More on Prikry forcings with non-normal ultrafilters.

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Abstract

We continue here the study of subforcing of the Prikry forcing started in [5] and then in [1].

1 Introduction.

We deal here with Prikry forcings with non-normal ultrafilters over κ (including tree Prikry forcings with different ultrafilters). Note that such forcing may add various new subsets to κ. For example start with κ which is a $\kappa-$compact cardinal. In [1], an example of a Prikry forcing which adds a Cohen generic over V subset was produced starting just from a measurable. Clearly, it cannot be equivalent to a Prikry forcing since the Cohen forcing preserves cofinalities and Prikry changes the cofinality of κ to ω.

Our aim here will be to study situations where in $V[A]$, κ changes its cofinality, for some set A of ordinals in a Prikry extension.

Let κ be a measurable cardinal and let $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega} \rangle$ be a tree consisting of $\kappa-$complete non-trivial ultrafilters over κ.

Denote by $P(\mathbb{U})$ the Prikry forcing with $\mathbb{U}$. Let C be a Prikry sequence for $P(\mathbb{U})$.

Our aim is to show the following:

Theorem 1.1 *Let A be a set of ordinals in $V[C]$ of size κ. Then the following are equivalent:*

1. *κ changes its cofinality in $V[A]$;*
2. *A is equivalent to a Prikry forcing with $\mathbb{W}$, for some tree $\mathbb{W}$ consisting of ultrafilters over κ Rudin-Keisler below some of those from $\mathbb{U}$.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious.
Our tusk will be to show (1) $\Rightarrow$ (2). So assume (1).

Clearly, the only possible value for the cofinality κ in $V[A]$ is ω, since $V[C]$ does not add new bounded subsets of κ. So, let $\langle \beta_n \mid n < \omega \rangle$ be a cofinal sequence to κ in $V[A]$.

2 Subsets of κ.

Let us assume first that $A \subseteq \kappa$.
Then for every $\xi < \kappa, A \cap \xi \in V$. In particular, for every $n < \omega$, $A \cap \beta_n \in V$, and so can be codded (in V) by an ordinal $\alpha_n < 2^{\beta_n}$.
Now, obviously, we have

$$V[A] = V[\langle \alpha_n \mid n < \omega \rangle].$$

Hence it is enough to prove (2) for $\langle \alpha_n \mid n < \omega \rangle$.

Let us use the following result from [1]:

Theorem 2.1 *Let $\langle \alpha_k \mid k < \omega \rangle \in V[C]$ be an increasing cofinal in κ sequence. Then $\langle \alpha_k \mid k < \omega \rangle$ is a Prikry sequence for a sequence in V of $\kappa-$complete ultrafilters which are Rudin -Keisler below $\langle U_n \mid n < \omega \rangle$.*
Moreover, there exist a non-decreasing sequence of natural numbers $\langle n_k \mid k < \omega \rangle$ and a sequence of functions $\langle F_k \mid k < \omega \rangle$ in V, $F_k : [\kappa]^{n_k} \to \kappa$, $(k < \omega)$, such that

1. *$\alpha_k = F_k(C \restriction n_k)$, for every $k < \omega$.*
2. *Let $\langle n_{k_i} \mid i < \omega \rangle$ be the increasing subsequence of $\langle n_k \mid k < \omega \rangle$ such that*
   (a) *$\{n_{k_i} \mid i < \omega\} = \{n_k \mid k < \omega\}$, and*
   (b) *$k_i = \min\{k \mid n_k = n_{k_i}\}$.*

   Set $\ell_i = |\{k \mid n_k = n_{k_i}\}|$. Then $\langle F_k(C \restriction n_{k_i}) \mid i < \omega, n_k = n_{k_i} \rangle$ will be a Prikry sequence for $\langle W_i \mid i < \omega \rangle$, i.e. for every sequence $\langle A_i \mid i < \omega \rangle \in V$, with $A_i \in W_i$, there is $i_0 < \omega$ such that for every $i > i_0$, $\langle F_k(C \restriction n_{k_i}) \mid i < \omega, n_k = n_{k_i} \rangle \in A_i$, where each W_i is an ultrafilter over $[\kappa]^{\ell_i}$ which is the projection of $U_{n_{k_i}}$ by $\langle F_{k_i}, ..., F_{k_i+\ell_i-1} \rangle$.

Let us replace functions F_k's by one to one functions.

