceq H'$. This is impossible. Contradiction.
□

Lemma 2.17 *Suppose that $E, H, H', M \preceq H(\theta^+)$, $H \in E \in M$, $H' \in M$,$M \cap E \subseteq H$,$H' \supseteq E \cup H$, $|H| = |H'|$ is a regular cardinal and H is closed. Then for every $x \in E$ and every regular $\tau \in E \cap |H| + 1$ there is a closed model $K_x \in E$ such that $|K_x| = \tau$ and $x \in K_x$.*

Proof. Follows from the previous lemma.
□

Similar argument can be used to show the following:

Lemma 2.18 *Suppose that $E, H, H', M \preceq H(\theta^+)$, $H \in E \in M$, $H' \in M$,$M \cap E \subseteq H$,$H' \supseteq E \cup H$, $|H| = |H'|$ is a regular cardinal and H has an $H-$sequence. Then for every $x \in E$ there is a model $H_x \in E$ with H_x-sequence such that $|H_x| = |H|$ and $x \in H_x$.*

We turn now to properness of $\mathcal{P}_{\theta\eta\delta}$.

Recall the following basic definition due to S. Shelah [16]:

Definition 2.19 Let $\mu \geq \omega$ be a regular cardinal and P a forcing notion. P is called $\mu-$proper iff for every $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \mu$, ${}^{\mu>}M \subseteq M$, $P, p \in M$ there is $p' \geq_P p$ such that for every dense open $D \subseteq P, D \in M$, $p' \Vdash$ "$D \cap \underset{\sim}{G} \cap M \neq \emptyset$." Such p' is called (M, P)-generic.

The following is obvious:

Lemma 2.20 *If P is μ-proper, then it preserves μ^+.*

Our next task will be to show that the forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$. Let us first prove some technical lemmas that allow us to add new models at places of specific type.

Lemma 2.21 *Let $Z \preceq H(\theta^+)$ and $E \in Z$ be an elementary submodel of $H(\theta^+)$ such that $E \supseteq |E|$ and $\mathrm{cof}(\sup(E \cap \theta^+)) > |Z|$. Assume that $\langle E_i \mid i < \mathrm{cof}(\sup(E \cap \theta^+))\rangle$ is an $\in$-increasing continuous sequence of elementary submodels of E such that*

1. $\bigcup_{i<\mathrm{cof}(\sup(E\cap\theta^+))} E_i = E$,

2. $\langle E_i \mid i < \mathrm{cof}(\sup(E \cap \theta^+))\rangle \in Z$,

3. *$E_i \in E$, for every $i < \mathrm{cof}(\sup(E \cap \theta^+))$.*

4. *$E_i \supseteq |E|$, for every $i < \mathrm{cof}(\sup(E \cap \theta^+))$.*

Let $i^* = \sup(Z \cap \mathrm{cof}(\sup(E \cap \theta^+)))$.
Then $E_{i^*} = cl((Z \cap E) \cup |E|)$.

Proof. For every $i' < i^*$ there is $i'' \in Z$, $i' < i'' < i^*$. Also, $i'' \in Z$ implies that $E_{i''} \in Z$, since $\langle E_i \mid i < \mathrm{cof}(\sup(E \cap \theta^+))\rangle \in Z$. Hence $E_{i''} \in Z \cap E$.
Then $E_{i^*} = \bigcup\{E_{i''} \mid i'' \in Z \cap cof(\sup(E \cap \theta^+))\}$.
Note that if $X \in cl((Z \cap E) \cup |E|)$ and $|X| \leq |E|$, then $X \subseteq cl((Z \cap E) \cup |E|)$, by elementarity. In particular, then $E_{i''} \subseteq cl((Z \cap E) \cup |E|)$, for every $i'' \in Z \cap cof(\sup(E \cap \theta^+))$, and so, $E_{i^*} \subseteq cl((Z \cap E) \cup |E|)$.

Let us show the opposite inclusion. Suppose that $x \in cl((Z \cap E) \cup |E|)$. Then there are a Skolem function h, $a \in Z \cap E$ and an ordinal $\nu < |E|$ such that $x = h(a, \nu)$.
Note that $Z \cap E \subseteq E_{i^*}$. Namely, if $y \in Z \cap E$, then the least i such that $y \in E_i$ is in Z.
So, $a \in E_{i^*}$. Also $\nu \in E_{i^*}$, since $E_{i^*} \supseteq |E|$. Hence, $x = h(a, \nu) \in E_{i^*}$, and we are done.
□

Lemma 2.22 *Let $\langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle$ be a $(\theta, \eta, \delta)-$wide piste. Let $X \in C^\mu, E \in C^\xi$, $\xi, \mu \in s, \xi > \mu$ and $E \in X$. Let $\langle E_i \mid i < cof(\sup(E \cap \theta^+))\rangle$ be an $E-$sequence*
Then $cl((X \cap E) \cup \xi) = E_{i_X}$, where $i_X = \sup(X \cap cof(\sup(E \cap \theta^+))$.

Proof. We have $E_{i_X} = \bigcup_{i<i_X} E_i$ and $E_i \in X$, if $i \in X$. Hence $cl((X \cap E) \cup \xi) \supseteq E_{i_X}$.
Let us show the opposite inclusion. Let $x \in cl((X \cap E) \cup \xi)$. Then $x = h(a, \nu)$, for a Skolem function h, $a \in X \cap E$ and $\nu < \xi$. There is $i_a \in X$ such that $a \in E_{i_a}$. Then $i_a < i_X$. So, $a \in E_{i_X}$, but then $x = h(a, \nu) \in E_{i_X}$ and we are done.
□

Lemma 2.23 *Let $\langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle$ be a $(\theta, \eta, \delta)-$wide piste. Suppose that for some $\mu, \xi \in s, \mu < \xi$ we have $M_1, M_2 \in C^\mu$ and $E^1, E^2 \in C^{\xi lim}, {}^{\mu\geq}E^1 \subseteq E^1, {}^{\mu\geq}E^2 \subseteq E^2$. Assume that $\xi \in M_1 \cap M_2$, E^1 is below M_1 and E^2 is below M_2, i.e.* $\sup(M_1 \cap \theta^+) > \sup(E_1 \cap \theta^+)$ *and* $\sup(M_2 \cap \theta^+) > \sup(E_2 \cap \theta^+)$.
Then either

1. $cl((M_1 \cap E^1) \cup \xi) = cl((M_2 \cap E^2) \cup \xi)$
   or

2. $cl((M_1 \cap E^1) \cup \xi) \subset cl((M_2 \cap E^2) \cup \xi)$ *, and then* $cl((M_1 \cap E^1) \cup \xi) \in cl((M_2 \cap E^2) \cup \xi)$
   or

3. $cl((M_2 \cap E^2) \cup \xi) \subset cl((M_1 \cap E^1) \cup \xi)$ *, and then* $cl((M_2 \cap E^2) \cup \xi) \in cl((M_1 \cap E^1) \cup \xi)$.

Proof. Without loss of generality, we can assume that $E^1 \in M_1$ and $E^2 \in M_2$, just otherwise cover E^1, E^2 by such models.
If $E^1 = E^2$, then the relation between M_1 and M_2 provides the conclusion. Similar, if $M_1 = M_2$.
Note that due to the closure of E^i's, we have $M_j \cap E^i \in E^i$, for any $i, j < 2$.
For example, if $M_2 \subseteq M_1$ and $E^2 \in E^1$, then $M_2 \cap E^2 \in E^2$. But $E^2 \in M_2 \cap E^1 \subseteq M_1 \cap E^1$, hence $cl((M_2 \cap E^2) \cup \xi) \subseteq E_2 \subseteq cl((M_1 \cap E^1) \cup \xi)$.
Similar, if $M_2 \in M_1$ and $E^2 \subseteq E^1$, then $M_2 \in M_1, E^2 \in M_2 \subseteq M_1$ imply $M_2 \cap E^2 \in M_1$. Also, $M_2 \cap E^2 \in E^2 \subseteq E^1$. Hence $M_2 \cap E^2 \in M_1 \cap E^1$, and then, $cl((M_2 \cap E^2) \cup \xi) \in M_1 \cap E^1 \subseteq cl((M_1 \cap E^1) \cup \xi)$.

Consider a nontrivial case: $E^2 \in E^1$ and $M_1 \in M_2$.
If $E^2 \in M_1$, then $E^2 \in M_1 \cap E^1$, and so, $E^2 \in cl((M_1 \cap E^1) \cup \xi)$. Hence, $E^2 \subset cl((M_1 \cap E^1) \cup \xi)$, and then $cl((M_2 \cap E^2) \cup \xi) \subset cl((M_1 \cap E^1) \cup \xi)$, since $cl((M_2 \cap E^2) \cup \xi) \subset E^2$.
By Lemma 2.22, $cl((M_2 \cap E^2) \cup \xi) \in E^2$ and $E^2 \subset cl((M_1 \cap E^1) \cup \xi)$. So, $cl((M_2 \cap E^2) \cup \xi) \in cl((M_1 \cap E^1) \cup \xi)$.
Suppose that $E^2 \notin M_1$.

If $M_1 \cap E^1 \subseteq E^2$, i.e. if E^1 is a cover of E^2 in M_1, then

$$cl((M_1 \cap E^1) \cup \xi) = cl((M_1 \cap E^2) \cup \xi) \subseteq cl((M_2 \cap E^2) \cup \xi).$$

Now, $M_1 \cap E^2 \in E^2$ due to the closure of E^2. Also, $M_1 \cap E^2 \in M_2$ since $M_1, E^2 \in M_2$. Hence, $M_1 \cap E^2 \in M_2 \cap E^2$, and so, $cl((M_1 \cap E^2) \cup \xi) \in M_2 \cap E^2 \subseteq cl((M_2 \cap E^2) \cup \xi)$.

Suppose that $M_1 \cap E^1 \not\subseteq E^2$. Pick then $E^* \in M_1 \cap E^1 \cap C^\xi$ such that $E^* \supset E^2$ and $M_1 \cap E^* \subseteq E^2$.

Now, $E^* \in M_1 \cap E^1$ implies $E^* \in cl((M_1 \cap E^1) \cup \xi)$, and then, $E^* \subset cl((M_1 \cap E^1) \cup \xi)$. So, $cl((M_2 \cap E^2) \cup \xi) \subset E^2 \subset E^* \subset cl((M_1 \cap E^1) \cup \xi)$.

By Lemma 2.22, $cl((M_2 \cap E^2) \cup \xi) \in E^2$ and $E^2 \subset cl((M_1 \cap E^1) \cup \xi)$. So, $cl((M_2 \cap E^2) \cup \xi) \in cl((M_1 \cap E^1) \cup \xi)$.

□

Lemma 2.24 *Let $\langle C^\tau, C^{\tau lim} \rangle \mid \tau \in s \rangle$ be a $(\theta, \eta, \delta)-$wide piste. Suppose that for some $\mu, \xi \in s, \mu < \xi$ we have $M_1, M_2 \in C^\mu$, $E^1, E^2 \in C^{\xi lim}$ such that $\xi \in M_1 \cap M_2$, E^1 is below M_1 and E^2 is below M_2.*

Then either

1. $cl((M_1 \cap E^1) \cup \xi) = cl((M_2 \cap E^2) \cup \xi)$
   or

2. $cl((M_1 \cap E^1) \cup \xi) \subset cl((M_2 \cap E^2) \cup \xi)$ *, and then* $cl((M_1 \cap E^1) \cup \xi) \in cl((M_2 \cap E^2) \cup \xi)$
   or

3. $cl((M_2 \cap E^2) \cup \xi) \subset cl((M_1 \cap E^1) \cup \xi)$ *, and then* $cl((M_2 \cap E^2) \cup \xi) \in cl((M_1 \cap E^1) \cup \xi)$.

Proof. Without loss of generality, we can assume that $E^1 \in M_1$ and $E^2 \in M_2$, just otherwise cover E^1, E^2 by such models.

By Lemma 2.22,

$cl((M_1 \cap E^1) \cup \xi) = E^1_{i_{M_1}}$ and $cl((M_2 \cap E^2) \cup \xi) = E^2_{i_{M_2}}$. Now, if $E^1 = E^2$, then $i_{M_1} < i_{M_2}$ will imply $E^1_{i_{M_1}} \in E^1_{i_{M_2}} = E^2_{i_{M_2}}$, and we are done.

Assume so that $E^1 \neq E^2$. For example, let $E^2 \in E^1$.

Pick then $E^* \in M^1 \cap C^\xi$ to be the cover of E^2.

If $E^* \in E^1$, then $cl((M_2 \cap E^2) \cup \xi) \subseteq E^2 \subseteq E^*$ and since $E^* \in M_1 \cap E^1$ we have $E^* \subseteq cl((M_1 \cap E^1) \cup \xi)$. Hence, $cl((M_2 \cap E^2) \cup \xi) \subseteq cl((M_1 \cap E^1) \cup \xi)$.

By Lemma 2.22, $cl((M_2 \cap E^2) \cup \xi) \in E^2$ or $cl((M_2 \cap E^2) \cup \xi) = E^2$. Also, $E^2 \in E^*$ or

$E^2 = E^*$. Hence, $cl((M_2 \cap E^2) \cup \xi) \in cl((M_1 \cap E^1) \cup \xi)$.
Suppose that $E^* \not\in E^1$. Then $E^* = E^1$ and also $M_1 \in M_2$, since $E^2 \in M_2$.
It follows then that $E^2 \supseteq E^1_{i_{M_1}}$. By Lemma 2.22, $E^1_{i_{M_1}} = cl((M_1 \cap E^1) \cup \xi)$. We have $M_1 \cap E^1_{i_{M_1}} = M_1 \cap E^1 = M_1 \cap E^2 \in M_2$.
Now by Definition 2.1(18), either $E^1_{i_{M_1}} \in E^2$ or $E^2 = E^1_{i_{M_1}}$. In the later case, $M_1 \cap \theta^+$ is unbounded in $\sup(E^2 \cap \theta^+)$. Then $M_2 \supseteq M_1$ is such as well. But then, by Lemma 2.22, $cl((M_2 \cap E^2) \cup \xi) = E^2$ and we are done.
Suppose finally that $E^1_{i_{M_1}} \in E^2$. Then $M_1 \cap E^2 = M_1 \cap E^1_{i_{M_1}} \subseteq E^1_{i_{M_1}} \in E^2$. Now, there is a successor model E^2_{j+1} on E^2-sequence such that $E^1_{i_{M_1}} \in E^2_{j+1}$, but successor models of E^2-sequence are closed under $< \xi-$sequences of its elements. In particular, $|M_1 \cap E^2| \leq |M_1| = \mu < \xi$, and so, $M_1 \cap E^2 \in E^2_{j+1} \subseteq E^2$. Clearly, $M_1 \cap E^2 \in M_2$. Hence, $M_1 \cap E^1 = M_1 \cap E^2 \in M_2 \cap E^2$, and then, $cl((M_1 \cap E^1) \cup \xi) \in cl((M_2 \cap E^2) \cup \xi)$.
$\square$

Let us prove now analogous statements for models M_1, M_2 of different cardinalities.

