A remark on subforcings of the Prikry forcing

Abstract

We will show that every subforcing of the basic Prikry forcing is either trivial or isomorphic to the Prikry forcing with the same ultrafilter.

Let κ be a measurable cardinal and U a normal ultrafilter over κ. We will denote by $P(U)$ the basic Prikry forcing with U. Let us recall the definition.

Definition 0.1 $P(U)$ is the set of all pairs $\langle p, A\rangle$ such that

1. p is a finite subset of κ,
2. $A \in U$, and
3. $\min(A) > \max(p)$.

It is convenient sometimes to view p as an increasing sequence of ordinals.

Definition 0.2 Let $\langle p, A\rangle, \langle q, B\rangle \in P(U)$. Then $\langle p, A\rangle \geq \langle q, B\rangle$ iff

1. $p \cap (\max(q) + 1) = q$,
2. $A \subseteq B$, and
3. $p \setminus q \subseteq B$.

If $\langle p, A\rangle \geq \langle q, B\rangle$ and $p = q$, then $\langle p, A\rangle$ is called a direct extension of $\langle q, B\rangle$ and denote this by $\langle p, A\rangle \geq^* \langle q, B\rangle$.

Let G be a generic for $\langle P(U), \leq\rangle$. Then

$$C = \cup\{p \mid \exists A \quad \langle p, A\rangle \in G\}$$

is called a Prikry sequence. It is easy to reconstruct G form C, just note that

$$G = \{\langle p, A\rangle \in P(U) \mid C \subseteq p \cup A\}.$$

So $V[G] = V[C]$. By Mathias [**?**], every infinite subsequence C' of C is a Prikry sequence as well for a generic set

$$G' = \{\langle p, A\rangle \in P(U) \mid C' \subseteq p \cup A\}.$$

Our aim will be to prove the following:

Theorem 0.3 *Every subforcing of the Prikry forcing is either trivial or it is isomorphic to the Prikry forcing (with the same ultrafilter).*

Proof. Let C be a Prikry sequence (over V). It is enough to show that for every set A of ordinals in $V[C]$ there is a subsequence C' of C such that $V[A] = V[C']$. We will show this by induction on $\sup(A)$. The Prikry forcing $P(U)$ preserves all the cardinals, so it is enough to deal with A's such that $\sup(A)$ is a cardinal. Also, recall that $P(U)$ does not add new bounded subsets to κ. Hence the first interesting case is $\sup(A) = \kappa$.
Let us deal first with the following simple partial case.

Lemma 0.4 *Let $\vec{\alpha} = \langle \alpha_n \mid n < \omega\rangle \in V[C]$ be an increasing cofinal in κ sequence. Then there exists an infinite subsequence C' of C such that $V[C'] = V[\vec{\alpha}]$.*

Proof. Work in V. Given a condition $\langle q, B\rangle$, construct by induction (using the Prikry property and the normality of U) a condition $\langle p, A\rangle$ and a non-decreasing sequence of natural numbers $\langle n_k \mid k < \omega\rangle$ such that for every $k < \omega$

1. $\langle q, B\rangle \leq^* \langle p, A\rangle$
2. for every $\langle \eta_1, ..., \eta_{n_k}\rangle \in [A]^{n_k}$ the condition $\langle p^\frown\langle \eta_1, ..., \eta_{n_k}\rangle, A \setminus \eta_{n_k} + 1\rangle$ decides the value of $\underset{\sim}{\alpha}_k$,
3. there is no $n, n_k \leq n < n_{k+1}$ such that for some $\langle \eta_1, ..., \eta_n\rangle \in [A]^n$ and $E \in U$ the condition $\langle p^\frown\langle \eta_1, ..., \eta_n\rangle, E\rangle$ decides the value of $\underset{\sim}{\alpha}_{n_{k+1}}$,

Define a function $F : [A]^{<\omega} \to [\kappa]^{<\omega}$ by setting $F(\eta_1, ..., \eta_n) = \langle\rangle$, if $n < n_0$ and $F(\eta_1, ..., \eta_n) = \langle \nu_1, ..., \nu_k\rangle$, if $n \geq n_0$, where $\langle \nu_1, ..., \nu_k\rangle$ is the sequence of the maximal length such that for every $i, 1 \leq i \leq k$,

$$\langle p^\frown\langle \eta_1, ..., \eta_n\rangle, A \setminus \eta_n + 1\rangle \Vdash \underset{\sim}{\alpha}_i = \nu_i.$$

