Intermediate Models of Magidor-Radin Forcing-Part II

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Abstract

We continue the work done in [6],[2],[3]. We prove that for every set A in a Magidor-Radin generic extension using a coherent sequence such that $o^{\vec{U}}(\kappa)=\kappa$, there is $C'\subseteq C_G$, such that $V[A]=V[C']$.

1 Introduction

In this paper we consider the version of Magidor-Radin forcing for $o^{\vec{U}}(\kappa)=\kappa$. The major difference when we let $o^{\vec{U}}(\kappa)=\kappa$, is that we cannot split $\mathbb{M}[\vec{U}]$ to the part below $o^{\vec{U}}(\kappa)$ and above it. As proven in [3], this decomposition provided the ability to run over all possible extension types. In terms of C_G this means that we cannot split C_G below κ in a way that will determine what are the measure which we use in the construction of C_G. The classic example for such a sequence is

$$\kappa_0,\kappa_{\kappa_0},\kappa_{\kappa_{\kappa_0}},\ldots$$

in which every element in the sequence is taken from a measure which depends of the previous element in the sequence. This suggest that some sort of tree construction is needed in order to refer to such sequences in the ground model.

In context of [3] and [2], we are working by induction on κ. In Sections $2,3$ we will assume that $o^{\vec{U}}(\kappa)<\kappa^+$ and in $4,5$ the assumption is that $o^{\vec{U}}(\kappa)=\kappa$. In the results of Sections 2,3 and $4.1,4.2$, there are no restrictions on $o^{\vec{U}}(\delta)$ for $\delta<\kappa$. In Sections $4.3,4.4,5$ we assume that $\forall\alpha\leq\kappa.\ o^{\vec{U}}(\alpha)\leq\alpha$. Most of the claims in those section are proven without the restriction on $o^{\vec{U}}(\delta)$, in order to provide basis for future work.

The main result of this paper is:

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Theorem 1.1 *Let $\vec{U}$ be a coherent sequence such that for every $\alpha \leq \kappa$, $o^{\vec{U}}(\alpha) \leq \alpha$. Then For every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$, and every $A \in V[G]$, there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

Distinguishing from the case where $o^{\vec{U}}(\kappa) < \min(\lambda \mid 0 < o^{\vec{U}}(\lambda))$, we do not have a classification of what are exactly the subforcings which generates the models $V[C']$. Let us give some examples of sub forcing of $\mathbb{M}[\vec{U}]$ in the case of $o^{\vec{U}}(\kappa) = \kappa$.

Example 1.2 Let G be a generic with ,C_G be the generic club added by $\mathbb{M}[\vec{U}]$, consider the increasing continuous enumeration of C_G, $\langle C_G(i) \mid i < \kappa\rangle$. Assume that $C_G(0) > 0$, and consider again the sequence $\langle \kappa_n \mid n < \omega\rangle$ which is defined as follows:

$$\kappa_0 = C_G(0),\ \ \kappa_{n+1} = C_G(\kappa_n)$$

Consider the following tree of measures:

$$\vec{W} = \langle W_{\vec{\alpha}} \mid \vec{\alpha} \in [\kappa]^{<\omega}\rangle$$

where $W_{\vec{\alpha}} = U(\kappa, \max(\vec{\alpha}))$. Note here the since $o^{\vec{U}}(\kappa) = \kappa$, this is well defined. It is not hard to check the Mathias criteria for the tree-Prikry forcing with $\vec{W}$, given in [1], to conclude that $\langle \kappa_n \mid n < \omega\rangle$ is a tree-Prikry generic sequence with respect to $\vec{W}$. Note that, since the sequence of measures $\langle U(\kappa, i) \mid i < \kappa\rangle$ is a discrete family of normal measure, this tree-Prikry forcing falls under the framework of [10] and therefore the model $V[\langle \kappa_n \mid n < \omega\rangle]$ is minimal above V. This phenomena does not occur in generic extensions of $\mathbb{M}[\vec{U}]$ with $o^{\vec{U}}(\kappa) < \kappa$.

Example 1.3 The previous example can be made more complex. Let $f : [\kappa]^{<\omega} \to \kappa$ be any function. Then $\langle \alpha_n \mid n < \omega\rangle$ is defined as follows: $\alpha_0 = C_G(\langle\rangle)$ and α_{n+1} is obtained by applying f to some finite $\vec{C}_n \in [C_G]^{<\omega}$ i.e. $\alpha_{n+1} = C_G(f(\vec{C}_n))$.

All the notations and basic definitions can be found in [3] section 2.

2 Fat Trees

Definition 2.1 Let $\vec{U}$ be a coherent sequence of normal measures and $\theta_1 \leq ... \leq \theta_n$ be measurables with $o^{\vec{U}}(\theta_i) > 0$. A $\vec{U} - fat\ tree$ on $\theta_1 \leq ... \leq \theta_n$ is a tree $\langle T, \leq_T\rangle$ such that

1. $T \subseteq [\theta_n]^{\leq n}$ and $\langle\ \rangle \in T$.
2. $\leq_T$ is end-extension i.e. $t \leq_T s \Leftrightarrow t = s \cap \max(t) + 1$
3. T is downward closed in end-extension.

4. For any $t \in T$ one of the following holds:
   (a) There is $\beta < o^{\vec{U}}(\theta_{|t|+1})$ such that $\{\alpha \mid t^\frown\langle\alpha\rangle \in T\} \in U(\theta_{|t|+1}, \beta)$.
   (b) $|t| = n$

We will use some usual notations of trees:

- $\text{Succ}_T(t) = \{\alpha \mid t^\frown\langle\alpha\rangle \in T\}$.
- Note that if the measures in $\vec{U}$ can be separated i.e. there are $\langle X(\alpha,\beta) \mid \langle\alpha,\beta\rangle \in Dom(\vec{U})\rangle$ such that $X_i \in U_i \wedge \forall j \neq i X_i \notin U_j$, then we can intersect each set of the form $\text{Succ}_T(t)$ with appropriate X_i and define $U_t^{(T)} = U(\theta_{|t|+1}, \beta)$ if $suc_T(t) \in U(\theta_{|t|+1}, \beta)$ (We drop the script (T) when there is no risk of confusion).
- We will assume that if $\theta_i < \theta_{i+1}$ then for every $t \in lev_i(T)$, $\min(suc_T(t)) > \theta_i$.
- $ht(t) = otp(s \in T \mid s <_T t)$
- $lev_i(T) = \{t \in T \mid ht(t) = i\}$.
- The height of a tree is $ht(T) = \max(\{n < \omega \mid lev_n(T) \neq \emptyset\})$.
- For $t \in T$ the tree above t is $T_t = \{s \in T \mid t \leq_T s\}$.
- The set of all maximal branches of T is denoted by $mb(T) = lev_{ht(T)}(T)$.
- Let $J \subseteq \{0, 1, ..., ht(T)\}$ then $T \restriction J = \bigcup_{j \in J} lev_j(T)$

Proposition 2.2 *Let T be a $\vec{U}$-fat tree on $\theta_1 \leq ... \leq \theta_n$, Then there is a θ_1-complete ultrafilter $\vec{U}_T$ on $\prod_{1 \leq i \leq ht(T)} \theta_i$ such that $mb(T) \in \vec{U}_T$.*

Proof. by induction on $ht(T)$, If $ht(T) = 1$ just take $suc_T(\langle\rangle)$. Let T be a tree with $ht(T) = n$, Let $\alpha \in suc_T(\langle\rangle)$ then by hypothesis there is $\vec{U}_{T_\alpha}$. For $X \subseteq \prod_{i=1}^{ht(T)} \theta_i$

$$X \in \vec{U}_T \Leftrightarrow \{\alpha < \theta_1 \mid X_\alpha \in \vec{U}_{T_\alpha}\} \in U_{\langle\rangle}$$

where $X_\alpha = X \cap (\{\alpha\} \times \prod_{i=2}^{ht(T)} \theta_i)$. By definition $mb(T) \in \vec{U}_{T_\alpha}$. It is routine to check that $\vec{U}_T$ is a θ_1-complete ultrafilter.
■

For any $t, t' \in mb(T)$, the of set $t \cup t'$ naturaly orders in one of finitely many orders. For example, if $t = \langle\alpha_1, .\alpha_2, \alpha_3\rangle$ and $t' = \langle\alpha'_1, \alpha'_2, \alpha'_3\rangle$ the following is a possible such interweaving:

$$\alpha_1 < \alpha'_1 = \alpha_2 < \alpha'_2 < \alpha'_3 < \alpha_3$$

Definition 2.3 An interweaving p is a pair of order embedding $\langle g, g' \rangle$ where $g, g' : ht(T) \to \{1, ..., k\}$ so that $Im(g) \cup Im(g') = \{1, ..., k\}$. Denote $A_p = Im(g), A'_p = Im(g')$ and $k = |p|$.

Define the iteration associated with p, j_p in the following way: the length of the iteration is $|p|$, start with the pair $\langle 0, 0 \rangle$, set $A_p^{\langle 0,0 \rangle} = A_p'^{\langle 0,0 \rangle} = \emptyset$. Assume that we are at some stage with the pair $\langle n_1, n_2 \rangle$ and we have performed the mth iteration with critical points $\kappa_1, ..., \kappa_m$. Also assume inductively that

$$\langle k_i \mid i \in A_p \cap m+1 \rangle \in j_m(T), \langle k_i \mid i \in A'_p \cap m+1 \rangle \in j_m(T)$$

If $m+1 \in A_p \setminus A'_p$, then perform the ultrapower by $j_m(\vec{U})_{\langle \kappa_i | i \in A_p \cap m+1 \rangle}$ which is an ultrafilter over $\kappa_{m+1} = j_m(\theta_{n_1})$ where $g(n_1) = m+1$. If $m+1 \in A'_p \setminus A_p$ we act in a similar manner. Note that it is impossible that κ_{m+1} is less then some κ_j by the assumption we made that for $t \in lev_j(T)$, $\min(suc_T(t)) > \theta_j$. If $m+1 = g(n_1) = g'(n_2)$ there are two possibilities, either

$$j_m(\vec{U})_{\langle \kappa_i | i \in A_p \cap m+1 \rangle} \neq j_m(\vec{U})_{\langle \kappa_j | j \in A'_p \cap m+1 \rangle}$$

In this case we stop, and declare that the iteration is undefined. Otherwise

$$j_m(\vec{U})_{\langle \kappa_i | i \in A_p \cap m+1 \rangle} = j_k(\vec{U})_{\langle \kappa_j | j \in A'_p \cap m+1 \rangle}$$

perform the ultrapower with this measure,

Proposition 2.4 *Let T be a $\vec{U}$-fat tree, where $\vec{U}$ is a discrete family of normal measures. For any interweaving p*

1. *If j_p is defined, then there is a $\vec{U}$-fat tree, S_p, with $ht(S_p) = |p|$ and for every $s \in mb(S_p)$, $s \restriction A_p, s \restriction A'_p \in mb(T)$ interweave as in p. Moreover, $U_r^{(S_p)} = U^{(T)}_{r \restriction A_p \cap ht(r)}$ or $U_r^{(s_p)} = U^{(T)}_{r \restriction A'_p \cap ht(r)}$.*

2. *For any formula $\Phi(X, y_1, ..., y_{|p|})$ and any parameter $f \in V$ we have*

$$M \models \Phi(j_p(f), \kappa_1, ..., \kappa_{|p|}) \Leftrightarrow \{\vec{\alpha} \in [\kappa]^{|p|} \mid \Phi(f, \vec{\alpha})\} \in \vec{U}_{S_p}$$

3. *We can shrink T to R such that $mb(R) \in \vec{U}_T$ and if $t, t' \in mb(R)$ interweave as in p then $t \cup t' \in S_p$*

4. *If in p $\alpha'_1 < \alpha_1$, then we can shrink T to R such that $mb(R) \in \vec{U}_T$ and for every $t \in mb(R)$ and $\alpha \in \mathrm{Succ}_R(\langle\rangle) \cap \min(t)$ there is $t' \in mb(T)$ such that t, t' interweave as p and $\min(t') = \alpha$.*

5. *If the iteration j_p is not defined then there is S such that $mb(S) \in \vec{U}_T$ and there are no $t, t' \in mb(S)$ interweaving as in p.*

Proof. Let θ_i be the meausrables of the interweaving i.e. $j_p(\theta_i)=\kappa_i$. Prove $1,2$ simultaneously, by induction on k we will define $S_p^{(k)}$- a tree of height k which correspond to step k of the iteration. $S_p^{(0)}=\{\emptyset\}$, and $S_p^{(1)}=T\restriction 1$ which satisfy $1,2$ by Los theorem. Assume that $S_p^{(m)}$ is defined and $1,2$ hold. Consider the mth step of the iteration. If $m+1\in A_p\setminus A'_p$ define $S_p^{(m+1)}\restriction m=S_p^{(m)}$ and for every $\vec{\alpha}\in Lev_m(S_p^{(m)})$ define

$$\mathrm{Succ}_{S_p^{(m+1)}}(\vec{\alpha})=suc_T(\vec{\alpha}\restriction A_p\cap m+1)\setminus(\max(\vec{\alpha})+1)\in U^{(T)}_{\vec{\alpha}\restriction A_p\cap m+1}$$

If $m+1\in A'_p\setminus A_p$ the definition is similar. Assume that $m+1=g(n_1)=g'(n_2)$, since j_p is defined

$$j_m(\vec{U})_{\langle\kappa_i|i\in A_p\cap m+1\rangle}=j_m(\vec{U})_{\langle\kappa_j|j\in A'_p\cap m+1\rangle}$$

By 2 of the induction hypothesis it follows the

$$Lev_m(S_p^{(m+1)}):=\{\vec{\alpha}\in Lev_k(S_p^{(m)})\mid \vec{U}_{\vec{\alpha}\restriction A_p\cap(m+1)}=\vec{U}_{\vec{\alpha}\restriction A'_p\cap(m+1)}\}\in U_{S_p^{(m)}}$$

For $\vec{\alpha}\in Lev_m(S_p^{(m+1)})$ define

$$\mathrm{Succ}_{S_p^{(m+1)}}(\vec{\alpha})=\mathrm{Succ}_T(\vec{\alpha}\restriction A_p\cap(m+1))\cap suc_T(\vec{\alpha}\restriction A'_p\cap(m+1))\in\vec{U}^{(T)}_{\vec{\alpha}\restriction A_p\cap(m+1)}$$

To see that $1,2$ hold, 1 follows directly from the definition of $S_p^{(m+1)}$. For 2, Let $\Phi(X,y_1,...,y_{m+1})$ be any formula and $f\in V$, then

$$M_{m+1}\models\Phi(j_{m+1}(f),\kappa_1,...,\kappa_{m+1})\leftrightarrow M_m\models\{\alpha<\kappa_{m+1}\mid\Phi(j_m(f),\kappa_1,...,\kappa_m,\alpha)\}\in j_m(\vec{U})_{\langle\kappa_i|i\in A_p\rangle}$$

$$\leftrightarrow\{\vec{\alpha}\in[\theta_m]^{\leq m}\mid\{\alpha<\theta_{m+1}\mid\Phi(f,\vec{\alpha},\alpha)\}\in U_{\vec{\alpha}\restriction A_p\cap(m+1)}\}\in U_{S_p^{(m)}}\leftrightarrow\{\vec{\alpha}^\frown\alpha\mid\Phi(f,\vec{\alpha},\alpha)\}\in U_{S_p^{(m+1)}}$$

Finally, $S_p=S_p^{(|p|)}$ is as wanted.
To see 3, for every $\vec{\alpha}\in Lev_{m+1}(S_p)$ define $t(\vec{\alpha})\in T$ to be $\vec{\alpha}\restriction A_p\cap m+1$ and $t'(\vec{\alpha})=\vec{\alpha}\restriction A'_p\cap m+1$. From 1 it follows that if $m+1\in A_p$ then $suc_{S_p}(\vec{\alpha})\in U_{t(\vec{\alpha})}$ and similarly for $m+1\in A'_p$. Define R inductively, let $k_1=\min(A_p),\ k_2=\min(A'_p)$ then

$$suc_R(\langle\rangle)=\Delta_{\vec{\alpha}\in Lev_{k_1}(S_p)}suc_{S_p}(\vec{\alpha})\cap\Delta_{\vec{\alpha}\in Lev_{k_2}(S_p)}suc_{S_p}(\vec{\alpha})\in U^{(T)}_{\langle\rangle}$$

Assume $r\in Lev_m(R)$ is defined, let $g(m)=n_1$ and $g'(m)=n_2$. Define

$$suc_R(r)=\underset{\vec{\alpha}\in Lev_{n_1}(S_p),t(\vec{\alpha})=r}{\Delta}suc_{S_p}(\vec{\alpha})\cap\underset{\vec{\alpha}\in Lev_{n_2}(S_p),t'(\vec{\alpha})=r}{\Delta}suc_{S_p}(\vec{\alpha})\in U^{(T)}_r$$

So $R\in\vec{U}_T$. To see 4, suppose that $\alpha'_1<\alpha_1$ and $A_p=\{n_1,...,n_k\}$. Define a sequence inductively, let $\vec{\eta}_1=\langle\beta_1,...,\beta_{n_1-1}\rangle\in S_p$. Then by 1, $suc_{S_p}(\vec{\eta}_1)\in U^{(T)}_{\langle\rangle}$, thus

$$\kappa_1\in j_1(suc_{S_p}(\vec{\eta}_1))=suc_{j_1(S_p)}(\vec{\eta}_1)$$

Consider $\vec{\eta_1}^\frown\langle\kappa_1\rangle \in lev_{n_1}(j_1(S_p))$, pick any $\vec{\eta_2}$ such that $\vec{\eta_1}^\frown\langle\kappa_1\rangle^\frown\vec{\eta_2} \in lev_{n_2-1}(j_1(S_p))$, then

$$suc_{j_1(S_p)}(\vec{\eta_1}^\frown\langle\kappa_1\rangle^\frown\vec{\eta_2}) \in j_1(\vec{U})^{(j_1(T))}_\kappa \Rightarrow \kappa_2 \in suc_{j_2(S_p)}(\vec{\eta_1}^\frown\langle\kappa_1\rangle^\frown\vec{\eta_2})$$

continuing in this fashion we end up with a witness for the statement

$$M_n \models \exists t \in mb(j_n(T)) \ s.t. \ \langle\kappa_1, ..., \kappa_n\rangle, t \text{ interweave as in } p$$

Since $\beta_1 \in suc_{S_p}(\langle\rangle) = suc_T(\langle\rangle)$ was arbitrary, it follows that

$$M_n \models \forall\beta \in suc_{j_1(T)}(\langle\rangle)\cap\kappa_1 \exists t \in mb(j_n(T)) \ s.t. \ min(t) = \beta \wedge \langle\kappa_1, ..., \kappa_n\rangle, t \text{ interweave as in } p$$

By $2, 3$, we can find R as wanted. To prove 5, we apply $1, 2, 3$ to first level of the iteration which is not defiend. ■

The following lemmas are generalizations of the combinatorical property that were proven in [2] for product of measures. They can be stated for more general trees but we will restrict our attention to our needs.