Lemma 2.25 *Let $\langle C^\tau, C^{\tau lim} \rangle \mid \tau \in s \rangle$ be a $(\theta, \eta, \delta)-$wide piste. Suppose that for some $\mu_1, \mu_2, \xi \in s, \mu_1 < \mu_2 < \xi$ we have $M_1 \in C^{\mu_1}, M_2 \in C^{\mu_2}$, $\xi \in M_1 \cap M_2$ and $E^1, E^2 \in C^{\xi lim}$, ${}^{\mu_1 \geq}E^1 \subseteq E^1$, ${}^{\mu_2 \geq}E^2 \subseteq E^2$.*
Assume that E^1 is below M_1 and E^2 is below M_2.
Assume in addition that if $M_2 \in M_1$, then $\sup(M_2 \cap \theta^+) \neq \sup(E^1 \cap \theta^+)$.
Then either

1. *$cl((M_1 \cap E^1) \cup \xi) = cl((M_2 \cap E^2) \cup \xi)$*
   or
2. *$cl((M_1 \cap E^1) \cup \xi) \subset cl((M_2 \cap E^2) \cup \xi)$, and then $cl((M_1 \cap E^1) \cup \xi) \in cl((M_2 \cap E^2) \cup \xi)$*
   or
3. *$cl((M_2 \cap E^2) \cup \xi) \subset cl((M_1 \cap E^1) \cup \xi)$, and then $cl((M_2 \cap E^2) \cup \xi) \in cl((M_1 \cap E^1) \cup \xi)$.*

Proof. Without loss of generality, we can assume that $E^1 \in M_1$ and $E^2 \in M_2$, just otherwise cover E^1, E^2 by such models.
Consider the new case that occurs here: $M_2 \in M_1, E^2 \in E^1$. Note that here we have $|M_2| > |M_1|$, and so, $M_2 \in M_1$ does not imply $M_2 \subseteq M_1$.

Case 1. $M_2 \cap E^1 \in E^1$.
Then $M_2 \cap E^1 \in M_1 \cap E^1$. We assumed that $E^2 \subset E^1$, so $M_2 \cap E^2 \subseteq M_2 \cap E^1 \in M_1 \cap E^1$. Hence,

$$cl((M_2 \cap E^2) \cup \xi) \subseteq cl((M_2 \cap E^1) \cup \xi) \in M_1 \cap E^1.$$

Then

$$cl((M_2 \cap E^2) \cup \xi) \subseteq cl((M_2 \cap E^1) \cup \xi) \subset cl((M_1 \cap E^1) \cup \xi).$$

Now, using the closure of successor models in an E^1-sequence, as in Lemma 2.24, we get

$$cl((M_2 \cap E^2) \cup \xi) \in cl((M_1 \cap E^1) \cup \xi).$$

Just for some $i < i^{E^1}_{M_1}$, $M_2 \cap E^2 \subseteq E^1_{i+1}$.

Case 2. $E^2 \in M_1$.
Then $M_2 \cap E^2 \in M_1$. Also, $M_2 \cap E^2 \in E^1$, since $M_2 \cap E^2 \subseteq E^2 \in E^1$, and picking an E^1-sequence, we will have for some $i < \mathrm{cof}(\sup(E^1 \cap \theta^+))$, $M_2 \cap E^2 \subseteq E^1_{i+1}$.
So, $M_2 \cap E^2 \in M^1 \cap E^1$.

Case 3. $E^2 \notin M_1$ and E^1 not the cover of E^2 in M_1.
Let F be the cover. Then $F \in E^1$. Also, as in the previous case, $M_2 \cap E^2 \in E^1$ and $M_2 \cap F \in E^1$. In addition, $M_2 \cap F \in M_1$, since both M_2 and F in M_1.
So, $M_2 \cap E^2 \subseteq M_2 \cap F \in M_1 \cap E^1$.

Case 4. $E^2 \notin M_1$ and E^1 is the cover of E^2 in M_1.
We can assume that $\sup(M_2 \cap \theta^+) > \sup(E^1 \cap \theta^+)$, since by the assumption of the lemma $\sup(M_2 \cap \theta^+) \neq \sup(E^1 \cap \theta^+)$ and Case 1 takes care of the situation in which $\sup(M_2 \cap \theta^+) < \sup(E^1 \cap \theta^+)$.
Let E^{1,M_2} be the cover of E^1 in M_2. Then $E^{1,M_2} \in M_1$, since $E^1, M_2 \in M_1$. By (18) of Definition 2.1, $E^1 = E^{1,M_2}_{i_{M_2}}$ or $E^{1,M_2}_{i_{M_2}} \in E^1$.
If $E^{1,M_2}_{i_{M_2}} \in E^1$, then we are in the situation of Case 1, since $M_2 \cap E^1 = M_2 \cap E^{1,M_2}_{i_{M_2}}$, $E^{1,M_2}_{i_{M_2}} \in E^1$ and we can use the closure of successor elements of E^1-sequence to argue that $M_2 \cap E^1 \in E^1$.
So, suppose that $E^1 = E^{1,M_2}_{i_{M_2}}$.
Set $i(M_1, E^1) = \sup(M_1 \cap \mathrm{cof}(E^1 \cap \theta^+))$.
Note that $\eta \leq \mu_1 < \mu_2$ implies that μ_2 is a regular uncountable cardinal. Then i_{M_2} has uncountable cofinality.
So, E^1-sequence agrees with the sequence $\langle E^{1,M_2}_i \mid i < i_{M_2} \rangle$ for $E^{1,M_2}_{i_{M_2}}$ on a club.
Suppose for simplicity that this two sequences agree everywhere.
Then by Lemma 2.22, applied to $E^1 = E^{1,M_2}_{i_{M_2}}$ and $\langle E^{1,M_2}_i \mid i < i_{M_2} \rangle$, we will have $E^{1,M_2}_{i(M_1,E^1)} = cl((M_1 \cap E^1) \cup \xi)$.
Now, we have $E^{1,M_2} \in M_1$ and so, $\langle E^{1,M_2}_i \mid i < \mathrm{cof}(E^{1,M_2} \cap \theta^+) \rangle \in M_1$.
Also $i_{M_2} \in M_1$, since $M_2 \in M_1$. Hence, $\langle E^{1,M_2}_i \mid i < i_{M_2} \rangle \in M_1$.

In particular for every $i < i(M_1, E^1) = \sup(M_1 \cap \mathrm{cof}(E^1 \cap \theta^+))$, there is $i^* \in M_1$ such that $E^{1,M_2}_{i^*} \in M_1$.
Remember that E^1 is a cover of E^2 in M_1, so $M_1 \cap E^1 = M_1 \cap E^2$.
Then, since $E^2 \notin M_1$ and $E^2 \in M_2$, we will have $E^{1,M_2}_{i(M_1,E_1)} = cl((M_1 \cap E^1) \cup \xi) \in E^2$ or $E^{1,M_2}_{i(M_1,E_1)} = cl((M_1 \cap E^1) \cup \xi) = E^2$.
The later possibility is impossible, since ${}^{\mu_1}E^2 \subseteq E^2$.
Deal with the former one. Then, since $E^2 \in M_2$ and $\langle E^{1,M_2}_i \mid i < \mathrm{cof}(E^{1,M_2} \cap \theta^+)\rangle \in M_2$ is a closed sequence, there will be the largest $j < \mathrm{cof}(E^{1,M_2} \cap \theta^+)$ such that $E^{1,M_2}_j \subseteq E^2$. Clearly $j \in M_2$. Then, $E^{1,M_2}_{i(M_1,E^1)} = cl((M_1 \cap E^1) \cup \xi) \subseteq E^{1,M_2}_j \in M_2$, and we are done.
□

The next three lemmas are most relevant if $\theta > \aleph_{\eta^+}$.

Lemma 2.26 *Let $X, E \preceq \langle H(\theta^+), \in, \leq, \delta, \eta\rangle$. Suppose that $\sup(X \cap \theta^+) > \sup(E \cap \theta^+)$, $\eta \leq |X| < |E|$. Assume that there is E^X a cover of E in X, i.e. $E^X \in X, E^X \preceq X, E^X \supseteq E$ and $X \cap E^X \subseteq E$.*
Then E^X is the cover of E in $cl(X \cup \rho)$, for any $\rho, \xi \leq \rho \leq \mu$.

Proof. Clearly, $E^X \in cl(X \cup \rho)$, since $cl(X \cup \rho) \supseteq X$ and $E^X \in X$.
Let us argue that $cl(X \cup \rho) \cap E^X \subseteq E$.
Let $x \in cl(X \cup \rho) \cap E^X$. Then $x = h(a, \nu)$, for some $a \in X \cap E^X, \nu < \rho$ and a Skolem function h. We have $X \cap E^X \subseteq E$. Hence, $a \in E$, but then $x = h(a, \nu) \in E$ as well.
□

Lemma 2.27 *Let $X, Y, Z \preceq \langle H(\theta^+), \in, \leq, \delta, \eta\rangle$. Suppose that $\sup(X \cap \theta^+) > \sup(Y \cap \theta^+) > \sup(Z \cap \theta^+)$, $\eta \leq |X| < |Y| < |Z|$. Assume that there is Z^* a cover of Z in X or in Y.*
Then $Z^* \cap cl((X \cap Y) \cup \rho) \subseteq Z$, for any $\rho, |X| \leq \rho \leq |Z|$. In particular, if $Z^* \in X \cap Y$, then Z^* is the cover of Z in $cl((X \cap Y) \cup \rho)$, for any $\rho, |X| \leq \rho \leq |Z|$.

Proof. Similar to the previous lemma.
□

Lemma 2.28 *Let $X, Y, Z \preceq \langle H(\theta^+), \in, \leq, \delta, \eta\rangle$. Suppose that the following hold:*

1. $\sup(X \cap Y \cap \theta^+) > \sup(Z \cap \theta^+)$, $\eta \leq |X| < |Y| < |Z|$.
2. *There is a $Y-$sequence $\langle Y_\nu \mid \nu < \mathrm{cof}(Y \cap \theta^+)\rangle$.*
3. *$Y \in X$.*

4. There is a cover Z^Y of Z in Y.

Then either $Z^Y \in X$ and then it is the cover of Z in $cl((X \cap Y) \cup \rho)$, for any $\rho, |X| \leq \rho \leq |Z|$, or
$Z^Y \notin X$ and then let $\nu^Z := \min(\{\nu \mid \nu < \operatorname{cof}(Y \cap \theta^+) \wedge Z^Y \in Y_\nu\})$.
Assume that $Z^Y \supseteq Y_{\nu_Z - 1}$ and $\nu_Z \notin X$.
Set $Z^* = cl(Y_{\nu^*} \cup \min(X \cap Y \cap \theta^+ \setminus |Z|)$, where $\nu^* = \min(X \cap \operatorname{cof}(Y \cap \theta^+) \setminus \nu^Z)$.
Then Z^* is the cover of Z in $cl((X \cap Y) \cup \rho)$, for any $\rho, |X| \leq \rho \leq |Z|$.

Proof. If $Z^Y \in X$, then it is the cover of Z in $cl((X \cap Y) \cup \rho)$, for any $\rho, |X| \leq \rho \leq |Z|$, by the previous lemma.
So, suppose that $Z^Y \notin X$ and $Z^Y \supseteq Y_{\nu_Z - 1}$ and $\nu_Z \notin X$, where $\nu^Z = \min(\{\nu \mid \nu < \operatorname{cof}(Y \cap \theta^+) \wedge Z^Y \in Y_\nu\})$.
Then $\sup(X \cap Y \cap \theta^+) > \sup(Y_{\nu_Z - 1} \cap \theta^+)$, since $\sup(X \cap Y \cap \theta^+) > \sup(Z \cap \theta^+)$ and Y_{ν^Z} is the first on the $Y-$sequence such that $\sup(Y_{\nu^Z} \cap \theta^+) > \sup(Z \cap \theta^+)$.
Then $X \cap \operatorname{cof}(Y \cap \theta^+) \setminus \nu^Z \neq \emptyset$. Let $\nu^* = \min(X \cap \operatorname{cof}(Y \cap \theta^+) \setminus \nu^Z)$.
Consider $Z^* = cl(Y_{\nu^*} \cup \min(X \cap Y \cap \theta^+ \setminus |Z|)$.
Clearly, $Z^* \in X \cap Y$.
We need to show that $Z^* \cap cl((X \cap Y) \cup \rho) \subseteq Z$.
Proceed as in Lemma 2.26.
Let $x \in Z^* \cap cl((X \cap Y) \cup \rho)$. Then $x = h(a, \nu)$, for some $a \in X \cap Y \cap Z^*, \nu < \rho$ and a Skolem function h.
$a \in X \cap Y \cap Z^*$ implies that $a = h'(a', \nu')$, for some $a' \in X \cap Y \cap Y_{\nu^*}$, $\nu' < \min(X \cap Y \cap \theta^+ \setminus |Z|), \nu' \in X \cap Y$ and a Skolem function h'.
Now, by the choice of ν^*, $X \cap Y_{\nu^*} = X \cap Y_{\nu_Z} = X \cap Y_{\nu_Z - 1}$.
We have $Z^Y \supseteq Y_{\nu_Z - 1}$, hence $a' \in Z^Y \cap Y$. But also, $Y \cap Z^Y \subseteq Z$.
Hence $a' \in Z$. Then $a' = h'(a', \nu') \in Z$, since $\nu' < \min(X \cap Y \cap \theta^+ \setminus |Z|), \nu' \in X \cap Y$ implies $\nu' < |Z|$, and so, $x = h(a, \nu) \in Z$, as well.
$\square$

Lemma 2.29 *Let $p = \langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle \in \mathcal{P}_{\theta\eta\delta}$ be a wide piste and B, D are potentially limit models of p such that $|B| = \tau$, for some regular $\tau \in s \cap \theta$, $|D| = \theta$ and $\sup(B \cap \theta^+) = D \cap \theta^+$.*
Let $\rho \in (\tau, \theta) \cap B$ be a regular cardinal. Then a model of cardinality ρ can be added to p between B and D such that the result remains a wide piste.

Proof. Suppose that there is no model of size ρ between B and D inside p. Without loss of generality we can assume that $\rho \in s$. Otherwise extend p by adding a single model of cardinality ρ and making it a potentially limit model.

Let E be the least elementary submodel of D such that

- $|E| = \rho$,
- $E \supseteq B$,
- $E \supseteq \rho + 1$,

So, E is the Skolem Hull of $B \cup \rho + 1$ in D.

We will add E to $C^{\rho lim}$. However, the addition of E may require that some other models will be added as well, in order to satisfy the covering properties of Definition 2.1.
Thus, for example, assume that $\theta < \aleph_{\eta^+}$ and that there is a model A in p such that $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \notin A$. Then we may need to add some $E' \in A$ of cardinality ρ such that $A \cap E' \subseteq E$.

Claim 1 Let B', D' be potentially limit models of p such that $|B'| < \theta$, $|D'| = \theta$ and $\sup(B' \cap \theta^+) = D' \cap \theta^+$. Let A be a model in p such that $\sup(A \cap \theta^+) > \sup(B' \cap \theta^+)$, $|A| < |B'|$ and $|B'| \in A$.
Suppose that B^* is a cover of B' in A, and $D^* \in C^\theta$ is a cover of D' in A. Then $\sup(B^* \cap \theta^+) = D^* \cap \theta^+$.

Proof. Note that by (23) of Definition 2.1, $B' \in A$ iff $D' \in A$.
If $B' \in A$ (or equivalently $D' \in A$), then $B^* = B'$, $D^* = D'$ and we are done.
Suppose that $B', D' \notin A$.