Using normality of U it is possible to find $A^* \in U$ and $I_n \subseteq n$, for each $n < \omega$, such that for every $n < \omega$ and $\langle \eta_0, ..., \eta_{n-1}\rangle, \langle \eta'_0, ..., \eta'_{n-1}\rangle \in [A^*]^n$ the following hold:

$$I_{n+1} \cap n \subseteq I_n$$

and

$$F(\eta_0, ..., \eta_{n-1}) = F(\eta'_0, ..., \eta'_{n-1}) \text{ iff } \eta_i = \eta'_i, \text{ whenever } i \in I_n,$$

i.e. $F \restriction [A^*]^n$ depends only on the coordinates in I_n and there it is one to one.

Now, using the density argument, we can find such $\langle p, A^* \rangle$ in the generic set. Consider a tree

$$T = \{\langle \eta_1, ..., \eta_n \rangle \in [A^*]^n \mid \exists k < \omega \quad F(\eta_1, ..., \eta_n) = \langle \alpha_1, ..., \alpha_k \rangle\}.$$

Set $J_n = \bigcap_{m<\omega}(I_m \cap n)$ and $J = \bigcup_{n<\omega} J_n$. Define $C' = C \restriction J$. We claim that $V[\vec{\alpha}] = V[C']$. Thus, given C' we can use F on A^* in order to define $\vec{\alpha}$. On the other hand given $\vec{\alpha}$ we use the tree T to reconstruct C'. Actually, for each $n \in J$ the n-th level of T consists of elements of the form $t^\frown\eta$, where t's from the level $n-1$ may vary, but η is always the same and it is the n-th element of C.

□

Lemma 0.5 *Suppose that $A \in V[C]$ is an unbounded subset of κ and κ is a singular cardinal in $V[A]$. Then there is a subsequence C' of C such that $V[A] = V[C']$.*

Proof. The cofinality of κ should be ω in $V[A]$, since $V[A] \subseteq V[C]$ and the Prikry forcing does not change cofinality of cardinals that differ from κ.

Let $\langle \eta_n \mid n < \omega \rangle \in V[A]$ be a cofinal sequence in κ. Let $\langle X_{ni} \mid i < \delta_n \rangle$ be the least in a fixed well ordering of V enumeration of $\mathcal{P}(\eta_n)$, for each $n < \omega$. Set i_n to be the least $i < \delta_n$ such that $A \cap \eta_n = X_{ni}$, for each $n < \omega$. Then $A \in V[\langle \eta_n \mid n < \omega \rangle, \langle i_n \mid n < \omega \rangle]$. So

$$V[A] = V[\langle \eta_n \mid n < \omega \rangle, \langle i_n \mid n < \omega \rangle],$$

and we can apply Lemma 0.4.

□

Lemma 0.6 *Suppose that $A \in V[C]$ is an unbounded subset of κ. Then there is a subsequence C' of C such that $V[A] = V[C']$.*

Proof. Without loss of generality let us assume that A is a new subset of κ and the weakest condition forces this.

Work in V. Let $\underset{\sim}{A}$ be a name of A and $\langle s, S \rangle \in P(U)$. Define by induction a subtree T of $[S]^{<\omega}$. For each $\nu \in S$ pick some $S_\nu \subseteq S, S_\nu \in U$ and $a_\nu \subseteq \nu$ such that

$$\langle s^\frown\nu, S_\nu \rangle \Vdash \underset{\sim}{A} \cap \nu = a_\nu.$$

Set

$$S(0) = S \cap \triangle_{\nu \in S} S_\nu.$$

Consider the function $\nu \to a_\nu, (\nu \in S(0))$. By normality of U it is easy to find $A(0) \subseteq \kappa$ and $T(0) \subseteq S(0), T(0) \in U$ such that $A(0) \cap \nu = a_\nu$, for every $\nu \in T(0)$. Set the first level of T to be $T(0)$.

