A very weak generalization of SPFA to higher cardinals.

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Abstract

Itay Neeman found in [7] a new way of iterating proper forcing notions and extended it in [8] to $\aleph_2$. In [5] his construction for $\aleph_1$ ([7]) was generalized to semi-proper forcing notions. We apply here finite structures with pistes in order extend the construction to higher cardinals. In the final model a very weak form of SPFA will hold.

1 Basic definitions and main results

The following two definitions are due to S. Shelah [9].

Definition 1.1 A forcing notion Q is called a $\{\eta\}$*–proper* iff for every $M \prec \langle H(\chi), \in, < \rangle$ of a size η with $Q \in M$ the following holds:
for every $q \in M$ there is $p \geq q$ which is (M, Q)-generic, i.e. $p \Vdash ((M[\underset{\sim}{G}])^V = M)$.
If Q is $\{\eta'\}$–proper for every regular cardinal $\eta' \leq \eta$, then we call Q a $\{\leq \eta\}$*–proper*.

Definition 1.2 A forcing notion Q is called a $\{\eta\}$*–semi-proper* iff for every $M \prec \langle H(\chi), \in, < \rangle$ of a size η with $Q \in M$ the following holds:
for every $q \in M$ there is $p \geq q$ which is (M, Q)–semi-generic, i.e. $p \Vdash (M[\underset{\sim}{G}] \cap \eta^+ = M \cap \eta^+)$.
If Q is $\{\eta'\}$–proper for every regular cardinal $\eta' \leq \eta$, then we call Q a $\{\leq \eta\}$*–semi-proper*.

Remark 1.3 Further we will use a bit weaker notions. Instead of arbitrary M's in Definitions 1.1,1.2 we restrict ourself to models closed under $< \eta$–sequences in GCH situations and once GCH breaks down - to models which are generic extensions of closed under $< \eta$–sequences models from the ground model which satisfies GCH.
It is possible to formulate this in terms of internal clubs as Neeman does.

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Dealing with finite $\in$-increasing sequences closed under intersections, as it was done in [6], [7] and worked fine at $\aleph_1$, seems to be problematic here in context of larger cardinals. The problems appear already at $\aleph_2$, i.e. once models of cardinalities $\aleph_0$ and $\aleph_1$ are around. The basic problem is with $\{\aleph_0\}$–properness. The proof of it requires kind of nice restrictions of conditions to a countable submodel which may not exist now.
We will follow the intuition of [4] and use instead of $\in$-increasing sequences closed under intersections – finite structures with pistes from [4], i.e. $\omega-$structures with pistes over ω of the length θ or members of $\mathcal{P}_{\theta\omega\omega}$, for some regular large enough θ.
Elements of $\mathcal{P}_{\theta\omega\omega}$ are of the form $\mathcal{A} = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$. Review briefly the nature of components of $\mathcal{A}$:

1. s is a finite set of regular cardinals $\leq \theta, \omega, \theta \in s$,
   for every $\tau \in s$,
2. $A^{1\tau}$ is a finite set of elementary submodels of size τ of $\langle H(\theta^+), \in, \leq \rangle$,
3. $A^{0\tau}$ is the largest element of $A^{1\tau}$,
4. $A^{1\tau lim} \subseteq A^{1\tau}$ is a set of potentially limit models of cardinality τ, basically the places where a non-end extension is allowed,
5. C^τ is the piste function for models of cardinality τ.

Definition 1.4 Let $\mathcal{A}$ be a finite structure with pistes and Q a forcing notion. We call a condition $p \in Q$ $(\mathcal{A}, Q)$*-generic* iff p is (A, Q)- generic for every $A \in \mathcal{A}$ with $Q \in A$.

Definition 1.5 Let Q be a $\leq \eta$–piste structures proper forcing, $\mathcal{A}$ a finite structure with pistes which consists of models of cardinalities $\leq \eta$ Let $p \in Q$ be $(\mathcal{A}, Q)$- generic. We call p *a minimal generic for* $\mathcal{A}$ if for every $\mathcal{B}$ which extends $\mathcal{A}$ (in sense of [4]) there is $q \geq p$ which is $(\mathcal{B}, Q)$-generic.