It is impossible to have $D^* \cap \theta^+ > \sup(B^* \cap \theta^+)$, since then $B^* \in D^*$, but then, $A \cap D^* \subseteq D'$ and $B^* \notin D'$.

Let us argue that it is impossible to have $D^* \cap \theta^+ < \sup(B^* \cap \theta^+)$. Thus suppose that this is the case. Recall that both B^* and D^* are in A. So,

$$A \models D^* \cap \theta^+ < \sup(B^* \cap \theta^+).$$

Hence,

$$A \models \exists \zeta \in B^* \cap \theta^+ (\zeta > D^* \cap \theta^+).$$

Pick such $\zeta \in A$. Then $\zeta \in A \cap B^* \subseteq B'$, but this impossible, since

$$\sup(B' \cap \theta^+) = D' \cap \theta^+ \leq D^* \cap \theta^+.$$

Hence, $D^* \cap \theta^+ = \sup(B^* \cap \theta^+)$.
$\square$ of the claim.

Suppose first that $\theta < \aleph_{\eta^+}$.

Let Z be the set of all pairs $\langle B', D' \rangle$ such that

1. B', D' are potentially limit models of p,
2. $|B'| = |B| = \tau$,
3. $\sup(B' \cap \theta^+) \geq \sup(B \cap \theta^+)$,
4. $|D'| = \theta$,
5. $\sup(B' \cap \theta^+) = D' \cap \theta^+$.

Now, for every $\langle B', D' \rangle \in Z$, let E' be the Skolem Hull of $B' \cup \rho + 1$ in D'.

Add all such models E' to p.

Let us argue that the result remains a wide piste. The first observation will be that all new models are linearly ordered by an $\in -$relation.

Claim 2 Suppose that $\langle B', D' \rangle, \langle B'', D'' \rangle \in Z$ and $D' \cap \theta^+ < D'' \cap \theta^+$. Let E' be the Skolem Hull of $B' \cup \rho + 1$ in D' and E'' be the Skolem Hull of $B'' \cup \rho + 1$ in D''. Then $E' \in E''$.

Proof. We have $B' \in B''$, since both are in C^τ and $\sup(B' \cap \theta^+) < \sup(B'' \cap \theta^+)$. Then, $D' \in B''$. Hence, E', which is the least elementary submodel of D' containing $B' \cup \rho + 1$, is in E'', since $B', \rho + 1, D' \in E''$.
$\square$ of the claim.

Let $A \in C^\xi$, for some $\xi \in s$ and $E' = cl(B' \cup \rho + 1)$ for some $\langle B', D' \rangle \in Z$.
We address now the covering property between A and E'.

Case 1. $\sup(A \cap \theta^+) > \sup(E' \cap \theta^+)$**.**
If $B' \in A$, then $E' \in A$ as well, and so we are done.
Suppose that $B' \notin A$. Apply (19) of Definition 2.1 to A, B', D'. Let $B^*, D^* \in A$ be the witnessing models. By the first claim, $\langle B^*, D^* \rangle \in Z$. Let $E^* = cl(B^* \cup \rho + 1)$.
Then $E^* \in A$.
Let show that $A \cap E^* \subseteq E$. Suppose $x \in A \cap E^*$. Then $x = h(a, \gamma)$, for some Skolem function h, $a \in A \cap B^*$ and $\gamma \leq \rho$. But $A \cap B^* \subseteq B'$, hence $a \in B'$. So, $x \in E'$, and we are done.

Case 2. $\sup(A \cap \theta^+) < \sup(E' \cap \theta^+)$.
If $A \in B'$, then we are done, since $B' \subseteq E'$.
Let $\xi = |A|$. Note that if $\xi \leq \tau$, then $A \in B'$.
Suppose that $A \notin B'$.
Then, by Definition 2.1(19), there is a cover $A^* \in B' \cap C^\xi$ of A in B'. But then, $A \in A^*$.
So, $A^* \in E'$, since $A^* \in B' \subseteq E'$.
If $\xi \leq \rho$, then $A^* \subseteq E'$. Hence $A \in E'$.
Assume now $\xi > \rho$ and $A \notin E'$.
Suppose first that $\mathrm{cof}(\sup(A^* \cap \theta^+)) = |A^*| = \xi$. Let $\langle A_i^* \mid i < \xi\rangle$ be the least (or just definable from A^*) $\in$ $-$increasing sequence of models such that for every $i < \xi$ the following holds:

1. $|A_i^*| = \xi$,
2. $A_i^* \preceq A^*$,
3. $A_i^* \in A^*$,
4. $\bigcup_{i<\xi} A_i^* = A^*$.

Set $i^* = \sup(B' \cap \xi)$. Then $A^* \supseteq A \supseteq A_{i^*}^*$, since $B' \cap A = B' \cap A^* = B' \cap A_{i^*}^*$ and $\langle A_i^* \mid i < \xi\rangle \in B'$.
But $E' = cl(B' \cup \rho + 1)$ and $\rho < \xi$, hence $\sup(E' \cap \xi) = \sup(B' \cap \xi) = i^*$.
So, A^* works for E' as well.

Suppose now that $\mathrm{cof}(\sup(A^* \cap \theta^+)) < |A^*| = \xi$. If $\mathrm{cof}(\sup(A^* \cap \theta^+)) > \rho$, then the argument above applies. Assume that $\mathrm{cof}(\sup(A^* \cap \theta^+)) \leq \rho$.
Then, apply (c) of (20) of Definition 2.1 to B' and A. Let $n^* < \omega$, $\langle G^n \mid n < n^*\rangle$, $\langle A^n \mid n \leq n^*\rangle$ be the witnesses. We have $A^0 = A^*$, $G^0 \in B'$, and either $A^0 = cl(G^0 \cup \xi)$ and ${}^{|G^0|>}G^0 \subseteq G^0$, or $A^0 = cl((\tilde{G}^0 \cap \tilde{F}^0) \cup \xi)$, for some models $\tilde{G}^0, \tilde{F}^0 \in B'$ such that G^0 is a core of $\tilde{G}^0$. Note that in both cases $\mathrm{cof}(A^0 \cap \theta^+) = |G^0|$, since $\mathrm{cof}(\sup(A^0 \cap \theta^+)) = \mathrm{cof}(\sup(G^0 \cap \theta^+))$ or $\mathrm{cof}(\sup(A^0 \cap \theta^+)) = \mathrm{cof}(\sup((\tilde{G}^0 \cap \tilde{F}^0) \cap \theta^+))$. So, the assumption $\mathrm{cof}(\sup(A^* \cap \theta^+)) \leq \rho$ implies that $|G^0| \leq \rho = |E'|$. Hence, $G^0 \in B' \subseteq E'$ implies $G^0 \subseteq E'$. In particular, $G^1 \in E'$, since $G^1 \in G^0$.

Suppose for simplicity that already $|G^1| > \rho$.
We claim then that $\langle G^n \mid 1 \leq n < n^*\rangle$, $\langle A^n \mid 1 \leq n \leq n^*\rangle$ witness (c) of (20) of Definition 2.1 for E' and A.

This holds, since $\rho = |E'| < |G^1| < |G^n|$ and
$\sup(B' \cap |G^n|) = \sup(E' \cap |G^n|)$, for every $n, 1 \leq n < n^*$.

Suppose now that $\theta > \aleph_{\eta^+}$.

We define Z to be as in the case $\theta < \aleph_{\eta^+}$, i.e. Z consists of all pairs $\langle B', D' \rangle$ such that

1. B', D' are potentially limit models of p,
2. $|B'| = |B| = \tau$,
3. $\sup(B' \cap \theta^+) \geq \sup(B \cap \theta^+)$,
4. $|D'| = \theta$,
5. $\sup(B' \cap \theta^+) = D' \cap \theta^+$.

Now, for every $\langle B', D' \rangle \in Z$, let E' be the Skolem Hull of $B' \cup \rho + 1$ in D'.

Add all such models E' to p.

Let $A \in C^\xi$, for some $\xi \in s$ and $E' = cl(B' \cup \rho + 1)$ for some $\langle B', D' \rangle \in Z$.
Address the covering property between A and E'.

Case 1. $\sup(A \cap \theta^+) > \sup(E' \cap \theta^+)$**.**
If $B' \in A$ and $\rho \in A$, then $E' \in A$ as well, and so we are done.
Suppose that $\rho \notin A$. Consider then $\rho^* = \min(A \cap On \setminus \rho)$. Let $E^* = cl(B' \cup \rho^* + 1)$. Clearly, $E^* \in A$.
Then, as before, $A \cap E^* \subseteq E$.

Suppose that $B' \notin A$. Apply Definition 2.1(21) to A and B', D'. Let $B^*, D^* \in A$ be the covering models. Then $D^* \in C^\theta(p)$, since θ belongs to every model of p, and so, it belongs to A. By Claim 1, we have $\sup(B^* \cap \theta^+) = D^* \cap \theta^+$.
Consider $E^* = cl(B^* \cup \min(A \cap \theta^+ \setminus \rho))$.
Then $E^* \in A$. Also, $B^* \subseteq E^* \subseteq D^*$, and so, $\sup(E^* \cap \theta^+) = \sup(B^* \cap \theta^+) = D^* \cap \theta^+$.
Let show that $A \cap E^* \subseteq E$. Suppose $x \in A \cap E^*$. Then $x = h(a, \gamma)$, for some Skolem function h, $a \in A \cap B^*$ and $\gamma \in A \cap \min(A \cap \theta^+ \setminus \rho)$. We have $A \cap B^* \subseteq B'$, hence $a \in B'$. In addition, $\gamma \in A \cap \min(A \cap \theta^+ \setminus \rho)$ implies that $\gamma < \rho$. So, $x \in E'$, and we are done.

Case 2. $\sup(A \cap \theta^+) < \sup(E' \cap \theta^+)$**.**
The argument is similar to those with $\theta < \aleph_{\eta^+}$, only (21) and (22) of Definition 2.1 should be used instead of (19) and (20).
□

Lemma 2.30 *Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle \in \mathcal{P}_{\theta\eta\delta}$ and B, D are models on the wide piste of p such that $|B| = \tau$, for some regular $\tau \in s \cap \theta$, $|D| = \theta$ and $\sup(B \cap \theta^+) = D \cap \theta^+$. Then for every regular cardinal $\rho \in (\tau, \theta) \cap B$ a model of cardinality ρ can be added to p between B and D.*

Proof. Let B, D be as in the statement of the lemma and $\rho \in (\tau, \theta) \cap B$ be a regular cardinal. Suppose that there is no model of size ρ between B and D inside p. Without loss of generality we can assume that $\rho \in s$. Otherwise extend p by adding a single model of cardinality ρ and making it potentially limit one.
Let E be the least elementary submodel of D such that

- $|E| = \rho$,
- $E \supseteq B$,
- $E \supseteq \rho + 1$,

Now we would like to add E to p. Namely we proceed as follows:

1. Apply Lemma 2.29 to the wide piste of p in order to add E.

2. Add all the images under $\Delta-$system triples isomorphisms of the models added to the wide piste.

3. Repeat the process of adding all the images under $\Delta-$system triples isomorphisms of the models added at previous stages.

After at most $\omega-$many steps the process terminates, since each model of p can be reached from the top by applying finitely many switches. Moreover, if p is finite then the number of steps will be finite and will be at most the number of splitting points of p.

The result will be as desired.

□

Lemma 2.31 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$.*

Proof. Let τ be a regular cardinal in the interval $[\eta, \theta]$. We would like to show that $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper.
Let $p \in \mathcal{P}_{\theta\eta\delta}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ regular large enough such that

1. $|\mathfrak{M}| = \tau$,

2. $\mathfrak{M} \supseteq \tau$,

3. $\mathcal{P}_{\theta\eta\delta}, p \in \mathfrak{M}$,

4. ${}^{\tau>}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\theta^+)$.
Clearly, M satisfies 2.1(3(b)). Moreover, using the elementarity of $\mathfrak{M}$, for every $x \in M$ there will be $Z \in M$ such that

- $Z \preceq H(\theta^+)$,
- $|Z| = \theta$,
- $Z \supseteq \theta$,
- ${}^{\theta>}Z \subseteq Z$,
- $x \in Z$.

This allows to find a chain $\langle N_i \mid i < \tau\rangle$ of models of size θ whose members are in M, which witnesses (3(b)) of Definition 2.1 for $N := \bigcup_{i<\tau} N_i$, and such that $N \supseteq M$.
Let $\langle M_i \mid i < \tau\rangle$ be a chain of models of size τ whose members are in M, which witnesses (3(b)) of Definition 2.1 for M. Then

$$N = \bigcup_{i<\tau} cl(M_i \cup \theta) = cl(M \cup \theta).$$

The same is true if we replace θ by any regular cardinal $\xi \in M \cap (\tau, \theta)$.

Extend p by adding M as a new $A^{0\tau}$ and N as a new $A^{0\theta}$. Require them to be potentially limit points. Denote the result by $p^\frown\{M, N\}$.
We claim that $p^\frown\{M, N\}$ is $(\mathcal{P}_{\theta\eta\delta}, \mathfrak{M})-$generic. So, let $p' \geq p^\frown\{M, N\}$ and $\bar{D} \in \mathfrak{M}$ be a dense open subset of $\mathcal{P}_{\theta\eta\delta}$.
Extending p' more if necessary, we can assume, without loss of generality, that $p' \in \bar{D}$. Extend p' further, if necessary, in order to insure that the following holds:

- maximal models of p' increase according their cardinalities,
- if $\tau > \eta$ and it is a successor of a regular cardinal, then its predecessor is in $s(p')$,
- if $\tau > \eta$ and it is an inaccessible cardinal, then $\tau \cap s(p')$ has a maximal element and it is regular,

- if $\tau > \eta$ and it is a successor of a singular cardinal τ^- of cofinality $\geq \delta$, then $\tau^- \cap s(p')$ has a maximal element and it is regular.

Let η^* be η, if $\tau = \eta$.[20] Suppose that $\tau > \eta$. Let then η^* denote the predecessor of τ, if τ is a successor of a regular cardinal or of a singular cardinal of cofinality $< \delta$, and if τ is an inaccessible cardinal, then let $\eta^* = \max(\tau \cap s(p'))$. If τ is a successor of a singular cardinal τ^- of cofinality $\geq \delta$, then let $\eta^* = \max(\tau^- \cap s(p'))$.

Extend p' further, by applying repeatedly Lemma 2.30, in order to achieve the following:

- for every $\xi \in s(p') \cap (\tau, \theta) \cap M \cap Regular$, there is a model B on the wide piste of p' of cardinality ξ such that $M \subseteq B \subseteq N$.

In particular, $\sup(M \cap \theta^+) = \sup(B \cap \theta^+) = N \cap \theta^+$. Denote such B by M_ξ.
Let us denote such extension of p' still by the same letter p'.

Pick now $A \preceq H(\theta^+)$ which satisfies the following:

1. $|A| = \eta^*$,

2. $A \supseteq \eta^* + 1$,

3. $A \cap \eta^{*+}$ is an ordinal,

4. ${}^{\mathrm{cof}(\eta^*)>}A \subseteq A$,

5. $p' \in A$.
   In particular every model of p' belongs to A.