Let now $\nu_0, \nu_1 \in T(0)$ and $\nu_1 > \nu_0$. Then, clearly, $\nu_1 \in S_{\nu_0}$. Find $S_{\nu_0,\nu_1} \subseteq S_{\nu_0}, S_{\nu_0,\nu_1} \in U$ and $a_{\nu_0,\nu_1} \subseteq \nu_1$ such that

$$\langle s^\frown\langle \nu_0, \nu_1\rangle, S_{\nu_0,\nu_1}\rangle \Vdash \underset{\sim}{A} \cap \nu_1 = a_{\nu_0,\nu_1}.$$

Set

$$S(\nu_0) = T(0) \cap \triangle_{\nu \in S_{\nu_0}} S_{\nu_0,\nu}.$$

Again, we consider the function $\nu \to a_\nu, (\nu \in S_{\nu_0})$. By normality of U it is easy to find $A(\nu_0) \subseteq \kappa$ and $T(\nu_0) \subseteq S_{\nu_0}, T(\nu_0) \in U$ such that $A(\nu_0) \cap \nu = a_{\nu_0,\nu}$, for every $\nu \in T(\nu_0)$. Define the set of the immediate successors of ν_0 to be $T(\nu_0)$, i.e. $Suc_T(\nu_0) = T(\nu_0)$.

This defines the second level of T. Continue similar to define further levels of T.

Now let us turn $\langle s, T\rangle$ into a condition in $P(U)$ by taking the diagonal intersections, i.e. set $T^* = \triangle_{t \in T} Suc_T(t)$ and consider $\langle s, T^*\rangle$. It has the following property:

(*) for every $\langle \eta_1, ..., \eta_n\rangle \in T^*$,

$$\langle s^\frown\langle \eta_1, ..., \eta_n\rangle, T^*_{\langle \eta_1,...,\eta_n\rangle}\rangle \Vdash \underset{\sim}{A} \cap \eta_n = A(\eta_1, ..., \eta_{n-1}) \cap \eta_n.$$

A simple density argument implies that there is a condition which satisfies (*) in the generic set. Assume for simplicity that already $\langle s, T^*\rangle$ is such a condition and $s = \langle\rangle$. Then, $C \subseteq T^*$. Let $\langle \kappa_n \mid n < \omega\rangle = C$. So, for every $n < \omega$,

$$A \cap \kappa_n = A(\kappa_0, ..., \kappa_{n-1}) \cap \kappa_n.$$

Let us work now in $V[A]$ and define by induction a sequence $\langle \eta_n \mid n < \omega\rangle$ as follows. Consider $A(0)$. It is a set in V, hence $A(0) \neq A$. So there is η such that for every $\nu \in T^* \setminus \eta$ we have $A \cap \nu \neq A(0) \cap \nu$. Set η_0 to be the least such η. Turn to η_1. Let $\xi \in T^* \cap \eta_0$. Consider $A(\xi)$. It is a set in V, hence $A(\xi) \neq A$. So there is η such that for every $\nu \in T^* \setminus \eta$ we have $A \cap \nu \neq A(\xi) \cap \nu$. Set $\eta(\xi)$ to be the least such η. Now define η_1 to be $\sup(\{\eta(\xi) \mid \xi < \eta_0\})$. Suppose that $\eta_0, ..., \eta_n$ are defined. Define η_{n+1}. Let $\xi_0 < \xi_1 < ... < \xi_n$ be in T^*. Consider $A(\xi_0, ..., \xi_n)$. It is a set in V, hence $A(\xi_0, ..., \xi_n) \neq A$. So there is η such that for every $\nu \in T^* \setminus \eta$ we have $A \cap \nu \neq A(\xi_0, ...\xi_n) \cap \nu$. Set $\eta(\xi_0, ...\xi_n)$ to be the least such η. Now define η_{n+1} to be $\sup(\{\eta(\xi_0, ...\xi_n) \mid \xi_0 < \eta_0, ..., \xi_n < \eta_n\})$.