Start with $i = 1$.
So, W_i is an ultrafilter over $[\kappa]^{\ell_1}$ which is the projection of $U_{n_{k_1}}$ by $\langle F_{k_1}, ..., F_{k_1+\ell_1-1} \rangle$.
Consider the elementary embeddings

$$j_1 : V \to N_1 \simeq {}^{n_{k_1}}V/U_{n_{k_1}}$$

and

$$j_1' : V \to N_1' \simeq {}^{n_{k_1}}V/W_1.$$

Define

$$\sigma_1 : N_1' \to N_1$$

by setting

$$\sigma_1(j_1'(g)([id]_{W_1}) = j_1(g)(\langle [F_{k_1}]_{U_{n_{k_1}}}, ..., [F_{k_1+\ell_1-1}]_{U_{n_{k_1}}} \rangle).$$

Find in N_1 the smallest set $\vec{a}_1$ of generators such that for some $g_1 : [\kappa]^{|\vec{a}|} \to [\kappa]^{\ell_1}$ we have

$$j_1(g_1)(\vec{a}) = \langle [F_{k_1}]_{U_{n_{k_1}}}, ..., [F_{k_1+\ell_1-1}]_{U_{n_{k_1}}} \rangle.$$

Set

$$U(1) = \{X \subseteq [\kappa]^{|\vec{a}|} \mid \vec{a} \in j_1(X)\}.$$

So, $U(1)$ is a $\kappa-$complete ultrafilter generated by $\vec{a}$ from $U_{n_{k_1}}$. Moreover, $U(1)$ is Rudin - Keisler equivalent to W_1, since they have the same ultrapower. In particular, the map g_1 can be taken one to one.

Continue now to W_2. We proceed in the similar fashion and find the smallest set of generators $\vec{a}_2$ of $U_{n_{k_2}}$ which define a $\kappa-$complete ultrafilter $U(2)$, Rudin - Keisler equivalent to W_2, as witnessed by a one to one function g_2 etc.

Let us now describe how to "unpack" a Prikry (tree) forcing from $U(n)$'s.
Let us deal with ultrafilters Rudin -Keisler below U_2. The general case is similar only notation are more complicated.
Consider the ultrapower by $U_{\langle\rangle}$:

$$i_{\langle\rangle} : V \to M_{\langle\rangle}.$$

The sequence $i_{\langle\rangle}(\langle U_{\langle\nu\rangle} \mid \nu < \kappa\rangle)$ will have the length $\kappa_1 := i_{\langle\rangle}(\kappa)$.
Let $U_{\langle [id]_{U_{\langle\rangle}}\rangle}$ be its $[id]_{U_{\langle\rangle}}$ ultrafilter in $M_{\langle\rangle}$ over $i_{\langle\rangle}(\kappa)$. Consider its ultrapower

$$i_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}} : M_{\langle\rangle} \to M_{\langle [id]_{U_{\langle\rangle}}\rangle}$$

Set

$$i_2 = i_{U_{\langle [id]_{U_{\langle\rangle}}\rangle}} \circ i_{\langle\rangle}.$$

Then

$$i_2 : V \to M_{\langle [id]_{U_{\langle\rangle}}\rangle}.$$

Let now

$$\vec{\rho}, \vec{\mu}, \vec{\rho} \leq [id]_{U_{\langle\rangle}} < \kappa_1 \leq \vec{\mu} \leq [id]_{U_{\langle [id]_{U_{\langle\rangle}} \rangle}}$$

be generators of i_2 and let

$$W = \{X \subseteq V_\kappa \mid \langle \vec{\rho}, \vec{\mu} \rangle \in i_2(X)\},$$

i.e. W is an ultrafilter below U_2 generated by $\langle \vec{\rho}, \vec{\mu} \rangle$. Consider in $M_{\langle\rangle}$ an ultrafilter over $[\kappa_1]^{|\vec{\mu}|}$ in $M_{\langle\rangle}$,

$$W' = \{Z \subseteq \kappa_1 \mid \vec{\mu} \in i_{U_{\langle [id]_{U_{\langle\rangle}} \rangle}}(Z)\}.$$

Pick the smallest possible set of generators $\vec{\rho'}$ of $i_{\langle\rangle}$ such that for some function h on $[\kappa]^{|\vec{\rho'}|}$ such that $i_{\langle\rangle}(h)(\vec{\rho'}) = W'$.
If $\vec{\rho'} < \kappa$, then h is a constant function mod $U_{\langle\rangle}$. So, $W' = i_{\langle\rangle}(W'')$. Let

$$W_{\vec{\rho}} = \{X \subseteq [\kappa]^{\vec{\rho}} \mid \vec{\rho} \in i_{\langle\rangle}(X)\}.$$

Then $W_{\vec{\rho}} \times W''$ will be as desired.
Suppose that $\vec{\rho'} \geq \kappa$. Then there is $E \in W_{\vec{\rho'}}$ such that for every $\nu \in A$, $h(\nu) = W^\nu$, for some $\kappa-$complete ultrafilter W^ν over $[\kappa]^{|\vec{\mu}|}$. Also, by the choice of $\vec{\rho'}$, h is one to one.
We are ready now to define the tree T of hight 2 which corresponds to W. Set its first level to be a set in $W_{\vec{\rho}^\frown\vec{\rho'}}$ which projection to $W_{\vec{\rho'}}$ is a subset of E. Now, for every $\tau \in Lev_1(T)$, let $Suc_T(\langle\tau\rangle)$ be a set in W_ν once τ projects to ν.