Extend p' to p'' by adding A as the new largest model of cardinality η^*. i.e. $p'' = p'^\frown A$, if η^* is a regular cardinal. If η^* is a singular cardinal (and, then by its definition, $\mathrm{cof}(\eta^*) < \delta$), then we add an increasing under inclusion and cardinality sequence of models with limit A instead, which correspond to a splitting point of type 2. Namely, fix an increasing sequence of regular cardinals $\langle \eta_\alpha \mid \alpha < \mathrm{cof}(\eta^*) \rangle$ cofinal in η^* and with $\eta_0 \geq \eta$. Pick a sequence of elementary submodels $\langle K_\alpha \mid \alpha < \mathrm{cof}(\eta^*) \rangle$ of $H(\theta^+)$ such that for every $\alpha < \mathrm{cof}(\eta^*)$ the following holds:

1. $|K_\alpha| = \eta_\alpha$,

[20] If $\tau = \eta$, then there will be no need to form a $\Delta-$system triple at the end. So the cardinality of the largest model does not actually matters in this case.

2. $K_\alpha \supseteq \eta_\alpha + 1$,

3. $K_\alpha \cap \eta_\alpha^+$ is an ordinal,

4. ${}^{\eta_\alpha >}K_\alpha \subseteq K_\alpha$,

5. $p' \in K_0$.

6. $\alpha < \beta$ implies $K_\alpha \in K_\beta$, and so, $K_\alpha \subseteq K_\beta$,

7. $A = \bigcup_{\alpha < \mathrm{cof}(\eta^*)} K_\alpha$.

Add $\langle K_\alpha \mid \alpha < \mathrm{cof}(\eta^*)\rangle$ to p'. Denote the result by p''. So, $K_\alpha = A^{0\eta_\alpha}(p'')$, for every $\alpha < \mathrm{cof}(\eta^*) < \delta$.

Let us reflect A down to $\mathfrak{M}$. Do it as follows.
If $\tau = \eta$, then let $i_M < \tau$ be large enough such that for every model U in p'', if U is below M, i.e. $\sup(U \cap \theta^+) < \sup(M \cap \theta^+)$, then U is already below M_{i_M}, and if $U \in M$, then $U \in M_{i_M}$.
We pick some A' and q in $M_{i_M+1} \setminus M_{i_M}$ which realize the same k-type (for some $k < \omega$ sufficiently big) over $\{U \in M \mid U \text{ is a model in } p''\}$ as A and p''.[21] Do this reflection in a rich enough language which includes $\bar{D}$ as well. [22]
In particular $q \in \bar{D} \cap M$.

Suppose that $\tau > \eta$.
Let $i_M < \tau$ be large enough such that

[21]The meaning is that A' and q satisfy the same formulas in $\mathfrak{M} \cap H(\theta^{+k})$ as A does with parameters from the set $W := \{U \in M \mid U \text{ is a model in } p''\}$ in $H(\theta^{+k})$.
Namely, consider the set

$$Z = \{\langle \ulcorner \varphi(v_0, v_1 ..., v_{n+2}) \urcorner, x_1, ..., x_n\rangle \mid n < \omega, \varphi(v_0, v_1 ..., v_{n+2}) \text{ is a formula in the language of set theory },$$

$$x_1, ..., x_n \in W \cup \{M_{i_M}, \bar{D}\} \text{ and } H(\theta^{+k}) \models \varphi(A, p'', x_1, ..., x_n)\},$$

where $\ulcorner \varphi(v_0, v_1 ..., v_{n+2}) \urcorner$ denotes the Gödel number of a formula $\varphi(v_0, v_1 ..., v_{n+2})$. Then $|Z| < \eta$, and so $Z \in M_{i_M+1}$, due to the closure of this model. Now,

$$H(\theta^{+k}) \models (\exists v_0 \exists v_1 \forall \langle \ulcorner \varphi(v_0, v_1 ..., v_{n+2}) \urcorner, x_1, ..., x_n\rangle \in Z)\varphi(v_0, v_1, x_1, ..., x_n).$$

Then, $\mathfrak{M}_{i_M+1} \cap H(\theta^{+k}) \preceq H(\theta^{+k})$ implies that there are $A, q \in M_{i_M+1} \setminus M_{i_M}$ such that

$$\mathfrak{M}_{i_M+1} \cap H(\theta^{+k}) \models (\forall \langle \ulcorner \varphi(v_0, v_1 ..., v_{n+2}) \urcorner, x_1, ..., x_n\rangle \in Z)\varphi(A', q, x_1, ..., x_n).$$

[22]We follow here a suggestion by Carmi Merimovich to include $\bar{D}$ into the language which simplifies the original argument considerably.

- for every model U in p'', if U is below M, i.e. $\sup(U \cap \theta^+) < \sup(M \cap \theta^+)$, then U is already below M_{i_M}, and if $U \in M$, then $U \in M_{i_M}$;
- for every $\rho \in s(p'') \cup \{\eta^*\}$,
  $\sup(A \cap \sup(M \cap \rho)) < \sup(M_{i_M} \cap \rho)$.
  Note that $\mathrm{cof}(\sup(M \cap \rho) = |M| = \tau$, since M is closed under $< \tau-$sequences of its elements, and so, $\sup(A \cap \sup(M \cap \rho)) < \sup(M \cap \rho)$.

Pick some A' and q in $M_{i_M+1} \setminus M_{i_M}$ which realize the same k-type (for some $k < \omega$ sufficiently big) over $\langle A \cap M, M_{i_M} \rangle$ as A and p''. Do this reflection in a rich enough language which includes $\bar{D}$ as well.
In particular $q \in \bar{D} \cap M$.

Note that if $\theta > \aleph_{\eta^+}$, then models of cardinalities in $s(p'') \setminus M$ (if there are such models) reflect to models in M of cardinalities in $M \setminus s(p'')$, or even in $(M_{i_M+1} \setminus M_{i_M}) \setminus s(p'')$, if $\tau > \eta$. If $\tau = \eta$, then every model G in p'' which is above M contains M, and hence, all cardinals added this way will be inside G. If $\tau > \eta$, then there may be models in p'' above M of cardinality smaller than τ. If G is such a model, then it may have a cardinal $\rho \in s(p'') \setminus M$, but ρ will reflect to some $\rho' \in M_{i_M+1} \setminus M_{i_M}$ which cannot be in G by the choice of i_M.

Extend p'' further, by applying repeatedly Lemma 2.30, in order to achieve the following:

- for every regular cardinal $\xi \in (s(p'') \cup s(q)) \cap M$, there is a model B on the wide piste of p'' of cardinality ξ such that $M \subseteq B \subseteq N$.

In particular, $\sup(M \cap \theta^+) = \sup(B \cap \theta^+) = N \cap \theta^+$. Denote such B by M_ξ. Note that $q \in M$, so $s(q) \in M$ and $s(q) \subseteq M$. Also, if $\theta < \aleph_{\eta^+}$, then $s(q) = s(p'')$. However, if $\theta > \aleph_{\eta^+}$, then the reflection process may add cardinals to $s(p'')$. All of this "new" cardinals come from M.
Let us denote such extension of p'' still by p''.

Now let us add to p'' more models. They will be of a form $cl((M \cap E) \cup |E|)$, where $E \in M$ is already in p'' and $\mathrm{cof}(\sup(E \cap \theta^+)) > |M|$.
The process will be similar to those used in Lemma 2.30.

Sublemma 2.32 *Let $E \in M$ be a model in $C^{\xi lim}(A^{0\xi}(p''))$ such that*

1. $\mathrm{cof}(\sup(E \cap \theta^+)) > |M|$
2. *there $E' \in C^{\xi}(A^{0\xi}(p'')) \cap E$, $E' \notin M$ and E is the least model of $C^{\xi}(A^{0\xi}(p'')) \cap M$ above E'.*

Then it is possible to add $cl((M \cap E) \cup |E|)$ to p''.

Proof. Deal with models on the wide piste of p''. Assume that $\theta < \aleph_{\eta^+}$. The treatment of the case $\theta > \aleph_{\eta^+}$ is similar.

Let B be a model on wide piste of p'' and suppose that B is above $cl((M \cap E) \cup |E|)$ and $cl((M \cap E) \cup |E|) \notin B$.

Case 1. B ***above*** M***.***

Let M^* be M if $M \in B$ and the least cover of it otherwise, i.e. $M^* \in B \cap C^{|M|}(A^{0|M|}(p''))$, $B \cap M^* \subseteq M$.

Let $\langle E_i \mid i < \mathrm{cof}(\sup(E \cap \theta^+))\rangle$ be an $E-$sequence.

Let $i_M = \sup(M \cap \mathrm{cof}(\sup(E \cap \theta^+)))$.

Then $E_{i_M} = cl((M \cap E) \cup |E|)$, by Lemma 2.21.

Subcase 1.1. $E \in B$***.***

Set $i_{M^*} = \sup(M^* \cap \mathrm{cof}(\sup(E \cap \theta^+)))$.

Then $E_{i_{M^*}} = cl((M^* \cap E) \cup |E|)$, by Lemma 2.21 with Z replaced by M^*. Also, $M^*, E \in B$ implies that $E_{i_{M^*}} \in B$. Now, $B \cap M^* \subseteq M$ implies that $B \cap E_{i_{M^*}} \subseteq E_{i_M}$. Suppose otherwise. Then there is $x \in (B \cap E_{i_{M^*}}) \setminus E_{i_M}$. Then the least $i, i_M < i < i_{M^*}$ is in B. We have $M^* \in B$ and $M^* \cap i_{M^*}$ is unbounded in i_{M^*}. So, there is $i' \in (B \cap M^* \cap i^*) \setminus i$, which is impossible since $B \cap M^* \subseteq M$ and now such i' in M.

Subcase 1.2. $E \notin B$***.***

Let $E^* \in B \cap C^{|E|}(A^{0|E|}(p''), E^* \cap B \subseteq E$ be the cover of E in B.

Consider the model $cl((M^* \cap E^*) \cup |E|)$. It is in B, since $M^*, E^* \in B$.[23]

Let us argue that $B \cap cl((M^* \cap E^*) \cup |E|) \subseteq E_{i_M}$.

Assume that $x \in cl((M^* \cap E^*) \cup |E|) \cap B$. Then there are a Skolem function h, $a \in M^* \cap E^*$ and $\nu < |E|$ such that $x = h(a, \nu)$. We have $M^*, E^* \in B$, hence by elementarity of B, there such $a, \nu \in B$.

So, $a \in B \cap M^* \cap E^* = (B \cap M^*) \cap (B \cap E^*) \subseteq M \cap E$.

Then $x = h(a, \nu) \in cl((M \cap E) \cup |E|) = E_{i_M}$.

Case 2. B ***below*** M***.***

Split into three subcases.

Subcase 2.1. $B \in M$ ***and*** $|B| < \mathrm{cof}(\sup(E \cap \theta^+))$***.***

Then $B \cap E \in M$ and, also, $B \cap E \in E$. Let $E^* \in B \cap C^{|E|}(A^{0|E|}(p''))$, $E^* \cap B \subseteq E$ be the cover of E in B.

Then $B \cap E^* = B \cap E \subseteq cl((M \cap E) \cup |E|) = E_{i_M}$.

[23]Note that $cl((M^* \cap E^*) \cup |E|)$ can be E^*. For example, if $E^* \in M^*$ and $\mathrm{cof}(\sup(E^* \cap \theta^+)) \leq |M^*|$.

Subcase 2.2. $B \notin M$ ***and*** $|B| < \mathrm{cof}(\sup(E \cap \theta^+))$.
Let $B^* \in M \cap C^{|B|}(A^{0|B|}(p'')), B^* \cap M \subseteq B$ be the cover of B in M.
Let $E^* \in B^* \cap C^{|E|}(A^{0|E|}(p'')), E^* \cap B^* \subseteq E$ be the cover of E in B^*.
Then $B^* \cap E^* = B^* \cap E \subseteq cl((M \cap E) \cup |E|)$, by the previous subcase.
Let $\tilde{E} \in B \cap C^{|E|}(A^{0|E|}(p'')), \tilde{E} \cap B \subseteq E$ be the cover of E in B.
Clearly, $B \cap \tilde{E} = B \cap E \subseteq B^* \cap E = B^* \cap E^* \subseteq cl((M \cap E) \cup |E|) = E_{i_M}$.

Subcase 2.3. $|B| \geq \mathrm{cof}(\sup(E \cap \theta^+))$.
If $E \in B$, then $\{E_i \mid i < \mathrm{cof}(\sup(E \cap \theta^+))\} \subseteq B$, and hence $E_{i_M} \in B$.
Suppose that $E \notin B$. Let $E^* \in B \cap C^{|E|}(A^{0|E|}(p'')), E^* \cap B \subseteq E$ be the cover of E in B.
Let $B^* = B$, if $B \in M$ and $B^* \in M \cap C^{|B|}(A^{0|B|}(p'')), B^* \cap M \subseteq B$ be the cover of B in M, if $B \notin M$.
Let also $E^*_{B^*} \in B^* \cap C^{|E|}(A^{0|E|}(p'')), E^*_{B^*} \cap B^* \subseteq E$ be the cover of E in B^*.
Then, by Definition 2.1 (16), $E^*_{B^*} \in M$. So, $E^*_{B^*} \in B^* \cap M \subseteq B$. We have $\mathrm{cof}(\sup(E^*_{B^*} \cap \theta^+)) > |B^*| = |B|$. Let $\langle E_i^{*B^*} \mid i < \mathrm{cof}(\sup(E^*_{B^*} \cap \theta^+))\rangle$ be an $E^*_{B^*}-$sequence.
Let $i_{B^*} = \sup(B^* \cap \mathrm{cof}(\sup(E^*_{B^*} \cap \theta^+)))$.
Then $E^{*B^*}_{i^*_B} \in M$ and below E. Hence, $E^{*B^*}_{i^*_B} \in E_{i_M}$.
Let now $\tilde{E} \in B \cap C^{|E|}(A^{0|E|}(p'')), \tilde{E} \cap B \subseteq E$ be the cover of E in B.
Then $B \cap \tilde{E} = B \cap E \subseteq B^* \cap E^*_{B^*} \subseteq E_{i_M}$.
$\square$ of Case 2.

Note that only in Case 1 we may need to add new models and they are of the form $cl((M^* \cap E^*) \cup |E^*|$ with $M^*, E^* \in B$.

The rest is as in Lemma 2.30.
$\square$ of sublemma.

Extend p'' further by applying repeatedly Sublemma 2.32 and adding models of the form $cl((M \cap E) \cup |E|)$.
Let us argue that q is compatible with p''.

If $\tau = \eta$, then we just combine p'' with q.
If $\tau > \eta$, then we add more models at the top in the following fashion.
Set $s = s(q) \cup s(p'')$. Let $\langle \xi_i \mid i < i^*\rangle$ be an increasing enumeration of s. Pick an $\in -$increasing sequence of models $\langle A_i \mid i < i^*\rangle$ such that for every $i < i^*$ the following hold:

1. $p'', q \in A_i$,
2. $|A_i| = \xi_i$,
3. A_i satisfies 2.3(2).

Set $A^{0\xi_i} = A_i$.
Finally let for every $\xi \in s$,

$$A^{1\xi} = \{A^{0\xi}\} \cup A^{1\xi}(p'') \cup A^{1\xi}(q).$$

Define $A^{1\xi lim}$ and $C^{\xi}(\xi \in s)$ in the obvious fashion now, but do not make $A^{0\xi}, \xi \in s$ potentially limit.