Definition 1.6 A forcing notion Q is called $\leq \eta$*–strongly piste structures proper (or just strongly piste proper)* iff

1. for every finite structure with pistes $\mathcal{A}'$ which consists of models of cardinalities $\leq \eta$ there exists $\mathcal{A} \geq \mathcal{A}'$ and $p_{\mathcal{A}} \in Q$ which is a minimal generic for $\mathcal{A}$.
2. Let $\mathcal{A}$ be a finite structure with pistes which consists of models of cardinalities $\leq \eta$ and $p_{\mathcal{A}} \in Q$ a minimal generic for $\mathcal{A}$.
   Then

(a) for every finite structure with pistes $\mathcal{A}'$ which extends $\mathcal{A}$ there is a minimal generic $p_{\mathcal{A}'} \geq p_{\mathcal{A}}$ for $\mathcal{A}'$,

(b) for every $p' \geq p_{\mathcal{A}}$ there are a finite structure with pistes $\mathcal{B}$ which extends $\mathcal{A}$ and $p_{\mathcal{B}} \geq p'$ such that $p_{\mathcal{B}}$ is a minimal generic for $\mathcal{B}$.

Remark 1.7 1. Note that in the original Neeman setting at $\aleph_1$ [7] or at those of [5] there was no problem to find a common generic or semi-generic condition for $\in$-increasing sequences of models, since only countable models or models of inaccessible cardinalities were involved. In present situation starting with $\aleph_2$, there are also models of size $\aleph_1$. This complicates the matter. Thus let for example Q be the Levy collapse of ω_3 to ω_2. Define a sequence A_0, A_1, A_2, A_3 of elementary submodels such that

(a) A_3 is countable,

(b) $A_i, i \leq 2$ are of size $\aleph_1$,

(c) $A_0 \in A_1 \in A_2 \in A_3$,

(d) $A_3 \cap A_2 \subseteq A_0$ and $\sup(A_3 \cap A_2 \cap \omega_2) = \sup(A_0 \cap \omega_2)$,

(e) there is no $p \in Q$ which is a generic simultaneously over each of A_i's or even A_1, A_2, A_3.

Assume CH. Pick any $A_2 \preceq H(\chi)$, for a regular χ large enough, which is a limit of increasing continuous sequence of the length $\aleph_1$ of elementary submodels of $H(\chi)$ each of size $\aleph_1$. Let $A_3 \prec H(\chi)$ with $A_2 \in A_3$ and the sequence in A_3 as well. Then $A_2 \cap A_3 \in A_2$ and there is a model A of the sequence such that $A_3 \cap A_2 \subseteq A$ and $\sup(A_3 \cap A_2 \cap \omega_2) = \sup(A \cap \omega_2)$. Let $A_0 = A$.
Now let choose A_1.
Pick a sequence $\langle A^i \mid i < \omega_1 \rangle \in A_2$ such that

- for every $i < \omega_1$, $|A^i| = \aleph_1$,
- for every $i < \omega_1$, $A_0 \in A^i$,
- for every $i, j < \omega_1$, $A_i \cap \omega_2 = A_j \cap \omega_2$
- for every $i, j < \omega_1$, $i \neq j \Rightarrow A_i \cap \omega_3 \neq A_j \cap \omega_3$.

Set $\delta = A^0 \cap \omega_2$. Consider the set $S = \{f''\delta \mid f \in Q \cap A_3\}$. Then S a countable set of subsets of ω_3. Pick $i < \omega_1$ such that $A^i \cap \omega_3 \notin S$. Set $A_1 = A^i$.
Now suppose that there is $p \in Q$ which is Q-generic over each $A_i, i \leq 3$. Then

$p \restriction A_2 \cap \omega_2 \in A_3$, since $A_2 \in A_3$. Hence $p''\delta \in S$, and so $p''\delta \neq A_1 \cap \omega_3$. This prevents p from being generic over A_1.

