A remark on large cardinals in "Cichon Maximum" by M. Gordstern, J. Kellner and S. Shelah.

Moti Gitik

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1 On elementary embeddings.

In [3], M. Gordstern, J. Kellner and S. Shelah isolated the following interesting notion:

Definition 1.1 An elementary embedding $j : V \to M$ with critical point κ is called *a BUP-embedding from κ to θ* (for some regular $\theta > \kappa$), if

1. $\mathrm{cof}(j(\kappa)) = |j(\kappa)| = \theta$,

2. ${}^{\kappa>}M \subseteq M$,

3. whenever S is $\leq \kappa-$directed partial order , then $j''S$ is cofinal in S.

Such embeddings where constructed in [3] using strongly compact cardinals.

The purpose of this note is to reduce the assumptions.

Let E be a $(\kappa, \delta)-$extender, i.e. $E = \langle E_a \mid a \in [\delta]^{<\omega} \rangle$, each E_a is a $\kappa-$complete ultrafilter over $[\kappa]^{|a|}$, $i_E : V \to M \simeq \mathrm{Ult}(V, E)$ is the canonical embedding

such that

1. $i_E(\kappa) = \delta$,

2. δ is a regular cardinal.

Assume GCH.

Claim 1. M is closed under $\kappa-$sequences of its elements.

Proof. It follows from regularity of δ. Actually $\mathrm{cof}(\delta) > \kappa$ is enough for this.

Claim 2. For every $x \in M$ there is a finite $a \in [\delta]^{|a|}$ and a function $f : [\kappa]^{|a|} \to V$ such that $x = i_E(f)(a)$.
Proof. Just M is a direct limit of ultrapowers $\langle \mathrm{Ult}(V, E_b) \mid b \in [\delta]^n, n < \omega \rangle$.

Claim 3. $crit(i_E) = \kappa$ and $i_E(\kappa) = \delta$.

Claim 4. If $|A| < \kappa$, then $i''_E A = i_E(A)$.
Proof. Follows from Claim 3.

Claim 5. If $\lambda > \kappa$ is regular, then $\max(\delta, \lambda) \leq i_E(\lambda) < \max(\delta, \lambda)^+$.
Proof. Let $\beta < i_E(\lambda)$. By Claim 2, there are $a_\beta \in [\delta]^{|a_\beta|}$ and a function $f_\beta : [\kappa]^{|a_\beta|} \to \lambda$ such that $\beta = i_E(f_\beta)(a_\beta)$. Counting the number of possibilities for a_β, f_β we obtain the bounds.

Claim 6. If S is a $< \lambda-$directed partial order, and $\kappa < \lambda$, then $i''_E S$ is cofinal in $i_E(S)$.
Proof. Let $s \in i_E(S)$. Use Claim 2 to find $a_s \in [\delta]^{|a_s|}$ and a function $f_s : [\kappa]^{|a_s|} \to S$ such that $s = i_E(f_s)(a_s)$. Now $\mathrm{ran}(f_s)$ has cardinality at most κ, so there is $s^* \in S$ above all members of the range. Then, by elementarity, s will be below $i_E(s^*)$.

Claim 7. If $\mathrm{cof}(\alpha) \neq \kappa$, then $i''_E \alpha$ is cofinal in $i_E(\alpha)$.
Proof. Clear, if $\mathrm{cof}(\alpha) < \kappa$. If $\mathrm{cof}(\alpha) > \kappa$, then use Claim 6.

Note that δ can be collapsed $Col(\theta, < \delta)$ to any regular $\theta, \kappa < \theta < \delta$ without effecting the extender E and its properties (Claims 1-7). Just such forcing does not add new subsets to κ, and so, each of the components E_a remains a $\kappa-$complete ultrafilter.

It follows:

Proposition 1.2 *The embedding i_E is a BUP-embedding from κ to δ.*

Let us address the question of what large cardinals are needed in order to have such embeddings.

The following statement combines results by W. Mitchell [4] and by the author [1]

Theorem 1.3

- *If there exists a BUP-embedding from κ to δ (even without Item 3 of 1.1), then $o(\kappa) \geq \kappa^{++}$ in the core model.*
- *If $o(\kappa) \geq \kappa^{++}$ in the core model, then in a cardinal preserving generic extension which satisfies GCH there is a $(\kappa, \kappa^{++})-$extender as above. In particular there exists a BUP-embedding from κ to κ^{++}.*

If we relax GCH assumptionsnd allow 2^κ to be large, then it is possible to get embeddings with targets (images of the critical point κ) larger than κ^{++}. See [2] for this type of constructions.