Set

$$p^* = \langle\langle A^{0\xi}, A^{1\xi}, A^{1\xi lim}, C^{\xi}\rangle \mid \xi \in s\rangle.$$

Then, in p^*, the triple $(A^{0\eta^*}, A^{0\eta^*}(p''), A^{0\eta^*}(q))$ will form a Δ–system triple relative to M and to the model which corresponds to M under the reflection, provided that η^* is a regular cardinal.
If η^* is a singular cardinal, then we have here a splitting point of type 2.

Let us check that the wide piste of p^* satisfies Definition 2.1. Suppose that it goes through $C^{\xi}(A^{0\xi})(p'')$, for each $\xi \in s \cap \tau$, i.e. via the part before the reflection.

Assume that we have two models B and E on the wide piste of p^*, with B above E.

Case 1. *$B \in C^{\rho}(A^{0\rho})$ is above M (i.e.* $\sup(B \cap \theta^+) > \sup(M \cap \theta^+)$*).*
If B is $A^{0\rho}$, then $p'', q \in B$, and so, every model which is below M is in B.
Suppose that $B \neq A^{0\rho}$. Then B is in p''.

Subcase 1.1 $\rho \in s \setminus \tau$.
By Definition 2.1(12(a)), for p'', we have $M \in B$. Hence, all models added by reflection are in B as well. In addition, by Definition 2.1(15), for p'', we have $N \in B$. So, by Definition 2.1(6), B cannot be minimal in $C^{\rho}(A^{0\rho})(p'')$, unless it is a potentially limit model. In addition, the least B on $C^{\rho}(A^{0\rho})(p'')$ which contains M should be a potentially limit point. So, adding new models of size ρ below M is legitimate.

Subcase 1.2. $\rho \in s \cap \tau$.
Then B is among models of p'' that reflect down to M.
Also, by the choice of A, $B \subseteq A$ and if $B \neq A$, then $B \in A$.
Suppose now that $E \in C^{\xi}(A^{0\xi})$ is below B. Assume that E does not appear in p''. Then E is below M and it is the image of a model of p'' under the reflection.
If E is on the wide piste of p^*, then $\xi \geq \tau$. Then there is a model E_{ξ} on the wide piste of p'' of cardinality ξ such that $M \subseteq E_{\xi} \subseteq N$. Clearly, $E \subseteq E_{\xi}$ and $E \in E_{\xi}$.
Let us show that the covering condition of Definition 2.1 hold for B and E. Let E' be a model of p'' which is the pre-image of E under the reflection. Then $|E'| \geq |E|$, since

$\min(M \cap \theta^+ \setminus |E'|)$ does not move by the the reflection because it is an element of $M \cap A$, in addition, $|E'| < \min(M \cap \theta^+ \setminus |E'|)$ implies that $|E| < \min(M \cap \theta^+ \setminus |E'|)$, but $|E| \in M$.

Subsubcase 1.2.1. $\theta < \aleph_{\eta^+}$.

Then $|E| = |E'|$, since $|E'| \in M$.

Let $E'' \in B \cap C^{\xi}(A^{0\xi}(p^*))$ be the least such that $E'' \supseteq E$. Then E'' appears already in p'', since E'' not in p'' implies that it is an image under the reflection. In particular, $E'' \in M$. So, $E'' \in B \cap M \subseteq A \cap M$, but then E'' does not move.

Suppose first that $E' \supseteq E_\xi$.

Then $E' \supseteq E_\xi \supset E$.

Note that if $B = A$, then E'' will be E_ξ.

If $E'' \supset E_\xi$, then E'' witnesses (20) of Definition 2.1 for B and E_ξ inside p''. Hence, $B \cap E'' \subseteq E_\xi$. If $E'' \not\supset E_\xi$, then $E'' \subseteq E_\xi$, and so, $B \cap E'' \subseteq E_\xi$.

Let us show that then $B \cap E'' \subseteq E$.

Set $M^* = M$, if $M \in B$, and let otherwise M^* witnesses (20) of Definition 2.1 for B and M inside p''. In particular, $B \cap M^* = B \cap M$.

Now, $B \cap M \subseteq M$, and hence, $B \cap M \in M$, since ${}^{|M|>}M \subseteq M$. Then $B \cap M \in E_\xi \subseteq E'$. Let us argue that $B \cap M \in A \cap M$. It is trivial $A = B$. If $B \neq A$, then $B \in A$. Also, $M \in A$, and so, $B \cap M \in A \cap M$.

Then $B \cap M \subseteq A \cap M$, since $|B| \leq |A|$. So, $B \cap M \in E$, since E' was reflected to E.

Suppose now that $x \in B \cap E''$. Then $x \in E_\xi$, and hence, $x = h(a, \nu)$, for some Skolem function h, $a \in M$ and an ordinal $\nu \leq \xi$. Pick $a \in M^*$ to be the least possible such that for some $\nu < \xi$ we have $x = h(a, \nu)$. Then $a, \nu \in B$, since $x, M^* \in B$ and $B \preceq H(\theta^+)$, where $\nu < \xi$ is the least possible such that $x = h(a, \nu)$. We have $B \cap M^* = B \cap M$, hence $a \in B \cap M$.

Then, $a \in E$. So, $x = h(a, \nu) \in E$ as well, and we are done.

Suppose now that $E' \not\supseteq E_\xi$**.**

Then $E' \in E_\xi$. Note that $E' \notin M$, since otherwise $E' \in M \cap A$, and so, it does not move by the reflection. Let $E^* \in M$ be the covering model of E' (exists by (20) of Definition 2.1 for M and E' inside p''). We have $\mathrm{cof}(\sup(E^* \cap \theta^+)) > |M|$, since $M \cap E^* \subseteq E'$. Also, $E^* \in M \cap A$, and hence, it does not move by the reflection.

Consider $E^*_\xi := cl((M \cap E^*) \cup \xi)$. Then $E' \supseteq E^*_\xi \supseteq E$.

Let us show that then $B \cap E'' \subseteq E$.

As above, set $M^* = M$, if $M \in B$, and let otherwise M^* witnesses (20) of Definition 2.1 for B and M inside p''. In particular, $B \cap M^* = B \cap M$.

Now, $B\cap M\cap E^*_\xi \subseteq M$, and hence, $B\cap M\cap E^*_\xi \in M$, since ${}^{|M|>}M \subseteq M$. Also, $B\cap M\cap E^*_\xi \subseteq E^*$, and hence, $B\cap M\cap E^*_\xi \in E^*$, since ${}^{|M|>}E^* \subseteq E^*$. Then $B\cap M\cap E^*_\xi \in M\cap E^* \subseteq E^*_\xi \subseteq E'$. We have $B, M, E^*_\xi \in A$, and so, $B\cap M\cap E^*_\xi \in A\cap M$. Then $B\cap M\cap E^*_\xi \subseteq A\cap M$, since $|B| \leq |A|$. So, $B\cap M\cap E^*_\xi \in E$, since E' was reflected to E.

We use here that E^*_ξ is in p'', as one of the models that were inside the original p'' or one the models added above. Hence, $B\cap E'' \subseteq E^*_\xi$.

Suppose now that $x \in B\cap E''$. Then $x \in E^*_\xi$, and hence, $x = h(a,\nu)$, for some Skolem function h, $a \in M\cap E^*_\xi$ and an ordinal $\nu \leq \xi$.

Set $E^{**}_\xi \in B$ to be E^*_ξ, if $E^*_\xi \in B$ and its covering model otherwise. We have then, $B\cap E^{**}_\xi = B\cap E^*_\xi$. Pick $a \in M^*\cap E^{**}_\xi$ to be the least possible such that for some $\nu < \xi$ we have $x = h(a,\nu)$. Then $a,\nu \in B$, since $x, M^*, E^{**}_\xi \in B$ and $B \preceq H(\theta^+)$, where $\nu < \xi$ is the least possible such that $x = h(a,\nu)$. We have $B\cap M^*\cap E^{**}_\xi = B\cap M\cap E^*_\xi$, hence $a \in B\cap M\cap E^*_\xi$ Then, $a \in E$. So, $x = h(a,\nu) \in E$ as well, and we are done.

Subsubcase 1.2.2. $\theta > \aleph_{\eta^+}$.

The argument is similar to those of Subcase 2.1. Let us deal only with a new possibilities that can occur now.

Thus suppose that $|E'| \neq |E|$. Then $|E'| \notin M$ and it reflects to $|E| \in M$.

Suppose first that B ***and*** E' ***are above*** M***.***

We have $|B| < |M| = \tau$ and by the choice of reflection then $|E| \notin B$. Denote $|E|$ by ξ. Let $\xi_B = \min((B\cap\theta^+)\setminus\xi)$.

Let M^B be M, if $M \in B$, or the cover of M in B otherwise. Similar, let N^B be N, if $N \in B$, or the cover of N in B otherwise. By the choice of M, N we have $\sup(M\cap\theta^+) = N\cap\theta^+$. If $M \notin B$, then by the first claim of Lemma 2.29, still $\sup(M^B\cap\theta^+) = N^B\cap\theta^+$.

Consider $cl(M^B\cup\xi_B)$. This model belongs to B, since it is just the smallest elementary submodel of N^B which includes M^B and ξ_B.

Let us argue that $B\cap cl(M^B\cup\xi_B) \subseteq E$.

Let x be in $B\cap cl(M^B\cup\xi_B)$. Then for some Skolem function h, $a \in B\cap M^B$ and $\nu \in B\cap\xi_B$, we have $x = h(a,\nu)$. Remember that $B\cap M^B \subseteq M$ and $B\cap\xi_B \subseteq \xi$.

Now, E' is above M, so $E' \supseteq M \supseteq M\cap B$. In addition $M\cap B \in M$, since $|B| < \tau$ and M is closed under $<\tau-$sequences of its elements. $B\cap M \subseteq A\cap M$ and so it does not move under the reflection. Hence, $E' \supseteq M\cap B$ implies $E \supseteq M\cap B$. In particular, $a \in E$. Also, $\nu \in E$, since $E \supseteq \xi$. Then $x = h(a,\nu)$ in E as well, and we are done.

Suppose now that B ***is above*** M***, but*** E' ***is below*** M***.***

$E' \notin M$, since otherwise it does not move under the reflection, i.e. $E' = E$. Let E^* be the

cover of E' in M. As above, denote $|E|$ by ξ and let $\xi_B = \min((B \cap \theta^+) \setminus \xi)$.
Define M^B, N^B as above.
Let $E^B \in B$ be the cover of E'.
We claim that $B \cap cl((E^B \cap M^B) \cup \xi_B) \subseteq E$.
Let x be in $B \cap cl(E^B \cap M^B \cup \xi_B)$. Then for some Skolem function h, $a \in B \cap E^B \cap M^B$ and $\nu \in B \cap \xi_B$, we have $x = h(a, \nu)$.
Now, $B \cap \xi_B \subseteq \xi$, $B \cap E^B \cap M^B = (B \cap E^B) \cap (B \cap M^B) = (B \cap E') \cap (B \cap M) = B \cap (E' \cap M) = E^* \cap M \cap B \subseteq E'$. Both E^* and $M \cap B$ do not move under the reflection. Hence, their intersection does not move. So, $E^* \cap M \cap B \subseteq E'$ implies that $E^* \cap M \cap B \subseteq E$. Then $a \in E, \nu \in E$. Hence $x = h(a, \nu) \in E$, and we are done.

Case 2. *B is below M, i.e.* $\sup(B \cap \theta^+) < \sup(M \cap \theta^+)$.
Let $B^* = B$, if $B \in M$, and B^* be the covering model in M of B otherwise. Then $B^* \in M \cap A$, and so, it does not move by the reflection. Hence $\sup(E \cap \theta^+) < \sup(B \cap \theta^+) \leq \sup(B^* \cap \theta^+)$ implies that $\sup(E' \cap \theta^+) < \sup(B^* \cap \theta^+)$.
Let $E'^* \in B^*$ be either E', if $E' \in B^*$ or the covering model in B^* of E' otherwise.
Let E'^{**} be the image of E'^* under the reflection.
Then $E'^{**} \in M \cap B^* \subseteq B$. By reflection, E'^{**} is either E, if $E \in B^*$ or the covering model in B^* of E otherwise. So, $E \supseteq E'^{**} \cap B^*$. Hence, $E \supseteq E'^{**} \cap B$, and then, E'^{**} works for B and E as well.

Let us argue that the covering property holds for models $\tilde{B}$ and $\tilde{D}$ on the wide piste such that $\tilde{B}$ is a reflection to M of a model B in p'' and $\tilde{D}$ in p'' is below $\tilde{B}$.
If $\tilde{D} \in M$, then this is clear, since both $\tilde{B}, \tilde{D}$ will be then inside the same condition q.
So, assume that $\tilde{D} \notin M$.
Apply the covering properties (19), if $\theta < \aleph_{\eta^+}$, or (21), if $\theta > \aleph_{\eta^+}$, of Definition 2.1 to M and $\tilde{D}$. Let $\tilde{E} \in M$ be the witnessing model. Split into two cases.

Case 1. *$\tilde{E}$ is a closed model, i.e.* ${}^{|\tilde{E}|>}\tilde{E} \subseteq \tilde{E}$.
Then $\sup(\tilde{B} \cap |\tilde{E}|) < \sup(M \cap |\tilde{E}|) < |\tilde{E}|$.
Let $\langle \tilde{E}_i \mid i < |\tilde{E}| \rangle$ be an $\tilde{E}-$sequence. Then $M \cap \tilde{E} = M \cap \tilde{D}$ implies that $\tilde{E}_{\sup(M \cap |\tilde{E}|)} \subseteq \tilde{D}$, since $\langle \tilde{E}_i \mid i < |\tilde{E}| \rangle$ is definable from $\tilde{E}$, and so, in M, in addition, $\tilde{E}_{\sup(M \cap |\tilde{E}|)} = \bigcup_{i \in M \cap |\tilde{E}|} \tilde{E}_i$.
Now, if $\tilde{E} \in \tilde{B}$, then $\tilde{B} \cap \tilde{E} \subseteq \bigcup_{i \in \tilde{B} \cap |\tilde{E}|} \tilde{E}_i = \bigcup_{i < \sup(\tilde{B} \cap |\tilde{E}|)} \tilde{E}_i \subseteq \tilde{E}_{\sup(M \cap |\tilde{E}|)} \subseteq \tilde{D}$.
If $\tilde{E} \notin \tilde{B}$, then still there is $i < |\tilde{E}|$ such that $\tilde{B} \cap \tilde{E} \subseteq \tilde{E}_i$. Let i^* be the least such i. Then, by elementarity, $\tilde{B}, \tilde{E} \in M$ imply $i^* \in M$. So, $\tilde{B} \cap \tilde{E} \subseteq \tilde{E}_{\sup(M \cap |\tilde{E}|)} \subseteq \tilde{D}$.
Both $\tilde{B}, \tilde{E}$ are in M, hence there is a cover $\tilde{E}^*$ of $\tilde{E}$ in $\tilde{B} \cap M$.
Then $\tilde{B} \cap \tilde{E}^* = \tilde{B} \cap \tilde{E} \subseteq \tilde{D}$, and we are done.

Case 2. *$\tilde{E}$ is not a closed model.*

Suppose first that $\theta < \aleph_{\eta^+}$.