This completes the definition of the sequence $\langle \eta_n \mid n < \omega \rangle$.
Let us argue that it is cofinal in κ. Then the lemma will follow by Lemma 0.5.
Suppose otherwise. Let k be the least such that $\kappa_k > \sup(\{\eta_n \mid n < \omega\})$. Then

$$A \cap \kappa_k = A(\kappa_0, ..., \kappa_{k-1}) \cap \kappa_k.$$

This is impossible, since $\eta_k < \kappa_k$.
□

Let now A be a subset of κ^+ in $V[C]$.

As a warm up let us show the following:

Lemma 0.7 *Suppose that $A \subseteq \kappa^+$ in $V[C]$ and $A \cap \alpha \in V$, for every $\alpha \in V$. Then $A \in V$.*

Proof. For each $\alpha < \kappa^+$ pick $\langle s_\alpha, S_\alpha \rangle \in G$ such that

$$\langle s_\alpha, S_\alpha \rangle \Vdash \underset{\sim}{A} \cap \alpha = A \cap \alpha.$$

There are an unbounded $E \subseteq \kappa^+$ and $s \in [\kappa]^{<\omega}$ such that for each $\alpha \in E$ we have $s = s_\alpha$. Now, in V, we consider

$$H = \{\langle s, T \rangle \in P(U) \mid \exists \alpha < \kappa^+ \exists a \subseteq \alpha \quad \langle s, T \rangle \Vdash \underset{\sim}{A} \cap \alpha = a\}.$$

Note that if $\langle s, T \rangle, \langle s, T' \rangle \in P(U)$ and for some $\alpha \leq \beta < \kappa^+$, $a \subseteq \alpha, b \subseteq \beta$ we have

$$\langle s, T \rangle \Vdash \underset{\sim}{A} \cap \alpha = a \text{ and } \langle s, T' \rangle \Vdash \underset{\sim}{A} \cap \beta = b,$$

then $b \cap \alpha = a$. Just conditions of this form are compatible, and so they cannot force contradictory information.
Apply this observation to H. Let

$$X = \{a \subseteq \kappa^+ \mid \exists \langle s, S \rangle \in H \quad \langle s, T \rangle \Vdash \underset{\sim}{A} \cap \alpha = a\}.$$

Then necessarily, $\bigcup X = A$.
□

So $A \notin V$ implies that some initial segments of A are not in V as well.

Work in $V[A]$. Fix some well ordering. By Lemma 0.6, for each $\alpha < \kappa^+$, we can pick the least Prikry sequence C_α for $P(U)$such that $V[C_\alpha] = V[A \cap \alpha]$. Note that C_α need not be a subsequence of C, but still $|C_\alpha \setminus C| < \aleph_0$. The number possibilities for C_α's is at most κ. So there is $\alpha^* < \kappa^+$ such that $C_\alpha = C_{\alpha^*}$, for every $\alpha, \alpha^* \leq \alpha < \kappa^+$. Set $C^* = C \cap C_{\alpha^*}$. Clearly, $C^* \in V[A]$ and

$$\forall\alpha(\alpha^* \leq \alpha < \kappa^+ \rightarrow V[C^*] = V[A \cap \alpha]).$$

It does not necessary means that $V[C^*] = V[A]$, since the sequence $\langle A \cap \alpha \mid \alpha < \kappa^+\rangle$ is probably not in $V[C^*]$. But let us argue that indeed $V[C^*] = V[A]$.

Suppose for simplicity that C^* is a sequence consisting of members of C standing at even places, i.e. $C^* = C_{even}$, where

$$C_{even} = \langle C(2n) \mid n < \omega\rangle.$$

Split the Prikry forcing with U, which we further denote by $P(U)$ into two parts the first adds the even part of the Prikry sequence and the second the rest of it.
For $S \in U$ let $S' = \{\nu \in S \mid S \cap \nu \text{ is unbounded in } \nu\}$. Let

$$D = \{\langle s_0, ..., s_k, S\rangle \in P(U) \mid k \text{ is even}\}.$$

Define a map $\pi : D \rightarrow P(U)$ as follows:

$$\pi(\langle s_0, ..., s_{2n}, S\rangle = \langle s_0, s_2, ..., s_{2k}, ...s_{2n}, S\rangle.$$

We would like to turn π into a projection map. In order to do so let us define a new order $\preceq$ over $P(U)$.