This basically completes the case of $A \subseteq \kappa$.

□

3 Larger sets, few generators.

We continue the argument of the previous section.
Suppose now that $A \subseteq \kappa^+$.
Assume that κ changes its cofinality already in $V[A \cap \kappa]$. Just otherwise, working in $V[A]$, we can rearrange A in order to make the above happen.

Note that at least starting from V of the form $L[V_\kappa, U]$, it is impossible that in each $V[A \cap \alpha]$ κ is regular and it changes its cofinality only in $V[A]$. The standard argument for $2^\kappa = \kappa^+$ shows this.
The next construction may be of some interest in this respect.
We will define an iteration of distributive forcing notions of size κ of given in advance length $\delta < \kappa^+$ of cofinality ω or κ in V, such that

1. κ remains regular at each intermediate stage of the iteration,

2. the full iteration collapses κ to ω,

3. the Prikry extension adds $A \subseteq \delta$ such that

   (a) κ is singular in $V[A]$,

   (b) for every $\alpha < \delta$, $A \cap \alpha$ codes in a very simple way a generic for the iteration up to α, and so, κ is regular in $V[A \cap \alpha]$.

Suppose for simplicity that $\delta = \kappa$. We proceed as follows.
Let

$$\vec{U} = \langle U(\eta, \delta) \mid \eta \in \mathrm{dom}(\vec{U}), \delta < o^{\vec{U}}(\eta) \rangle$$

be a coherent sequence of ultrafilters such that

1. $\kappa = \max(\mathrm{dom}(\vec{U}))$,

2. $o^{\vec{U}}(\kappa) = \kappa \cdot \kappa$,

3. $U(\kappa, 0)$ concentrates on η's which are η^+-supercompact.[1]

Now we iterate in Backward Easton way the forcings which change cofinalities below κ according to $o^{\vec{U}}$ and also on a set of η's of $U(\kappa, 0)$ measure one, we change both cofinalities of η and η^+ to ω using the η^+-supercompactness of η. We refer to [2] for this type of iteration. Now, for every $\alpha < \kappa$, let

$$S_\alpha = \{\eta < \kappa \mid o^{\vec{U}}(\eta) \in [\kappa \cdot \alpha, \kappa \cdot (\alpha + 1))\}.$$

Set

$$S_{-1} = \{\eta < \kappa \mid \eta \notin \mathrm{dom}(\vec{U})\}.$$

Let G be a generic. Then, by $\kappa-$c.c. each S_α will be stationary and fat.
Our main interest will be in the extension $\mathcal{U} := \bar{U}(\kappa, 0)$ of $U(\kappa, 0)$.

Claim 1 *In $V[G]^{P(\mathcal{U})}$, for every $\alpha < \kappa$, there is C_α such that C_α is a club in $V[G]$ generic over $V[G]$ for the natural forcing of adding a club that turns all $S_\beta, \beta \leq \alpha$ into non-stationary and leaves all S_β's with $\beta > \alpha$ stationary.*

[1]It will work with η^+-strongly compact cardinal or, even, with $\eta-$compact cardinal instead.

Proof. Such forcing is distributive of size κ, so using the supercompact part of the iteration, it is not hard to construct such C_α's.
$\square$ of the claim.

Now set

$$A_\alpha = \kappa \cdot \alpha \cup \{\kappa \cdot \alpha + \xi \mid \xi \in C_\alpha\},$$

for every $\alpha < \kappa$. Set

$$A = \bigcup_{\alpha<\kappa} A_\alpha.$$

Claim 2 κ is regular in $V[A \cap \alpha]$ for every $\alpha < \kappa$.

Proof. Just $A \cap \alpha$ is a generic (after the obvious decoding) for a $\kappa-$distributive forcing.
square of the claim.

Claim 3 κ has cofinality ω in $V[A]$.