Then we apply (c) of (20) of Definition 2.1 to M and $\tilde{D}$. Let $n^* < \omega$, $\langle G^n \mid n < n^* \rangle$, $\langle K^n \mid n \leq n^* \rangle$ be the witnesses.

Suppose for simplicity that $n^* = 2$.

We have $K^0 = \tilde{E}$, $G^0 \in M$, and either $K^0 = cl(G^0 \cup |\tilde{E}|)$ and ${}^{|G^0|>}G^0 \subseteq G^0$, or $K^0 = cl((\tilde{G'}^0 \cap \tilde{F'}^0) \cup |\tilde{E}|)$, for some models $\tilde{G'}^0, \tilde{F'}^0 \in M$ such that G^0 is a core of $\tilde{G'}^0$. Note that in both cases $\mathrm{cof}(K^0 \cap \theta^+) = |G_0|$, since $\mathrm{cof}(\sup(K^0 \cap \theta^+)) = \mathrm{cof}(\sup(G^0 \cap \theta^+))$ or $\mathrm{cof}(\sup(K^0 \cap \theta^+)) = \mathrm{cof}(\sup((\tilde{G'}^0 \cap \tilde{F'}^0) \cap \theta^+))$.

We have $\tilde{B}$ above $\tilde{D}$, so G^n, K^n's will be below $\tilde{B}$ as well.

Now, if $|\tilde{B}| \geq |G^0|, |G^1|$, then $\tilde{B} \supseteq G^0, G^1$, and so, $\tilde{D} \in \tilde{B}$.

If $|\tilde{B}| < |G^0|$, we proceed as in Case 1.

Suppose that $|\tilde{B}| \geq |G^0|$ and $|\tilde{B}| < |G^1|$.

Then $G^1 \in \tilde{B}$, since $G^1 \in G^0 \subseteq \tilde{B}$.

Let $\langle G^1_i \mid i < |G^1| \rangle$ be a G^1-sequence. Then it belongs to $\tilde{B}$.

Let $i_{\tilde{B}} = \sup(\tilde{B} \cap |G^1|)$. Then $i_{\tilde{B}} < i_M = \sup(M \cap |G^1|)$, since $\tilde{B} \in M$ and $|\tilde{B}| < |G^1|$.

Now, $K^2 = \tilde{D} \in G^1 \setminus G^1_{i_M+1}$, and so, $K^2 = \tilde{D} \in G^1 \setminus G^1_{i_{\tilde{B}}+1}$.

Hence, the sequences $\langle G^0, G^1 \rangle, \langle K^0, K^1, K^2 \rangle$ witness (c) of (20) of Definition 2.1 for $\tilde{B}$ and $\tilde{D}$.

Suppose now that $\theta > \aleph_{\eta^+}$.

The argument is similar to the one above, only we apply (22) of Definition 2.1 instead of (20). The crucial thing is that again we have $i_{\tilde{B}} < i_M$ in the main case of the argument.

Let us turn to Definition 2.3. The only non-trivial thing to check here is what happens if we change the wide piste to the one that replaces the part of p^* that was reflected by its reflection, according to 2.3(9) .

So, suppose that such switching between the reflecting part and its reflection was made. We need to argue that the result still satisfies Definition 2.1. The issue is the covering. Namely, the conditions (19)- (22) of Definition 2.1.

We need to deal with images of models which appear in q under isomorphisms of models in p'' and with images of models which appear in p'' under isomorphisms of models in q. Split into two cases. Deal first with images of models which appear in q under an isomorphism of models in p''.

Let U denote a model like this.

Case 1. *There is a splitting point X with immediate predecessors X_0, X_1 inside p'' and*

a model B inside q such that $B \in X_0$ and $U = \pi_{X_0X_1}(B)$.
Without loss of generality, assume that $X \notin M$.

Subcase 1.1 $\sup(X_0 \cap \theta^+) > \sup(M \cap \theta^+)$ and $|X| < |M|$.
Then $|M| = \tau > \eta$. By the choice of the model A in the beginning of the proof, $A \supseteq X$. Then $X_0 \cap M \subseteq A \cap M$. So $X_0 \cap M$ does not move by the reflection, moreover the same is true for any of its members. In particular, if B' is the model that reflects to B, $B \in X_0 \cap M$ implies $B' \in X_0 \cap M$, and so B' does not move by the reflection. This means that $B' = B$ and so B in p''. But then also $U = \pi_{X_0X_1}(B)$ in p'', and we are done.

Subcase 1.2 $\sup(X_0 \cap \theta^+) < \sup(M \cap \theta^+)$ and $|X| < |M|$.
Still $|M| = \tau > \eta$, and so $A \supseteq X$.
$|X| < |M|$ implies that $X_0 \subseteq M$. Hence, $X_0 \subseteq M \cap A$.
Now we proceed as in the previous subcase and deduce that B is its own pre-image under the reflection. This implies that B and U are both in p''.

Subcase 1.3 $\sup(X \cap \theta^+) > \sup(M \cap \theta^+)$ and $|X| \geq |M|$.
Then $X \supseteq M$, provided that X is on the wide piste.
Then by Definition 2.1(4) $M \subseteq X_0$ or $M \subseteq X_1$, since M is a potentially limit point. Now, we just replace M by $\pi_{X_0X_1}(M)$ and proceed as above.[24]

It is not hard using switching above X to move the wide piste such that the model X will be on it. M may be moved by this process to an isomorphic model M' inside p', but A does not move. Replace M by M' and proceed as above.

Subcase 1.4 $\sup(X \cap \theta^+) = \sup(M \cap \theta^+)$.
It is impossible, since then X must be a potentially limit point.

Subcase 1.5 $\sup(X \cap \theta^+) < \sup(M \cap \theta^+)$.

Suppose first that $\theta < \aleph_{\eta^+}$.
Apply the covering property (19) of Definition 2.1 and pick the cover $X^* \in M \cap A^{1|X|}(p'')$.
Making switches (below M), if necessary, we may assume that X^*, X, X_0 are on the wide piste of p'.
Note that then by Definition 2.1(8), if $X \in M$, then X_0 in M as well, and $X_0 \in M$ implies that $X \in M$ by Definition 2.1(17).
So, if $X^* \neq X$, then $M \cap X^* \subseteq X_0$.
Consider now the X_1-wide piste (see Definition 2.3).
It is obtained by applying one more switch which turns the wide piste from X_0 to X_1 leaving

[24] Just the wide piste switched to the one that goes through $X_1, \pi_{X_0X_1}(M)$ and the previous arguments apply to it.

everything from X above unchanged. By (9) of Definition 2.3 the X_1-wide piste is a wide piste, i.e. it satisfies Definition 2.1. In particular, X^* will remain as before for this new wide piste. So, if $X^* \neq X$, then $M \cap X^* \subseteq X_1$.

This means that either $X, X_0, X_1 \in M$, and then in q, or $M \cap X^* \subseteq X_0 \cap X_1$ and then $\pi_{X_0 X_1}(B) = B$.

Suppose that $\theta > \aleph_{\eta^+}$.

The argument is similar to the one used for the case $\theta < \aleph_{\eta^+}$, (21, 22) of Definition 2.1 are used instead of (19).

$\square$ of case 1.

The next case deals with the situation in which a model D is an image of a model which appears in p'' under an isomorphism of models in q.

Case 2. *There is a splitting point X with immediate predecessors X_0, X_1 inside q and a model Z inside p'' such that $Z \in X_0$ and $D = \pi_{X_0 X_1}(Z)$.*

We may assume that Z does not appear in q, since otherwise also D will be in q.

Let U be a model in p''. The principle case is when $X \in U$ and $|X| > |U|$.

Then X_0, X_1 and $\pi_{X_0 X_1}$ are in U. So, if $Z \in U$, then $D = \pi_{X_0 X_1}(Z) \in U$, as well. Assume that $Z \notin U$.

We showed already that the covering properties of Definition 2.1 for U and Z inside p^* hold.

Let Z^* be the cover in U of Z. Then $\sup(Z^* \cap \theta^+) \leq \sup(X_0 \cap \theta^+)$, since $\sup(X_0 \cap \theta^+) \in U$ and $U \cap Z^* \subseteq Z$.

Note that $|X_0| \geq \mathrm{cof}(\sup(Z^* \cap \theta^+))$, since $Z \in X_0 \in U$ and, if $|X_0| < \mathrm{cof}(\sup(Z^* \cap \theta^+))$, then $\sup(X_0 \cap Z^* \cap \theta^+) \in U \cap \sup(Z^* \cap \theta^+)$, and hence, $\sup(X_0 \cap Z^* \cap \theta^+) < \sup(U \cap Z^* \cap \theta^+) < \sup(Z \cap \theta^+)$, which is impossible.

Let $\langle Z_i \mid i < \mathrm{cof}(\sup(Z^* \cap \theta^+))\rangle$ be a Z^*-sequence.

Then $\langle Z_i \mid i < \mathrm{cof}(\sup(Z^* \cap \theta^+))\rangle \in U$. Let $i^* = \sup(\{i < \mathrm{cof}(\sup(Z^* \cap \theta^+)) \mid i \in U\})$. Clearly, i^* is a limit ordinal and $U \cap Z^* \subseteq Z_{i^*}$. Just if $x \in U \cap Z^* \setminus Z_{i^*}$, then the least $i' < i^*$ such that $x \in Z_{i'}$ is in U, and then $Z_{i'} \in U$ as well, but $i' > i^*$, which is impossible.

Also, we have $Z_{i^*} \subseteq Z$. Thus, for each $i < i^*$ there is $i' \in U, i < i' < i^*$. Then $Z_{i'} \in U$ as well. Hence, $Z_{i'} \in U \cap Z^* \subseteq Z$. So, $Z_{i'} \subseteq Z$. Then,

$$Z_{i^*} = \bigcup_{i<i^*} Z_i = \bigcup_{i'<i^*, i' \in U} Z_{i'} \subseteq Z.$$

Now, if $Z^* \in X_0$, then $\langle Z_i \mid i < \mathrm{cof}(\sup(Z^* \cap \theta^+))\rangle \in X_0$ and, hence,

$\{Z_i \mid i < \text{cof}(\sup(Z^* \cap \theta^+))\} \subseteq X_0$. So, $Z_{i^*} \in X_0$. Use $\pi_{X_0X_1}$ and move everything to X_1. We have $|X_1| = |X_0| \geq \text{cof}(\sup(Z^* \cap \theta^+))$, hence $\pi_{X_0X_1}(i) = i$, for every $i < \text{cof}(\sup(Z^* \cap \theta^+))$. Denote $\pi_{X_0X_1}(Z^*)$ by Z' and $\pi_{X_0X_1}(\langle Z_i \mid i < \text{cof}(\sup(Z^* \cap \theta^+))\rangle)$ by $\langle Z'_i \mid i < \text{cof}(\sup(Z^* \cap \theta^+))\rangle$.
Note that $U \cap Z' \subseteq Z'_{i^*}$, since otherwise there is $z \in U \cap Z', z \notin Z'_{i^*}$. Let i' be the least such that $z \in Z'_{i'}$. Then $i' > i^*$. In addition, $i' \in U$, since $Z' \in U$, and so, $\langle Z'_i \mid i < \text{cof}(\sup(Z^* \cap \theta^+))\rangle \in U$. But then also $Z_{i'} \in U$, since $\langle Z'_i \mid i < \text{cof}(\sup(Z^* \cap \theta^+))\rangle \in U$. This contradicts the choice of i^*.
Now,

$$Z' \in U \text{ and } U \cap Z' \subseteq Z'_{i^*} \subseteq \pi_{X_0X_1}(Z) = D.$$

So we are done.

Assume that $Z^* \notin X_0$.
Let Z^{**} be the cover of Z^* in X_0. Then $\text{cof}(Z^{**} \cap \theta^+) > |X_0|$. Also, $Z^{**} \in U$, since X_0 and Z^* are in U.
Let $\langle Z_i^{**} \mid i < \text{cof}(\sup(Z^{**} \cap \theta^+))\rangle$ be a $Z^{**}-$ sequence. Then $\langle Z_i^{**} \mid i < \text{cof}(\sup(Z^{**} \cap \theta^+))\rangle \in U \cap X_0$. Let $i^{**} = \sup(\{i < \text{cof}(\sup(Z^{**} \cap \theta^+)) \mid i \in X_0\})$. Clearly, i^{**} is a limit ordinal and $X_0 \cap Z^{**} \subseteq Z_{i^*}$.
Note that $\text{cof}(i^{**}) = |X^0|$, since X_0 is closed under $< |X_0|-$sequences of its elements.
We have $\sup(Z^{**}_{i^{**}} \cap \theta^+) \leq \sup(Z^* \cap \theta^+)$, since $X_0 \cap Z^{**} \subseteq Z^*$. Also, $\sup(Z^{**}_{i^{**}} \cap \theta^+) > \sup(Z \cap \theta^+)$, since $Z \in X_0, Z \in Z^{**}$. But, $X_0, Z^{**} \in U$, so $i^* \in U$, and then, also, $Z^{**}_{i^{**}} \in U$. But, $Z^{**}_{i^{**}} \supseteq Z$. So, it is the cover of Z in U, and hence $Z^{**}_{i^{**}} = Z^*$.
Now we proceed as before. Let $Z^{'**} = \pi_{X_0X_1}(Z^{**})$ and $\pi_{X_0X_1}(\langle Z_i^{**} \mid i < \text{cof}(\sup(Z^{**} \cap \theta^+))\rangle) = \langle Z_i^{'**} \mid i < \text{cof}(\sup(Z^{**} \cap \theta^+))\rangle$.
Note that $Z^{'**}_{i^{**}} = \bigcup_{i<i^{**}} Z_i^{'**} = Z^{'*}$, since $Z^{**}_{i^{**}} = Z^*$ and $\text{cof}(i^{**}) = |X^0|$.
Let us argue that $U \cap Z^{'**}_{i^{**}} = U \cap D$. Clearly $U \cap Z^{'**}_{i^{**}} \supseteq U \cap D$. Let us show the other inclusion. Suppose otherwise. Then there is $z \in (U \cap Z^{'**}_{i^{**}}) \setminus D$. Pick the least $j < i^{**}, j \in X_0$ such that $z \in Z_j^{'**}$. Then $j \in U$, and so, $Z_j^{**} \in U$. Now, $j \in U$, $j < i^{**}$ and $Z^{**}_{i^{**}} = Z^*$ imply $Z_j^{**} \subseteq Z$. Then $Z_j^{'**} \subseteq D$, which is impossible since there is $z \in Z_j^{'**} \setminus D$.
□

The next lemma is straightforward.

Lemma 2.33 *The forcing notion* $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ *is* $< \delta-$*strategically closed.*

Proof. Use the strategy to switch each time back to the same (extended) wide piste. Take unions along the wide piste at limit stages. Note that 2.1(19, 21) will hold with such limit models, since non-limit ones are closed at least under $< \eta-$sequences and in particular, once including members of an increasing sequence (which length is less than $\delta \leq \eta$) will have the limit inside.

$\square$

Combining together Lemmas 2.31, 2.33, we obtain the following:

Theorem 2.34 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ preserves all cardinals $\leq \delta$ and all cardinals $> \eta$.*
In particular, if $\delta = \eta$, then all cardinals are preserved.