Lemma 2.5 *Let $\vec{U}$ be a sequence of normal measures and let T be a $\vec{U}$-fat tree on $\theta_1 \leq \theta_2 \leq ... \leq \theta_n$. For any $\lambda < \theta_1$ and $f : mb(T) \rightarrow \lambda$ there is a $\vec{U}$-fat tree $T' \subseteq T$ such that $mb(T') \in \vec{U}_T$ and $f \restriction mb(T') = const$.*

Proof. By induction on the heigth of a tree. If $ht(T) = 1$ it is the case of one measure, $U_{\langle\rangle}$, which is well known. Assume the lemma holds for n and fix T, f such that $ht(T) = n + 1$. For $\vec{\alpha} = \langle\alpha_0, ...\alpha_{n-1}\rangle \in lev_n(T)$ consider $suc_T(\vec{\alpha}) \in U_{\vec{\alpha}}$. Define $f_{\vec{\alpha}} : suc_T(\vec{\alpha}) \rightarrow \lambda$ by $f_{\vec{\alpha}}(\beta) = f(\vec{\alpha}^\frown\beta)$. Then there exist $H_{\vec{\alpha}} \in U_{\vec{\alpha}}$ homogeneous for $f_{\vec{\alpha}}$ with color $c_{\vec{\alpha}}$. Consider the function

$$g : mb(T \restriction n + 1) \rightarrow \lambda \quad g(\vec{\alpha}) = c_{\vec{\alpha}}$$

Since $ht(T \restriction n + 1) = n$ we can apply the induction hypothesis to g, so let $T' \subseteq T \restriction n + 1$ be an homogeneous $\vec{U}$-fat subtree. Extend T' by adjoining $H_{\vec{\alpha}}$ as the successors of $\vec{\alpha} \in mb(T')$, denote the resulting tree by T^*. Note that by the induction, T^* is a $\vec{U}$-fat tree with $ht(T^*) = n + 1$. It is routine to check that T^* is as wanted.
■

Lemma 2.6 *Let T be a $\vec{U}$-fat tree on $\theta_1 \leq ... \leq \theta_n$ and $f : mb(T) \rightarrow B$ where B is any set. Then there is a $\vec{U}$-fat tree $T' \subseteq T$, with $mb(T') \in \vec{U}_T$ $I \subseteq \{1, ..., ht(T)\}$ such that for any $t, t' \in mb(T')$*

$$t \restriction I = t' \restriction I \Leftrightarrow f(t) = f(t')$$

We call the set I- a set of important coordinates.

Note that the condition $t \restriction I = t' \restriction I \Leftrightarrow f(t) = f(t')$ ensures that f is well defined modulo this relation and the induced function is $1-1$. We denote this function by f_I.

Proof. Again we go by induction on $ht(T)$. For $ht(T) = 1$ it is well known. Assume $ht(T) = n+1$ and fix $\alpha \in lev_1(T)$ consider the function

$$f_\alpha : mb(T_{\langle\alpha\rangle}) \to B \quad f_\alpha(\vec{\beta}) = f(\alpha^\frown\vec{\beta})$$

By the induction hypothesis there is $T'_\alpha \subseteq T_{\langle\alpha\rangle}$ and $I_\alpha \subseteq \{2, ..., n+1\}$ for which the lemma holds. Shrink $lev_1(T)$ to $H \in U_{\langle\rangle}$ so there is I' such that $I_\alpha = I'$ for $\alpha \in H$. Let S be the tree with $lev_1(S) = H$ and for every $\alpha \in H$, $(S)_\alpha = T'_\alpha$. Our strategy is to go over all possible interweaving of counter examples for the lemma and shrink the tree S to try eliminate them. A counter example is two elements $t = \langle\alpha_1, ..., \alpha_{n+1}\rangle, t' = \langle\alpha'_1, ..., \alpha'_{n+1}\rangle \in mb(S)$, such that

$$t \restriction I' \cup \{1\} \neq t' \restriction I' \cup \{1\} \wedge f(t) = f(t')$$

Note that if $min(t) = min(t')$ then by the construction of S, t, t' cannot be a counter example, hence a counter example is one with $min(t) \neq min(t')$. Fix any interweaving p with $\alpha_1 \neq \alpha'_1$, and consider the corresponding iteration, j_p. If this iteration is undefined then by 2.4(5) we can shrink S such that we have eliminated this kind interweaving. If the iteration is defined, compare $j_p(f)(\langle\kappa_i \mid i \in A_p\rangle), j_p(f)(\langle\kappa_j \mid j \in A'_p\rangle)$. Suppose the interweaving is such that for some $i \in I'$, $\alpha_i \neq \alpha'_i$ we claim that

$$j_p(f)(\langle\kappa_i \mid i \in A_p\rangle) \neq j_p(f)(\langle\kappa_j \mid j \in A'_p\rangle)$$

Otherwise by 2.4(2), we can shrink S so that any t, t' which interweaves as p, satisfy $f(t) = f(t')$. WLOG suppose that $\alpha'_i < \alpha_i$and let $q = g(i)$, in particular $q \in A_p$. We construct recursively two maximal branches of S_p, pick any element in $t \in lev_{q-1}(S_p)$, pick $t_q < r_q \in suc_{S_p}(t)$. Assume that t_k, r_k are defined such that for any $j \in A'_p \cap (k+1)$, $t_j = r_j$. If $k+1 \in A'_p$ then $\vec{U}^{(S_p)}_{t^\frown\langle t_l ... t_k\rangle}$, $\vec{U}^{(S_p)}_{t^\frown\langle r_l ... r_k\rangle}$ depends only on $t^\frown\langle t_l, ..., t_k\rangle \restriction A'_p = t^\frown\langle r_l, ..., r_k\rangle \restriction A'_p$ so we can choose

$$t_{k+1} = r_{k+1} \in suc_{S_p}(t^\frown\langle t_l ... t_k\rangle) \cap suc_{S_p}(t^\frown\langle r_l ... r_k\rangle)$$

If $k+1 \in A_p$, extend t_k, r_k randomly. Eventually we obtain $t^*, r^* \in mb(S_p)$ with $t^* \restriction A'_p = r^* \restriction A'_p = \vec{\alpha}'$ and $\min(t^*) = \min(r^*) = \min(t)$. Hence $t^* \restriction A_p, r^* \restriction A_p, \vec{\alpha}' \in mb(S)$, note that both $t^* \restriction A_p, \vec{\alpha}'$ and $r^* \restriction A_p, \vec{\alpha}'$ interweave as in p. Consequently,

$$f(t^* \restriction A_p) = f(\vec{\alpha}') = f(r^* \restriction A_p)$$

This means we found a counter example with the same first coordinate with is a contradiction, concluding that $j_p(f)(\langle\kappa_i \mid i \in A_p\rangle) \neq j_p(f)(\langle\kappa_j \mid j \in A'_p\rangle)$. By 2.4(2) we can shrink S so that for every t, t' which interweaves as p, $f(t) \neq f(t')$, in other words, we have eliminated all such counter examples which coresponds to p. Next, consider p in which $\alpha_i = \alpha'_i$ for every $i \in I'$. If

$$j_p(f)(\langle \kappa_i \mid i \in A_p \rangle) = j_p(f)(\langle \kappa_j \mid j \in A'_p \rangle)$$

then we can shrink S so that whenever $t, t' \in mb(S)$ interweave as p, $f(t) = f(t')$. By 2.4 (4) we can shrink S further to S^* so that for evert $t \in mb(S^*)$ and $\alpha < \min(t)$ there is $s \in mb(S)$ so that $\min(s) = \alpha \wedge t, s$ interweave as in p. We claim that 1 is not an important coordinate i.e. $I' = I$ is a set of important coordinate. To see this, assume that $t, t' \in mb(S^*)$. WLOG assume that $\min(t') = \alpha < \min(t)$, by the construction of S^*, there is $t'' \in mb(S)$ such that t, t'' interweave as in p and $\min(t') = \alpha = \min(t'')$, also $t \restriction I = t'' \restriction I$. Hence $f(t) = f(t'')$ and

$$f(t) = f(t') \Leftrightarrow f(t'') = f(t') \Leftrightarrow t'' \restriction I = t' \restriction I \Leftrightarrow t \restriction I = t' \restriction I$$

Finally if $j_p(f)(\langle \kappa_i \mid i \in A_p \rangle) \neq j_p(f)(\langle \kappa_j \mid j \in A'_p \rangle)$ then we shrink S and eliminate counter examples of the form p. Obviously, if we went through all possible interweaving of a counter examples and eliminated them, then $I = I' \cup \{1\}$ will be a set of important coordinate. ■

Given $F : mb(T) \to \kappa$ as in the last lemma, and important coordinates I, it is possible that the reason for a specific i to be in I is that for every $\vec{\alpha}, \vec{\alpha}' \in mb(T)$ if $\alpha_i \neq \alpha'_i$ then for some $j \in I$, $i \neq j$ and $\alpha_j \neq \alpha'_j$. In this case it is possible to drop i from I since

$$\vec{\alpha} \restriction I = \vec{\alpha}' \restriction I \Leftrightarrow \vec{\alpha} \restriction I \setminus \{i\} = \vec{\alpha}' \restriction I \setminus \{i\}$$

In general we can pick I so that no matter how we shrink T to S with $mb(S) \in \vec{U}_T$, for every $i \in I$ there are $\vec{\alpha}, \vec{\alpha}' \in mb(S)$ such that $\vec{\alpha} \restriction I \setminus \{i\} = \vec{\alpha}' \restriction I \setminus \{i\}$ and $\alpha_i \neq \alpha'_i$. Call such I *minimal set of important coordinates*. In the next lemma we will also need the following definition $mb(T) \restriction I = \{t \restriction I \mid t \in mb(T)\}$.

Lemma 2.7 *Let T and S be $\vec{U}$-fat trees on $\theta_1 \leq ... \leq \theta_n$, $\kappa_1 \leq ... \leq \kappa_m$ respectively. Suppose $F : mb(T) \to \kappa$ and $G : mb(S) \to \kappa$ are any functions with minimal sets of important coordinates I, J respectively. Then there exists $\vec{U}$-fat subtrees T^*, S^* with $mb(T^*) \in \vec{U}_T$ and $mb(S^*) \in \vec{U}_S$ such that one of the following holds:*

1. *$mb(T^*) \restriction I = mb(S^*) \restriction J$, $(F \restriction mb(T^*))_I, (G \restriction mb(S^*))_J$ are well defined on this set and*
$$(F \restriction mb(T^*))_I = (G \restriction mb(S^*))_J$$
2. *$Im(F \restriction mb(T^*)) \cap Im(G \restriction mb(S^*)) = \emptyset$*

Proof. The proof is similar to case of product of measures. By induction on $\langle ht(T), ht(S) \rangle$, If $ht(T) = ht(S) = 1$ then we are in the case of product of measures. If $\kappa_1 < \theta_1$ assume that $\min(suc_T(\langle\rangle)) > \kappa_1$ and if $\theta_1 < \kappa_1$ assume that $\min(suc_T(\langle\rangle)) > \kappa_1$. assume that $\langle ht(T), ht(S) \rangle >_{LEX} \langle 1, 1 \rangle$. Assume WLOG that $\kappa_1 \geq \theta_1$, if $ht(T) = 1$ define

$$H_1 : suc_T(\langle\rangle) \times mb(S) \to \{0, 1\}, \quad H_1(\alpha, \vec{\beta}) = 1 \Leftrightarrow F(\alpha) = G(\vec{\beta})$$

Since $suc_T(\langle\rangle) \times S$ is again a $\vec{U}$-fat tree we can shrink S, T to trees so that H_1 is constantly c_1. As in the case of product of sets, if $c_1 = 1$ then F, G are constant on large sets, thus $I = J = \emptyset$ and we are done. Assume that $c_1 = 0$. If $ht(T) > 1$, for every $\alpha \in suc_T(\langle\rangle)$ define the functions

$$F_\alpha : mb(T_\alpha) \to X, \quad F_\alpha(\vec{\alpha}) = F(\alpha, \vec{\alpha})$$

Use the induction hypothesis for F_α, G (which have important coordinates $I^* = I \setminus \{1\}, J^* = J$) to obtain T^*_α, S^α for which $mb(T^*_\alpha) \in \vec{U}_{T_\alpha}, mb(S^\alpha) \in \vec{U}_S$ such that one of the following:

1. $mb(T^*_\alpha) \restriction I^* = mb(S^\alpha) \restriction J^*$ and $(F_\alpha \restriction mb(T^*_\alpha))_{I^*} = (G \restriction mb(S^*))_{J^*}$.
2. $Im(F_\alpha \restriction mb(T^*_\alpha)) \cap Im(G \restriction mb(S^\alpha)) = \emptyset$.

denote by $i_\alpha \in \{1, 2\}$ the relevant case. There is $H \subseteq suc_T(\langle\rangle)$ $U_{\langle\rangle}$-large such that i_α is constantly i^*. Let T' be the tree such that $suc_{T'}(\langle\rangle) = H$ and $T'_\alpha = T^*_\alpha$ then $T' \in \vec{U}_T$. Let S' be the tree define as follows:

$$suc_{S'}(\langle\rangle) = \underset{\alpha \in A_1}{\Delta} suc_{S^\alpha}(\langle\rangle)$$

(Since $\theta_1 \leq \kappa_1$ we can take the diagonal intersection) and for every $\beta \in suc_{S'}(\langle\rangle)$

$$S'_\beta = \cap_{\alpha < \beta} (S^\alpha)_\beta$$

then $S' \in \vec{U}_S$. If $i^* = 1$ the we can shrink $suc_T(\langle\rangle)$ even more to stabilize the value of $I_\alpha = I^*, J_\alpha = J^*$. Note that $I = I^*$ and $J^* \subseteq J$, to see this is suffices to prove that $1 \notin I$, otherwise, fix $\alpha \neq \alpha' \in suc_{T'}(\langle\rangle)$, by the assumption

$$mb(T^*_\alpha) \restriction I^* = mb(S^\alpha) \restriction J^*$$

take some $\vec{\beta} \in mb(S_\alpha) \cap mb(S_{\alpha'})$ thus $\vec{\beta} \restriction J^* \in mb(T^*_\alpha) \restriction I^* \cap mb(T^*_{\alpha'}) \restriction I^*$ there are $\vec{\alpha} \in mb(T^*_\alpha)$, $\vec{\alpha'} \in mb(T^*_{\vec{\alpha}'})$ such that $\vec{\alpha} \restriction I^* = \vec{\alpha'} \restriction I^* = \vec{\beta} \restriction J^*$. On one hand, since 1 is an important coordinate, $\alpha \neq \alpha' \to F(\vec{\alpha}) \neq F(\vec{\alpha}')$. On the other hand, $\vec{\alpha} \restriction I^* = \vec{\alpha}' \restriction I^*$ and $F(\vec{\alpha}) = G(\vec{\beta}) = F(\vec{\alpha}')$, contradiction. Thus $1 \notin I$. Let $\langle \alpha, \vec{\alpha} \rangle, \langle \alpha', \vec{\alpha}' \rangle \in mb(T')$ with $\vec{\alpha} \restriction I^* = \vec{\alpha}' \restriction I^*$, then $\vec{\alpha} \restriction I^* \in mb(S') \restriction J^*$ and

$$F(\alpha, \vec{\alpha}) = F_\alpha(\vec{\alpha}) = (F_\alpha)_{I^*}(\vec{\alpha} \restriction I^*) = G_{J^*}(\vec{\alpha} \restriction I^*) = (F_{\alpha'})_{I^*}(\vec{\alpha}' \restriction I^*) = F_{\alpha'}(\vec{\alpha}) = F(\alpha', \vec{\alpha}')$$

consequently F_{I^*} is a well defined function on $mb(T')$ and $(F \restriction mb(T'))_{I^*} = (G \restriction mb(S'))_{J^*}$, so we may assume that assume $i^* = 2$. We repeat the same process only this time we use G_β and fixing F, denoting j_β the relevant case, shrink the sets so that j^* is constant. In case $j^* = 1$ the proof is the same as $i^* = 1$. So we assume that $i^* = j^* = 2$, meaning that for every $\langle \alpha, \vec{\alpha} \rangle \in mb(T'), \langle \beta, \vec{\beta} \rangle \in mb(S')$ if $\alpha < \beta$ then $\langle \beta, \vec{\beta} \rangle \in mb(S^\alpha)$ and $\vec{\alpha} \in mb(T^*_\alpha)$, by $i^* = 2$

$$F(\alpha, \vec{\alpha}) = F_\alpha(\vec{\alpha}) \neq G(\beta, \vec{\beta})$$

Similarly, if $\beta < \alpha$ then $\langle\alpha,\vec{\alpha}\rangle \in mb(T^{\beta})$ and $\vec{\beta} \in mb(S^*_{\beta})$, hence $F(\alpha,\vec{\alpha}) \neq G(\beta,\vec{\beta})$ by $j^* = 2$, so the only possibility for equality is $\alpha = \beta$. If the measures $U^{(T)}_{\langle\rangle}, U^{(S)}_{\langle\rangle}$ are different we can just separate the sets $suc_{T'}(\langle\rangle), suc_{S'}(\langle\rangle)$ and avoid the case $\alpha = \beta$, we conclude that

$$Im(F \restriction mb(T')) \cap Im(G \restriction mb(S')) = \emptyset$$

If $U^{(T)}_{\langle\rangle} = U^{(S)}_{\langle\rangle}$ we can shrink to $suc_{T'}(\langle\rangle) = suc_{S'}(\langle\rangle)$ and for every $\alpha \in suc_{T'}(\langle\rangle)$ we apply the induction hypothesis to the functions F_α, G_α, this time denoting the cases by r^*. If $r^* = 2$, then we have eliminated the possibility of $F(\alpha,\vec{\alpha}) = G(\alpha,\vec{\beta})$, together with $i^* = 2, j^* = 2$ we are done. Finally, assume $r^* = 1$, namely that for $I \setminus \{1\} = I^* \subseteq \{2,...,ht(T)\}, J \setminus \{1\} = J^* \subseteq \{2,...,ht(S)\}$, and every $\alpha \in suc_{T'}(\langle\rangle)$

$$mb(T'_\alpha) \restriction I^* = mb(S'_\alpha) \restriction J^* \ \wedge\ (F_\alpha \restriction mb(T'_\alpha))_{I^*} = (G_\alpha \restriction mb(S'_\alpha))_{J^*}$$

It follows that

$$mb(T') \restriction I^* \cup \{1\} = \cup_{\alpha \in suc_{T'}(\langle\rangle)} mb(T'_\alpha) \restriction I^* = \cup_{\alpha \in suc_{S'}(\langle\rangle)} mb(S'_\alpha) \restriction J^* = mb(S') \restriction J^* \cup \{1\}$$

Moreover, for every $\langle\alpha,\vec{\alpha}\rangle \restriction I^* \cup \{1\} \in mb(T') \restriction I^* \cup \{1\}$,

$$F_{I^* \cup \{1\}}(\alpha, \vec{\alpha} \restriction I^* \cup \{1\}) = (F_\alpha)_{I^*}(\vec{\alpha} \restriction I^*) = (G_\alpha)_{J^*}(\vec{\alpha} \restriction I^*) = G_{J^* \cup \{1\}}(\alpha, \vec{\alpha} \restriction I^* \cup \{1\})$$

If $1 \notin I$ then $I = I^*$ and F_I is well defined. We claim that $1 \notin J$, to see this, take some $\vec{\beta}, \vec{\beta}' \in mb(S')$ such that $\beta_1 \neq \beta'_1$ and $\vec{\beta} \restriction J^* = \vec{\beta}' \restriction J^*$. there exists such by minimality of J. It follows that $G(\vec{\beta}) \neq G(\vec{\beta}')$. Moreover,

$$\vec{\beta} \restriction J^* \in mb(S'_{\beta_1}) \restriction J^* \cap mb(S'_{\beta'_1}) \restriction J^* = mb(T'_{\beta_1}) \restriction I \cap mb(T'_{\beta'_1})$$

there are $\vec{\alpha} \in mb(T'_{\beta_1}), \vec{\alpha} \in mb(T'_{\beta'_1})$ such that $\vec{\alpha} \restriction I = \vec{\beta} \restriction J^*$ and $\vec{\alpha}' \restriction I = \vec{\beta}' \restriction J^*$. Since F is well defined on I it follows that $F(\vec{\alpha}) = F(\vec{\alpha}')$ which is impossible since $F(\vec{\alpha}) = G(\vec{\beta})$ and $F(\vec{\alpha}') = G(\vec{\beta}')$. So $1 \notin J$. In a similar way, we conclude that $1 \in I$ iff $1 \in J$. In any case we are done. ■

3 The proof for short sequences

Let us turn to the theorem for Magidor forcing with $o^{\vec{U}}(\kappa) = \kappa$. The analog of the set $X(p)$ would be the notion of tree of extensions.