2. It is possible to weaken a little applying restrictions of 1.3.

A combination of Neeman's ideas from [7] with a models produced in [4] allows to show the following:

Theorem 1.8 *Let κ be a supercompact cardinal and $\eta < \kappa$ be a regular cardinal. Then in a forcing extension which preserves all the cardinals $\leq \eta^+$ and turns κ into η^{++} the $\leq \eta$–strongly piste structures PFA holds, i.e. for every $\leq \eta$–strongly piste structures proper forcing notion Q and for every collection $\mathcal{D}$ of $\leq \eta^+$ dense subsets of Q there is a filter on Q that meets all of them.*

We will proceed here by replacing piste structures properness by a certain a parallel variation of semi-properness.

Definition 1.9 Let $\mathcal{A}$ be a finite structure with pistes and Q a forcing notion. We call a condition $p \in Q$ $(\mathcal{A}, Q)$*-semi-generic* iff p is (A, Q)- semi-generic for every $A \in \mathcal{A}$ with $Q \in A$.

Definition 1.10 Let Q be a $\leq \eta$–piste structures proper forcing, $\mathcal{A}$ a finite structure with pistes which consists of models of cardinalities $\leq \eta$ Let $p \in Q$ be $(\mathcal{A}, Q)$- semi-generic. We call p *a minimal semi-generic for* $\mathcal{A}$ if for every $\mathcal{B}$ which extends $\mathcal{A}$ (in sense of [4]) there is $q \geq p$ which is $(\mathcal{B}, Q)$-semi-generic.

Definition 1.11 A forcing notion Q is called $\leq \eta$*–strongly piste structures semi-proper (or just strongly piste semi-proper)* iff

1. for every finite structure with pistes $\mathcal{A}'$ which consists of models of cardinalities $\leq \eta$ there exists $\mathcal{A} \geq \mathcal{A}'$ and $p_{\mathcal{A}} \in Q$ which is a minimal semi-generic for $\mathcal{A}$.

2. Let $\mathcal{A}$ be a finite structure with pistes which consists of models of cardinalities $\leq \eta$ and $p_{\mathcal{A}} \in Q$ a minimal semi-generic for $\mathcal{A}$.
   Then

   (a) for every finite structure with pistes $\mathcal{A}'$ which extends $\mathcal{A}$ there is a minimal semi-generic $p_{\mathcal{A}'} \geq p_{\mathcal{A}}$ for $\mathcal{A}'$,

(b) for every $p' \geq p_{\mathcal{A}}$ there are a finite structure with pistes $\mathcal{B}$ which extends $\mathcal{A}$ and $p_{\mathcal{B}} \geq p'$ such that $p_{\mathcal{B}}$ is a minimal semi-generic for $\mathcal{B}$.

Our purpose will be to show the following:

Theorem 1.12 *Let κ be a supercompact cardinal and $\eta < \kappa$ be a regular cardinal. Then in a forcing extension which preserves all the cardinals $\leq \eta^+$ and turns κ into η^{++} the $\{\leq \eta\}$–strongly piste structures semi-proper SPFA holds, i.e. for every $\{\leq \eta\}$–strongly piste structures semi-proper forcing notion Q and for every collection $\mathcal{D}$ of $\leq \eta^+$ dense subsets of Q there is a filter on Q that meets all of them.*

2 The iteration.

The iteration is organized as in [5], only $\in$ −increasing sequences are replaced here with finite structures with pistes. The treatment of reflection, which was an issue in [5], in the present context is well incorporated into such structures.

Let us repeat the settings of [5].

Let κ be a Mahlo cardinal. Fix an increasing continuous chain $\langle \mathfrak{M}_\alpha \mid \alpha < \kappa \rangle$ of elementary submodels of $\langle V_{\kappa+1}, \in, \trianglelefteq \rangle$ such that

1. $|\mathfrak{M}_\alpha| < \kappa$,
2. $\mathfrak{M}_\alpha \cap V_\kappa = V_{\kappa_\alpha}$, for some $\kappa_\alpha < \kappa$,
3. κ_0 and each $\kappa_{\alpha+1}$ are inaccessible cardinals,
4. $\mathfrak{M}_{\alpha+1}$ is the $\trianglelefteq$ −least elementary submodel of $\langle V_{\kappa+1}, \in, \trianglelefteq \rangle$, which contains $\{\mathfrak{M}_\beta \mid \beta \leq \alpha\}$ and such that $\mathfrak{M}_{\alpha+1} \cap V_\kappa = V_{\kappa_{\alpha+1}}$, for some regular cardinal $\kappa_{\alpha+1} < \kappa$.