We are not going to force with $\mathcal{P}_{\theta\eta\delta}$ in the cardinal arithmetic applications, but rather to use its members as domains of conditions of a further forcing. However, the forcing with it is of an interest. Thus, for example, $\mathcal{P}_{\theta\omega\omega}$, where $\theta > \omega$ is a regular cardinal, may be of an interest on its own since the forcing with it will add a closed unbounded subset to θ^+ by finite conditions which runs away from every countable set in the ground model.

The next two lemmas will insure that generic clubs produced by $\mathcal{P}_{\theta\eta\delta}$ run away from old sets.

Lemma 2.35 *Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be an element of $\mathcal{P}_{\theta\eta\delta}$. Let $X \in A^{1\rho lim}$, for some $\rho \in s$.*
Suppose that for every $t \in X$ there is $D \preceq X$ such that

1. *$D \in X$,*
2. *$t \in D$,*
3. *$|D| = \rho$,*
4. *$D \supseteq \rho$*
5. *${}^{\rho>}D \subseteq D$,*
6. *D is a union of a chain of its elementary submodels which are members of D and satisfy items 1-5* [25].

Then for every $\beta < \sup(X \cap \theta^+)$ there is T of size ρ with $\sup(T \cap \theta^+) > \beta, T \in X$ such that adding T as a potentially limit point and reflecting it through $\Delta-$system type triples[26] *gives an extension of p.*

Proof. Let $\tau = \mathrm{cof}(\sup(X \cap \theta^+))$.
Use the assumption of the lemma and construct an increasing continuous sequence $\langle X_i \mid i < \tau\rangle$ of elementary submodels of X such that $\bigcup_{i<\tau} X_i = X$, for every $i < \tau$, $X_i \in M$, ${}^{\rho>}X_{i+1} \subseteq X_{i+1}$ and X_{i+1} is a union of a chain of its elementary submodels which satisfy items 1,3-5 above.
Pick now T to be one of X_{i+1}, such that

[25] The issue here is to satisfy 2.1(3(b)).
[26] By "reflecting through $\Delta-$system type triples", we mean that whenever $Z \in \bigcup_{\zeta\in s} A^{1\zeta}$ is a splitting point with immediate predecessors Z_0, Z_1 and $T \in Z_0$, then the image $\pi_{Z_0Z_1}(T)$ of T under the isomorphism $\pi_{Z_0Z_1}$ is added together with T.

1. $\sup(T \cap \theta^+) > \beta$,
2. for every model $E \in X$ which appears in p or is a covering model of a model of p, require that $E \in T$,
3. for every model Z which appears in p and has cardinality $< \tau$, we require that $Z \cap X \in T$.[27]
4. Let D, D_0, D_1 from p form a $\Delta-$system triple, i.e. D is a splitting point and D_0, D_1 are its immediate predecessors. Suppose that $D_0 \supseteq X$.
   - for every model $E \in \pi_{D_0,D_1}(X)$ which appears in p or is a covering model of a model of p,

     require that $E \in \pi_{D_0,D_1}(T)$,
   - for every model Z which appears in p and has cardinality $< \tau$, we require that $Z \cap \pi_{D_0,D_1}(X) \in \pi_{D_0,D_1}(T)$.

Let us argue that T is as desired.
We need to check that adding T as a potentially limit point and reflecting it through $\Delta-$system type triples gives an extension of p. The only thing that may go wrong here is that T (or its images through $\Delta-$system type triples) interferes badly with models of p.
First note that if D is one of the models of p and $\sup(D \cap \theta^+) < \sup(X \cap \theta^+)$, then $\sup(D \cap \theta^+) < \sup(T \cap \theta^+)$.
It follows by (2) for models which are in X.
So, suppose that $D \notin X$. Changing the wide piste of p if necessary and using (5) above, we can assume that both X and D are on the same wide piste. Apply Definition 2.1(19) or (22) to X and D. Let $D^* \in X$be the covering model. But then by (2) above, we have $D^* \in T$.
The requirement (3) ensures the items (20), (22) of Definition 2.1.

Let us argue now that adding T does not cause any harm once moving through $\Delta-$system type triples. Let D, D_0, D_1 from p form a $\Delta-$system triple, i.e. D is a splitting point and D_0, D_1 are its immediate predecessors.
Note that by (3) above $T \notin Z$, for any Z in p of cardinality $< \tau$. So, let us assume that $|D| \geq \tau$.
As before, X and D are on the same wide piste.

[27]Recall that ${}^{\tau>}X \subseteq X$ by 2.1(3(c(i))). So, $Z \cap X \in X$, and hence, $Z \cap X \in X_i$ for every large enough $i < \tau$.

By the argument above, it is enough to deal with the case $\sup(D \cap \theta^+) > \sup(X \cap \theta^+)$.
Note that $\sup(D \cap \theta^+) = \sup(X \cap \theta^+)$ is impossible since D is not potentially limit model.
If $|D| < \rho$, then, by the definition of the $\Delta-$system triple, π_{D_0,D_1} would not move X, since $|X| = \rho$.
Suppose that $|D| \geq \rho$.
If $X \not\subseteq D_0$ and $X \not\subseteq D_1$, then let us argue that T is not in the domain of π_{D_0,D_1} and π_{D_1,D_0}, and so, does not move. For this apply the intersection property $ip(X, D_0)$. Then

$$X \cap D_0 = X \cap T_1 \cap ... \cap T_n,$$

for some $T_1, ..., T_n \in X$. Hence, by (2) above, $T_1, ..., T_n \in T$. So T cannot be in $X \cap D_0$.

Now, if $X \subseteq D_0$, then the condition (4) above provides the desired conclusion.

□

Lemma 2.36 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ *be an element of* $\mathcal{P}_{\theta\eta\delta}$. *Let* $X \in A^{1\rho lim}$, *for some* $\rho \in s$.
Suppose that for every $t \in X$ *there is* $D \preceq X$ *such that*

1. $D \in X$,
2. $t \in D$,
3. $|D| = \rho$,
4. $D \supseteq \rho$
5. ${}^{\rho>}D \subseteq D$,
6. *D is a union of a chain of its elementary submodels which satisfy items 1-5.*

Let $\beta < \sup(X \cap \theta^+)$ *and* T *be a potentially limit point of size* ρ *with* $\sup(T \cap \theta^+) > \beta, T \in X$ *added by the previous lemma 2.35. Then for every* $\gamma, \sup(T \cap \theta^+) < \gamma < \sup(X \cap \theta^+)$ *there is* T' *of size* ρ *with* $\sup(T' \cap \theta^+) > \gamma, T' \in X$ *such that adding* T' *as a non-potentially limit point and reflecting it through* $\Delta-$*system type triples gives an extension of the previous condition.*

Proof. The proof repeats that of 2.35. The purpose of first adding T and only then T' is to satisfy Item (17) of Definition 2.1. Thus we add first a potentially limit point T above everything relevant, then we are free to add above it a non-potentially limit point T'.

□

Let $G \subseteq \mathcal{P}_{\theta\eta\delta}$ be a generic. Set

$$C = \{X \cap \theta^+ \mid \exists p \in G(X \in A^{1\theta lim}(p))\}$$

and let

$$C' = \{\alpha < \theta^+ \mid \alpha \text{ is a limit of points in } C\}.^{28}$$

Lemma 2.37 *Let $\alpha \in C'$ be of cofinality θ, then $\alpha \in C$.*

Proof. Suppose otherwise. Let $p \in \mathcal{P}_{\theta\eta\delta}$ and $\alpha < \theta^+$ be of cofinality θ and

$$p \Vdash \alpha \notin \underset{\sim}{C} \text{ and } \underset{\sim}{C} \text{ is unbounded in } \alpha.$$

Split into two cases.

Case 1. *There is no model A in p with* $\sup(A \cap \theta^+) = \alpha$.
Pick then B to be the least model on the wide piste of p with $\sup(B \cap \theta^+) > \alpha$. Let $A \in B$ be given by 2.1(7). Then, $\sup(A \cap \theta^+) < \alpha$.
If B is not a potentially limit point, then, by 2.13(6), if B has a unique immediate predecessor, or by 2.13(5), if B is a splitting point, no models can be added between A and B, which contradicts unboundedness of C in α.
If B is a potentially limit point, then we would like to use 2.35 and 2.36 to add a potentially limit point T and a non potentially limit model T' such that

1. $A \in T \in T' \in B$,
2. $T, T' \in C^{|B|}(B)$,
3. $\sup(T \cap \theta^+) < \alpha < \sup(T' \cap \theta^+)$.

This will provide a contradiction, since no element of C then will be inside the interval $(\sup(T \cap \theta^+), \sup(T' \cap \theta^+))$.

By the assumption $\operatorname{cof}(\alpha) = \theta$, $|p| < \delta$ and

$$p \Vdash \underset{\sim}{C} \text{ is unbounded in } \alpha,$$

we can find such T and then by 2.36 also T'.

[28] Note that the set C is not closed. Thus, densely often there are conditions $p \in \mathcal{P}_{\theta\eta\delta}$ with models A, B inside such that both are on the wide piste of p and are potentially limit, $|A| = \eta, |B| = \theta$, $B \in A$, ${}^{\eta>}A \subseteq A$, ${}^{\theta>}B \subseteq B$ and $\sup(A \cap B)$ is a limit of models of size θ (clearly most of them not in p, due to $|p| < \delta$) that may be added to p.

Case 2. *There is a model B in p with* $\sup(B \cap \theta^+) = \alpha$.
B is then in $A^{1\theta}$, since $\mathrm{cof}(\alpha) = \theta$. In addition, B must be a potentially limit model, since otherwise, by the argument of the previous case, it $B \cap \theta^+$ cannot be a limit of elements of C.
Hence $\alpha \in C$. Contradiction.
□

Lemma 2.38 *Let $\alpha \in C'$ be of cofinality $< \delta$, then $\alpha \in C$.*

Proof. Let $p \in \mathcal{P}_{\theta\eta\delta}$ and $\alpha < \theta^+$ be of cofinality $< \delta$ and

$$p \Vdash \underset{\sim}{C} \text{ is unbounded in } \alpha.$$

We construct, using the strategic closure of the forcing, an increasing sequence of extensions of p of length $\mathrm{cof}(\alpha) < \delta$ having an upper bound which decide unboundedly many elements of C below α. Let q be such an upper bound. Then by 2.1(2(e)), there will be $A \in A^{1\theta}(q)$ with $A \cap \theta^+ = \alpha$.
□

Lemma 2.39 *Let $\alpha \in C'$ be such that $\delta \leq \mathrm{cof}(\alpha) < \eta$. Then $\alpha \notin C$.*

Proof. Let us argue that it is impossible to have a model A in a condition $p \in \mathcal{P}_{\theta\eta\delta}$ such that $\sup(A \cap \theta^+) = \alpha$. Suppose for a moment that this is the case. Without loss of generality assume that A is on the wide piste of p. A cannot be a limit point since $\sup(A \cap \theta^+) = \alpha$, $\delta \leq \mathrm{cof}(\alpha) < \eta$. Also, A cannot be a successor non-potentially limit model, since this will contradict to the unboundedness of C in α. So, A must be a potentially limit point. But then ${}^{\eta>}A \subset A$, which rules out the possibility that $\mathrm{cof}(\alpha) < \eta$. Contradiction.
□.

Lemma 2.40 *Let $x \in V$ be a subset of θ^+ of cardinality δ. Then $x \not\subseteq C'$.*

Proof. Let $x \in V$ be a set of cardinality δ and $p \in \mathcal{P}_{\theta\eta\delta}$. Without loss of generality assume that $\sup(x)$ is a limit ordinal of cofinality δ. Set $\nu = \sup(x)$.
If

$$p \Vdash (\underset{\sim}{C}' \text{ is bounded in } \nu),$$

then we are done. So, suppose that

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Split models of p into two groups. Set

$$H_0 = \{A \mid A \text{ appears in } p \text{ and } \sup(A \cap \nu) < \nu\}$$

and

$$H_1 = \{A \mid A \text{ appears in } p \text{ and } \sup(A \cap \nu) = \nu\}.$$

Note that since the total number of models in p is less than δ and δ is regular, we have $\bigcup_{A \in H_0} \sup(A \cap \nu) < \nu$. Also, if $\delta < \eta$, then $\nu \in A$, for every $A \in H_1$.
Let $\nu^* = \bigcup_{A \in H_0} \sup(A \cap \nu)$.
Take now any $B \in H_1$.

Claim. *For every $\beta, \nu^* < \beta < \nu$, there is $D \in B$ such that*

1. $|D| = \theta$,
2. $D \subseteq \theta$,
3. ${}^{\theta>}D \subseteq D$,
4. $\beta < D \cap \theta^+ < \nu$,
5. there exists a $D-$sequence.

Proof. Assume that $\beta \in B$, since $B \in H_1$ just replacing it by a bigger ordinal if necessary. We have

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Let $p \in G$. So in $V[G]$, there is $D' \in A^{1\theta lim}$ such that ${}^{\theta>}D' \subseteq D'$,$D' \cap \theta^+ \in C$ and $\beta < D' \cap \theta^+ < \nu$ [29]. Then there is an extension q of p with D' inside. If $D' \in B$, then we are done. Suppose otherwise. Apply the covering properties 2.1((19) or (21)) to B, D' for q. Then there will be $D'' \in B \cap A^{1\theta lim}$ which includes D' and $D' \cap \theta^+ < D'' \cap \theta^+ < \nu$, since $B \in H_1$.
Now,

$$H(\theta^+) \models \exists D \in D''(\beta \in D \text{ and it satisfies conditions (1)-(5) of the claim}).$$

Just D' witnesses this. By elementarity then

$$B \models \exists D \in D''(\beta \in D \text{ and it satisfies conditions (1)-(5) of the claim}).$$

[29] We just use a density argument to find such D' which is a closed model, i.e. ${}^{\theta>}D' \subseteq D'$.

$\square$ of the claim.

Suppose that the following holds in $V[G]$:

(*) There is an elementary $\in -$chain $\langle D_i \mid i<\delta\rangle$ of elementary submodels of $\langle H(\theta^+), \in, \leq, \delta, \eta\rangle$ such that

- for every $A \in H_1$,$\{D_i \mid i<\delta\} \subseteq A$,
- each D_i satisfies the items (1),(2),(4)(without β),(5) of the claim,
- $\{D_i \cap \theta^+ \mid i<\delta\}$ is unbounded in ν.

Without loss of generality we can assume that $\langle D_i \mid i<\delta\rangle$ is a closed chain.
Recall that

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Hence for every $i<\delta$ there is the least $i', i<i'<\delta$ such that

$$D_{i'+1} \models \exists D \preceq D_{i'}(D \text{ satisfies the items (1)-(5) of the claim, with (4)without } \beta \wedge D_i \in D).$$

Pick the least such D and make it the new D_{i+1}.
Let p' be an extension of p which adds a model X in C above ν^*, but below ν. Change, if necessary, our sequence $\langle D_i \mid i<\delta\rangle$ by removing an initial segment such that $X \in D_1$.
Now we pick some $\xi \in x, D_1 \cap \theta^+ < \xi < D_{i^*+1} \cap \theta^+$, for some $i^*<\delta$. Next, add D_1, D_{i^*+1} to p, D_1 as a potentially limit point and D_{i^*+1} as a non-limit and non-potentially limit point. The requirement 2.1(17) will hold, since D_1 is a potentially limit point above X (this for models in H_0) and D_1, D_{i^*+1} are in every model of H_1 on the wide piste. Reflect them through all relevant splittings. Let q be the result. Then

$$q \Vdash x \not\subseteq \underset{\sim}{C},$$

since nothing can be added between D_1 and D_{i^*+1}.