Definition 3.1 Let $p \in \mathbb{M}[\vec{U}]$ be a condition. As usual assume that the large sets in the condition are separated i.e. $B_i(p) = \biguplus_{j < o^{\vec{U}}(\kappa_i(p))} B_{i,j}(p)$. A tree of extension of p is a $\vec{U}$-fat tree T such that each $t \in T$ is a legal extension of p i.e. $p^\frown t \in \mathbb{M}[\vec{U}]$. Moreover we require that for every $t \in T \setminus Lev_{ht(T)}(T)$ there exist $i \leq l(p) + 1$ and $j < o^{\vec{U}}(\kappa_i(p))$ such that $\mathrm{Succ}_T(t) \subseteq B_{i,j}(p)$

Note that by definition of tree of extensions, if $t_1, t_2 \in mb(T)$ are different then $p^\frown t_1, p^\frown t_2$ are incompatible. To see this, assume $t_1 \restriction i = t_2 \restriction i$ and with out loss of generality $t_1(i) < t_2(i)$. Then there are $j \leq l(p)+1$ and $\xi < o^{\vec{U}}(\kappa_j(p))$ such that $t_1(i), t_2(i) \in \mathrm{Succ}_T(t_1 \restriction i) \subseteq B_{j,\xi}(p)$, in particular $o^{\vec{U}}(t_2(i)) = o^{\vec{U}}(t_1(i)) = \xi$. Thus $t_1(i) \notin (\biguplus_{j<\xi} B_{i^*,j}(p)) \cap (t_2(i-1), t_2(i))$ and therefore $p^\frown t_1, p^\frown t_2$ are incompatible.

Proposition 3.2 *Let $p \in \mathbb{M}[\vec{U}]$ be a condition and T a tree of extensions for p. Then there exists $p^* \geq^* p$ such that T is also a tree of extensions for p^* and*

$$D_T = \{p^{*\frown} t \mid t \in mb(T)\}$$

is a maximal antichain above p^*. In particular, for any generic G with $p^* \in G$, $|G \cap D_T| = 1$.

Proof. Fix $\langle \nu, A \rangle$ in p and $i < o^{\vec{U}}(\nu)$. For every $t \in T \cap [\nu]^{<\omega}$ if $suc_T(t) \in U(\nu, i)$ then let $B_t = suc_T(t)$, otherwise $B_t = A$. Define

$$A_i^* = \Delta_{t \in T \cap [\nu]^{<\omega}} B_t \cap A_i \in U(\nu, i)$$

also let $A^* = \cup_{i < o^{\vec{U}}(\nu)} A_i^*$. Extend $\langle \nu, A \rangle$ to $\langle \nu, A^* \rangle$, doing so for every ν in p defines $p \leq^* p^*$. Now we turn to the proof that D_t is a maximal antichain above p^*. In the discussion preceding this lemma we saw that D_T forms a antichain. To see it's maximality we will use induction on $ht(T)$. For $ht(T) = 1$ we are in the case of extension types. Assume that it holds for $ht(T) = k$, And let T be a tree of height $k+1$. Set

$$T' = T \setminus mb(T)$$

then T' is a tree of height k. Let $p^* \leq q$, by induction hypothesis there exists $t \in T'$ and $r \in \mathbb{M}[\vec{U}]$ such that $p^{*\frown} t, q \leq r$. Consider i, ν such that $suc_T(t) \in U(\nu, i)$ and let $\nu' > \max(t)$ be minimal appearing in r such that $i \leq o^{\vec{U}}(\nu')$, obviously, $\nu' \leq \nu$. If $o^{\vec{U}}(\nu') = i$, then $\nu' \in A_i^*$ therefore $\nu' \in suc_T(t)$ so we can take $t^\frown \nu' \in T$. If follows that $q, p^{*\frown}(t^\frown \nu') \leq r$. If $o^{\vec{U}}(\nu') > i$, there is B such that $\langle \nu', B \rangle$ in r and $B \in \cap \vec{U}(\nu')$, in particular

$$B_i \subseteq A_i^* \cap (\max(t), \nu) \subseteq suc_T(t)$$

then any choice $\gamma \in B_i$ will witness that $p^{*\frown}(t^\frown \gamma), q \leq r^\frown \gamma$.
■

Proposition 3.3 *Let T be a $\vec{U}$-fat tree of extensions of p. Suppose that for every $t \in mb(T)$ there is a condition $p_t \geq^* p^\frown t$. Then there is $p^* \geq^* p$ and T^* with $mb(T^*) \in U_T$ such that for every $t \in mb(T^*)$, every $q \geq p^{*\frown} t$ is compatible with p_t.*

Proof. The proof is similar to the case $o^{\vec{U}}(\kappa) < \kappa$, by induction on $ht(T)$. For $ht(T) = 1$, this is simply what we have already proved in [3]. The proof of the induction step is the same. ■

Lemma 3.4 *Let $p \in \mathbb{M}[\vec{U}]$ and $\langle \lambda, B\rangle$ in the steam of p. Consider the decomposition, $p = \langle q, r\rangle$, where $q \in \mathbb{M}[\vec{U}] \restriction \lambda \wedge r \in \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$. Let $\underset{\sim}{x}$ be a name for an ordinal. Then there is $r \leq^* r^*$ such that for any $q' \geq q$ if*

$$\exists \vec{\alpha}\ \exists r' \geq^* r^{*\frown}\langle\vec{\alpha}\rangle.\ \langle q', r'\rangle\ ||\underset{\sim}{x}$$

then there is a tree of extensions of r^*, $T_{q'}$, such that

$$\forall t \in mb(T_{q'}).\ \langle q', r^{*\frown}t\rangle\ ||\underset{\sim}{x}$$

Proof. In order to simplify notation assume p has empty steam i.e. $p = \langle\langle\kappa, A\rangle\rangle$ where $A = \biguplus_{i<o^{\vec{U}}(\kappa)} A_i$. Fix $q' \geq q$ and $n < \omega$. Let $\vec{\alpha} = \langle\alpha_1, ..., \alpha_n\rangle \in [\kappa]^{<\omega}$, such that $r^\frown\langle\vec{\alpha}\rangle$ is a condition and $i < o^{\vec{U}}(\kappa)$. Set

$$A^0_i(\vec{\alpha}) = \{\alpha \in A_i \setminus (\alpha_n + 1) \mid \exists r' \geq^* r^\frown\langle\vec{\alpha}, \alpha\rangle\ (\langle q', r'\rangle\ ||\ \underset{\sim}{x})\},\quad A^1_i(\vec{\alpha}) = A_i \setminus A^0_i(\vec{\alpha})$$

only one of $A^0_i(\vec{\alpha}), A^1_i(\vec{\alpha})$ is in $U(\kappa, i)$. Denote it by $A_i(\vec{\alpha})$ and let $C_i(\vec{\alpha}) \in \{0, 1\}$ such that $A_i(\vec{\alpha}) = A_i^{C_i(\vec{\alpha})}(\vec{\alpha})$. Define

$$A^*_i = \underset{\vec{\alpha}\in[\kappa]^{<\omega}}{\Delta} A_i(\vec{\alpha}) \cap A \in U(\kappa, i)$$

so far A^*_i has the property that for $\vec{\alpha} \in [\kappa]^{<\omega}$ if $\exists \alpha \in A^*_i$ and $r' \geq^* r^\frown\langle\vec{\alpha}, \alpha\rangle$ deciding $\underset{\sim}{x}$ then every $\alpha \in A^*_i$ there is r' deciding $\underset{\sim}{x}$. For every $j < o^{\vec{U}}(\kappa)$ define $D^{(1)}_j(\alpha_1, ..., \alpha_{n-1}, *) : A^*_j \to \{0, 1\}$ by

$$D^{(1)}_j(\alpha_1, ..., \alpha_{n-1}, \alpha) = 0 \Leftrightarrow \exists i < o^{\vec{U}}(\kappa)\ C_i(\alpha_1, .., \alpha_{n-1}, \alpha) = 0$$

There is an homogeneous $A^{(1)}_j(\alpha_1, ..., \alpha_{n-1}) \in U(\kappa, j)$ with color $C^{(1)}_j(\alpha_1, ..., \alpha_{n-1})$, as before, denote the diagonal intersection over all sequences of length $n-1$ by $A^{*(1)}_j \in U(\kappa, j)$. In similar fashion, define recursively for $k \leq n$

$$D^{(k)}_j(\alpha_1, ..., \alpha_{n-k}, \alpha) = 0 \Leftrightarrow \exists i < o^{\vec{U}}(\kappa)\ C^{(k-1)}_i(\alpha_1, .., \alpha_{n-k}, \alpha) = 0$$

find homogeneous $A^{(k)}_j \in U(\kappa, j)$ with color $C^{(k)}_j(\alpha_1, ..., \alpha_{n-k})$. Eventually, set

$$A'_{i,n} = \bigcap_{k\leq n} A^{*(k)}_i,\ A'_i = \bigcap_{n<\omega} A'_{i,n} \in U(\kappa, i) \text{ and } A' = \biguplus_{i<o^{\vec{U}}(\kappa)} A'_i$$

Let $r'_{q'} = \langle\langle\kappa, A'\rangle\rangle \geq^* r$. Assume that there exists $r'' \geq r'_{q'}$ such that $\langle q', r''\rangle||\underset{\sim}{x}$. There is $\langle\vec{\alpha}, \alpha\rangle \in [A']^{<\omega}$ such that $r'^\frown_{q'}\langle\vec{\alpha}, \alpha\rangle \leq^* r''$. Thus, for some $i < o^{\vec{U}}(\kappa)$, $\alpha \in A'_i$ and $i_1, .., i_n$ such that $\vec{\alpha} = \langle\alpha_1, ..., \alpha_{n-1}\rangle \in A'_{i_1} \times A'_{i_2} \times ... \times A'_{i_n}$. It follows that $A_i(\vec{\alpha}) = A^0_i(\vec{\alpha})$. Hence,

$$C_i(\vec{\alpha}) = 0 \Rightarrow D^{(1)}_{i_n}(\alpha_1, .., \alpha_n) = 0 \Rightarrow C^{(1)}_{i_n}(\alpha_1, .., \alpha_{n-1}) = 0 \Rightarrow D^{(2)}_{i_{n-1}}(\alpha_1, .., \alpha_{n-1}) = 0 \Rightarrow$$

$$C^{(2)}_{i_{n-1}}(\alpha_1,...,\alpha_{n-2})=0\Rightarrow ...\Rightarrow D^{(n)}_{i_1}(\alpha_1)=0\Rightarrow C^{(n)}_{i_1}(\langle\rangle)=0$$

Define the tree $T'_{q'}$: $ht(T'_{q'})=n+1$ we initiate the definition by $\mathrm{Succ}_{T'_{q'}}(\langle\rangle)=A'_{i_1}$. Since A'_{i_1} is homogeneous, $D^{(n)}_{i_1}(\alpha)=0$ for every $\alpha\in A'_{i_1}$, hence, there is i_{α_1} such that $D^{(n-1)}_{i_{\alpha_1}}(\alpha_1,*)$ takes the color 0 on $A'_{i_{\alpha_1}}$. Let $\mathrm{Succ}_{T'_{q'}}(\alpha_1)=A'_{i_{\alpha_1}}$ keep defining the other levels similarly. The tree $T'_{q'}$ has the property that for every $t\in mb(T'_{q'})$ there is $r_t\geq^* r'_{q'}{}^\frown t$ such that $\langle q',r_t\rangle$ decide $\underset{\sim}{x}$. By proposition 3.3 we can amalgamate all those r_t's and find $r^*_q\geq^* r'_q$ and shrink T'_q to T^*_q such that for every $t\in T^*_q$, every $q\geq r^*_q{}^\frown t$ is compatible with r_t. It follows that $r^*_q{}^\frown t||\underset{\sim}{x}$. Finally, since $|Q|$ is small enough we can find $r^*\geq r^*_q$ and shrink T^*_q accordingly.
$\blacksquare$

Lemma 3.5 *Assume that $|A|<\kappa$. Then there exists $C'\subseteq C_G$ such that $V[A]=V[C']$.*

Proof. Let $A=\langle a_i\mid i<\lambda\rangle$ where $\lambda=|A|<\kappa$ be an enumeration of A. In V, Pick a name for A, $\langle\underset{\sim}{a}_i\mid i<\lambda\rangle$. We proceed by a density argument, let $p\in\mathbb{M}[\vec{U}]\restriction(\lambda,\kappa)$ be any condition, using lemma 3.4, find an $\leq^*$-increasing sequence $\langle p_i\mid i<\lambda\rangle$ above p and maximal antichains $Z_i\subseteq\mathbb{M}[\vec{U}]\restriction\lambda$ such that for every $q\in Z_i$ there is a $\vec{U}-fat$ tree $T_{q,i}$ such that any extension of p_i from $mb(T_{q,i})$ together with q decides $\underset{\sim}{a}_i$. Since $p_i\in\mathbb{M}[\vec{U}]\restriction(\lambda,\kappa)$, we can find p^* such that for every $i<\lambda$ $p_i\leq^* p^*$. Define the function $F_{q,i}:mb(T_{q,i})\to On$ by:

$$F_{q,i}(\vec{\alpha})=\gamma\quad\Leftrightarrow\quad\langle q,p^{*\frown}\vec{\alpha}\rangle\Vdash\underset{\sim}{a}_i=\check{\gamma}$$

By lemma 2.6, we can find $p^*\leq^* p^{**}$ and restrict $F_{q,i}$ to extensions from p^{**} such that there exists $I_{q,i}\subseteq\{1,...,ht(T_{q,i})\}$, minimal sets of important coordinates of $F_{q,i}$. For any $q,q'\in Z_i$ use lemma 2.7 for the functions $F_{q,i},F_{q',i}$ and shrink $T_{q,i},T_{q',i}$ so that either $Im(F_{q,i})\cap Im(F_{q',i})=\emptyset$ or $mb(T_{q,i})\restriction I_{q,i}=mb(T_{q',i})\restriction J_{q',i}$ and $(F_{q,i})_{I_{q,i}}=(F_{q',i})_{I_{q',i}}$. Extend $p^{**}\leq^* p_{q,q'}$ using proposition 3.2. Since $|\mathbb{M}[\vec{U}]\restriction\lambda|$ is small enough there is p' above these $p_{q,q'}$. By density find such $p'\in G$, the continuation is identical to the proof for $o^{\vec{U}}(\kappa)<\kappa$.
$\blacksquare$

Corollary 3.6 *Suppose that $p\in\mathbb{M}[\vec{U}]$ and $\underset{\sim}{x}$ is a name such that $p\Vdash\underset{\sim}{x}\in\underset{\sim}{C}_G$. Then there is $p^*\geq^* p$ either $p^*||\underset{\sim}{x}$ or there is a a $\vec{U}$-fat tree, T such that $\forall\vec{b}\in mb(T)$ $p^\frown\vec{b}\Vdash\underset{\sim}{x}=\max(\vec{b})$. Moreover, in the later case, if $o^{\vec{U}}(\kappa)=\kappa$ then for every $\vec{b}\in mb(T)$*

$$p^\frown\langle\vec{b}\setminus\{\max\vec{b}\}\rangle||o^{\vec{U}}(\underset{\sim}{x})$$

Proof. Assume that there is no $p^*\geq^* p$ which decides $\underset{\sim}{x}$. By 3.4 find T with minimal $ht(T)$ so that for every $t\in mb(T)$ $p^\frown t||\underset{\sim}{x}$. Assume that $\{\nu_1,...,\nu_n\}$ are the ordinals appearing in

p, denote x_t the forced value and shrink T so that the function

$$f(t)=\begin{cases} i & x_t=\nu_i \\ n+1 & x_t\notin\{\nu_1,...,\nu_n\}\end{cases}$$

is constant. if f would be constantly some $i\leq n$ then there is $p^*\geq^* p$ such that $p^*\Vdash \underset{\sim}{x}=\nu_i$, contradiction. So we may assume that $x_t\notin\{\nu_1,...,\nu_n\}$. Keep shrinking T so that there is a unique $i\leq ht(T)$, such that $x_t\in[t(i),t(i+1))$ (where $t(ht(T)+1)=\kappa$). If $i<ht(T)$ then for every $t\in Lev_i(t)$, the function $g_t: mb((T)_t)\rightarrow\kappa$, define by $g_t(s)=x_{t^\frown s}$ is regressive and therefore can be stabilized on some $S_t\subseteq T_t$. so that for every $t\in S_t$, $x_{t^\frown s}=y_t$, depending only on t. Thus the tree $T\restriction i$ already decides $\underset{\sim}{x}$, contradiction the minimality of $ht(T)$. Hence it most be that $x_t\geq t(ht(T))=\max(t)$. Again we shrink the tree so that $x_t>\max(t)$ or $x_t=\max(t)$. Toward a contradiction, assume that $x_t>\max(t)$, then $x_t\notin\{\nu_1,...,\nu_n\}\cup t$ so we can shrink the sets in $p^\frown t$ so that $p^\frown t\Vdash \underset{\sim}{x}=x_t\notin \underset{\sim}{C}_G$, contradiction. Hence for every $t\in mb(T)$ $p^\frown t\Vdash \underset{sim}{x}=\max(t)$. Finally, if $o^{\vec{U}}(\kappa)=\kappa$ then the measures in $\vec{U}$ are separated using $o^{\vec{U}}(\nu)$, so for every $t\in mb(T)$

$$o^{\max(t)}=\gamma\leftrightarrow suc_T(t\setminus\{\max(t)\})\in U(\kappa,\gamma)$$

and therefore

$$p^\frown\langle\vec{b}\setminus\{\max\vec{b}\}\rangle || o^{\vec{U}}(\underset{\sim}{x})$$

The following lemma is analogous to a lemma proven in [1] for Prikry forcing.