Definition 0.8 Let $p = \langle t_1, ..., t_n, T\rangle, q = \langle r_1, ..., r_m, R\rangle \in P(U)$. Set $q \preceq p$ iff

1. $T \subseteq R$,
2. $\langle t_1, ..., t_n\rangle$ end extends $\langle r_1, ..., r_m\rangle$,
3. for each $k, m < k \leq n$ we have $t_k \in R$,
4. (a) if $m = 0$, i.e. the sequence of q is empty, then $R \cap t_1 \neq \emptyset$ and for each $k, 1 < k \leq n$ we have $R \cap (t_{k-1}, t_k) \neq \emptyset$,

   (b) if $m > 0$, then for each $k, m < k \leq n$ we have $R \cap (t_{k-1}, t_k) \neq \emptyset$.

Lemma 0.9 *π projects the forcing $\langle P(U), \leq\rangle$ onto the forcing $\langle P(U), \preceq\rangle$.*

Proof. Let $p = \langle s_0, s_1, ..., s_{2n}, S\rangle$, $\langle t_1, ..., t_i\rangle \in [S']^i$ and $T \subseteq S'$ with $\min(T) > t_i$. We need to find an extension q of p which π projects above $\langle s_0, s_2, ..., s_{2k}, ..., s_{2n}, t_1, ..., t_i, T\rangle$. It is easy to arrange. Thus pick some $r_1, ..., r_i \in S$ such that

$$r_1 < t_1 < r_2 < ... < r_i < t_i.$$

Consider

$$q = \langle s_0, s_1, ..., s_{2n}, r_1, t_1, ..., r_i, t_i, T\rangle.$$

Then $\pi(q) = \langle s_0, s_2, ..., s_{2k}, ..., s_{2n}, t_1, ..., t_i, T'\rangle$ and we are done.
□

Lemma 0.10 *$\langle P(U), \preceq, \leq^* \rangle$ is a Prikry type forcing notion, where $\leq^*$ is the usual direct extension order on $P(U)$.*

Proof. The standard argument for the Prikry forcing works here.
□

Let $G \subseteq P(U)$ be a $\langle P(U), \preceq \rangle$ generic. Denote by E the set

$$\bigcup\{\{t_1, ..., t_n\} \mid \exists T \in U \quad \langle t_1, ..., t_n, T\rangle \in G\}.$$

Clearly, E is just a Prikry sequence for U. Let $\langle e_n \mid n < \omega\rangle$ be the increasing enumeration of E.

Note that it is possible to reconstruct G from E.

Thus set

$$G' = \{\langle t_1, ..., t_n, T\rangle \mid \langle t_1, ..., t_n\rangle = \langle e_1, ..., e_n\rangle, \forall k \geq n \quad T \cap (e_k, e_{k+1}) \neq \emptyset\}.$$

Lemma 0.11 *$G = G'$.*

Proof. Assume first that $p = \langle t_1, ..., t_n, T\rangle$ is in G. Then clearly $\langle t_1, ..., t_n\rangle = \langle e_1, ..., e_n\rangle$. Suppose that for some $k \geq n$ we have $T \cap (e_k, e_{k+1}) = \emptyset$. There is a condition $q = \langle e_1, ..., e_m, S\rangle \in G$ for some $m > k$. Both p and q in G, so there is $r = \langle e_1, ..., e_l, R\rangle \in G$ stronger than both p, q. So $l \geq m$. Then $p \preceq r$ implies by 0.8, $T \cap (e_k, e_{k+1}) \neq \emptyset$. Contradiction. Hence $p \in G'$.

Suppose now that $p = \langle t_1, ..., t_n, T\rangle$ is in G'. Then $\langle t_1, ..., t_n\rangle = \langle e_1, ..., e_n\rangle$ and for all $k \geq n$ we have $T \cap (e_k, e_{k+1}) \neq \emptyset$. It is enough to show that p is compatible ($\preceq$) with every member of G. Let $q = \langle e_1, ..., e_m, S\rangle \in G$. Extending if necessary we can assume that $n = m$.