Proof. Suppose otherwise. Let S be a stationary subset of κ. Define a regressive function f on S as follows:

$$f(\nu) = 0, \text{ if } \nu \notin \operatorname{dom}(\vec{U}),$$

$$f(\nu) = \alpha, \text{ if } \nu \in \operatorname{dom}(\vec{U}) \text{ and for some } \mu < \kappa, o(\nu) = \kappa \cdot \alpha + \mu.$$

It is a regressive function since there is no $\eta < \kappa$ with $o^{\vec{U}}(\eta) = \eta \cdot \eta$. Then there are $S' \subseteq S$ stationary and $\alpha' < \kappa$ such that for every $\nu \in S'$, $f(\nu) = \alpha'$. But this is impossible, since the club $C_{\alpha'+1}$ is disjoint to S'.
Contrudiction.
$\square$ of the claim.

3.1 Larger sets, few generators.

Suppose that the number of generators of each $U_n, n < \omega$ is less than κ, then it is possible to stabilize the $\omega-$sequence for $A \cap \alpha$'s. Then the continuation is as in [3]. So we obtain the following:

Theorem 3.1 *Suppose that non of the ultrafilters in $\mathbb{U}$ has more than $\kappa-$generators. Let A be a set of ordinals in $V[C]$. Then the following are equivalent:*

1. *κ changes its cofinality in $V[A]$;*

2. *A is equivalent to a Prikry forcing with $\mathbb{W}$, for some tree $\mathbb{W}$ consisting of ultrafilters over κ Rudin-Keisler below some of those from $\mathbb{U}$.*

In view of [4], in order to have κ or more than $\kappa-$many generators, the strength $\kappa = \sup(\{o(\beta) \mid \beta < \kappa\})$ is needed. So, we have:

Theorem 3.2 *Suppose that there is no inner model in which $\kappa = \sup(\{o(\beta) \mid \beta < \kappa\})$. Let A be a set of ordinals in $V[C]$. Then the following are equivalent:*

1. *κ changes its cofinality in $V[A]$;*
2. *A is equivalent to a Prikry forcing with $\mathbb{W}$, for some tree $\mathbb{W}$ consisting of ultrafilters over κ Rudin-Keisler below some of those from $\mathbb{U}$.*

3.2 Larger sets, at least $\kappa-$many generators.

Each $A\cap\alpha$, for $\alpha < \kappa^+$ is equivalent to some subforcing of $P(\mathbb{U})$. If this subforcings stabilize, then the arguments of [3] apply.
Assume that this does not happen.
We deal with a special, but typical case of such situation. Suppose for simplicity that we have a single $\kappa-$complete ultrafilter $\mathcal{U}$ over κ instead of $\mathbb{U}$.
Let $\langle \rho_\alpha \mid \alpha < \kappa\rangle$ be increasing sequence of generators of $\mathcal{U}$ such that the ultrafilters $\mathcal{U}_{\rho_\alpha} := \{X \subseteq \kappa \mid \rho_\alpha \in i_{\mathcal{U}}(X)\}$ is Rudin-Keisler increasing.
Suppose that $\kappa = \rho_0$, i.e. $\mathcal{U}_{\rho_0}$ is the smallest normal measure.
Assume that its Prikry sequence $\langle \kappa_n^{nor} \mid n < \omega\rangle$ appears in $V[A \cap \kappa]$.
Finally, the forcing equivalent to $A \cap \alpha, \kappa \leq \alpha < \kappa^+$, is determined by a function $f_\alpha : \kappa \to \kappa, f_\alpha \in V$ as follows:

It is a tree Prikry forcing with trees T such that

1. $Lev_0(T) \in \mathcal{U}_{\rho_{f_\alpha(0)}}$,
2. if $\langle \nu_1, ..., \nu_n\rangle \in T$, then $Suc_T(\langle \nu_1, ..., \nu_n\rangle) \in \mathcal{U}_{\rho_{f_\alpha(\nu_n^{nor})}}$, where ν_n^{nor} is the projection of ν_n to the least normal measure.

If $\alpha < \beta < \kappa^+$, then $A\cap\alpha$ is in $V[A\cap\beta]$. So the forcing equivalent to $A\cap\alpha$ is a subforcing of the forcing equivalent to $A \cap \beta$.
We assume that just $f_\alpha < f_\beta$ mod $\mathcal{U}_\kappa$.
Now, the exact upper bound of

$$\langle\langle f_\alpha(\kappa_n) \mid n < \omega\rangle\alpha < \kappa^+\rangle$$

is some $\langle \lambda_n \mid n < \omega \rangle$ which corresponds to non-generators or to generators say of normal measure (measures).
This eliminates the possibility that there is an obvious subforcing equivalent to A.
From here the case of $\kappa-$generators is handled as those with κ^+-generators.