We will use here finite structures with pistes $\mathcal{P}_{\theta\omega\omega}$ of [4] with the following minor differences:

1. models of the form $V_\delta \preceq V_\kappa$, for inaccessible δ's below κ will replace models of cardinality θ,
2. no non-transitive models of cardinalities above η.
   Such models are not required here since all the cardinals between η^+ and κ will be collapsed.

3. $A^{0\omega}(p)$ is maximal under $\in$ among models of p, where $p \in \mathcal{P}_{\theta\omega\omega}$.

We refer to [5], for definitions of reachablity and $A[G]$.

Definition 2.1 Define by induction on $\tau \leq \kappa$ an iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \tau, \beta < \tau\rangle$ and the projection maps $\langle \pi_{\alpha\gamma} \mid \gamma \leq \alpha \leq \tau\rangle$, where $\pi_{\alpha\gamma}$ will be a projection map from the complete Boolean algebra $RO(P_\alpha)$ onto $RO(P_\gamma)$.

1. For each $\gamma < \tau$ we set $\underset{\sim}{Q}_\gamma$ to be the trivial forcing unless κ_γ is a regular cardinal.
2. Suppose that τ is a limit ordinal or $\tau = \tau' + 1$ and $\kappa_{\tau'}$ is a regular cardinal. Then $p \in P_\tau$ iff
   (a) for each $\gamma < \tau$, with κ_γ regular, the following hold.
   Let Q be the $\gamma-$th set in the fixed in advance well ordering of V_κ. Set $\underset{\sim}{Q}_\gamma$ to be the trivial forcing unless Q is a $P_\gamma-$name and
   $0_{P_\gamma} \Vdash_{P_\gamma} Q$ is a $\leq \eta-$strongly piste structures semi-proper forcing notion and
   $Q \in V_{\kappa_{\gamma+1}+1}[\underset{\sim}{G}_\gamma]$.
   We set, in the later case, $\underset{\sim}{Q}_\gamma = Q$.
   (b) There are two finite sets $s(p)$ and $m(p)$ such that
      i. $s(p) \subseteq \tau$ called the *support* of p.
      A set $\underset{\sim}{p}_\gamma$ is attached to every $\gamma \in s(p)$. We require that
      $0_{P_\gamma} \Vdash \underset{\sim}{p}_\gamma \in \underset{\sim}{Q}_\gamma$,
      ii. $m(p)$ is a fine set called the *models* of p.
      It will be just a finite structure with pistes in a proper forcing context. In a semi-proper context some complications may occur due to reachability that sometimes require addition of new models.
      Let us state the requirements on $m(p)$. Let $A \in m(p)$. Then the following hold.
         A. $|A|$ is a regular cardinal $\leq \eta$ or $A = V_\delta$ for some inaccessible $\delta < \kappa_\tau$,
         B. $A \prec V_{\kappa_\tau}$, if κ_τ is an inaccessible,
         and $A \prec V_{\kappa_{\tau+1}}$, otherwise, i.e. whenever τ is a limit ordinal and κ_τ is a singular cardinal.
         C. there is $A \in m(p)$ which is countable and for every $B \in m(p)$ if $B \neq A$, then either $B \in A$ or for some inaccessible $\delta < \kappa_\tau$ in A we have $V_\delta \in m(p)$ and B realizes the same type over $V_{\sup(A\cap\delta)}$ in V_δ as A in V_κ.

Further we refer to such B's as *reflections of A*. Note that if B is a reflection of A, then $\sup(B \cap On) < \sup(A \cap On)$ and $otp(B \cap On) = otp(A \cap On)$. Moreover, the order isomorphism is the identity on $A \cap B$ [1].

Denote such A by $max(m(p))$.

We would like to attach to every $\gamma < \tau$ a sequence of models based on members of $m(p)$ which, at least intuitively, will form the $\gamma-$th coordinate of p. In order to do so, let us first define $q = \pi_{\tau\gamma}(p)$ which will be in $RO(P_\gamma)$.