Lemma 3.7 *Let $\{d_i\mid i<\lambda<\kappa\}\in V[C_G]$ be some set of ordinals such that*

$$C_G\cap\{d_i\mid i<\lambda\}=\emptyset$$

then there is $X\in\bigcap_{i<o(\kappa)}U(\kappa,i)$ such that

$$X\cap\{d_i\mid i<\lambda\}=\emptyset$$

Proof. Let us start with a single name of an ordinal $\underset{\sim}{x}$ and $p\in G$ such that $p\Vdash\underset{\sim}{x}\notin\underset{\sim}{C}_G$. Assume that $p=\langle r,\langle\kappa,A\rangle\rangle$, then by 3.4 there is $A^*_q\subseteq A$ and a maximal antichain $Z\subseteq\mathbb{M}[\vec{U}\restriction\max(q)]$ such that for every $q\in Z$ there is a tree T_q for which every $\vec{b}\in mb(T_q)$,

$$\langle q,A^*_q\rangle^\frown\vec{b}\Vdash\underset{\sim}{x}=f_q(\vec{b})$$

For every $\vec{b}$, $f_q(\vec{b})\notin\vec{b}$ hence it falls in one of the intervals

$$(0,b_1),(b_1,b_2),...,(b_{ht(T_q)},\kappa)$$

let $n_{\vec{b}}$ be the number of this interval. Using lemma 2.5 there we can take $A^{**}_q\subseteq A^*_q$ and a tree $T^*_q\subseteq T_q$ on which the value $n_{\vec{b}}$ is constantly n^* now we can find A^{***}_q such that $f_q(\vec{b})$

depends only on $b_1, ..., b_{n^*}$ and $f_q(b_1, ..., b_{n^*}) > b_{n^*}$ hence $T^*_q \restriction n^* = S_q$ decides $\underset{\sim}{x}$. Finally we can find $B_q \subseteq A^{***}_q$, such that every $\vec{\alpha} \in [B_q]^{<\omega}$ is a legal extension. It must be that $f_q(b_1, ..., b_{n^*}) \notin B_q \setminus b_{n^*}$, otherwise, the condition $\langle q, \langle \kappa, B_q\rangle\rangle^\frown\langle b_1, ..., b_{n^*}, f_q(b_1, ..., b_{n^*})\rangle \Vdash \underset{\sim}{x} \in C_G$ contradiction. Also $f_q(b_1, ..., b_{n^*}) > b_{n^*}$ and we conclude that $f_q(b_1, ..., b_{n^*}) \notin B_q$. Let $A_x = \bigcap\limits_{q\in Z} B_q$, we claim that

$$p \leq^* \langle r, \langle \kappa, A_x\rangle\rangle \Vdash \underset{\sim}{x} \notin A_x$$

Otherwise, there is $q \in Z$, $\vec{b} \in mb(S_q)$ and p' such that

$$\langle q, \langle \kappa, A_x\rangle\rangle^\frown\vec{b} \leq p' \Vdash \underset{\sim}{x} \in A_x$$

but also $p' \Vdash \underset{\sim}{x} = f_q(\vec{b})$ so $f_q(b_1, ..., b_{n^*}) \in A_x \subseteq B_q$ which is a contradiction. Now the lemma follows easily, Let $\{d_i \mid i < \lambda < \kappa\} \in V[C_G]$ be some set of ordinals such that

$$C_G \cap \{d_i \mid i < \lambda\} = \emptyset$$

then we can take names $\{\underset{\sim}{d_i} \mid i < \lambda\}$ and some p forcing $\forall i < \lambda \underset{\sim}{d_i} \notin \underset{\sim}{C}_G$, as before we can define the sets A_i and find an increasing $\leq^*$ sequence $\langle r, p_i\rangle$, find p^* which bounds all of them and $A^* = \bigcap\limits_{i<\lambda} A_i$, then p^* forces that $\forall i < \lambda\ \underset{\sim}{d_i} \notin A^*$. By density argument we can find such in G.

■

4 The proof for subsets of κ

In the proof of $o^{\vec{U}}(\kappa) < \kappa$ we use the fact that $|C_G| < \kappa$, which is no longer true if $o^{\vec{U}}(\kappa) \geq \kappa$.

4.1 Stabilization of Subsets of κ

Let us start by proving a lemma which will help us code the information we need into one sequence.

Proposition 4.1 *If $C^* \subseteq C_G$ be any subset and $C'' \subseteq C_G$ be countable, then there is $C' \subseteq C_G$ such that $C^* \cup C'' \subseteq C'$ and $C^*, C'' \in V[C']$*

Proof. To find such C', we start with names $\underset{\sim}{C}^*, \langle \underset{\sim}{c}_n \mid n < \omega\rangle$ for C^* and C'' respectively and

$$p' \Vdash \underset{\sim}{C}^*, \langle \underset{\sim}{c}_n \mid n < \omega\rangle \subseteq \underset{\sim}{C}_G$$

We proceed by a density argument, let $p \in \mathbb{M}[\vec{U}]$ denote by $\underset{\sim}{\gamma}_n^{(0)} = \underset{\sim}{c}_n$, $\underset{\sim}{\gamma}_n^{(k+1)}$ be a name such that

$$p \Vdash \underset{\sim}{\gamma}_n^{(k+1)} = \sup(\{x \in \underset{\sim}{C}^* \cap \underset{\sim}{\gamma}_n^{(k)} \mid o^{\vec{U}}(x) \geq o^{\vec{U}}(\underset{\sim}{\gamma}_n^{(k)})\} \cup \{0\})$$

enumerate $\langle \underset{\sim}{\gamma}_n^{(k)} \mid n, k < \omega \rangle = \langle \underset{\sim}{\delta}_n^0 \mid n < \omega \rangle$. Note that for every $n, k < \omega$

$$p \Vdash \underset{\sim}{\gamma}_n^{(k)} = 0 \vee \underset{\sim}{\gamma}_n^{(k)} \in \underset{\sim}{C}_G \text{ and } \underset{\sim}{\gamma}_n^{(k+1)} < \underset{\sim}{\gamma}_n^{(k)}$$

This is since $\underset{\sim}{C}_G$ is closed and for every $\alpha \in C_G$ there is $\xi < \alpha$ such that for every $\gamma \in C_G \cap (\xi, \alpha)$, $o^{\vec{U}}(\gamma) < o^{\vec{U}}(\alpha)$. Suppose that $\langle \underset{\sim}{\delta}_n^{(k)} \mid n < \omega \rangle$ is defined and there is $p \leq^* p_k$ such that

$$\forall n < \omega \ p_k \Vdash \underset{\sim}{\delta}_n^{(k)} \in \underset{\sim}{C}_G$$

Use corollary 3.6 to find $p_k \leq^* p_{k+1}$, trees $T_n^{(k)}$ such that for every $\vec{b} \in mb(T_n^{(k)})$ $p_{k+1}^\frown \vec{b} \Vdash \underset{\sim}{\delta}_n^{(k)} = \max(\vec{b})$. consider the function $F_n^{(k)} : Lev_{ht(T_n^{(k)})-1}(T_n^{(k)}) \to \kappa$

$$F_n^{(k)}(\vec{\alpha}) = \beta \leftrightarrow p_{k+1}^\frown \vec{\alpha} \Vdash o^{\vec{U}}(\underset{\sim}{\delta}_n^{(k)}) = \beta$$

Shrink $T_n^{(k)}$, extend p_{k+1} accordingly and find important coordinates $I_n^{(k)}$. For every $j \in I_n^{(k)}$ let $\underset{\sim}{\gamma}_{n,j}^{(0)}$ be a name for the uniqe jth ordinal in a branch that ends with $\underset{\sim}{\delta}_n^{(k)}$ in the tree $T_n^{(k)}$. As before let $\underset{\sim}{\gamma}_{n,j}^{(m+1)}$ be a name name such that

$$p_{k+1} \Vdash \underset{\sim}{\gamma}_{n,j}^{(m+1)} = \sup(\{x \in \underset{\sim}{C}^* \cap \underset{\sim}{\gamma}_{n,j}^{(m)} \mid o^{\vec{U}}(x) \geq o^{\vec{U}}(\underset{\sim}{\gamma}_{n,j}^{(m)})\} \cup \{0\})$$

enumerate $\langle \underset{\sim}{\gamma}_{n,j}^{(m)} \mid n, j, m < \omega \rangle$ as $\langle \underset{\sim}{\delta}_n^{(k+1)} \mid n < \omega \rangle$. Note that $p_{k+1} \Vdash \underset{\sim}{\gamma}_{n,j}^{(m)} \in \underset{\sim}{C}_G$. Use σ-closure to find $p_n \leq^* p_\omega$ and shrink all the trees to be extension tree of p_ω. By density there is such $p_\omega \in G$. Define

$$C_* = \langle (\underset{\sim}{\delta}_n^{(k)})_G \mid n, k < \omega \rangle$$

Since $p_\omega \Vdash \underset{\sim}{\delta}_n^{(k)} \in \underset{\sim}{C}_G$, $C_* \subseteq C_G$. We claim that $C_* \in V[A]$. Work inside $V[A]$, recall that $C'', C^* \in V[A]$ therefore $\langle (\underset{\sim}{\delta}_n^{(0)})_G \mid n < \omega \rangle$ is definable in $V[A]$. Assume we have defines $\langle \delta_n^{(k)} \mid n < \omega \rangle$, choose $D_n \in F_n^{(0),-1}[\{(\underset{\sim}{\delta}_n^{(k)})_G\}]$ (definable in $V[A]$). Similar to 3.5, it follows that $j \in I_n^{(k)}$ $(\underset{\sim}{\gamma}_{n,j})_G = D_n(j)$. Again, since $C^* \in V[A]$ it follows that $\langle (\underset{\sim}{\delta}_n^{(k+1)})_G \mid n < \omega \rangle$ is definable in $V[A]$. So we conclude that $C_* \in V[A]$. Define

$$C' = C^* \cup C_* \in V[A]$$

We claim that $C_* \in V[C']$, So it remains to prove that $C_* \in V[C']$ let $\langle \lambda_i \mid i < otp(C_*) \rangle$, be a countable increasing enumeration of C^* and let $\langle \underset{\sim}{\lambda}_i \mid i < otp(C_*) \rangle$ re numerate

$$\langle \underset{\sim}{\delta}_n^{(m)} \mid n, m < \omega \rangle$$

accordingly. Note that this order is in V since no new reals are added. More over the relation of $\underset{\sim}{\lambda}_{i_1}, ..., \underset{\sim}{\lambda}_{i_k}$ are the branch of $\underset{\sim}{\lambda}_i$ and $\underset{\sim}{\lambda}_j$ is a name for $\sup(\{x \in \underset{\sim}{C}^* \cap \underset{\sim}{\lambda}_i \mid o^{\vec{U}}(x) \geq$

$o^{\vec{U}}(\underset{\sim}{\lambda}_i)\} \cup \{0\})$ can be coded as a real hence we can extend p_ω to a condition $p \in G$ that forces all this information. Also let $I = I(C_* \setminus C^*, C_*) \subseteq otp(C_*)$, so $I \in V$. Work in $V[C']$, Inductively we will define $\langle \beta_i \mid i < \text{otp}(C_*)\rangle$. $\beta_0 = \lambda_0$, Assume that $\langle \beta_j \mid j < i\rangle$ is defined, in particular the indices of the branch of $\underset{\sim}{\lambda}_i = \delta_n^{(k)}$, is $i_1 < ...i_k < i$ and $i^* < i$ is the index of the supremum. Define

$$\beta_i = \min(\{x \in C' \setminus \{\beta_j \mid j \in I \cap i\} \mid x > \beta_{i^*} \wedge o^{\vec{U}}(x) \geq (F_n^{(k)})_{I_n^{(k)}}(\beta_{i_1}, ..., \beta_{i_k})\} \cup \{0\})$$

This is a legitimate definition in $V[C']$. Let us prove that $\beta_i = \lambda_i$, inductively assume that $\langle \beta_j \mid j < i\rangle = \langle \lambda_j \mid j < i\rangle$ then

$$\{\beta_j \mid j \in I \cap i\} = (C_* \setminus C^*) \cap \lambda_i$$

and therefore $\lambda_i \in C' \setminus \{\beta_i \mid i \in I \cap p\}$, also $\lambda_i > \lambda_{i_*} = \beta_{i^*}$ and $F_n^{(k)}(\beta_{i_1}, ..., \beta_{i_k}) = o^{\vec{U}}(\lambda_i)$ hence $\lambda_i \geq \beta_i$. If $\beta_i < \lambda_i$ then $\beta_i \in (\lambda_{i^*}, \lambda_i)$ with $o^{\vec{U}}(\beta_i) \geq o^{\vec{U}}(\lambda_i)$ which is a contradiction to the definition of λ_{i^*}. Thus $C_* \in V[C']$. From this it will follow that $C_* \setminus C^*, C'' \in V[C']$ since they are all subset of a countable set in $V[C']$, therefore $C^* = C' \setminus (C^* \setminus C_*) \in V[C']$ which is what we needed.

■

Lemma 4.2 *Assume* $o^{\vec{U}}(\kappa) = \kappa$ *and let* $A \in V[G]$, $\sup(A) = \kappa$. *Assume that* $\exists C^* \subseteq C_G$ *such that*

1. $C^* \in V[A]$ *and* $\forall \alpha < \kappa$ $A \cap \alpha \in V[C^*]$
2. $cf^{V[A]}(\kappa) < \kappa$

Then $\exists C' \subseteq C_G$ *such that* $V[A] = V[C']$.

Proof. If $|C^*| < \kappa$ then we can proceed as in the case when $o^{\vec{U}}(\kappa) < \kappa$. Assume that $|C^*| = \kappa$, since $C^* \subseteq C_G$ and $o^{\vec{U}}(\kappa) = \kappa$, we can construct a cofinal sequence $\langle \alpha_n \mid n < \omega\rangle \in V[C^*]$ unbounded and cofinal in κ. we define in $V[A]$ as before $\langle \delta_n \mid n < \omega\rangle$ that codes $A \cap \alpha_n$ in $V[C^*]$, by the previous section we can find $C'' \subseteq C_G$ such that $|C''| = \omega$ such that $V[\langle \delta_n \mid n < \omega\rangle] = V[C'']$. By proposition 4.1, in $V[A]$ we can find some $C' \subseteq C_G$ such that $C^*, C'' \in V[C']$, then $V[A] = V[C']$. ■

Consider the crucial set

$$X_A = \{\nu < \kappa \mid \nu \textit{ is a cardinal } cf^V(\nu) > cf^{V[A]}(\nu)\}$$

which is defined in $V[A]$. Note also that $X_A \subseteq Lim(C_G)$, and that is not necessarily closed:

Example 4.3 If there is $\alpha \in C_G$ such that $o^{\vec{U}}(\alpha) = \alpha^+$, then α stays regular in $V[G]$. Set $A = C_G$, then $X_A \cap \alpha$ will be unbounded in α, but $\alpha \notin X_A$.

However, a final segment of X_A is closed:

Lemma 4.4 *Suppose that $o^{\vec{U}}(\kappa) = \kappa$, and let $\kappa^* < \kappa$ be such that for every $\xi \in C_G \cap (\kappa^*, \kappa)$, $o^{\vec{U}}(\xi) \leq \xi$[1], then $X_A \setminus \kappa^*$ is closed.*

Proof. Since the only cardinals that changed cofinality in $V[G]$ are limit points of the Magidor club, $X_A \subseteq Lim(C_G)$. Moving to $V[G]$, assume that

$$C_G = \langle \kappa_\alpha \mid \alpha < \kappa \rangle$$

is the increasing enumeration. For every $\alpha \leq \kappa$ with $o^{\vec{U}}(\alpha) = \alpha$, there is in C_G a maximal member $\alpha^* < \alpha$ such that $o^{\vec{U}}(\kappa_{\alpha^*}) \geq \kappa_{\alpha^*}$, hence if we define the sequence:

$$\alpha_0 = k_{\alpha^*+1}, \quad \alpha_{n+1} = \kappa_{\alpha_n}$$

it must be unbounded in α, otherwise it's limit, α', would be a point of the Magidor club which satisfy $o^{\vec{U}}(\alpha') \geq \alpha'$ contradicting the maximality of α^*. If $\alpha \in C_G$ and $o^{\vec{U}}(\alpha) < \alpha$, then $otp(C_G \cap \kappa_\rho) < \kappa_\rho$. We will prove that $X_A \setminus \kappa_\rho$ is a club. To it is closed, note that is $\sup(X_A \cap (\kappa_\rho, \alpha)) = \alpha$, then $\alpha \in Lim(C_G)$ and therefore $\alpha \in C_G \setminus \kappa_\rho$, hence $otp(X_A \cap \alpha) \leq otp(C_G \cap \alpha) < \alpha$, hence $cf^{V[A]}(\alpha) < \alpha$ as witnnessed by $X_a \cap \alpha$, which implies that $\alpha \in X_A$. ■

There are trivial examples in which the set X_A is bounded. We will use a new kind of "freshness" of sets $A \in V[G]$, to see get that X_A is unbounded.

Definition 4.5 Let $A \subseteq On$, we say that A stabilizes if there is $\beta < \kappa$ such that $\forall \alpha < \sup(A)$, $A \cap \alpha \in V[G \restriction \beta]$

Proposition 4.6 *Suppose that $o^{\vec{U}}(\kappa) = \kappa$, $A \in V[G]$ is a set of ordinals and $\langle A \cap \alpha \mid \alpha < \sup(A) \rangle$ does not stabilize and for every $\alpha < \sup(A)$ there is $C' \subseteq C_G$ such that $V[A \cap \alpha] = V[C']$[2]. Then X_A is unbounded in κ. In particular, then $cf^{V[A]}(\kappa) < \kappa$.*

Proof. To see that it is unbounded, let $\delta < \kappa$, take some β such that $A \cap \beta \notin V[G \restriction \max(\delta, \kappa_\rho)$ which exists by our assumption that A does not stabilize, By the inductive assumption, there exists $C' \subseteq C_G$ such that

$$V[C'] = V[A \cap \beta] \subseteq V[A]$$

[1] In fact, we can also prove it for $o^{\vec{U}}(\xi) < \xi^+$

[2] This assumption is simply an inductive assumption about $\sup(A)$. We have this assumption if $A \subseteq \kappa$ by the part for short sequences.

It is impossible that $C' \setminus (C_G \cap \max(\delta, \kappa_\rho))$ is finite, otherwise

$$A \cap \beta \in V[C'] \subseteq V[G \restriction \max(\delta, \kappa_\rho)]$$

which contradicts the choice of β. There is a limit point γ of C' above $\max(\delta, \kappa_\rho)$, it is clear that $\gamma \in X_A \setminus \kappa_\rho$. If $otp(X_A) < k$ then κ changes cofinality in $V[A]$, otherwise

$$X_A = \langle x_\beta \mid \beta < \kappa \rangle$$

As before define inductively

$$y_0 = min(X_A \setminus \kappa^*), \quad y_{n+1} = x_{y_n}$$

So $\sup(y_n \mid n < \omega) = \kappa$ and $cf^{V[A]}(\kappa) = \omega$.

4.2 Subsets of κ which stabilizes

In this section we assume that $A \subseteq \kappa$ and $|A| = \kappa$ and the sequence $\langle A \cap \alpha \mid \alpha < \kappa \rangle$ stabilizes, which means that there is $\kappa^* < \kappa$ such that

$$\forall \alpha < \kappa \ A \cap \alpha \in V[C_G \cap \kappa^*]$$

Note that if $A \in V[C_G \cap \lambda]$ for some $\lambda < \kappa$ then we can use the induction, so we also assume that A is fresh with respect to the models $V[C_G \cap \lambda]$. We will use freshness and work a little bit to prove $cf^{V[A]}(\kappa) < \kappa$ while finding C' is easy. Then we use of lemma 4.2.