3.3 Larger sets, at least κ^+-many generators.

Assume, so that $A\cap\alpha$'s never stabilize, and hence, the number of generators of U_n's is above κ.

Let $\alpha, \kappa \leq \alpha < \kappa^+$. We first attach to $A \cap \alpha$ an $\omega-$sequence $\langle \eta_n^\alpha \mid n < \omega \rangle$ in the following canonical fashion:
first use the least in some fixed in advance well ordering of a large enough portion of V map $r_\alpha : \alpha \leftrightarrow \kappa$. Then consider $A_\alpha = r_\alpha{}'' A \cap \alpha$. Now use the initial sequence $\langle \beta_n \mid n < \omega \rangle$ to code A_α into an $\omega-$sequence, as it was done for subsets of κ in the beginning of the proof. Set $\langle \eta_n^\alpha \mid n < \omega \rangle$ to be such sequence.
Then we have

$$V[\langle \beta_n \mid n < \omega \rangle, A_\alpha] = V[\langle \eta_n^\alpha \mid n < \omega \rangle].$$

Consider now

$$\vec{\eta} = \langle \langle \eta_n^\alpha \mid n < \omega \rangle \mid \kappa \leq \alpha < \kappa^+ \rangle.$$

Clearly, we have

$$V[A] \supseteq V[\vec{\eta}] \supseteq \bigcup_{\alpha < \kappa^+} V[A_\alpha],$$

since the definition of $\vec{\eta}$ carried out inside $V[A]$.

Note also that for every $n_0 < \omega, X \subseteq \kappa^+, X \in V[A]$ unbounded we have

$$V[\langle \langle \eta_n^\alpha \mid n_0 \leq n < \omega \rangle \mid \kappa \leq \alpha \in X \rangle] \supseteq \bigcup_{\alpha < \kappa^+} V[A_\alpha].$$

Now, in $V[C]$, for every $\alpha < \kappa^+$, there is $n(\alpha)$, such that $\langle C(n) \mid n(\alpha) \leq n < \omega \rangle$ projects onto $\langle \eta_n^\alpha \mid n(\alpha) \leq n < \omega \rangle$, by the corresponding projections of U_n's.
Find $n_0 < \omega$, $X \subseteq \kappa^+$ stationary such that for every $\alpha \in X$ we have $n(\alpha) = n_0$.
Assume that there is $X^* \subseteq X, |X^*| = \kappa^+$ and $X^* \in V[A]$. By [6], it is consistent to have such X^* already in V.
Return back to $V[A]$. Then the following holds there:

for every $n \geq n_0$, there is $\xi_n < \kappa$ such that for every $\alpha \in X^*$

$$\pi_\alpha(\xi_n) = \eta_n^\alpha,$$

where π_α is the canonical projection to the sequence (i.e. to the measures of) $\langle \eta^\alpha_n \mid n < \omega \rangle$. Then

$$\bigcup_{\gamma < \kappa^+} V[\langle \beta_n \mid n < \omega \rangle, A \cap \gamma] \subseteq V[\langle \xi_n \mid n_0 \leq n < \omega \rangle] \subseteq V[A].$$

If $V[\langle \xi_n \mid n_0 \leq n < \omega \rangle] \neq V[A]$, then we proceed as in [3] and derive a contradiction.

So, we have the following conclusion:

Theorem 3.3 *Let A be a set of ordinals in $V[C]$ of size κ^+.*
Assume that for every $X \subseteq \kappa^+, |X| = \kappa^+$, in $V[C]$, there is $X^* \subseteq X, |X^*| = \kappa^+$ and $X^* \in V[A]$.
Then the following are equivalent:

1. *κ changes its cofinality in $V[A]$;*
2. *A is equivalent to a Prikry forcing with $\mathbb{W}$, for some tree $\mathbb{W}$ consisting of ultrafilters over κ Rudin-Keisler below some of those from $\mathbb{U}$.*

Let us continue further without the assumption of 3.3.

Consider the sequence

$$\vec{\eta} = \langle \langle \eta^\alpha_n \mid n < \omega \rangle \mid \kappa \leq \alpha < \kappa^+ \rangle$$

defined above. By the Shelah Trichotomy Theorem [7], it has an exact upper bound. Let $\vec{\eta}^* := \langle \eta^*_n \mid n < \omega \rangle$ be such a bound in $V[A]$.
Note that probably in $V[C]$ the exact upper bound for $\vec{\eta}$ is different (smaller).
Now, if $A \in V[\vec{\eta}^*]$ or $A \in V[\vec{\eta}^*, B]$ for a set B of size κ, then we are done.
Let us describe particular cases when this situation occurs.