Set $s(q) = s(p) \cap \kappa_\gamma$, if κ_γ is a regular and $s(q) = s(p) \cap \kappa_{\gamma+1}$, otherwise.

Split into cases.

Case 1.[2] *κ_γ is an accessible (and then singular) cardinal and $\kappa_{\gamma+1} \in A$, where $A = max(m(p))$.*

If there is $A' \in m(p)$ which is a reflection of A to $V_{\kappa_{\gamma+1}}$ over $V_{\sup(A\cap\kappa_{\gamma+1})}$, then set $m(q) = \{B \in m(p) \mid B \in A' \text{ or } B = A' \text{ or } B \text{ is a reflection of } A'\}$. Assume by induction that such defined q is in P_γ.

If there is no reflection of A to $V_{\kappa_{\gamma+1}}$ over $V_{\sup(A\cap\kappa_{\gamma+1})}$ in $m(p)$, then we consider all possible reflections A' of A to $V_{\kappa_{\gamma+1}}$ over $V_{\sup(A\cap\kappa_{\gamma+1})}$. For every such A' define a condition in P_γ as above, and set q to be their join in $RO(P_\gamma)$.

Case 2. *$\kappa_\gamma \in A$ and κ_γ is an inaccessible cardinal.*

Proceed as in the previous case, only look for reflections of A to V_{κ_γ} over $V_{\sup(A\cap\kappa_\gamma)}$.

Case 3. *$\kappa_\gamma \notin A$.*

Then, let C be a countable model such that $C \prec V_{\kappa_\tau}$, if κ_τ is an inaccessible, and $C \prec V_{\kappa_{\tau+1}}$, otherwise, be so that $m(p) \in C$, $\kappa_\gamma \in C$ and if κ_γ is a singular cardinal, then also $\kappa_{\gamma+1} \in C$. Now reflect C to $V_{\kappa_{\gamma+1}}$ over $V_{\sup(C\cap\kappa_{\gamma+1})}$, if κ_γ is an accessible cardinal and V_{κ_γ} over $C \cap V_{\kappa_\gamma}$, if κ_γ is an inaccessible cardinal.

Take the join over all the possibilities. Denote it by q_C. Finally, let q be the join of q_C's over all C's as above.

Let us continue now with the requirements on p.

(c) For each $\gamma < \tau$ we specify a sequence of models from $m(p)$ which are relevant for (or stand over) the coordinate γ.

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[1] I.e. A, B form a splitting pair. We do not require here the third model X which is the immediate successor of A and B, and such that $\langle X, A, B\rangle$ form a splitting triple. Allowing such pair simplifies a bit the definition of a forcing projection below.

[2] Note that if the transitive collapse of $\mathfrak{M}_{\gamma+1}$ belongs to A, then $\kappa_{\gamma+1} \in A$, and actually $\langle \kappa_\xi \mid \xi \leq \gamma\rangle \in A$.

If κ_γ is a singular cardinal, then Q_γ is a trivial forcing, then let this sequence be empty.
Suppose that κ_γ is regular and, hence, inaccessible.
Let G_γ be a generic subset of P_γ with $\pi_{\tau\gamma}(p) \in G_\gamma$.
Consider the set $Z_\gamma(p) := \{A[G_\gamma] \mid A \in m(p)$ and $P_\gamma, Q_\gamma \in A[G_\gamma]\}$.
We require that the set $Z_\gamma(p) \restriction \gamma := \{A[G_\gamma] \cap V_{\kappa_{\gamma+1}+1}[G_\gamma] \mid A[G_\gamma] \in Z_\gamma\}$ satisfy the following conditions:

i. $Z_\gamma(p) \restriction \gamma$ forms a finite structure with pistes,

ii. if $\gamma \in s(p)$, then p_γ is a minimal $(Z_\gamma(p) \restriction \gamma, Q_\gamma)-$semi-generic.