Set $R = S \cap T \setminus e_n + 1$. Consider now $r = \langle e_1, ..., e_n, R \rangle$. Then $p, q \preceq r$ and we are done.
□

Consider now

$$P^* = \{\langle p_e, p \rangle \in P(U) \times P(U) \mid p_e \Vdash p \in P(U)/\underset{\sim}{C}_{even}\}.$$

Clearly, P^* is isomorphic to $P(U)$.

Lemma 0.12 *The following two conditions are equivalent:*

$$\langle \nu_0, ..., \nu_n, S \rangle \Vdash \langle \eta_0, \eta_1, \eta_2, \eta_3, ..., \eta_{2m}, T \rangle \in P(U)/\underset{\sim}{C}_{even}$$

and

1. $\langle \eta_0, \eta_2, ..., \eta_{2m-2}, \eta_{2m} \rangle$ *is an initial segment (probably not proper) of* $\langle \nu_0, ..., \nu_n \rangle$,
2. $T \supseteq S$,
3. *if* $\tau_1, \tau_2 \in S$ *or* τ_1, τ_2 *are members of the sequence* $\langle \nu_0, ..., \nu_n \rangle$ *above* η_{2m}, *then* $\tau_1 < \tau_2 \rightarrow (\tau_1, \tau_2) \cap T \neq \emptyset$.

Proof. Suppose otherwise. Let for example $T \not\supseteq S$. Pick then some $\nu \in S \setminus T$ and extend $\langle \nu_0, ..., \nu_n, S \rangle$ to $\langle \nu_0, ..., \nu_n, \nu, S \setminus \nu + 1 \rangle$. Then for each generic G_{even} with $\langle \nu_0, ..., \nu_n, \nu, S \setminus \nu + 1 \rangle \in G_{even}$ we will have that $\nu \in C_{even}$. But $\nu \notin C$ for any G with $\langle \eta_0, ..., \eta_{2m}, T \rangle \in G$.
Now suppose that for some $\tau_1, \tau_2 \in S \quad \tau_1 < \tau_2$ but $(\tau_1, \tau_2) \cap T = \emptyset$. Extend $\langle \nu_0, ..., \nu_n, S \rangle$ to $\langle \nu_0, ..., \nu_n, \tau_1, \tau_2, S \setminus \tau_2 \rangle$.
□

Lemma 0.13 *The forcing* $P(U)/G_{even}$ *satisfies* κ^+*-c.c..*

Proof. Let $\{p_\alpha \mid \alpha < \kappa^+\} \subseteq P(U)/G_{even}$.
Work in V. For each $\alpha < \kappa^+$ pick some $q_\alpha = \langle \langle s_{\alpha,0}, ..., s_{\alpha,2n_\alpha} \rangle, S_\alpha \rangle$ and $\langle \vec{\eta}_\alpha, T_\alpha \rangle$ such that

$$q_\alpha \Vdash \underset{\sim}{p}_\alpha = \langle \vec{\eta}_\alpha, T_\alpha \rangle.$$

By shrinking if necessary, we can assume for some $n < \omega$ and some sequence $\langle s_0, ..., s_{2n} \rangle$ $n_\alpha = n$ and $\langle s_{\alpha,0}, ..., s_{\alpha,2n_\alpha} \rangle = \langle s_0, ..., s_{2n} \rangle$, for every $\alpha < \kappa^+$. By shrinking more and extending if necessary, using Lemma 0.12 we may assume that $\vec{\eta}_\alpha = \langle s_0, s_1, ..., s_{2n-1}, s_{2n} \rangle$, for some sequence $\langle s_1, s_3, ..., s_{2n-1} \rangle$, for each $\alpha < \kappa^+$.

Let now $\alpha, \beta < \kappa^+$. Consider $S = S_\alpha \cap S_\beta$. Let $T = (T_\alpha \cap T_\beta)'$. Clearly $T \in U$. Finally, let $S^* = S \cap T$. Then

$$\langle\langle s_0, ..., s_{2n}\rangle, S^*\rangle \Vdash \langle\langle s_0, s_1, ..., s_{2n-1}, s_{2n}\rangle, T\rangle \in P(U)/\underset{\sim}{G}_{even},$$

by Lemma 0.12.