Suppose the following:
W is a $\kappa-$complete ultrafilter over κ which has among its generators the following increasing sequence $\langle \theta_\alpha \mid \alpha < \kappa^+ \rangle$ with the property that if $\theta := \bigcup_{\alpha < \kappa^+} \theta_\alpha$, then in the ultrapower N_W of V by W there is Z of size κ^+ there such that $Z \supseteq \{\theta_\alpha \mid \alpha < \kappa^+\}$.
It is note hard to arrange this type situation using a $(\kappa, \kappa^{++})-$extender, etc.
Force now with $P(W)$. Then the Prikry sequence $\vec{\theta}$ for θ will be the exact upper bound of the Prikry sequences $\vec{\theta}_\alpha$ for θ_α's. In addition, using the canonical functions it is easy to see that each $\vec{\theta}_\alpha$ is in $V[\vec{\theta}]$.
The same phenomenon holds once, for example, $2^\kappa = \kappa^{++}$ and κ^+ above is replaced by κ^{++}. Only instead of the canonical functions, we use those that represent ordinals below κ^{++} in the ultrapower by the normal measure of W.

Let us sketch now two forcing construction below such that in the first we have an exact upper bound (in $V[C]$) for κ^+-many Prikry sequences which catches all of them and without the covering property in the ultrapower.
In the second the exact upper bound (in $V[C]$) for κ^+-many Prikry sequences does not catch any of them.

The first construction.

Start with a GCH model with an increasing Rudin - Keisler sequence $\langle W_\alpha \mid \alpha < \kappa^+\rangle$ of ultrafilters over κ. Assume that W_0 is a normal one.

Let $i_0 : V \to N_0$ be the elementary embedding by W_0, $i : V \to N$ the elementary embedding into the direct limit of $\langle W_\alpha \mid \alpha < \kappa^+\rangle$.
Denote by $k_0 : N_0 \to N$ the canonical embedding.
Take additional ultrapower. Apply $i_0(W_0)$ to N_0 and $i(W_0)$ to N.
Let $i_0^1 : V \to N_0^1$ be the result of the first and $i^1 : V \to N^1$ of the second. Denote by k_0^1 the obvious embedding of N_0^1 to N^1.
Now, force (with preparations below $G_{<\kappa}$) Cohen functions $g_\xi : \kappa \to \kappa, \xi < \kappa^+$. Let $g := \langle g_\xi \mid \xi < \kappa^+\rangle$.
We extend i_0 to $i_0^* : V[G_{<\kappa}, g] \to N_0[G_{<\kappa}, g, G_{[\kappa^+, i_0(\kappa))}]$. Next, extend i and k.
So, we will have

$$i^* : V[G_{<\kappa}, g] \to N[[G_{<\kappa}, g, G_{[\kappa^+, i_0(\kappa))}, G_{[i_0(\kappa), i(\kappa))}],$$

$$k^* : N_0[G_{<\kappa}, g, G_{[\kappa^+, i_0(\kappa))}] \to N[[G_{<\kappa}, g, G_{[\kappa^+, i_0(\kappa))}, G_{[i_0(\kappa), i(\kappa))}].$$

Now deal with the additional ultrapowers. We extend first i_0^1 to

$$i_0^{1*} : V[G_{<\kappa}, g] \to N_0^1[G_{<\kappa}, g, G_{[\kappa^+, i_0(\kappa))}, G_{[i_0(\kappa), i_0^1(\kappa))}].$$

Then use k_0^1 and the point wise image of $G_{[i_0(\kappa), i_0^1(\kappa)]}$ to generate
$N[[G_{<\kappa}, g, G_{[\kappa^+, i_0(\kappa))}, G_{[i_0(\kappa), i(\kappa))}]-$generic set in the interval $[i(\kappa), i^1(\kappa)]$. So we will have an extension of i^1:

$$i^{1*} : V[G_{<\kappa}, g] \to N[[G_{<\kappa}, g, G_{[\kappa^+, i_0(\kappa))}, G_{[i_0(\kappa), i(\kappa))}, G_{[i(\kappa), i^1(\kappa))}].$$

Consider $i^{1*}(g_\xi) : i^1(\kappa) \to i^1(\kappa)$, for every $\xi < \kappa^+$. Change one value of each of this functions by sending $i(\kappa)$ to the generator of W_ξ. Let j denotes the resulting embedding. Consider

$$U = \{X \subseteq \kappa \mid i(\kappa) \in j(X)\}.$$

Force with $P(U)$. The Prikry sequence for U will be the exact upper bound for Prikry sequence of extensions of W_ξ's and using g_ξ's one obtains each of them from those of U.