Lemma 4.7 *There is $C' \subseteq C_G$ such that $C' \in V[A]$ and $\forall \alpha < \kappa \ A \cap \alpha \in V[C']$.*

Proof. Let $\langle \alpha_i \mid i < cf^{V[A]}(\kappa) \rangle \in V[A]$ be unbounded in κ. Pick $\langle D_i \mid i < cf^{V[A]}(\kappa) \rangle \in V[A]$ such that $V[D_i] = V[A \cap \alpha_i]$ and each D_i is generic. Then, $D_i \subseteq^* C_G \cap \kappa^*$. Assume that $D_i \subseteq \kappa^*$. If κ is singular in $V[A]$, we can code the sequence $\langle \alpha_i \mid i < cf^{V[A]}(\kappa) \rangle$ as a bounded subset of κ and use previous results. If κ is regular in $V[A]$ (In the rest of this section we will see that this situation is impossible), then there is $E \subseteq \kappa$ unbounded and $D_* \subseteq \kappa^*$ generic such that for every $i \in E$, $D_i = D_*$. It is routine to check that $C' = D_* \cap C_G \in V[A]$ is as wanted. ∎

It remains to prove that κ changes cofinality in $V[A]$. Let us settle first a simple case:

Lemma 4.8 *Assume that A is such that $\forall \beta < \kappa$ there is $\alpha < \kappa$ such that $A \cap \alpha \notin V[A \cap \beta]$, then $cf^{V[A]}(\kappa) < \kappa$.*

Proof. Define a sequence $\langle \alpha_i \mid i < \theta \rangle$, in the following way, $\alpha_0 = 0$, for limit δ $\alpha_\delta = \sup(\alpha_\beta \mid \beta < \delta)$. In successor stage let

$$\alpha_{\beta+1} = \min(\gamma \mid A \cap \gamma \notin V[A \cap \alpha_\beta])$$

this is well defined by our assumption about A. In $V[G]$, each $i < \theta$ can be mapped to $C_{\alpha_i} \subseteq C_G \cap \kappa^*$ thus $|\theta| \leq 2^{\kappa^*} < \kappa$.
■

For general A, we fix $C' \subseteq \kappa^* \cap C_G$ such that $\forall \beta < \kappa$ $A \cap \beta \in V[C']$. Find a subforcing $\mathbb{P}$ of $\mathbb{M}[\vec{U}] \restriction \kappa^*$ for which C' is generic [8] and let $\mathbb{Q} = (\mathbb{M}[\vec{U}] \restriction \kappa^*)/C'$. It remains to force above $V[C']$ with $\mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$. Note for every $\kappa^* \leq \alpha < \kappa$ with $o^{\vec{U}}(\alpha) > 0$ we have

$$|\mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \alpha)| < \min(\nu > \alpha \mid o^{\vec{U}}(\nu) = 1)$$

Let $\underset{\sim}{A}$ be a name for A in the forcing $\mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$ and assume that $\Vdash$ "$\forall \alpha$ $\underset{\sim}{A} \cap \alpha$ *is old*".

Lemma 4.9 *Let $p \in \mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$ and $\vec{\alpha}^\frown \alpha \in [\kappa]^{<\omega}$ such that $p^\frown \vec{\alpha}^\frown \alpha \in \mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$ and $o^{\vec{U}}(\alpha) = 0$. Consider the decomposition of*

$$p^\frown \vec{\alpha}^\frown \alpha = \langle p_{<\alpha}, \alpha, p_{>\alpha} \rangle$$

then there is $Z_{\vec{\alpha},\alpha} \subseteq \mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \max(\vec{\alpha}))$ Maximal Anti chain and $p_{>\alpha} \leq^* p_{\vec{\alpha},\alpha}$ such that

$$\forall q \in Z_{\vec{\alpha},\alpha} \ \langle q, \alpha, p_{\vec{\alpha},\alpha} \rangle || \ \underset{\sim}{A} \cap \alpha$$

Proof. Fix $\vec{\alpha}, \alpha$ as in the statement, For each $q \in \mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \max(\vec{\alpha}))$ we will find $q \leq q'$ and $p_{<\alpha} \leq^* p_q \in \mathbb{M}[\vec{U}] \restriction (\alpha, \kappa)$ such that $\langle q', \alpha, p_q \rangle || \underset{\sim}{A} \cap \alpha$. Take some generic

$$H \subseteq \mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \max(\vec{\alpha}))$$

with $q \in H$, and denote $(\underset{\sim}{A})_H$ the $\mathbb{M}[\vec{U}] \restriction (\alpha, \kappa)$-name in $V[C'][H]$ derived from $\underset{\sim}{A}$. We use the Prikry condition to find $p_{>\alpha} \leq^* p_q$ such that $p_q \Vdash (\underset{\sim}{A})_H \cap \alpha = X$ for some $X \in V[C'][H]$, it is possible to find such X since above α, the order $\leq^*$ is α-closed. Note that, $X \in V[C']$ since we assumed $\Vdash$ "$\forall \alpha$ $\underset{\sim}{A} \cap \alpha$ *is old*" Hence there is $q \leq q'$ such that

$$q' \Vdash (p_q \Vdash \underset{\sim}{A} \cap \alpha = X)$$

Thus $\langle q', \alpha, p_q \rangle \Vdash \underset{\sim}{A} \cap \alpha = X$. Again by α-closure of $\leq^*$ above α and since $\mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \max(\vec{\alpha}))$ is of small cardinality, we can find $p_{\vec{\alpha},\alpha}$ such that for $p_q \leq^* p_{\vec{\alpha},\alpha}$ for every q. The definition of $Z_{\vec{\alpha},\alpha}$ is a simple use of by Zorn's lemma and the property of $p_{\vec{\alpha},\alpha}$.
■

Assume that $p_{\vec{\alpha},\alpha} = \langle \langle \nu_1, B_1 \rangle, ..., \langle \nu_n, B_n \rangle, \langle \kappa, B \rangle \rangle$ and that $\nu_i < \max(\vec{\alpha}) < \nu_{i+1}$, then for every $i + 1 \leq j$ shrink the sets $B_j(0)$ so that $Z_{\vec{\alpha},\alpha}$ does not depend on α nut only on $\vec{\alpha}$ and j. We denote these antichains by $Z_{\vec{\alpha},j}$.

Lemma 4.10 *For every $p \in \mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$ there is $p \leq^* p^*$ such that for all $\vec{\alpha}^\frown\alpha \in [\kappa]^{<\omega}$ such that $p^{*\frown}\vec{\alpha}^\frown\alpha \in \mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$ with $o^{\vec{U}}(\alpha) = 0$, $p_{\vec{\alpha},\alpha} \leq^* p^*_{>\alpha}$. In particular*

$$\forall q \in Z_{\vec{\alpha},j} \forall \alpha \in B_j(0)\ \langle q, \alpha, p^*_{>\alpha}\rangle || \ \underset{\sim}{A} \cap \alpha$$

Proof. Assume $p = \langle\langle\nu_1, B_1\rangle, ..., \langle\nu_n, B_n\rangle\langle\kappa, B\rangle\rangle$. For every $\vec{\alpha}, \alpha < \nu_i$ there is a pair $\langle\nu_i, B_i(\vec{\alpha}, \alpha)\rangle$ in $p_{\vec{\alpha},\alpha}$, define

$$B^*_i = \underset{\vec{\alpha},\alpha<\nu}{\Delta} B_i(\vec{\alpha}, \alpha) \in \cap\vec{U}(\nu_i)$$

it follows that $p^* = \langle\langle\nu_1, B^*_1\rangle, ...\langle\nu_n, B^*_n\rangle, \langle\kappa, B^*\rangle\rangle$ is as wanted.
■

Lemma 4.11 *There is $p^* \leq^* p^{**} = \langle\langle\nu_1, B_1\rangle, ..., \langle\nu_n, B_n\rangle, \langle\kappa, B\rangle\rangle$ and sets $A_j(q, \vec{\alpha})$ for $\vec{\alpha} \in [\kappa]^{<\omega}$ and $q \in Z_{\vec{\alpha},j}$, such that for every $\alpha \in B_j(0)$, such that $p^{**\frown}\vec{\alpha}^\frown\alpha \in \mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$ and*

$$\langle q, \alpha, p^{**}_{>\alpha}\rangle \Vdash \ \underset{\sim}{A} \cap \alpha = A_j(q, \vec{\alpha}) \cap \alpha$$

Proof. Fix $\vec{\alpha}$ and $q \in Z_{\vec{\alpha},j}$. By the previous lemma, for every $\alpha \in B^*_j(0)$, we can find $a_j(q, \vec{\alpha}, \alpha) \subseteq \alpha$ such that

$$\langle q, \alpha, p^*_{>\alpha}\rangle \Vdash \underset{\sim}{A} \cap \alpha = a_j(q, \vec{\alpha}, \alpha)$$

By ineffability of ν_j there is $B'_j(0) \subseteq B^*_j(0)$ in $U(\nu_j, 0)$ and $A_j(q, \vec{\alpha})$ such that

$$\forall \alpha \in B'_j(0)\ A_j(q, \vec{\alpha}) \cap \alpha = a_j(q, \vec{\alpha}, \alpha)$$

Shrink B^*_j to B'_j so obtain p^{**}.
■

Lemma 4.12 *Assume that $o^{\vec{U}}(\kappa) = \kappa$, then $cf^{V[A]}(\kappa) < \kappa$.*

Proof. Work in $V[A]$, by density find $p^{**} \in G \restriction (\kappa^*, \kappa)$ with the properties described in lemma 4.11. There is Some $\xi < \kappa$ such that

$$0_{\mathbb{M}[\vec{U}]} \Vdash \forall \alpha \in \underset{\sim}{C}_G \setminus \xi,\ o^{\vec{U}}(\alpha) < \alpha$$

Assume that $\xi = 0$, otherwise just work above ξ. In $V[G]$, let $\langle\kappa_i \mid i < \kappa\rangle$ be the enumeration of C_G, let us define in $V[A]$ a sequence $\langle\gamma_\lambda \mid \lambda < \theta\rangle$ where $\theta \leq \kappa$. $\gamma_0 = \kappa_0 + 1$, at limit point $\delta < \kappa$, denote by $\gamma'_\delta = \sup(\gamma_\lambda \mid \lambda < \delta) \leq \kappa$. if $\gamma'_\delta = \kappa$ then define $\theta = \delta$ and stop the definition, in this case we are done since $cf^{V[A]}(\kappa) \leq \delta < \kappa$. Assume that $\gamma'_\delta < \kappa$, define $\gamma_\delta = \gamma'_\delta + 1$. At successor stage, assume that $\delta = \lambda + 1$, let

$$\eta(q, \vec{\alpha}) = \max(\min(A\Delta A_1(q, \vec{\alpha})), ..., \min(A\Delta A_n(q, \vec{\alpha}))) < \kappa$$

this is well define since $A_i(q) \in V[C']$ and by freshness assumption on A, $A \notin V[C']$, in particular it must be that $A \neq A_i(q, \vec{\alpha})$. Define

$$\gamma'_\delta = \sup(\eta(q, \vec{\alpha}) \mid q \in Z_{\vec{\alpha},i}, \vec{\alpha} \in [\gamma_\lambda]^{<\omega}) \leq \kappa$$

If $\gamma'_\delta = \kappa$ as before we stop the definition since we found a short cofinal sequence in κ, otherwise, define $\gamma_\delta = \gamma'_\delta + 1$. Assume that the definition goes up to κ and $\langle \gamma_\lambda \mid \lambda < \kappa \rangle$ is defined. Let us show that $\forall \lambda < \kappa\ \gamma_\lambda > \kappa_\lambda$. At 0 and limit stage it is an clear from the definition and continuity. Assume that $\gamma_\lambda > \kappa_\lambda$, since $\kappa_{\lambda+1}$ is successor in C_G, $o^{\vec{U}}(\kappa_{\lambda+1}) = 0$, so find $\vec{C}_*$ such that $p^{**\frown}\vec{C}_*^\frown\kappa_{\lambda+1} \in G \restriction (\kappa^*, \kappa)$. There is j such that $\kappa_{\lambda+1} \in B_j^{**}(0)$

$$Z_{\vec{C}_*,j} \subseteq \mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \max(\vec{C}_*))$$

By the induction hypothesis $\max(\vec{C}_*) \leq \kappa_\lambda < \gamma_\lambda$. $Z_{\vec{C}_*,j}$ is a maximal anti chain so there is $q \in Z_{\vec{C}_*,j}$ such that $t^* = \langle q, \kappa_{\lambda+1}, P^{**}_{>\kappa_{\lambda+1}} \rangle \in G$ and $t^* \Vdash \underset{\sim}{A} \cap \kappa_{\lambda+1} = A_j(q, \vec{C}_*) \cap \kappa_{\lambda+1}$ but then it must be that $A \cap \kappa_{\lambda+1} = A_j(q, \vec{\alpha}) \cap \kappa_{\lambda+1}$. This means that $\kappa_{\lambda+1} \leq \eta(q, \vec{C}_*) \leq \gamma'_{\lambda+1} < \gamma_{\lambda+1}$ as wanted. Let us define an ω-sequence unbounded in κ, $\alpha_0 = \gamma_0$ and $\alpha_{n+1} = \gamma_{\alpha_n}$, by the assumption about ξ at the beginning of the proof, it follows that $\kappa = \sup(\kappa_{\alpha_n} \mid n < \omega) \leq \sup(\gamma_{\alpha_n} \mid n < \omega) \leq \kappa$.
■

4.3 Subsets of κ which does not stabilize

Assume that A does not stabilize. By proposition 4.6, since we assume that $o^{\vec{U}}(\kappa) = \kappa$, then X_A is a club, and κ changes cofinality in $V[A]$.

It doesn't have to be the case that $cf^{V[A]}(\kappa) = \omega$, but $cf^{V[A]}(\kappa)$ most be some member of the generic club that will eventually change it's cofinality to ω. For example, using the enumeration $C_G = \langle \kappa_i \mid i < \kappa \rangle$ and the canonical sequence α_n that was defined in the last lemma, we can define in $V[G]$ the set

$$A = \bigcup_{n<\omega} \{\kappa_{\alpha_n + \alpha} \mid \alpha < \kappa_n\}$$

then A does not stabilize. Moreover, we cannot construct the sequence $\langle \alpha_n \mid n < \omega \rangle$ or any other ω-sequence unbounded in κ inside $V[A]$ since A is generic for the forcing $\mathbb{M}[\vec{U} \restriction (\kappa, \kappa_\omega)]$ which does not change the cofinality of κ to ω. For this kind of examples the case $o^{\vec{U}}(\kappa) < \kappa$ suffices.

Definition 4.13 A set D is generic if for every $\delta \in Lim(D)$, $\delta \in X_A$ and for every $Y \in \bigcap \vec{U}(\delta)$ there is $\xi < \delta$ such that $D \cap (\xi, \delta) \subseteq Y$.

Lemma 4.14 *If $D \in V[G]$ is generic then $D \setminus C_G$ is finite. In particular, if $\sup(D)$ is limit, then $\sup(D) \in Lim(C_G)$.*

Proof. Otherwise there let $\delta \leq sup(D)$ be minimal such that $|D \cap \delta \setminus C_G| \geq \omega$ then $\delta \in Lim(D)$. So there most be some infinite $\{d_n \mid n < \omega\} \subseteq D \setminus C_G$ unbounded in δ. By 3.7 there is $Y \in \bigcap \vec{U}(\delta)$ such that $Y \cap \{d_n \mid n < \omega\} = \emptyset$ contradicting the condition of D.■

We denote $X \subseteq^* Y$ if $X \setminus Y$ is finite. Also define $X =^* Y$ if $X \subseteq^* Y \wedge Y \subseteq^* X$, equivalently, if $X \triangle Y$ is finite.

Lemma 4.15 *Let $\langle D_i \mid i < \theta \rangle \in V[A]$ be a sequence of generic subsets of κ such that for every $i < \theta$ $min(D_i) \geq \theta$ and θ is regular in $V[A]$. Then there is $\langle D_i^* \mid i < \theta \rangle \in V[A]$ such that:*

1. $\bigcup_{i<\theta} D_i^*$ *is generic.*
2. $\forall i < \theta, D_i =^* D_i^* \subseteq \sup(D_i)$.

Proof. By removing finitely many elements from every D_i, we can assume that $otp(D_i)$ is a limit ordinal and therefore $\sup(D_i) \in X_A$. Denote $D = \bigcup_{i<\lambda} D_i$ and $\nu^* = \sup(D) > \theta$. Note that $\nu^* \in X_A$, since C_G is closed, and $\nu^* = \sup(\sup(D_i) \mid i < \lambda)$. Proceed by induction on ν^*, By lemma 4.14, $D_i \setminus C_G$ is finite, it follows that $|D \setminus C_G| \leq \theta$. We would like to remove the noise in D by intersecting it with a large set, define a sequence $\langle Y_\alpha \mid \alpha < \lambda < \nu^{*+} \rangle$ of sets such that

1. $\forall i < \nu^{*+}$ $Y_i \in \bigcap \vec{U}(\nu^*)$.
2. For every $i < j < \lambda$ $Y_j \setminus Y_i$ is bounded in ν^*.

If for every $Y \in \cap \vec{U}(\nu^*)$ $D \setminus Y$ is bounded in ν^*, define $\lambda = 0$. Otherwise, let $Y_0 \in \cap \vec{U}(\nu^*)$ such that $D \setminus Y_0$ is unbounded in ν^*. Assume that $\langle Y_\alpha \mid \alpha < \beta \rangle$ is defined and satisfy $1, 2$ for some $\beta < \nu^{*+}$. If $\alpha + 1 = \beta$ let $Y' = Y_\beta$. If β is limit, find $\langle \beta_i \mid i < cf^V(\beta) \leq \nu^* \rangle \in V$ a sequence cofinal in β. The sequence $\langle Y_{\beta_i} \mid i < cf^V(\beta) \rangle$, might not be in V, but by ν^{*+}-c.c. there is a sequence $\langle Z_i \mid i < \nu^* \rangle$ covering it. In particular,

$$Y' = \underset{i<\nu^*}{\triangle} Z_i \in \cap \vec{U}(\nu^*)$$

Note that for every $j < \beta$ there is β_i such that $j < \beta_i$, hence $Y_{\beta_i} \setminus Y_j$ is bounded by some $\alpha < \nu^*$. Moreover, there is $\rho < \nu^*$ such that $Y_{\beta_i} = Z_\rho$, and by the definition of diagonal intersection, $Y' \setminus Y_{\beta_i} \subset \rho < \nu^*$. It follows that

$$Y' \setminus Y_j = [(Y' \cap Y_{\beta_i}) \setminus Y_j] \cup [(Y' \setminus Y_{\beta_i}) \setminus Y_j] \subseteq (Y_{\beta_i} \setminus Y_j) \cup (Y' \setminus Y_{\beta_i}) \subseteq \max(\alpha, i) < \nu^*$$

thus $Y' \setminus Y_j$ is bounded in ν^*. If for every $Y \in \cap\vec{U}(\nu^*)$, $D \cap Y' \setminus Y$ is bounded in ν^*, stop the recursion at $\lambda = \beta$, else let Y be some set witnessing the opposite, define $Y_\beta = Y \cap Y'$. To see that this process reaches the halting condition before ν^{*+}, assume otherwise, then we have defined a sequence $\langle Y_i \mid i < \nu^{*+}\rangle$. For every $\alpha < \nu^{*+}$, $D \cap Y_\alpha \setminus Y_{\alpha+1}$ is unbounded in ν^*. Also there is an ordinal $\xi < \nu^*$ such that $C_G \cap (\xi, \nu^*) \subseteq Y_{\alpha+1}$ thus

$$D \cap Y_\alpha \setminus Y_{\alpha+1} \subseteq \xi \cup (D \setminus C_G)$$

There only ν^* many such subsets, since $|D \setminus C_G| \leq \theta$ and $\nu^* \in Lim(C_G)$ and thus a strong limit. Moreover, the the function

$$\alpha \mapsto D \cap Y_\alpha \setminus Y_{\alpha+1}$$

is $1-1$ since if $\alpha < \beta < \nu^{*,+}$ then $Y_\beta \setminus Y_{\alpha+1}$ is bounded by some $\xi < \nu^*$. Take some $\gamma \in D \cap Y_\alpha \setminus Y_{\alpha+1}$ above ξ then $\gamma \notin Y_\beta$ and in particular not in $D \cap Y_\beta \setminus Y_{\beta+1}$, contradiction. So the halting condition must be reached at some $\lambda < \nu^{*+}$. Let $\langle \lambda_\alpha \mid \alpha < cf^V(\lambda)\rangle \in V$ be some sequence cofinal in λ. Find a covering sequence $\langle Z_i \mid i < \nu^*\rangle$, and define

$$Y^* = \underset{i<\nu^*}{\triangle} Z_i \in \cap\vec{U}(\nu^*), \ D^* = D \cap Y^*$$

By the definition of the halting condition:

Claim 1 For every $Y \in \bigcap\vec{U}(\nu^*)$, $D^* \setminus Y$ is bounded in ν^*

Claim 2 $D^* \setminus C_G$ is bounded in ν^*.