2 Four cardinals.

In the construction of [3], M. Gordstern, J. Kellner and S. Shelah used the following:

Assumption. $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ such that

1. For $\ell = 6, 7, 8, 9$, there is a BUP embedding j_ℓ from κ_ℓ to λ_ℓ.
2. All λ_i are regular and $\lambda_3 = \chi^+$ with $\chi^{\aleph_0} = \chi$.
3. $\lambda_2^{<\lambda_2} = \lambda_2, \lambda_4^{\aleph_0} = \lambda_4, \lambda_5^{<\lambda_4} = \lambda_5$.

Four strongly compacts were used in order to satisfy this assumption.
Let us show that variations of superstrong cardinals can be used instead.

Recall that a cardinal κ is called *a superstrong* iff there is an elementary embedding $j : V \to M$ with a critical point κ such that $M \supseteq V_{j(\kappa)}$.
Let us call a cardinal κ *a superstrong with a target* λ iff there is an elementary embedding $j : V \to M$ with a critical point κ such that $M \supseteq V_{j(\kappa)}$ and $j(\kappa) = \lambda$.
Note that such λ need not be a regular. Moreover, it will have cofinality ω, if κ is the least superstrong.
Our interest will be in embeddings with regular (and then necessary inaccessible) targets.
The embedding j can be replaced by ultrapower embedding by extender, as in the previous section.
In particular, assuming GCH, we will have a BUP embedding from κ to λ.

Proposition 2.1 *Let κ be a superstrong cardinal with a target λ.*
Suppose that κ is a limit of superstrong cardinals η with targets $\eta^* < \kappa$.
Then there is a superstrong cardinal κ' with a target λ' such that $\kappa < \kappa' < \lambda' < \lambda$.

Proof. Follows by elementarity.
□

The next proposition is similar:

Proposition 2.2 *Let κ be a superstrong cardinal with a target λ.*
Suppose that κ is a limit of superstrong cardinals η with regular targets $\eta^* < \kappa$.
Then there is a superstrong cardinal κ' with a regular target λ' such that $\kappa < \kappa' < \lambda' < \lambda$.

Let us now iterate the process.

Definition 2.3 Let κ be a superstrong cardinal with a regular target λ.
Say then that *a degree* $d(\kappa) \geq 1$.
Set $d(\kappa) \geq n+1$ iff κ is a limit superstrong cardinals η with regular targets $\eta^* < \kappa$ and $d(\eta) \geq n$.

It follows then:

Proposition 2.4 *Let $n, 1 \leq n < \omega$ and κ be a superstrong cardinal with a target λ with $d(\kappa) \geq n+1$. Then there are superstrong cardinals κ_i with regular targets λ_i,$1 \leq i \leq n$ such that $\kappa < \kappa_1 < ... < \kappa_n < \lambda_n < ... < \lambda_1 < \lambda$.*

In conclusion let us state the following:

Proposition 2.5 *Suppose that κ is a superstrong cardinal with a Mahlo target λ. Then for every $n < \omega$, $d(\kappa) \geq n$.*

Proof. Let $j : V \to M$ be a witnessing embedding by a $(\kappa, \lambda)-$extender E.
Let $f : [\kappa]^k \to \kappa$, for some $k < \omega$.
Consider

$$C_f = \{\nu < \kappa \mid f \restriction [\nu]^k : [\nu]^k \to \nu\}.$$

It is clearly a club. Then $j(C_f)$ is a club in λ.
Then

$$C := \bigcap \{j(C_f) \mid f : [\kappa]^k \to \kappa, k < \omega\}$$

is also a club in λ.
Let η be any inaccessible cardinal in C. Then $E \restriction \eta$ will be a $(\kappa, \eta)-$extender witnessing that κ is a strong cardinal with a target η.
Clearly, $E \restriction \eta \in M$. By elementarity, then, κ is a limit of superstrong cardinals ν with regular targets $\nu^* < \kappa$. Again by elementarity, the same is true in M (and so in V) with κ replaced by $j(\kappa) = \lambda$.

Next we can pick an inaccessible cardinal μ in C above η. Then $E \restriction \eta$ will be in the ultrapower by $E \restriction \mu$. We can repeat the argument above, with $E \restriction \mu$ replacing E, and argue that μ (the target of κ under $E \restriction \mu$) is a limit of superstrong cardinals ν with regular targets $\nu^* < \mu$.

Continue by induction.

$\square$

References

[1] M. Gitik, Negation of SCH from $o(\kappa) = \kappa^{++}$, APAL

[2] M. Gitik, On measurable cardinals violating GCH, APAL

[3] M. Gordstern, J. Kellner and S. Shelah, Cichon Maximum.

[4] W. Mitchell, Core model for sequences of measures I,