Let us now argue that (*) holds.
Split into three cases.

Case 1. $\eta > \delta$.
Then $\nu \in A$, for every $A \in H_1$, since if A is a non-limit, then it is closed under $<\eta-$sequences of its elements. If A is a limit model, then it is a union of less than δ non-limit models. So the regularity of δ implies that a final segment of them is in H_1. But then ν belongs to each non-limit model of this final segment, and hence it belongs to A as well.

Pick the least sequence $\langle \nu_i \mid i < \delta \rangle$ that is cofinal in ν. Then $\langle \nu_i \mid i < \delta \rangle \in A$ and $\{\nu_i \mid i < \delta\} \subseteq A$ for every $A \in H_1$ by elementarity.
Now it is easy to construct $\langle D_i \mid i < \delta \rangle$ which satisfies (*). Just proceed by induction and apply the claim with β there replaced by ν_i's.

Case 2. $\delta > \omega$.
Work in $V[G]$ with $p \in G$. Let $\langle E_i \mid i < \delta \rangle$ be an increasing sequence of models with $E_i \cap \theta^+ \in C \cap \nu$, for every $i < \delta$, and $\langle E_i \cap \theta^+ \mid i < \delta \rangle$ unbounded in ν.
Let $A \in H_1$ be a non-limit model of p and $i < \delta$. Pick an extension $p_i \in G$ of p such that E_i appears in p_i and A is on the wide piste of p_i. Then, by the covering properties Definition 2.1(19)or (21), there is $E_i^A \in A \cap C^\theta(A^{0\theta})(p_i)$ the least which contains E_i (recall that θ is in every model). We have $A \in H_1$, so $E_i^A \cap \theta^+ < \nu$.
Consider $\{E_i^A \mid i < \delta\}$. Adding limit models if necessary we can assume that it is a closed chain. A is a non-limit, hence it is closed under less than $\delta-$sequence of its elements, so, $\{E_i^A \mid i < \delta\} \subseteq A$. Set

$$Y^A := \{E_i^A \cap \theta^+ \mid i < \delta\}.$$

Consider

$$Y := \bigcap \{Y^A \mid A \in H_1 \text{ non-limit}\}.$$

Then Y is an intersection of fewer than δ clubs, and hence is a club in ν. Now, $Y \subseteq A$, for every $A \in H_1$ non-limit . If B is a limit model of p and $B \in H_1$, then B is an increasing union of less than δ non-limit models. Then, the final segment of them is in H_1, and hence contains Y. So, $Y \subseteq B$.
Finally note that if E, E' two models which appear in $A^{1\theta}(r)$, for some r, and $E \cap \theta^+ = E' \cap \theta^+$, then $E = E'$, by Definition 2.1(2). So, take any $A \in H_1$ non-limit , consider the sequence

$$\langle E_i^A \mid i < \delta, E_i^A \cap \theta^+ \in Y \rangle.$$

It will witness (*).

Case 3. $\delta = \omega = \eta$.
Work in V. Conditions are finite now.
We have

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Let $\beta < \nu$.
Extend p to some p' such that there is $D' \in A^{1\theta}(p'), \beta < D' \cap \theta^+ < \nu$.

Suppose that there is $A \in H_1$ such that $D' \notin A$. Replace p' by an equivalent condition, if necessary, to ensure that A is on the wide piste. Then by Definition 2.1(19)or (21), there is $D^A \in A \cap C^\theta(A^{0\theta})(p')$ the least which contains D' (recall that θ is in every model). Then $D^A \cap \theta^+ < \nu$, since $A \in H_1$ and $A \cap D^A \subseteq D'$.
Now, if there is $B \in H_1$ such that $D^A \notin B$, then we repeat the process with D', A replaced by D^A, B. Remember that p' is finite, so after finitely many steps there will be $D^* \in C^\theta(A^{0\theta})(p'), D' \subseteq D^*$ such that $D^* \in Z$, for every $Z \in H_1$.
More formally - set $A_0 = A$ and $D_0 = D^A$. Proceed by recursion. Suppose that $A_k \in H_1$, $D_k \in A_k \cap C^\theta(A^{0\theta})(p'), D_{k-1} \subsetneqq D_k$ are defined and $D_k \cap \theta^+ < \nu$. If there is no $B \in H_1$ such that $D_k \notin B$, then we stop and set $D^* = D_k$. Otherwise, let A_{k+1} be some such B.[30] Apply Definition 2.1(19)or (21) to A_{k+1} and D_k. Set D_{k+1} to be the witnessing model.
p' is finite, so after finitely many steps the process must terminate.

The ordinal $\beta < \nu$ was arbitrary, so there is a cofinal in ν sequence $\langle \nu_n \mid n < \omega \rangle$ such that $\{\nu_n \mid n < \omega\} \subseteq A$, for every $A \in H_1$.
Now, we proceed as in Case 1 and use $\langle \nu_n \mid n < \omega \rangle$ in order to define $\langle D_n \mid n < \omega \rangle$ which satisfies (*).
$\square$

In particular, taking $\delta = \eta = \omega$, we obtain the following generalization of corresponding results by U. Abraham [1], J. Baumgartner, S. Friedman [2] and by W. Mitchell [14] to higher cardinals[31]:

Corollary 2.41 *The forcing $\mathcal{P}_{\theta\omega\omega}$ is cardinals preserving forcing adding a club in θ^+ which avoids old countable sets using finite conditions.*

Remark 2.42 Given a stationary subset S of θ^+ such that for every $\alpha < \theta^+$ if $\mathrm{cof}(\alpha) < \theta$, then $\alpha \in S$, it is easy to modify the forcing $\mathcal{P}_{\theta\eta\delta}$ such that it will add a club through S. Only require that for every X of cardinality θ in a condition we have $X \cap \theta^+ \in S$.

[30] It is possible to have $A_{k+1} = A_i$, for some $i < k$.
[31] Note that Magidor and Radin forcings ([11],[15]) also add clubs by finite conditions.

3 Suitable structures.

We reorganize here the structures with pistes of the previous section in order to allow isomorphisms of them over different cardinals.

Definition 3.1 Let $\delta < \kappa < \theta$ be cardinals, with δ and θ regular. A structure $\mathfrak{X} = \langle X, E, E^{lim}, C, S, \in, \subseteq \rangle$, where $E \subseteq [X]^2$ and $C \subseteq [X]^3$ is called a $\delta-$suitable structure with pistes over κ of length θ iff there is a δ structure with pistes over κ^+ of length θ
$p(\mathfrak{X}) = \langle\langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), A^{1\tau lim}(\mathfrak{X}), C^{\tau}(\mathfrak{X})\rangle \mid \tau \in s(\mathfrak{X})\rangle$ such that

1. $X = A^{0\eta}(\mathfrak{X}) \cup \{A^{0\eta}(\mathfrak{X})\}$, where $\eta \in s(\mathfrak{X})$ is such that for every $\tau \in s(\mathfrak{X})$ we have then $A^{0\tau}(\mathfrak{X}) \in X$ or $A^{0\tau}(\mathfrak{X}) \subseteq X$,
2. $S = s(\mathfrak{X})$,
3. $\langle a, b\rangle \in E$ iff $a \in S$ and $b \in A^{1a}(\mathfrak{X})$,
4. $\langle a, b\rangle \in E^{lim}$ iff $a \in S$ and $b \in A^{1alim}(\mathfrak{X})$,
5. $\langle a, b, d\rangle \in C$ iff $a \in S, b \in A^{1a}(\mathfrak{X})$ and $d \in C^{a}(\mathfrak{X})(b)$.

Let us refer to $\mathfrak{X}$ for shortness as a $\delta-$suitable structure if κ, θ are fixed.

Note that $p(\mathfrak{X})$ is uniquely defined from $\mathfrak{X}$. Also, it is easy to define a δ-suitable structure from $p \in \mathcal{P}_{\theta\kappa^+\delta}$.

Definition 3.2 Let $\mathfrak{X}, \mathfrak{Y}$ be δ-suitable structures. Set $\mathfrak{X} \leq \mathfrak{Y}$ iff $p(\mathfrak{X}) \leq p(\mathfrak{Y})$.

3.1 Forcing conditions.

Let κ be a limit of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega\rangle$ with each κ_n being strong up to λ_n^+ for the least Mahlo cardinal λ_n above κ_n as witnessed by an extender E_n.
Let $\theta > \kappa$ be a regular cardinal.

The definitions below follow closely the corresponding definitions starting with [4], Chapter 2, then [3],[8], and finally [6], Chapter 2.4.

For every $n < \omega$ define Q_{n0}.

Definition 3.3 Let Q_{n0} be the set of the triples $\langle a, A, f\rangle$ so that:

1. f is a partial function from θ^+ to κ_n of cardinality at most κ,

2. a is an isomorphism between a κ_n–suitable structure $\mathfrak{X}$ over κ of length θ and a κ_n–suitable structure $\mathfrak{X}'$ over κ_n^{+n} of length λ_n such that

   (a) X' is above every model which appears in $(\bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')) \setminus \{X'\}$, in the order $\leq_{E_n}$, (or actually after coding X' by an ordinal by using a canonical well ordering of $H(\lambda_n^+)$),[32]

   (b) if $t \in \bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')$,then for some $k, 2 < k < \omega$, $t \prec H(\chi^{+k})$, with χ big enough fixed in advance.
   The way to compare such models $t_1 \prec H(\chi^{+k_1})$, $t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
   move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{nX'}$,

4. for every ordinals α, β, γ which code models in $\bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')$, we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \text{ implies}$$

$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho)),$$

   for every $\rho \in \pi^{E_n}_{X'\alpha}{}''A$.

Definition 3.4 Let $\langle a, A, f\rangle, \langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $\mathrm{dom}(a) \geq \mathrm{dom}(b)$ in the order of suitable structures (Definition 3.2),

2. $\mathrm{ran}(a) \geq \mathrm{ran}(b)$ in the order of suitable structures (Definition 3.2),

3. $a \supseteq b$,

4. $f \supseteq g$,

5. $\pi^{E_n}_{\max(\mathrm{ran}(a)),\max(\mathrm{ran}(b)}$“$A \subseteq B$.

Definition 3.5 Q_{n1} consists of all partial functions $f : \theta^+ \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

[32] $\leq_{E_n}$ is the order of the extender E_n, see [4], Chapter 2.

Definition 3.6 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq_n^* = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f \rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown \nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{ran}(a)),a(\alpha)}(\nu)\rangle \mid \alpha \in A^{1\theta}(\mathrm{dom}(a)) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $A^{1\theta}(\mathrm{dom}(a))$ are used to produce non direct extensions, the rest of models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq_n^* q$ or $p \in Q_{n1}, q = \langle b, B, g \rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown \nu$.

Definition 3.7 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

1. for every $n < \omega$, $\quad p_n \in Q_n$,
2. there is $\ell(p) < \omega$ such that
   (a) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,
   (b) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$,
   (c) if $\ell(p) \leq n \leq m$, then $\mathrm{dom}(a_n) \leq \mathrm{dom}(a_m)$ in the order of suitable structures. In particular $\mathrm{dom}(a_n) \subseteq \mathrm{dom}(a_m)$.
   (d) If $\ell(p) \leq n \leq m$, then $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$.
3. for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite. (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).[33]
4. For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.
5. There is a κ-structure with pistes $\mathfrak{p}$ over κ such that
   (a) $\mathfrak{p} \geq \mathrm{dom}(a_n)$, for every $n, \ell(p) \leq n < \omega$,
   (b) if a model A appears in $\mathfrak{p}$, then A appears in $\mathrm{dom}(a_n)$ for some $n, \ell(p) \leq n < \omega$ (and then in a final segment of them),

[33]See [3] Definition 2.8, where this idea appears for the first time.

(c) $\max(\mathrm{dom}(a_n)) = \max(\mathfrak{p})$ (actually this follows from the previous condition).

Note that $\mathfrak{p}$ of 3.7(5) is uniquely determined by p. Let us refer to it further as the κ-structure with pistes over κ of p.

The forcing order $\leq$ and the direct extension orders $\leq^*$ are defined on $\mathcal{P}$ as follows:

Definition 3.8 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle$ be in $\mathcal{P}$. We define $p \geq q (p \geq^* q)$ iff for every $n < \omega$, $p_n \geq_n q_n (p_n \geq^*_n q_n)$.

The proofs of the next lemmas repeat those of corresponding lemmas in [3](2.10, 1.11,2.14),[5](1.8,1.14,1.15),[6] Chapter 1.

Lemma 3.9 *$\langle Q_{n0}, \leq_{n0} \rangle$ is $< \kappa_n$-strategically closed.*

Lemma 3.10 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of length $< \kappa_0$.*

Lemma 3.11 *$\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Lemma 3.12 *Let $p \in \mathcal{P}$ and $\alpha < \theta^+$, then there are $q \geq^* p$ and $\beta, \alpha < \beta < \theta^+$ such that $\beta = M \cap \theta^+$, for some M which appears in the κ-structure with pistes over κ of q.*

Proof. Pick some $M \prec H(\theta^+)$ of size θ which is above the maximal model of $\mathfrak{p}$ (say $\mathfrak{p} \in M$) and such that $M \cap \theta^+ > \alpha$. Add it to κ-structure with pistes over κ of p. Then, for every $n, \ell(p) \leq n < \omega$, similarly find a model M_n and extend a_n by mapping M to M_n. Let q be the resulting condition. Then it is as desired.
$\square$

The next lemma follows now:

Lemma 3.13 *Let G be a generic subset of $\langle \mathcal{P}, \leq \rangle$. Then in $V[G]$ there are $\mathrm{cof}((\theta^+)^V)$–many ω–sequences of ordinals below κ.*

Define $\rightarrow$ and $\leftrightarrow$ on $\mathcal{P}$ as in [3] or [5].

κ^{++}–c.c. and even θ^+–c.c. break down here for the forcing $\langle \mathcal{P}, \rightarrow \rangle$.
Following C. Merimovich [13] we replace them by properness.

3.2 Properness.

We will turn now to the properness of the forcing.

Lemma 3.14 *$\langle \mathcal{P}, \rightarrow \rangle$ is η-proper, for every regular $\eta, \kappa^+ < \eta \leq \theta$.*

The proof repeats almost completely those of Lemma 2.31. The only additional ingredient is to put new models that were added below κ in the process of extension of conditions inside old ones. As usual, in [6], we use $\longleftrightarrow$ for this purpose and pass to equivalent models.
In the proof of Lemma 2.31 the reflection was made over κ into the model $\mathfrak{M}$. Here we will need in addition to reflect into the images M_n of $\mathfrak{M} \cap H(\theta^+)$. It is possible for every $n < \omega$ large enough due to (4) of Definition 3.7.

Finally, combining together Lemmas 3.10, 3.11, 3.13, 3.14, we obtain the following:

Theorem 3.15 *Let G be a generic subset of $\langle \mathcal{P}, \to \rangle$. Then $V[G]$ is cofinalities preserving extension of V in which $2^\kappa = \kappa^\omega = \theta^+$.*

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