Proof. Toward a contradiction, assume there is an infinite $\{\beta_i \mid i < cf^{V[A]}(\nu^*)\} \subseteq D^* \setminus C_G$ and $\sup(\beta_i \mid i < cf^{V[A]}(\nu^*)) = \nu^*$. Since $\nu^* \in X_A$, $cf^{V[A]}(\nu^*) < \nu^*$ and by 3.7, there most be $Z_0 \in \bigcap\vec{U}(\nu^*)$ such that $Z_0 \cap \{\beta_i \mid i < \nu^*\} = \emptyset$. But then $\{\beta_i \mid i < \nu^*\} \subseteq D^* \setminus Z_0$ which contradicts claim 1.

$\blacksquare_{claim\ 2}$

In $V[A]$, let $\xi_1 < \nu^*$ be an ordinal such that $D^* \setminus \xi_1$ is generic. By claim 2 such a ξ_1 exists. Consider the set

$$Z^{(0)} = \{\nu < \nu^* \mid Y^* \cap \nu \in \cap\vec{U}(\nu)\}$$

to see that $Z^{(0)} \in \cap\vec{U}(\nu^*)$, let $i < o^{\vec{U}}(\nu^*)$, then $j_{U(\nu^*,i)}(Y^*) \cap \nu^* = Y^* \in \bigcap_{\xi<i} U(\nu^*,\xi)$. By coherency, the order of ν^* in $j_{U(\nu^*,i)}(\vec{U})$ is i, which implies that

$$\bigcap_{\xi<i} U(\nu^*,\xi) = \cap j(\vec{U})(\nu^*)$$

By definition $\nu^* \in j(Z^{(0)})$ thus $Z^{(0)} \in U(\nu^*, i)$ for every $i < o^{\vec{U}}(\nu^*)$ and $Z^{(0)} \in \bigcap\vec{U}(\nu^*)$. By claim 2, we can find $\xi_2 < \nu^*$ such that $Lim(D^*) \setminus \xi_2 \subseteq Z^{(0)}$. Let $\eta_0 = \max(\xi_1, \xi_2) < \nu^*$. The sets $D_i \cap \eta_0$ are also generic, so we may apply the induction hypothesis to the sequence $\langle D_i \cap \eta_0 \mid i < \theta\rangle$ to find $\langle D'_i \mid i < \theta\rangle$ such that

1. $\bigcup\limits_{i<\theta} D'_i$ is generic.

2. $D_i \cap \eta_0 =^* D'_i \subseteq \eta_0$.

Define

$$D^*_i = D'_i \uplus (D_i \cap Y^* \setminus \eta_0)$$

Claim 3 $D_i =^* D^*_i \subseteq \sup(D_i)$ for every $i < \lambda$.

Proof. It is clear that $D^*_i \subseteq^* D_i$. Toward a contradiction,assume that there is $i < \theta$ and $\delta \leq \sup(D_i)$ minimal such that

$$|(D_i \cap \delta) \setminus (D^*_i \cap \delta)| \geq \omega$$

By the definition of D^*_i, $\delta > \eta_0$ and $\delta \in Lim(D_i)$. By the definition of η_0, $\delta \in Z^{(0)}$ which means that $\delta \cap Y^* \in \bigcap \vec{U}(\delta)$. Since D_i is generic, there is $\xi < \delta$ such that $D_i \cap (\xi, \delta) \subseteq Y^*$, in particular

$$D_i \cap (\xi, \delta) = D_i \cap Y^* \cap (\xi, \delta) = D^*_i \cap (\xi, \delta)$$

So $(D_i \cap \delta) \setminus (D^*_i \cap \delta) = (D_i \cap \xi) \setminus (D^*_i \cap \xi)$, this is a contradiction to the minimality of δ.
$\blacksquare_{claim\ 3}$
$\bigcup\limits_{i<\theta} D^*_i = D^* \setminus \eta_0 \cup (\bigcup\limits_{i<\theta} D'_i)$ is generic as the union of two generics.
$\blacksquare_{lemma 4.15}$

Lemma 4.16 *Assume that $\lambda = cf^{V[A]}(\kappa)$ and for every $0 < \alpha < \kappa$, $o(\alpha) < \alpha$. Then there is a sequence $\langle \beta_i \mid i < \lambda \rangle \in V[A]$ such that*

1. *$\langle \beta_i \mid i < \lambda \rangle$ is increasing and continuous of elements of X_A.*

2. *$\beta_0 = \min(X_A)$, $\sup(\beta_i \mid i < \lambda) = \kappa$.*

3. *If $D \subseteq \beta_i$ is generic then for any $Y \subseteq \mathrm{otp}(D)$, $Y \in V[A]$, there is $j < i$ and $D_Y \subseteq \beta_j$ such that $V[Y] = V[D_Y]$.*

4. *$\beta_j \in X_A$ (Recall that X_A is the set of all measurables in V that changed cofinality in $V[A]$).*

Proof. Fix in $V[A]$ some cofinal sequence $\langle \alpha_i \mid i < \lambda \rangle$ such that $\{\alpha_i \mid i < \lambda\} \subseteq X_A$. Since we assumed $\forall 0 < \alpha < \kappa,\ o(\alpha) < \alpha$, we have that $\mathrm{otp}(C_G \cap \alpha_i) < \alpha_i$. We would like to bound in $V[A]$ the order type of D's which deviate from $C_G \cap \alpha$ at finitely many places. For every α denote

$$\tilde{\alpha} = \sup(\mathrm{otp}(D) \mid D \subseteq \alpha \textit{ is generic})$$

By lemma 4.14, every D participating in the sup deviates from C_G at finitely many places, and therefore $\tilde{\alpha} \leq \text{otp}(C_G \cap \alpha) + \omega$, if $\alpha \in Lim(C_G)$ then $\tilde{\alpha} \leq \text{otp}(C_G \cap \alpha) < \alpha$. Define in $V[A]$

$$\alpha^{-1} = \max((\tilde{\alpha} + 1) \cap X_A) \leq \tilde{\alpha}$$

$$\alpha^{-(k+1)} = \max((\widetilde{\alpha^{-k}} + 1) \cap X_A) \leq \widetilde{\alpha^{-k}}$$

Since for every k, $\alpha^{-k} \in X_A \subseteq Lim(C_G)$ we have $\alpha^{-(k+1)} < \alpha^{-k}$ and this definitions reach 0 after finitely many steps. Define a new sequence: $\beta_0 = \min(X_A)$. From α_0 find the last k_0 such that $\alpha_0^{-k_0} > \beta_0$ and define

$$\beta_1 = \alpha_0^{-k_0}, \ \ldots, \ \beta_{k_0} = \alpha_0^{-1}, \ \beta_{k_0+1} = \alpha_0$$

Continuing in this fashion, assume $\alpha_i = \beta_j$ is defined such that $i \leq j < i + \omega$, let $k_i < \omega$ be last such that $\alpha_{i+1}^{-k_i} > \alpha_i$ and define

$$\beta_{j+1} = \alpha_{i+1}^{-k_i}, \ \ldots, \ \beta_{j+k_i} = \alpha_{i+1}^{-1}, \ \beta_{j+k_i+1} = \alpha_{i+1}$$

and $i + 1 \leq j < i + 1 + \omega$. At limit points we want to stay continuous so β_i is defined to be the limit, it is clear $\alpha_i = \beta_i$. Claim that $\langle \beta_i \mid i < \lambda \rangle$ is as wanted. $(1), (2), (4)$ are trivial, assume $D \subseteq \beta_i$ is generic, then $\text{otp}(D) \leq \beta_i^{-1} < \beta_i$. If i is limit then $\text{otp}(D) \leq \beta_j$ for some $j < i$ and we use can use the last section. Otherwise $i = j + 1$, again, if $\text{otp}(D) \leq \beta_j$ we are done, so assume that $\text{otp}(D) > \beta_j$, it follows that from the definition of β_{j+1} that

$$\max(X_A \cap \text{otp}(D)) = \beta_j$$

let $Y \subseteq \text{otp}(D)$, there is $C \subseteq C_G \cap \text{otp}(D)$, such that $V[Y] = V[C]$. There cannot be a limit point γ of C above β_j since $C \in V[Y] \subseteq V[A]$ thus $\gamma \in X_A$, which contradicts the definition of β_j. So removing finitely many ordinals we can assume that $C \subseteq \beta_j$ is a suitable generic.
■

From now on the sequence $\langle \beta_i \mid i < \lambda \rangle$ is fixed.

Proposition 4.17 *Let $D, D' \in V[A]$ be generic sets, bounded in κ. Then there is $D^* \subseteq \sup(D \cup D')$ also generic such that $D^* \in V[A]$, $D, D' \in V[D^*]$ and $D \cup D' \subseteq D^*$.*

Proof. By induction on $sup(D \cup D') = \nu < \kappa$. For $\nu \leq \beta_0$ this is trivial since we can just take the union and the indices are in V. Assume that it is true for every $\alpha < \nu$, define $D_0 = D \cup D' \in V[A]$ and consider $I(D, D_0), I(D', D_0) \in V[A]$. Let $i < \lambda$ be minimal such that $\beta_i \geq \nu$ By the property in lemma 4.16 of β_i and genericity of D_0, there is some $j < i$ and a generic $E \subseteq \beta_j$ such that $I(D, D_0), I(D', D_0) \in V[E] \subseteq V[A]$. Use the induction hypothesis to find $D_1 \in V[A]$ good for $E, D_0 \cap \beta_j$. Define $D^* = D_1 \cup (D_0 \setminus \beta_j) \in V[A]$, then $\sup(D^*) = \nu$. It is routine to see that D^* is as wanted.
■

Lemma 4.18 *Assume that $\theta < \kappa$ is a regular cardinal in $V[A]$ and $\langle D_i \mid i < \theta\rangle \in V[A]$ is a sequence of generic sets such that $D_i \subseteq \theta_i < \kappa$ and θ_i's are non decreasing. Then there is $\langle D^*_i \mid i < \theta\rangle \in V[A]$ such that*

1. $\bigcup\limits_{i<\theta} D^*_i$ *is generic.*
2. $i < i' \to D^*_i \subseteq^* D^*_{i'}$.
3. $D_i \in V[D^*_i]$ *and* $D_i \subseteq^* D^*_i \subseteq \theta_i$.

Proof. Work in $V[A]$, define $D^*_0 = D_0$. At successor stage, define use proposition 4.17 to find $D^*_{\alpha+1}$ generic such that $D^*_\alpha \cup D_{\alpha+1} \subseteq D^*_{\alpha+1}$ and $D_{\alpha+1} \in V[D^*_{\alpha+1}]$. At limit stage δ, consider $\delta' = cf^{V[A]}(\delta)$ and $\delta^* = \max(X_A \cap \delta')$. Since $cf^{V[A]}(\delta)$ is regular in $V[A]$ it follows that $\delta^* < \delta'$. Let $\langle \delta_i \mid i < \delta'\rangle$ be cofinal in δ. Then $D^*_{\delta_i} \cap \delta^*$ stabilizes in $=^*$ at some i^*. To see this, note that $|C_G \cap \delta^*| < \delta^*$ and δ^* stays strong limit in $V[G]$, hence

$$|\{D^{\delta_i}_j \cap \delta^* \mid i < \delta'\}| \leq 2^{|C_G \cap \delta^*|} \cdot [\delta^*]^{<\omega} = \delta^* < \delta'$$

Thus there is a value $D_* = D^*_{\delta_{i^*}} \cap \delta^*$ repeating cofinally many times. Since the sequence is $\subseteq^*$-increasing, for every $i \geq i^*$ there is $j > i$ such that $D^*_{\delta_j} \cap \delta^* = D_*$, therefore

$$D_* \subseteq^* D^*_i \cap \delta^* \subseteq^* D^*_{\delta_j} \cap \delta^* =^* D_*$$

So $D^*_i \cap \delta^* =^* D_*$. Use lemma 4.15 for the sequence $\langle D_{\delta_i} \setminus \delta' \mid i < \delta'\rangle$ and obtain the sequence $\langle E_i \mid i < \delta'\rangle$. Note that by genericity, $D^*_{\delta_i} \cap (\delta^*, \delta')$ must be finite, otherwise there is dome limit point of D_i in the interval (δ^*, δ'). Now limit points of D_i are also in X_A, this is a contradiction to the definition of δ^*. Define $E = \bigcup\limits_{i<\delta'} E_i \subseteq \sup(\theta_{\delta_i} \mid i < \delta') \leq \theta_\delta$, then E is generic and for every $i < \delta'$, $D^*_{\delta_i} \setminus \delta' \subseteq^* E \subseteq \beta_\delta$. Let

$$D' = D_* \uplus E \subseteq \theta_\delta$$

Then D' is generic and $D^*_{\delta_i} \subseteq^* D'$. Finally, D^*_δ is defined by proposition 4.17 and the generics D', D_δ. It is clear that $D_\delta \subseteq^* D^*_\delta \subseteq \theta_\delta$, that it is generic and that $D_\delta \in V[D^*_\delta]$. Let $\alpha < \delta$, find $\alpha \leq \delta_i < \delta$ then

$$D^*_\alpha \subseteq^* D^*_{\delta_i} = D^*_{\delta_i} \cap \delta^* \uplus D^*_{\delta_i} \setminus \delta^* \subseteq^* D' \cap \delta^* \uplus E_i \subseteq^* D^*_\delta$$

So the sequence $\langle D^*_i \mid i < \theta\rangle$ is defined. The union my not be generic so we use lemma 4.15 again in the same way as in the limit stage, let $\theta^* = \sup(X_A \cap \theta)$. Since $V[A] \models \theta$ is regular, $\theta^* < \theta$. Consider the $\subseteq^*$-increasing sequence $\langle D^*_i \cap \theta^* \mid i < \theta\rangle$, it $=^*$-stabilizes from some i^* on the va;ue D_*. Use lemma 4.15 for the sequence $\langle D^*_i \setminus \theta \mid i < \theta\rangle$ which yield the sequence $\langle E_i \mid i < \theta\rangle$. For every $i < \theta$ define

$$F^*_i = D^*_i \cap D_* \uplus E_i$$

We note that $D_i^* =^* F_i^*$. So $V[D_i^*] = V[F_i^*]$, $D_i \subseteq^* F_i^* \subseteq \theta_i$ and

$$\bigcup_{i<\theta} F_i^* = D_* \uplus (\bigcup_{j<\theta} E_j)$$

which is the union of two generics. The sequence $\langle F_i^* \mid i < \theta\rangle$ is still $\subseteq^*$-increasing since $D_i^* =^* F_i^*$. ■

The following theorem is what we need to finish the proof of the main result for subsets of κ which does not stabilize.

Theorem 4.19 *Assume that for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Let $\langle D_i \mid i < \lambda\rangle \in V[A]$ be a sequence of generics such that for every $i < \lambda$, $D_i \subseteq \beta_i$. Then there is $\langle D_i^* \mid i < \lambda\rangle \in V[A]$ such that*

1. $\forall i < \lambda,\ D_i, D_i^* \in V[\bigcup_{i<\lambda} D_i^*]$.
2. $\bigcup_{i<\lambda} D_i^*$ *is generic.*
3. $D_i \subseteq^* D_i^* \subseteq \beta_i$.
4. $\langle D_i^* \mid i < \lambda\rangle$ *is $\subseteq^*$-increasing.*

Proof. Use 4.18 to get $\langle D_i^0 \mid i < \lambda\rangle$ such

1. $i < i' \to D_i^0 \subseteq^* D_{i'}^0$
2. $D_i \subseteq^* D_i^0 \subseteq \beta_i$
3. $\bigcup_{i<\lambda} D_i^0$ is generic.
4. $D_i \in V[D_i^0]$.

Define sequences $\langle D_i^\xi \mid i < \lambda\rangle$ for $\xi < \kappa^+$ recursively such that for every $i < \lambda$,

1. $\xi_1 < \xi_2 \to D_i^{\xi_1} \subseteq^* D_i^{\xi_2}$
2. $\forall i \leq j . D_i^\xi \subseteq^* D_j^\xi$
3. $D_i^\xi \subseteq \beta_i$
4. $\bigcup_{j<\lambda} D_j^\xi$ is generic.

5. $D_i^0 \in V[D_i^\xi]$.

6. there is $\rho_i < i$ such that $I(D_i^\xi, \underset{j<\lambda}{\cup} D_j^\xi \cap \beta_j) \in V[D_{\rho_i}^{\xi+1}]$

At successor stage, assume $\langle D_i^\alpha \mid i < \lambda\rangle$ is defined, since D_j^α is generic, there is $\rho_j < j$ such that $I(D_j^\alpha, (\bigcup_{i<\lambda} D_i^\alpha) \cap \beta_j)$ is coded by a generic subset $Y_j^\alpha \subseteq \beta_{\rho_j}$. We need to pick ρ_j carefully since at β_{ρ_j} we wont be able to code more the β_{ρ_j} many sets. To do this, we make use of the following subsequence of the β_i's:

$$\gamma_0 = \beta_0,\ \gamma_{n+1} = \beta_{\gamma_n}$$

the sequence γ_n reaches λ after finitely many steps, otherwise we would have found some point below κ with $o^{\vec{U}}(\alpha) \geq \alpha$ which i a contradiction. Now for the choice of ρ_j, for successor j $\rho_j = j-1$. For limit j, assume $\gamma_k < j \leq \gamma_{k+1}$ then choose $\alpha_k < \rho_j < j$. Note that for a specific $\gamma_k < i \leq \gamma_{k+1}$, $\{j \mid \rho_j = i\} \subseteq \gamma_{k+1} = \beta_{\gamma_k} < \beta_i$. Code $\langle Y_j^\alpha \mid \rho_j = i\rangle$ as a single subset X_i of β_i. Pick some sequence $\langle E_i \mid i < \lambda\rangle$ of generic sets $E_i \subseteq \beta_i$ and $V[X_i, D_i^*] = V[E_i]$'s and find for each $i < \lambda$ using 4.17 a generic F_i such that $E_i \cup D_i^\alpha \subseteq F_i$ and $E_i, D_i^\alpha \in V[F_i]$. By lemma 4.18 we may find $\langle D_i^{\alpha+1} \mid i < \lambda\rangle$ such that

1. $\bigcup_{i<\lambda} D_i^{\alpha+1}$ is generic.