(d) Let us state now more requirements that are relevant for limit stages.
It is easier to formulate it here then it was in [5], since instead of $\in -$increasing sequences, finite structures with pistes are used now.

i. If $s(p) = \emptyset$, then $m(p)$ is a finite structure with pistes.

ii. If $s(p) \neq \emptyset$, then $m(p)$ is an increasing union of $\langle m(p)_\gamma \mid \gamma \in s \cup \{0\}\rangle$ such that

A. $m(p)_0$ is a finite structure with pistes,

B. for every $\gamma \in s$ let γ' be the first element of $s(p)$ above γ, if it exists. If there is no such γ', i.e. if γ is the last element of $s(p)$, then we replace below $\kappa_{\gamma'+1}$ by κ.
We require that the set $\{A[G_\gamma] \cap V_{\kappa_{\gamma'}+1}[G_\gamma] \mid A \in m(p)_\gamma$ and $P_\gamma, Q_\gamma \in A[G_\gamma]\}$ is a finite structure with pistes in $V[G_\gamma]$[3].

Define the order on P_τ in the obvious fashion.

Definition 2.2 Let $\tau \leq \kappa$ and $p, p' \in P_\tau$. Set $p \geq p'$ iff

1. $m(p) \supseteq m(p')$,

2. $s(p) \supseteq s(p')$,

3. for every $\gamma \in s(p') \cup \{0\}, s(p')_\gamma \subseteq s(p)_\gamma$,

4. if $\delta \in s(p) \setminus s(p')$ and γ, γ' are two successive members of $s(p') \cup \{0, \kappa\}$ such that $\gamma < \delta < \gamma'$, then $s(p')_\gamma \subseteq s(p)_\delta$,

[3]Instead of closure of models required in [4], we require the closure of corresponding members of $m(p)$. This are models in V and $V \models$GCH. Alternatively, approachability, internal clubs can be used as such replacement.

5. for every $\beta \in s(p')$ we have $\pi_{\tau\beta}(p) \Vdash_{P_\beta} \underset{\sim}{p}_\beta \geq_{\underset{\sim}{Q}_\beta} \underset{\sim}{p}'_\beta$.

The next lemma repeats those of [5].

Lemma 2.3 *Let $\rho \leq \tau \leq \kappa$. Then the forcing P_ρ is a complete subforcing of P_τ.*

The arguments that show that the iteration is $\leq \eta-$semi-proper follow mostly those of [5]. Let us address only two new points that appear in the present context, i.e. once non-transitive models of different sizes are around.

Lemma 2.4 *The forcing P_α is $\{\omega\}-$semi-proper (i.e. semi-proper), for every $\alpha \leq \kappa$.*

Lemma 2.5 *Let $\eta', \omega < \eta' \leq \eta$ be a regular cardinal. Then the forcing P_α is $\{\eta'\}$–semi-proper, for every $\alpha \leq \kappa$.*

Proof of 2.4. The proof repeats those of [5]. Thus, a countable $M \prec \langle V_\chi, \in, \trianglelefteq, \kappa \rangle$ is picked, for some $\chi > \kappa$ large enough. A condition $r \in P_\alpha$, with $r, P_\alpha \in M$, is extended basically by adding M. Further extension, which is supposed to decide a value of a name (in M) of a countable ordinal, made inside M. The new point here is that such extension may have new uncountable non-transitive models $X \in M$. It is possible that some $\gamma \in s(p)$ which is not in M and even not reachable from M is in one this models X. Such situation may already occur once $\alpha = \omega_1$, since $M \cap \omega_1 < \omega_1$, but every relevant uncountable model $X \in M$ includes ω_1.
In order to overcome this difficulty, we use that p_γ is minimal semi-generic, which means in particular that after adding X's, p_γ extends to a minimal semi-generic condition over a larger finite structure with pistes, by 1.11(2a).
□
Proof of 2.5. Elementary submodel M of size η' is used instead of a countable. It is added to a condition as in the countable case. A value of a name $\underset{\sim}{\mu} \in M$ of an ordinal $< \eta'^{+}$ is decided and then everything is reflected into M. The reflection process adds new models. It may add new models of sizes $> \eta'$, as well. Clearly, such models cannot be contained in M (they only belong to it). So some elements of $s(p) \setminus M$ can appear in this models. Here we appeal again to the minimal semi-genericity condition over a larger finite structure with pistes, by 1.11(2a).
□

Lemma 2.6 *The forcing P_κ preserves κ.*

Proof. It repeats the proof of Lemma 1.8 of [5] only replace a countable model A there by a model of cardinality η.
□

Now, if κ is a supercompact and a Laver function $F : \kappa \to V_\kappa$ supplies semi-proper forcings, then $\leq \eta$–strongly piste structure semi-proper SPFA will hold in $V[G(P_\kappa)]$.