□

Let show a bit stronger statement.

Lemma 0.14 *Let G_{odd} be a generic subset of $P(U)/G_{even}$. Then the forcing $P(U)/G_{even}$ satisfies κ^+-c.c. in $V[G_{even}, G_{odd}]$.*

Proof. Let $\{p_\alpha \mid \alpha < \kappa^+\} \subseteq P(U)/G_{even}$ in $V[G_{even}, G_{odd}]$.
Work in V. For each $\alpha < \kappa^+$ pick some $q_\alpha = \langle\langle s_{\alpha,0}, ..., s_{\alpha,2n_\alpha}\rangle, S_\alpha\rangle$, $\langle\vec{\nu}_\alpha, R_\alpha\rangle$ and $\langle\vec{\eta}_\alpha, T_\alpha\rangle$ such that

$$q_\alpha \Vdash \langle\vec{\nu}_\alpha, R_\alpha\rangle \in P(U)/\underset{\sim}{G}_{even}$$

and

$$\langle q_\alpha, \langle\vec{\nu}_\alpha, R_\alpha\rangle\rangle \Vdash \underset{\sim}{p}_\alpha = \langle\vec{\eta}_\alpha, T_\alpha\rangle.$$

By shrinking if necessary, we can assume for some $n < \omega$ and some sequence $\langle s_0, ..., s_{2n}\rangle$ $n_\alpha = n$ and $\langle s_{\alpha,0}, ..., s_{\alpha,2n_\alpha}\rangle = \langle s_0, ..., s_{2n}\rangle$, for every $\alpha < \kappa^+$. By shrinking more and extending if necessary, using Lemma 0.12 we may assume that $\vec{\nu} = \langle s_0, r_1, s_2, ..., r_{2n-1}, s_{2n}\rangle$ and $\vec{\eta}_\alpha = \langle s_0, s_1, ..., s_{2n-1}, s_{2n}\rangle$, for some sequences $\langle r_1, r_3, ..., r_{2n-1}\rangle$, $\langle s_1, s_3, ..., s_{2n-1}\rangle$, for each $\alpha < \kappa^+$.
Let now $\alpha, \beta < \kappa^+$. Consider $S = S_\alpha \cap S_\beta$. Let $T = (T_\alpha \cap T_\beta)', R = (R_\alpha \cap R_\beta)'$. Clearly $T, R \in U$. Finally, let $S^* = S \cap T \cap R$. Then

$$\langle\langle s_0, ..., s_{2n}\rangle, S^*\rangle \Vdash \langle\langle s_0, r_1, ..., r_{2n-1}, s_{2n}\rangle, R\rangle \in P(U)/\underset{\sim}{G}_{even}$$

and

$$\langle\langle s_0, ..., s_{2n}\rangle, S^*\rangle \Vdash \langle\langle s_0, s_1, ..., s_{2n-1}, s_{2n}\rangle, T\rangle \in P(U)/\underset{\sim}{G}_{even},$$

by Lemma 0.12. Hence

$\langle\langle\langle s_0, ..., s_{2n}\rangle, S^*\rangle, \langle\langle s_0, r_1, ..., r_{2n-1}, s_{2n}\rangle, R\rangle\rangle \Vdash \underset{\sim}{p}_\alpha, \underset{\sim}{p}_\beta$ are compatible conditions in $P(U)/\underset{\sim}{G}_{even}$.

□

Lemma 0.15 $A \in V[G_{even}]$.