2. $F_i \in V[D_i^{\alpha+1}]$.

3. $F_i \subseteq^* D_i^{\alpha+1} \subseteq \beta_i$

4. $i < i' \rightarrow D_i^{\alpha+1} \subseteq^* D_{i'}^{\alpha+1}$

For limit $\delta < \kappa^+$, pick a cofinal sequence $\langle \delta_i \mid i < cf^{V[A]}(\delta)\rangle$. Note that $\delta' = cf^{V[A]}(\delta) < \kappa$. For every $j < \lambda$ apply lemma 4.18 to the sequence $\langle D_j^{\delta_\xi} \mid \xi < \delta'\rangle$ and obtain $\langle D_{j,\xi} \mid \xi < \delta'\rangle$ and let

$$F_j = \underset{\xi<\delta'}{\cup} D_{j,\xi}$$

Then $F_j \subseteq \beta_j$ is generic. As in the successor stage, apply proposition 4.17 to F_j^δ and D_j^0 and get $\langle G_j \mid j < \lambda\rangle$ generic such that $F_j \subseteq G_j \subseteq \beta_j$ and $D_j^0 \in V[G_j]$. Define $\langle D_j^\delta \mid j < \lambda\rangle$ using 4.18 on the sequence $\langle G_j \mid j < \lambda\rangle$. Let $\alpha < \delta$, there is $\xi < \delta'$ such that $j \leq \delta_\xi$ then

$$D_j^\alpha \subseteq^* D_j^{\delta_\xi} \subseteq^* D_{j,\xi} \subseteq F_j \subseteq G_j \subseteq^* D_j^\delta$$

Hence the sequence $\langle D_j^\xi \mid j < \lambda\rangle$ is defined. For every $j < \lambda$, $\langle D_j^\xi \mid \xi < \kappa^+\rangle$ is a $\subseteq^*$-increasing sequence of subsets of β_i, thus there is $\xi_j < \kappa^+$ from which this sequence stabilizes. Let $\xi^* = \sup(\xi_j \mid j < \lambda) < \kappa^+$.

Denote by $D^*_i = D^{\xi^*}_i$, and let us prove that D^*_i is as wanted. By the construction of the sequence $(2),(3),(4)$ of the theorem follows directly. To see (1), for every $\xi^* \leq \xi' < \kappa^+$ and for every $i < \lambda$, $D^*_i =^* D^{\xi'}_i$. In particular $D^{\xi+1}_i =^* D^*_i$. By induction we will show that $D^*_j \in V[\bigcup\limits_{i<\lambda} D^*_i]$. For $j = 0$ this is trivial since $I(D^*_0, \bigcup\limits_{i<\lambda} D^*_i \cap \beta_0) \in V$. Assume that $D^*_j \in V[\bigcup\limits_{i<\lambda} D^*_i]$, then $D^{\xi+1}_j \in V[\bigcup\limits_{i<\lambda} D^*_i]$ and therefore $I(D^*_{j+1}, \bigcup\limits_{i<\lambda} D^*_i \cap \beta_{j+1}) \in V[\bigcup\limits_{i<\lambda} D^*_i]$. If j is limit, $\rho_j < j$ is such that $I(D^*_j, (\bigcup\limits_{i<\lambda} D^*_i) \cap \beta_j)$ is coded by $D^{\xi+1}_{\rho_j}$, by induction $D^*_{\rho_j} \in V[\bigcup\limits_{i<\lambda} D^*_i]$ and therefore $D^{\xi+1}_{\rho_j} \in V[\bigcup\limits_{i<\lambda} D^*_i]$ which finishes the induction. Finally, for every $i < \lambda$, $D^0_i \in V[D^*_i] \subseteq V[\bigcup\limits_{i<\lambda} D^*_i]$ so $\langle D^*_i \mid i < \lambda\rangle$ is as wanted.
$\blacksquare$

Corollary 4.20 *If $A \subseteq \kappa$, such that $A \in V[G]$ and $A \cap \alpha$ does not stabilize, then there is a generic $C' \subseteq C_G$ such that $\forall \alpha < \kappa. A \cap \alpha \in V[C']$.*

Proof. By the previous section, find generic sets $\langle D_i \mid i < \lambda\rangle \in V[A]$ such that $V[D_i] = V[A \cap \beta_i]$ and $D_i \subseteq \beta_i$. Use 4.19 to find $\langle D^*_i \mid i < \lambda\rangle$ and set $D^* = \underset{i<\lambda}{\cup} D^*_i$. Then D^* is generic and therefore $D^* \subseteq^* C_G$. Let $C' = C_G \cap D^*$. Hence $C' =^* D^*$ and therefore $V[C'] = V[D^*]$. Now for every $\alpha < \kappa$, find $i < \lambda$ such that $\alpha < \beta_i$. By the properties of D^*, $D_i \in V[D^*]$, hence, $A \cap \beta_i \in V[D^*]$. Note that $A \cap \alpha = (A \cap \beta_i) \cap \alpha$ and therefore $A \cap \alpha \in V[D^*] = V[C']$ as wanted.$\blacksquare$

4.4 Removing the assumption that κ is the first such that $o^{\vec{U}}(\kappa) \geq \kappa$

So far we have proved that if $o^{\vec{U}}(\kappa) = \kappa$, and for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every $A \subseteq \kappa$, there is $C' \subseteq C_G$ such that $V[A] = V[C']$. We can use the techniques of this section to inductively remove the assumption that every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. More precisely, we will assume that $\forall \alpha \leq \kappa. o^{\vec{U}}(\alpha) \leq \alpha$[3] We will use the fact that if $o^{\vec{U}}(\kappa) = \kappa$, then there are only finitely many points $\alpha \in C_G$ such that $o^{\vec{U}}(\alpha) = \alpha$.

We prove by induction on the number of point $\alpha < \kappa$, such that $o^{\vec{U}}(\alpha) = \alpha$, that for every $A \subseteq \kappa$, there is $V[C']$ such that $V[A] = V[C']$. The first lemma in which we assumed that κ is the first was 4.16, we will change this definition, and then prove the rest of the claims 4.17-4.20 with will be our inductive assumption.

What we proved so far in the induction basis, when κ is the first such point. Note that

[3]most of the results are actually true under the assumption that $o^{\vec{U}}(\kappa) = \kappa$, and for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \alpha^+$.

beside 4.16-4.20 we did not restrict $o^{\vec{U}}(\kappa)$ to be the first so we will can use them without having to prove them again.

Let $\kappa^* < \kappa$ be the last such that $\kappa^* \in C_G \wedge o^{\vec{U}}(\kappa^*) = \kappa^*$. Note that κ^* satisfy the induction hypothesis.

The definition of the sequence $\langle \beta_i \mid i < \lambda \rangle$ will start above κ^* instead $\min(X_A)$:

Lemma 4.21 *In $V[A]$ there is a sequence $\langle \beta_i \mid i < \lambda \rangle$, such that:*

1. *$\langle \beta_i \mid i < \lambda \rangle$ is increasing and continuous of elements of X_A.*
2. *$\beta_0 = \min(X_A \setminus \kappa^*)$, $\sup(\beta_i \mid i < \lambda) = \kappa$.*
3. *If $D \subseteq \beta_i$ is generic then for any $Y \subseteq \mathrm{otp}(D)$, $Y \in V[A]$, there is $j < i$ and $D_Y \subseteq \beta_j$ such that $V[Y] = V[D_Y]$.*
4. *$\beta_j \in X_A$ (Recall that X_A is the set of all measurables in V that changed cofinality in $V[A]$).*

By the induction hypothesis, lemma 4.17 holds for κ^*, let us prove that the lemma holds also for unbounded generics of κ^*. Note that once we finish the induction, this lemma will also hold for κ.

Lemma 4.22 *Let $D, E \in V[A]$ be generic subsets of κ^*. Then there is $F \in V[A]$ generic such that $D \cup E \subseteq^* F$ and $D, E \in V[F]$.*

Proof. If $|D|, |E| < \kappa^*$ then $|D \cup E| < \kappa^*$ and therefore $I(D, D \cup E), I(D \cup E)$ is bounded in κ^*. Therefore, there is a bounded in κ^* generic T such that

$$V[T] = V[I(D, D \cup E), I(D \cup E)]$$

Let $\nu = \sup(T) < \kappa^*$. By proposition 4.17 applied to κ^*, we can find a generic $F_* \subseteq \nu$ such that

$$[(D \cup E) \cap \nu] \cup T \subseteq F_* \text{ and } (D \cup E) \cap \nu, T \in V[F_*]$$

Let $F = [(D \cup E) \setminus \nu] \cup F_*$. Then $F \in V[A]$ generic, and $(D \cup E) \setminus \nu, T \in V[F]$. Moreover, $(D \cup E) \cap \nu, T \in V[F]$. Hence $I(D, D \cup E), I(D \cup E), D \cup E \in V[F]$. It follows that $D, E \in V[F]$ and obviously, $D \cup E \subseteq F$.

If $|D| = \kappa^* \vee |E| = \kappa^*$, then necessarily $cf^{V[A]}(\kappa^*) = \omega$. Let $\langle \alpha_n \mid n < \omega \rangle \in V[A]$ be cofinal in κ^*, and consider $D \cap \alpha_n, E \cap \alpha_n$. Again by proposition 4.17, there is $F_n \subseteq \alpha_n$ generic with

$$D \cap \alpha_n, E \cap \alpha_n \in V[F_n] \text{ and } (D \cap \alpha_n) \cup (E \cap \alpha_n) \subseteq F_n$$

By 4.19, we can find a generic F such that for every $n < \omega$, $F_n \subseteq^* F$ and $F_n \in V[F]$. It follows that $\cup_{n<\omega} F_n \subseteq^*_{\aleph_0} F$ i.e. $F \setminus (\cup_{n<\omega} F_n)$ is at most countable. Moreover, $D \cup E \subseteq \cup_{n<\omega} F_n$, hence $|(D \cup E) \setminus F| \leq \aleph_0$. Denote this set by $\langle \gamma_n \mid n < \omega \rangle$. Now proposition 4.1, we can find a generic H such that

$$F, (D \cup E) \setminus F \in V[H], \text{ and } F \cup D \cup E \subseteq H$$

In $V[H]$, we have F, and therefore we have F_n's (not as a sequence), therefore we have $D \cap \alpha_n$ and $E \cap \alpha_n$ for every $n < \omega$. As usually, to have the sequences

$$\langle D \cap \alpha_n \mid n < \omega \rangle, \langle E \cap \alpha_n \mid n < \omega \rangle$$

Code these subsets by ordinals $\langle \delta_n \mid n < \omega \rangle$ and $\langle \rho_n \mid n < \omega \rangle$, then we use 4.1 again, to find a generic $D^* \in V[A]$ such that

$$\langle \delta_n \mid n < \omega \rangle, \ \langle \rho_n \mid n < \omega \rangle \in V[D^*]$$

and $H \subseteq D^*$. So in $V[D^*]$ we can find also D, E ad wanted.$\blacksquare$

Now we can prove 4.17 for κ:

Corollary 4.23 *Let $D, E \subseteq \kappa$ be bounded such that $D, E \in V[A]$. Then there is $F \in V[A]$ generic, such that*

$$D \cup E \subseteq F \subseteq \sup(D \cup E)$$

and $D, E \in V[F]$.

Proof. If $\sup(D \cup E) \leq \kappa^*$ then we use 4.22. Then the induction step is the same as 4.17.$\blacksquare$

Now lemma 4.18 follows, since the proof only used lemma 4.17 and 4.15, which are known at this point for κ. Finally, let us prove 4.19:

Theorem 4.24 *Assume that for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Let $\langle D_i \mid i < \lambda \rangle \in V[A]$ be a sequence of generics such that for every $i < \lambda$, $D_i \subseteq \beta_i$. Then there is $\langle D_i^* \mid i < \lambda \rangle \in V[A]$ such that*

1. $\forall i < \lambda, \ D_i, D_i^* \in V[\bigcup_{i<\lambda} D_i^*]$.
2. $\bigcup_{i<\lambda} D_i^*$ *is generic.*
3. $D_i \subseteq^* D_i^* \subseteq \beta_i$.
4. $\langle D_i^* \mid i < \lambda \rangle$ *is $\subseteq^*$-increasing.*

Proof. Work in $V[A]$, list all the generic sets $D \subseteq \kappa^*$. This is definable in $V[A]$ and so the list is in $V[A]$. The is a list of length at most 2^{κ^*}. Since $\beta_1 \in X_A$, and $\beta_1 > \beta_0 = \kappa^*$, it is a strong limit, hence $2^{\kappa^*} < \beta_1$. Hence there is $C' \subseteq C_G$ bounded in β_1 such that every generic $D \subseteq \kappa^*$ in $V[A]$ belongs to $V[C']$, and $C' \in V[A]$.

As in 4.19, we construct for every $\xi < \kappa^+$, a sequence of generics $\langle D_i^\xi \mid i < \lambda \rangle$ such that for every $i < \lambda$:

1. $\xi_1 < \xi_2 \to D_i^{\xi_1} \subseteq^* D_i^{\xi_2}$
2. $\forall i \leq j . D_i^\xi \subseteq^* D_j^\xi$
3. $D_i^\xi \subseteq \beta_i$
4. $\bigcup\limits_{j<\lambda} D_j^\xi$ is generic.
5. $D_i^0 \in V[D_i^\xi]$.
6. there is $\rho_i < i$ such that $I(D_i^\xi, \underset{j<\lambda}{\cup} D_j^\xi \cap \beta_j) \in V[D_{\rho_i}^{\xi+1}]$

This can be done since we use already proved everything used in this lemma. Again by regularity of κ^*, we can find $\xi^* < \kappa^+$, such that for every $\xi^* \leq \xi' < \kappa^+$, and for every $i < \lambda$, $D_i^{\xi^*} =^* D_i^{\xi'}$. Let $\nu^* = \sup(C') < \beta_1$, use 4.22 to find $D_* \subseteq \nu^*$ such that that $\cup_{i<\lambda} D_i^* \cap \nu^*$, $C' \in V[D_*]$, and $\cup_{i<\lambda} D_i^* \cap \nu^* \cup C' \subseteq D_*$. Define $D_{0,*} = D_* \cap \kappa^*$ and for $0 < i < \lambda$,

$$D_{i,*} = D_* \cup (D_i^* \setminus \nu^*)$$

Let us show that $D_{*,i}$ is as wanted, $(2), (3), (4)$ are clear by the definition. To see (1), first we denote $D^* = \cup_{i<\lambda} D_{*,i}$. Note that since $D^* \cap \nu^* = D_*$ and $D^* \setminus \nu^* = \cup_{i<\lambda} D_i^* \setminus \nu^*$ so in $V[D^*]$ we have D_* and therefore we have C' and $\cup_{i<\lambda} D_i^* \cap \nu^*$. It follows that

$$\cup_{i<\lambda} D_i^* = \Big(\cup_{i<\lambda} D_i^* \cap \nu^* \Big) \cup \Big(\cup_{i<\lambda} D_i^* \setminus \nu^* \Big) \in V[D^*]$$

We claim that $D_i^* \in V[D^*]$ for every $i < \lambda$ (and therefore also D_i and $D_{*,i}$). For $i = 0$, note that $I(D_0^*, \cup_{i<\lambda} D_i^* \cap \kappa^*) \in V[A]$ and is coded by a generic subset of κ^* in $V[A]$. Thus $I(D_0^*, \cup_{i<\lambda} D_i^* \cap \kappa^*) \in V[C']$ and also in $V[D^*]$. So in $V[D^*]$ we have both $I(D_0^*, \cup_{i<\lambda} D_i^* \cap \kappa^*)$ and $\cup_{i<\lambda} D_i^* \cap \kappa^*$, which implies that $D_0^* \in V[D^*]$. The prove for $0 < i < \lambda$ is exactly as in 4.19.$\blacksquare$

This proves the induction step. We conclude as in 4.20 the following:

Theorem 4.25 *If for every $\alpha \leq \kappa$, $o^{\vec{U}}(\alpha) \leq \alpha$, then for every $A \subseteq \kappa$ in $V[G]$, there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

5 The proof for subsets of κ^+

We start with have two easy observations:

Proposition 5.1 *If $A \in V[G]$, such that $A \subseteq \kappa^+$ is of cardinality κ, then there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

Proof. Let $\sup(A) < \theta < \kappa^+$. In V find a bijection from $\pi : \kappa \to \theta$. Consider $A_\kappa = \pi^{-1''}[A] \subseteq \kappa$. Apply the last section to A_κ, then there is $C' \subseteq C_G$ such that $V[A_\kappa] = V[C']$. Since $\pi \in V$, it is clear that $V[A] = V[A_\kappa]$, hence C' is as wanted.$\blacksquare$

Lemma 5.2 *If there is $\beta < \kappa$ such that for every $\alpha < \kappa^+$, $A \cap \alpha \in V[G \restriction \beta]$, then there is $C' \subseteq C_G$ such that $V[A] = V[C']$*

Proof. In this situation we claim that $A \in V[G \restriction \beta]$ as well. To see this, Note that the forcing completing $V[G \restriction \beta]$ to $V[G]$ is simply $\mathbb{M}[\vec{U}] \restriction (\beta, \kappa)$ which is κ^+-c.c. in $V[G]$ (since κ^+ is regular in $V[G]$). Therefore, A cannot be a fresh set with respect to the models $V[G \restriction \beta] \subseteq V[G]$.$\blacksquare$

Claim 4 If $A \cap \alpha$ does not stabilize, then $cf^{V[A]}(\kappa) < \kappa$.

Proof. Simple corollary of 4.4.$\blacksquare$

Proposition 5.3 *Let $A \in V[G]$ be any subset of κ^+ such that $A \cap \alpha$ does not stabilize. Then there is a sequence $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ such that:*

1. $\langle D_\alpha \mid \alpha < \kappa^+ \rangle \in V[A]$.
2. $\forall \alpha < \kappa^+$, $D_\alpha \subseteq^* C_G$.
3. $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ *is $\subseteq^*$-increasing.*
4. $A \cap \alpha \in V[D_\alpha]$

Proof. Work in $V[A]$. For every $\alpha < \kappa^+$, by the last section, there is a generic set $D'_\alpha \subseteq C_G$ such that $V[A \cap \alpha] = V[D'_\alpha]$. Then $1, 2, 4$ hold but 3 might fail. Let us construct the sequence $\langle D_\alpha \mid \alpha_0 \leq \alpha < \kappa^+ \rangle$ more carefully to insure condition (3): We go by induction on $\beta < \kappa^+$ Assume the sequence $\langle D_\alpha \mid \alpha < \beta \rangle$ is defined. If $\beta = \alpha + 1$, then use lemma 4.22 with D_α and D'_β to find $D_{\beta+1}$ such that $D_\alpha \subseteq D_\beta$ and $D'_\beta \in V[D_\beta]$. If β is limit, let $\lambda = cf^{V[A]}(\beta)$. Since κ is singular in $V[A]$, then $\lambda < \kappa$. By lemma 4.18, for every $\alpha < \beta$, $D_\alpha \cap \lambda$ is bounded in λ and we can find $D_* \subseteq \lambda$ such that the sequence $\langle D_\alpha \cap \lambda \mid \alpha < \beta\rangle$, $=^*$-stabilizes of D_*. As for the sequence $\langle D_\alpha \setminus \lambda \mid \alpha < \beta\rangle$ we can use 4.14 to find a single $D^* \in V[A]$ generic, such that $D_\alpha \setminus \lambda \subseteq^* D^*$.