3 Examples of strongly piste structures semi-proper.

Let $\eta \geq \omega_1$ be a regular cardinal.

Recall that a set with two partial orders $\langle \mathcal{P}, \leq, \leq^* \rangle$ is called a *Prikry type forcing notion* ([2]) iff it satisfies the following two conditions:

1. $\leq \subseteq \leq^*$.

2. (The Prikry condition) For every $p \in \mathcal{P}$ and a statement σ of the forcing language of $\langle \mathcal{P}, \leq \rangle$ there is $p^* \geq^* p$ deciding σ.

   Let us add two more conditions in order to insure $\{\leq \eta\}$–strongly piste structures semi-properness.

3. $\langle \mathcal{P}, \leq^* \rangle$ is η^+-closed,

4. for every $p \in \mathcal{P}$ if $p_1, p_2 \geq^* p$ then p_1, p_2 are $\leq^* -$compatible, i.e. there is $q \geq^* p_1, p_2$.

Let us call $\langle \mathcal{P}, \leq, \leq^* \rangle$ which satisfies (1)-(4) above a *strongly Prikry type forcing notion* .

Clearly, Prikry, Magidor, Radin, their supercompact and tree versions are strongly Prikry type forcing notion, as well the Magidor iterations of such forcing notions, once the measures involved are at least η^+-complete.
Let us argue that such forcings are $\{\leq \eta\}$–strongly piste structures semi-proper forcings.

Lemma 3.1 *Strongly Prikry type forcing notions are $\{\leq \eta\}$–strongly piste structures semi-proper.*

Proof. Let $\langle \mathcal{P}, \leq, \leq^* \rangle$ be a strongly Prikry type forcing notion, $p \in \mathcal{P}$ and $\mathcal{A}$ be a finite structure with pistes with models of cardinalities $\leq \eta$. Let $p \in \mathcal{P}$ be such that for every model A, if $\langle \mathcal{P}, \leq, \leq^* \rangle \in A$, then $p \in A$, as well. For example, take p to be the weakest element $0_{\mathcal{P}}$.

Pick a model A in $\mathcal{A}$ with $\langle \mathcal{P}, \leq, \leq^* \rangle \in A$. Let $\underset{\sim}{\xi} \in A$ be a name of an ordinal $\leq \eta$. Then, by elementarity, there is $p_{\underset{\sim}{\xi} A} \in A, p_{\underset{\sim}{\xi} A} \geq^* p$ which decides $\underset{\sim}{\xi}$. Denote the decided value by ξ. So, $\xi \in A$.
Do this for every $\underset{\sim}{\xi} \in A$. Then we will have a sequence $\langle p_{\underset{\sim}{\xi} A} \mid \underset{\sim}{\xi} \in A\rangle$ of the length η of $\leq^*$ −extensions of p.
Let q_A be a $\leq^*$ −upper bound of it. Such q_A exists by (3),(4) of the definition of strongly Prikry type forcing notion. Clearly, it is $(A, \mathcal{P})$−semi-generic.
Do this for each A in $\mathcal{A}$. Let q be a $\leq^*$ −upper bound of such q_A's.
Then q will be $(\mathcal{A}, \mathcal{P})$−semi-generic.

We like to show now that q is also a minimal semi-generic for $\mathcal{A}$.
Let $\mathcal{B}$ be an extension of $\mathcal{A}$. Suppose B is a model of $\mathcal{B}$ with $\langle \mathcal{P}, \leq, \leq^* \rangle \in B$ that does not appear in $\mathcal{A}$. Then $p \in B$, since p.Define q_B exactly as above. Let r a $\leq^*$ −upper bound of such q_B's. Then r and q are $\leq^*$ −compatible as they both are $\leq^*$ −extensions of p. Let $s \geq^* q, r$. Then s is $(\mathcal{B}, \mathcal{P})$−semi-generic.
Moreover, the same argument shows that s is minimal $(\mathcal{B}, \mathcal{P})$−semi-generic.