Proof. Suppose that $A \notin V[G_{even}]$.
Work in $V[G_{even}]$. For each $\alpha < \kappa^+$ consider the set

$$X_\alpha = \{B \subseteq \alpha \mid \|\underset{\sim}{A} \cap \alpha = B\| \neq 0\},$$

where $\underset{\sim}{A}$ is a name of A in $P(U)/C_{even}$ and the boolean value is taken in $RO(P(U)/C_{even})$.
For $B \in X_\alpha$ we denote

$$\|\underset{\sim}{A} \cap \alpha = B\| \text{ by } b(B).$$

Note that by Lemma 0.13 each X_α has cardinality at most κ. Also, for every $\alpha \leq \beta < \kappa^+$ and $B \in X_\alpha$ there is $B' \in X_\beta$ such that $B' \cap \alpha = B$. In addition, if $B' \in X_\beta$ and $B' \cap \alpha = B$ then $b(B') \leq b(B)$. Clearly, that if $b(B') < b(B)$, then there is $p \in P(U)/G_{even}$ stronger than $b(B)$ but incompatible with $b(B')$. Just any p stronger than $b(B) \setminus b(B')$ will work.
Now force with $P(U)/G_{even}$. Let G_{odd} be a generic. For each $\alpha < \kappa^+$ let $A_\alpha = A \cap \alpha$. By our assumptions, each $A_\alpha \in V[G_{even}]$. Clearly, $A_\alpha \in X_\alpha$, for every $\alpha < \kappa^+$. Set $b_\alpha = b(A_\alpha)$. Then

$$b_\beta \leq b_\alpha,$$

for every $\alpha \leq \beta < \kappa^+$. The sequence of b_α's cannot stabilize since A not in $V[G_{even}]$, by the assumption. Hence there will be a strictly decreasing subsequence $\langle b_{\alpha_i} \mid i < \kappa^+\rangle$ of the sequence $\langle b_\alpha \mid \alpha < \kappa^+\rangle$. But then

$$\langle b_{\alpha_i} \setminus b_{\alpha_{i+1}} \mid i < \kappa^+\rangle$$

will be an antichain of the length κ^+ which is impossible by Lemma 0.14. Contradiction.
□

Now, using induction we can go up and show that for every cardinal $\lambda > \kappa^+$ and a set $A \subseteq \lambda$ in $V[C]$ there is a subsequence C^* of C such that $V[A] = V[C^*]$.
Thus, if $\operatorname{cof}(\lambda) > \kappa$, then the argument of Lemma 0.15 applies.
Suppose that $\delta = \operatorname{cof}^V(\lambda) \leq \kappa$. Pick in V a cofinal in λ sequence $\langle \lambda_\alpha \mid \alpha < \delta\rangle$ consisting of regular cardinals. Find a subsequence C' of C which is in $V[A]$ and such that $V[C'] \supseteq V[A \cap \lambda_\alpha]$, for each $\alpha < \delta$. Thus for each $\alpha < \delta$ pick C_α to be the least Prikry sequence for $P(U)$ (in a fixed well ordering of $V[A]$) such that $V[A \cap \lambda_\alpha] = V[C_\alpha]$. Consider $\langle C_\alpha \mid \alpha < \delta\rangle$. Clearly, this sequence is in $V[A]$. It can be coded as a subset of κ. Hence, by Lemma 0.6, there is a Prikry sequence $C'' \in V[A]$ for $P(U)$ such that $V[C''] = V[\langle C_\alpha \mid \alpha < \delta\rangle]$. Set $C' = C \cap C''$. Still $C' \in V[A]$ and $V[C''] = V[\langle C_\alpha \mid \alpha < \delta\rangle]$. Note that, if $\delta \notin \{\omega, \kappa\}$, then it

is possible to find C' such that $V[C'] = V[A \cap \lambda_\alpha]$ for a final segment of α's.
Suppose that $V[C'] \neq V[C]$. Work in $V[C']$. For each $\alpha < \delta$ let

$$X_\alpha = \{B \subseteq \lambda_\alpha \mid \|\underset{\sim}{A} \cap \lambda_\alpha = B\| \neq 0\}.$$

By Lemma 0.13 each X_α has cardinality at most κ. Hence we can code $\langle X_\alpha \mid \alpha < \delta \leq \kappa \rangle$ as a subset of κ. So, over $V[C']$, adding A is equivalent to adding of a subset of κ. Let H denote such a subset. Then

$$V[A] = V[C'][H] = V[C', H].$$

But $C' \times H$ can be coded again into a subset of κ and it in turn is equivalent to a subsequence C^* of C, i.e. $V[C', H] = V[C^*]$. Hence, $V[A] = V[C^*]$.

This completes the proof of the theorem.

□