The second construction.

Let us modify the first construction a little.
Thus, instead of one additional ultrapower, we take now two. I.e. apply $i_0^1(W_0)$ to N_0^1 and $i^1(W_0)$ to N^1.
Let $i_0^2 : V \to N_0^2$ be the result of the first and $i^2 : V \to N^2$ of the second.
Then we proceed as in the first example - add generic Cohen function and extend the embeddings.
Only at the final stage, let us change one value of each of the Cohen functions by sending $i^1(\kappa)$ (instead of $i(\kappa)$) to the generator of W_ξ. Let j' denotes the resulting embedding. Consider

$$U' = \{X \subseteq \kappa \mid i^1(\kappa) \in j'(X)\}$$

and

$$U = \{X \subseteq \kappa \mid i(\kappa) \in j'(X)\}.$$

Force with $P(U')$. The Prikry sequence for U (not the main one for U') will be the exact upper bound for Prikry sequence of extensions of W_ξ's. However, now we will be unable to reconstruct the Prikry sequences of extensions of W_ξ's from the Prikry sequence for U.
The reason is that due to our particular extension of the initial embeddings, U is Rudin-Keisler equivalent to the extension of W_0 which strictly below each of the extensions of $W_\xi, 0 < \xi < \kappa^+$.

Back to the argument.

In V, for every $\alpha < \kappa^+$, there are $n_\alpha < \omega$ and T_α such that for every $t \in T_\alpha$ of the length n_α we have

$$\langle t, T_\alpha \rangle \Vdash \forall n \geq n_\alpha(\pi_\alpha(\underset{\sim}{C}(n)) = \underset{\sim}{\eta}_n^\alpha).$$

There will be a set $Z \subseteq \kappa^+$ consisting of κ^+-many α's with same n_α. Suppose for simplicity that this constant value is just 0.

Our assumption is that $V[A] \neq V[C]$.
Consider $P(\mathbb{U})/A$.
So it is a non-trivial forcing (over $V[A]$).

Then we will have conditions $\langle t, T \rangle \in P(\mathbb{U})/A$ such that for some $\nu \neq \nu'$,

$$\langle t^\frown \nu, T_{t^\frown \nu} \rangle \in P(\mathbb{U})/A, \langle t^\frown \nu', T_{t^\frown \nu'} \rangle \in P(\mathbb{U})/A,$$

$$\langle t^\frown \nu, T_{t^\frown \nu} \rangle \geq \langle t, T \rangle, \langle t^\frown \nu', T_{t^\frown \nu'} \rangle \geq \langle t, T \rangle.$$

Note that $\nu^{nor} = \nu'^{nor}$, where ξ^{nor} is the projection of ξ to the least normal measure of the corresponding level. Just $\langle C(n)^{nor} \mid n < \omega \rangle \in V[A]$.
Suppose for simplicity that t is just the empty sequence.

Now back in V, for almost all $\nu < \kappa$ there will be a name $\underset{\sim}{x}_\nu$ and a condition $p_\nu = \langle \langle \nu \rangle, R_\nu \rangle$ such that

$$p_\nu \Vdash \underset{\sim}{x}_\nu \text{ is the set of all } \nu'$$

as above (i.e. the set of all possible replacements of ν which do not effect $V[A]$).

Note that each $x_\nu \in V[A]$, since it is just definable there. Also, this are subsets of κ, hence there is a single Prikry sequence in $V[A]$ which adds all of them.

Suppose for a moment that x_ν's are in V, as well as the function $\nu \mapsto x_\nu$.
Define a projection map

$$\nu \mapsto \min(x_\nu).$$

So the Prikry sequence for the projection will be in $V[A]$, since the corresponding forcing over A will be trivial.
Assuming that there is no largest Prikry sequence in $V[A]$(i.e. one that catches every initial segment of A), we will have it below a final segment of sequences of $V[A]$.
Now pick two elements $\alpha < \beta$ of Z from this final segment. Shrink T_α and T_β if necessary. For every $\gamma < \alpha$ there will be $\nu \neq \nu'$ such that $\pi_\gamma(\nu) = \pi_\gamma(\nu')$ and $\pi_\alpha(\nu) = \pi_\alpha(\nu')$, but

$\pi_\beta(\nu) \neq \pi_\beta(\nu')$. Which is impossible. The existence of such ν, ν' follows due to the fact that the ultrafilters generated by γ and α are strictly below (in R-K order) the ultrafilter generated by β. So, in every set of measure one for β there will be elements like ν, ν'.