Suppose now that p is a minimal $(\mathcal{A}, \mathcal{P})$−semi-generic. Let $q \geq p$. We extend then $\mathcal{A} = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ to $\mathcal{A}_1 = \langle\langle A_1^{0\tau}, A_1^{1\tau}, A_1^{1\tau lim}, C_1^\tau\rangle \mid \tau \in s\rangle$, where $\mathcal{A}_1$ is an end-extension of $\mathcal{A}$ such that for every $\alpha, \beta \in s$

- $A_1^{0\alpha}$ a non-limit and not potentially limit model,
- $\alpha < \beta$ implies $A_1^{0\alpha} \in A_1^{0\beta}$,
- $q \in A_1^{0\omega}$.

Define now $q^* \geq^* q$ to be a $\leq^*$ −upper bound of such $q_{A_1^{0\alpha}}$'s, $\alpha \in s$, where $q_{A_1^{0\alpha}}$ is defined as q_A above only working above q instead of p.
We claim that q^* is a minimal semi-generic for $\mathcal{A}_1$.
Let $\mathcal{B}$ be an extension of $\mathcal{A}_1$. Let B be a new model in $\mathcal{B}$. By the definition of extension of structures with pistes, then either B is contained in a model of $\mathcal{A}$ or $A_1^{0\alpha} \in B$, for every $\alpha \in s$, and then, in particular, $A_1^{0\omega} \subseteq B$.

Claim. *Assume that B is contained in a model of $\mathcal{A}$. Then p is $(B, \mathcal{P})$−semi-generic.*
Proof. Let $\underset{\sim}{\xi} \in B$ be a name of an ordinal $\leq \eta$. Let A be a model of $\mathcal{A}$ with $B \subseteq A$. Then $\underset{\sim}{\xi} \in A$, and so, for some $\xi \in A$, $p \Vdash \underset{\sim}{\xi} = \check{\xi}$.
There is $r \geq p$ which is $(\mathcal{B}, \mathcal{P})$−semi-generic, by minimality of p. In particular, r is $(B, \mathcal{P})$−semi-generic. So, $r \Vdash \underset{\sim}{\xi} = \check{\xi}'$, for some $\xi' \in B$. But $r \geq p$, hence $\xi' = \xi$. In

particular, $\xi \in B$.
Since $\underset{\sim}{\xi} \in B$ arbitrary, the above shows that indeed p is $(B, \mathcal{P})-$semi-generic.
$\square$ of the claim.

Suppose now that $A_1^{0\omega} \subseteq B$.
Recall that $q \in A_1^{0\omega}$ and so, $q \in B$.
Proceed now as above and find $q_B \geq^* q$ which is $(B, \mathcal{P})-$semi-generic.
Let t be a $\leq^*$ $-$upper bound of such q_B's with B as in Case 2. Then $t \geq^* q$, also $q^* \geq q$.
Hence by t, q^* are $\leq^*$ $-$compatible. Let $q^{**} \geq^* t, q^*$.
In addition, $q^{**} \geq p$, since $q \geq p$.
Then q^{**} will be $(\mathcal{B}, \mathcal{P})-$semi-generic.
$\square$

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References

[1] M. Foreman, M. Magidor and S. Shelah, Martin Maximum, Ann. of Math.

[2] M. Gitik, Prikry-Type forcing notions, in Handbook of Set Theory, Foreman, Kanamory eds, Springer 2010, pp.1351-1447.

[3] M. Gitik, Changing cofinality and the Nonstationary Ideal, Israel Journal of Math.

[4] M. Gitik, Short extenders forcings–doing without preparations, www.math.tau.ac.il/~gitik/.

[5] M. Gitik and M. Magidor, SPFA by finite conditions, Archive Math. Logic

[6] W. Mitchell, Adding closed unbounded subsets of ω_2 with finite conditions, Norte Dame Journal of Formal Logic, 46(3), 2005, pp.357-371.

[7] I. Neeman, Forcing with sequences of models of two types.

[8] I. Neeman, Forcing with side conditions, Oberwolfach, 2011.

[9] S. Shelah, Proper and Improper forcing, Springer 1998.

[10] S. Shelah, Cardinal Arithmetic.