In the general case, i.e. once x_ν's are not in V, we will use the same idea. Proceed as follows: extend first p_ν to $p^*_\nu = \langle\langle\nu\rangle, R^*_\nu\rangle$ such that for some $y_\nu \subseteq \nu$,

$$p^*_\nu \Vdash y_\nu = \underset{\sim}{x}_\nu \cap \nu.$$

Claim 4 *Let $\rho \in y_\nu$. Then for every $\alpha < \kappa^+, n < \omega, \xi < \kappa, r, R \subseteq R^*_\nu$,*

$$\langle\langle\nu\rangle^\frown r, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi \text{ iff } \langle\langle\rho\rangle^\frown r, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi,$$

Proof. Suppose first that $\langle\langle\nu\rangle^\frown r, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi$.
If $\langle\langle\rho\rangle^\frown r, R\rangle \not\Vdash \underset{\sim}{\eta}^\alpha_n = \xi$, then for some r', R' with $\langle\langle\rho\rangle^\frown r^\frown r', R'\rangle \geq \langle\langle\rho\rangle^\frown r, R\rangle$ and $\xi' \neq \xi$,

$$\langle\langle\rho\rangle^\frown r^\frown r', R'\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi'.$$

Clearly, $\langle\langle\nu\rangle^\frown r^\frown r', R'\rangle \geq \langle\langle\nu\rangle^\frown r, R\rangle$. So,

$$\langle\langle\nu\rangle^\frown r^\frown r', R'\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi.$$

But, $\rho \in y_\nu$, hence the value of $\underset{\sim}{\eta}^\alpha_n$ cannot be effected by replacing ν with ρ. Contradiction.

The opposite direction is similar.

$\square$ of the claim.

Define a projection map

$$\nu \mapsto \min(y_\nu).$$

Note that $\rho \in y_\nu \cap y_\mu$, for some ν, μ, then for every $\alpha < \kappa^+, n < \omega, \xi < \kappa, r, R \subseteq R^*_\nu \cap R^*_\mu$,

$$(*)\langle\langle\nu\rangle^\frown r, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi \text{ iff } \langle\langle\rho\rangle^\frown r, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi \text{ iff } \langle\langle\mu\rangle^\frown r, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi.$$

Again the Prikry sequence for the projection will be in $V[A]$, since the corresponding forcing over A will be trivial.
Assuming that there is no largest Prikry sequence in $V[A]$(i.e. one that catches every initial segment of A), we will have it below a final segment of sequences of $V[A]$.

Denote the generator of this projection by γ.

Let now β be an element of Z above γ.
Find some $\nu \neq \nu'$ such that

1. $\nu, \mu \in T_\beta$,

2. $\pi_\gamma(\nu) = \pi_\gamma(\mu)$,

3. $\pi_\beta(\nu) \neq \pi_\beta(\mu)$.

There must be such ν, μ, since the ultrafilter generated by β is strictly above those of γ, so in each set of measure one there will be such elements.

Consider now two conditions $\langle\langle\nu\rangle, T_\beta \cap R^*_\nu\rangle$ and $\langle\langle\mu\rangle, T_\beta \cap R^*_\mu\rangle$. Then

$$\min(y_\nu) = \pi_\gamma(\nu) = \pi_\gamma(\mu) = \min(y_\mu).$$

It follows by (*) above that for every $\alpha < \kappa^+, n < \omega, \xi < \kappa, R \subseteq R^*_\nu \cap R^*_\mu$,

$$\langle\langle\nu\rangle, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi \text{ iff } \langle\langle\mu\rangle, R\rangle \Vdash \underset{\sim}{\eta}^\alpha_n = \xi.$$

In particular,

$$\langle\langle\nu\rangle, T_\beta \cap R^*_\nu \cap R^*_\mu\rangle \Vdash \underset{\sim}{\eta}^\beta_n = \xi \text{ iff } \langle\langle\mu\rangle, T_\beta \cap R^*_\nu \cap R^*_\mu\rangle \Vdash \underset{\sim}{\eta}^\beta_n = \xi.$$

Take now $n = 0$, then $\nu, \mu \in T_\beta$ implies that

$$\langle\langle\nu\rangle, T_\beta\rangle \Vdash \underset{\sim}{\eta}^\beta_0 = \pi_\beta(\nu) \text{ and } \langle\langle\mu\rangle, T_\beta\rangle \Vdash \underset{\sim}{\eta}^\beta_0 = \pi_\beta(\mu).$$

However, we have $\pi_\beta(\nu) \neq \pi_\beta(\mu)$. Which is impossible.
Contradiction to the assumption that γ is below of $\alpha's$ less than κ^+.

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