Consider now the two sets $D^* \cup D_*$ and D'_β. Use lemma 4.22 to find D_β such that $D^* \cup D_* \subseteq D_\beta$ and $D'_\beta \in V[D_\beta]$. Clearly the sequence $\langle D_\alpha \mid \alpha < \kappa^+\rangle$ is as wanted.■

In the next theorem we will prove that the sequence $\langle D_i \mid i < \kappa^+\rangle$ must stabilize, we will use the Erdös-Rado theorem[11], which is stated here for the convenience of the reader.

Theorem 5.4 *If θ is a regular cardinal then for every $\rho < \theta$*

$$(2^{<\theta})^+ \to (\theta)^2_\rho$$

i.e. for every function $f : [(2^{<\theta})^+]^2 \to \rho$ there is $H \subseteq \theta$ such that $|H| = \theta$ such that $f \restriction [H]^2$ is constant.

Proof. see [9, Theorem 7.3].■

The next theorem is stated in general settings but will be used for the specific sequence defined in 5.3.

Theorem 5.5 *Let κ be a singular strong limit cardinal, and $\langle D_\alpha \mid \alpha < \kappa^+\rangle$ be any $\subseteq^*$-increasing sequence of subsets of κ. Then the sequence $=^*$-stabilizes i.e. there is $\alpha^* < \kappa^+$ such that for every $\alpha^* \leq \alpha < \kappa^+$, $D_\alpha =^* D_{\alpha^*}$.*

Proof. Toward a contradiction, assume that the theorem fails, then there is $Y \subseteq \kappa^+$ such that $|Y| = \kappa^+$ and for every $\alpha, \beta \in Y$, if $\alpha < \beta$ then $D_\alpha \subseteq^* D_\beta$ and $|D_\beta \setminus D_\alpha| \geq \omega$. Denote $\lambda = cf(\kappa) < \kappa$. Fix $\langle \eta_i \mid i < \lambda\rangle$ be cofinal in κ. For every $i < \lambda$, there is $E_i \subseteq C_G \cap \eta_i$ such that The set

$$X_i = \{\nu < \kappa^+ \mid D_\nu \cap \eta_i = E_i\}$$

is unbounded in κ^+, set $\alpha_i := \min(X_i)$. Since D_i is $\subseteq^*$-increasing, for every $\alpha_i \leq \alpha < \kappa^+$, $D_\alpha \cap \eta_i =^* E_i$. To see this, find $\beta \in X_i$ such that $\alpha_i \leq \alpha \leq \beta$, then

$$D_{\alpha_i} \subseteq^* D_\alpha \subseteq^* D_\beta$$

Hence

$$E_i = D_{\alpha_i} \cap \eta_i \subseteq^* D_\alpha \cap \eta_i \subseteq^* D_\beta \cap \eta_i = E_i$$

Therefore, $E_i =^* D_\alpha \cap [\eta_i, \eta_{i+1})$.

Set $E^* = \cup_{i<\lambda} E_i$ and $\alpha^* = \sup(\alpha_i \mid i < \lambda)$. Clearly, $\alpha^* < \kappa^+$.

Claim 5 For every $\delta < \kappa$ and every $\alpha^* \leq \beta_1 < \beta_2 < \kappa^+$, $|(D_{\beta_1} \cap \delta)\Delta(D_{\beta_1} \cap \delta)| < \omega$

Proof. (of claim 5) Let $i < \lambda$ be such that $\eta_i \geq \delta$. Since $\beta_1, \beta_2 \geq \alpha^* \geq \alpha_i$,

$$|(D_{\beta_1} \cap \delta)\Delta(D_{\beta_2} \cap \delta)| \leq |(D_{\beta_1} \cap \eta_i)\Delta(D_{\beta_2} \cap \eta_i)| \leq |\Big((D_{\beta_1} \cap \eta_i)\Delta E_i\Big) \cup \Big((D_{\beta_2} \cap \eta_i \delta)E_i\Big)| < \omega$$

$\blacksquare_{claim6}$

Claim 6 For every $\alpha^* \leq \beta_1 < \beta_2 < \kappa^+$, $D_{\beta_1} =^*_{\aleph_0} D_{\beta_2}$.

Proof. (of claim 6) Otherwise there are β_1, β_2 such that $|D_{\beta_1} \Delta D_{\beta_2}| \geq \aleph_1$. Then there is $\delta < \kappa$ such that $|D_{\beta_1} \cap \delta \Delta D_{\beta_2} \cap \delta| \geq \omega$. Contradiction to the last claim. $\blacksquare_{claim5}$

Let $\chi = (2^{\lambda^+})^+$ and let $X \subseteq Y$ be such that $|X| = \chi$. Enumerate X, $\langle D_{\alpha_i} \mid i < \chi \rangle$.

Define the partition $f : [\chi]^2 \to \lambda$:

Let $i < j < \chi$. Since $D_i \subseteq^* D_j$, there is $\gamma_{i,j} < \lambda$ such that $(D_{\alpha_i} \setminus \eta_{\gamma_{i,j}}) \subseteq (D_{\alpha_j} \setminus \eta_{\gamma_{i,j}})$. Simply pick some $\eta_{\gamma_{i,j}}$ above finitely many elements in $D_{\alpha_i} \setminus D_{\alpha_j}$. Then set

$$f(i,j) = \gamma_{i,j}$$

By Erdös-Rado theorem, we can find $I \subseteq \chi$ such that $|X| = \lambda^+$ which is homogeneous with color $\gamma^* < \lambda$. This means that for any $i < j$ in I, $D_{\alpha_i} \setminus \eta_{\gamma^*} \subseteq D_{\alpha_j} \setminus \eta_{\gamma^*}$.

Let $\langle i_\rho \mid \rho < \lambda^+ \rangle$ be the increasing enumeration of I. We will prove that $|D_{\alpha_{i_{\omega_1}}} \setminus D_{\alpha_{i_0}}| \geq \omega_1$, and since $\alpha_{i_1}, \alpha_{i_{\omega_1}} \geq \alpha^*$, this is a contradiction to claim 6.

Indeed for every $\xi < \omega_1$, pick any $\delta_\xi \in (D_{\alpha_{i_{\xi+1}}} \setminus \eta_{\gamma^*}) \setminus (D_{\alpha_{i_\xi}} \setminus \eta_{\gamma^*})$. Such δ_ξ exists, since by claim 5, $D_{\alpha_{i_{\xi+1}}} \cap \eta_{\gamma^*} =^* D_{\alpha_{i_\xi}} \cap \eta_{\gamma^*}$. Since $\alpha_{i_\xi}, \alpha_{i_{\xi+1}} \in Y$, then $\omega \leq |D_{\alpha_{i_{\xi+1}}} \Delta D_{\alpha_{i_\xi}}|$. So $\omega \leq |(D_{\alpha_{i_{\xi+1}}} \setminus \eta_{\gamma^*})\Delta(D_{\alpha_{i_\xi}} \setminus \eta_{\gamma^*})|$. Since $i_\xi, i_{\xi+1} \in I$, $D_{\alpha_{i_\xi}} \setminus \eta_{\gamma^*} \subseteq D_{\alpha_{i_{\xi+1}}} \setminus \eta_{\gamma^*}$, it follows that $|(D_{\alpha_{i_{\xi+1}}} \setminus \eta_{\gamma^*}) \setminus (D_{\alpha_{i_\xi}} \setminus \eta_{\gamma^*})| \geq \omega$. The map $\xi \mapsto \delta_\xi$ is a bijection from ω_1 to $D_{\alpha_{i_{\omega_1}}} \setminus D_{\alpha_{i_0}}$, contradiction.$\blacksquare$

Corollary 5.6 *There is $C' \subseteq C_G$ such that $C' \in V[A]$ for every $\alpha < \kappa^+$, $A \cap \alpha \in V[C']$*

Proof. Consider the sequence $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ from proposition 5.3, then use theorem 5.5 to find $\alpha^* < \kappa^+$ such that for every $\alpha^* \leq \beta < \kappa^+$, $D_\beta =^* D_\alpha$. In particular, $V[D_\beta] = V[D_\alpha]$. Define $C' = D_{\alpha^*} \cap C_G$, let us prove that C' is as wanted. Since D_{α^*} is generic, $C' =^* D_{\alpha^*}$, then

$$V[C'] = V[D_{\alpha^*}] = V[A \cap \alpha^*] \subseteq V[A]$$

Let $\alpha < \kappa^+$, if $\alpha \leq \alpha^*$, then

$$A \cap \alpha \in V[A \cap \alpha^*] = V[D_{\alpha^*}] = V[C']$$

If $\alpha > \alpha^*$, then $D_\alpha =^* D_{\alpha^*} =^* C'$ and therefore

$$A \cap \alpha \in V[D_\alpha] = V[C']$$

■

As usual we would like to conclude that A cannot be fresh with respect to the models $V[C'] \subseteq V[C_G]$, and for this we need to dill with the quotient forcing.

Definition 5.7 Let $\underset{\sim}{C}'$ be a $\mathbb{M}[\vec{U}]$-name such that $\underset{\sim}{C}'_G = C'$. Define $\mathbb{P}_{\underset{\sim}{C}'}$, the complete subalgebra of $RO(\mathbb{M}[\vec{U}])$ generated by the conditions $X = \{||\alpha \in \underset{\sim}{C}'|| \mid \alpha < \kappa\}$.

By [8, 15.42], $V[C'] = V[H]$ for some V-generic filet H of $\mathbb{P}_{\underset{\sim}{C}'}$. In fact

$$C' = \{\alpha < \kappa \mid ||\alpha \in \underset{\sim}{C}'|| \in X \cap G\}$$

Definition 5.8 Define the function $\pi : \mathbb{M}[\vec{U}] \to \mathbb{P}_{\underset{\sim}{C}'}$ by

$$\pi(p) = inf(b \in \mathbb{P}_{\underset{\sim}{C}'} \mid b \geq p)$$

It not hard to check that π is a projection i.e.

1. π is order preserving.
2. $\forall p \in \mathbb{M}[\vec{U}] \forall \pi(p) \leq q \exists p' \geq p. \pi(p') \geq q$.
3. $Im(\pi)$ is dense in $\mathbb{P}_{\underset{\sim}{C}'}$.

Definition 5.9 Let $\pi : \mathbb{P} \to \mathbb{Q}$ be any projection, let $H \subseteq \mathbb{Q}$ be V-generic, define

$$\mathbb{P}/H = \pi^{-1''} H$$

We abuse notation by defining $\mathbb{M}[\vec{U}]/C' = \mathbb{M}[\vec{U}]/H$, where H is some generic for $\mathbb{P}_{\underset{\sim}{C}'}$ such that $V[H] = V[C']$.

It is known that G is $V[C']$-generic for $\mathbb{M}[\vec{U}]/C'$ and $V[G] = V[C'][G]$.

It is important to note that $\mathbb{M}[\vec{U}]/C'$ depends of the choice of the name $\underset{\sim}{C}'$.

Example 5.10 It is tempting to try and discard this name and define $\mathbb{M}[\vec{U}]/C'$ to consist of all p such that there is a V-generic $H \subseteq \mathbb{M}[\vec{U}]$, with $p \in H$ and $C' \subseteq C_H$. Such a forcing is not κ^+- c.c. even above $V[C']$. Indeed, we take for example any $\{c_n \mid n < \omega\} \subseteq C_G$ unbounded in κ, such that for every n, $o^{\vec{U}}(c_n) = 0$. Basically, it is a Prikry sequence for the measure $U(\kappa, 0)$. Now $V[C'] \models \kappa^\omega = \kappa^+$ so let $\langle f_i \mid i < \kappa^+ \rangle \in V[C']$ be an enumeration of all function from ω to κ. we can factor the forcing to first pick $i < \kappa^+$, then the rest of the forcing ensures that $C_G(f_i(n)+1) = c_n$, this means that f_i, determined the places of c_n's in the sequence C_G. Since no choice of $i \neq j$ can be compatible, the first part is not κ^+-c.c. and therefore also the product.

Example 5.11 Let us consider another possible simplification of $\mathbb{M}[\vec{U}]/C'$,

$$\mathbb{M}[\vec{U}]' = \{q \in \mathbb{M}[\vec{U}] \text{ for every finite } a \subseteq \kappa \text{ there is } q_a \geq q, q_a \Vdash \underset{\sim}{c}'_\alpha = \check{c}'_\alpha, \text{ for every } \alpha \in a\}$$

First we define an $\mathbb{M}[\vec{U}]-$name $\underset{\sim}{C}'$ of a subset $\{c'_\alpha \mid \alpha < \kappa\}$ of a generic sequence C_G. For every $\alpha < \kappa$, let

$$X_\alpha = \{\nu < \kappa \mid o^{\vec{U}}(\nu) = \alpha\}.$$

Pick some different $\rho^0, \rho^1 \in X_0$. The play would be between two conditions

$$p^0 = \langle \rho^0, \langle \kappa, \kappa \setminus \rho^0 + 1 \rangle \rangle \text{ and } p^1 = \langle \rho^1, \langle \kappa, \kappa \setminus \rho^1 + 1 \rangle \rangle$$

Above p^0 we do something simple - for example, let $\underset{\sim}{c}'_\alpha$ be a name just the first element of X_α in the generic sequence C_G.

Now above p^1, let us do something more sophisticated. We will build a $\kappa-$tree with each of its branches corresponding to an extension of p^1 and such conditions will be incompatible in $\mathbb{M}[\vec{U}]/C'$, where $C' := \underset{\sim}{C}'_H$ and $H \subseteq \mathbb{M}[\vec{U}]$ is a V-generic filter with $p^0 \in H$.

Start with a description of the first level:
Fix $Y_1 \in U(\kappa, 1)$, such that $Y_1 \subseteq X_1$ and $Z_1 = X_1 \setminus Y_1$ has cardinality κ. Split Z_1 into two disjoint non-empty sets $Z_{1,0}, Z_{1,1}$.

Now, let p^1 extended by an element of Y_1 produces $\underset{\sim}{c}'_1$ to be different from those which p^0 defines, for example, let it be the the first element of X_2 in C_G.

For $i = 0, 1$, let p^1 extended by an element of $Z_{1,i}$ produces $\underset{\sim}{c}'_1$ to be the same as $\underset{\sim}{c}'_1$ by p^0.

The idea behind is to insure that for every i, $p^1 {}^\frown Z_{1i} \cup Y_1$ will be in $M[\vec{U}]/C'$, but only because of Z_{1i}. So, if $i \neq j$ are different then we will have incompatibility since Z_{1i} and Z_{1j} are disjoint. Continue in a similar fashion to define the rest of the levels, the α-th level we Take $Y_\alpha \subseteq X_\alpha$ such that $Z_\alpha := X_\alpha \setminus Y_\alpha$ has size κ, and we split Z_α into two disjoint non empty sets $Z_{\alpha,0}, Z_{\alpha_1}$. The definition of $\underset{\sim}{c}'_\alpha$ is such that p^1 extended by elements of Y_α forces $\underset{\sim}{c}'_\alpha$ to be the first member of $X_{\alpha+1}$ in C_G. While p^1 extended by elements of Z_α will force the same value as p^0 did.

Note that the construction is completely inside V.

Finally, there are κ^+-branches of length κ in T. Let p^h denotes an extension of p^1 which corresponds to a $\kappa-$branch h i.e. $p^h = \langle \rho_1, \langle \kappa, \bigcup_{\alpha<\kappa} Y_\alpha \uplus Z_{\alpha,h(\alpha)} \rangle\rangle$.

Let h_1, h_2 be two different branches. Let $\alpha < \kappa$ be the least such that $h_1(\alpha) \neq h_2(\alpha)$. Then p^{h_1} and p^{h_2} are incompatible in $\mathbb{M}[\vec{U}]/C'$. This follows from the choice of $\underset{\sim}{c}'_\alpha$ and the definitions of conditions at the level α.

Note that every p^h is in $\mathbb{M}[\vec{U}]'$, since for every finite $a \subseteq \kappa$, we can extend p^h to some q_a using the elements from $Z_{\alpha,h(\alpha)}$.

The problem here however is that the conditions p^h are not in $\mathbb{M}[\vec{U}]/C'$, thus $\mathbb{M}[\vec{U}]' \neq \mathbb{M}[\vec{U}]/C'$. Otherwise, by the next proposition, there is a generic H such that $\{(\underset{\sim}{c}'_\alpha)_H \mid \alpha < \kappa\} = C'$ with $p^h \in H$. Since $Y^* := \bigcup_{\alpha<\kappa} Y_\alpha \in \cap\vec{U}(\kappa)$, then by the Mathias criteria there is $\xi < \kappa$ such that $C_H \setminus \xi \subseteq Y^*$. It follows that the interpretation $(\underset{\sim}{c}'_\alpha)_H$ must be different from the one p^0 made, contradiction

Proposition 5.12 *For every $q \in \mathbb{M}[\vec{U}]$, $q \in \mathbb{M}[\vec{U}]/C'$ iff there is a generic G' for $\mathbb{M}[\vec{U}]$ such that $\underset{\sim}{C}'_{G'} = C'$.*

Proof. Let $q \in \mathbb{M}[\vec{U}]/H$, let G' be any $V[C']$-generic for $\mathbb{M}[\vec{U}]/H$ with $q \in G'$, then $G' \subseteq \mathbb{M}[\vec{U}]$ is a V-generic filter. To see that $\underset{\sim}{C}'_{G'} = C'$, denote $C'' := \underset{\sim}{C}'_{G'}$, toward a contradiction assume that $s \in C' \setminus C''$, then there is $q \leq q' \in G'$ such that $q' \Vdash s \notin \underset{\sim}{C}'$, hence $\pi(q') \notin H$, this is a contradiction since $G' \subseteq \mathbb{M}[\vec{U}]/H$. Also if $s \in C'' \setminus C'$, then there is $q \leq q' \in G$ such that $q' \Vdash s \in \underset{\sim}{C}'$. Since $s \notin C'$, then $||s \in \underset{\sim}{C}'|| \notin H$, hence $q' \notin H$ which is again a contradiction.

For the other direction, if $q \in G'$ for some G' for $\mathbb{M}[\vec{U}]$ such that $\underset{\sim}{C}'_{G'} = C'$, then $X \cap G' = X \cap G$. Let $a \in G'$, if $\pi(a) \notin H$, then there is $\alpha \in C'$ such that $\pi(a)$ and $||\alpha \in \underset{\sim}{C}'||$ are incompatible, hence $||\alpha \in \underset{\sim}{C}'|| \notin G'$, but $||\alpha \in \underset{\sim}{C}'|| \in X \cap G$, contradiction.■

Definition 5.13 A uniform ultrafilter on a regular cardinal κ is called p-point, if for every function $f : \kappa \to \kappa$ which is not constant $(modU)$ is almost $1-1$ $(modU)$ i.e. There is $A \in U$ such that for every $\delta < \kappa$,

$$|\{\nu < \kappa \mid f(\nu) = \delta\}| < \kappa$$

Proposition 5.14 *Let U be a p-point ultrafilter on κ, let $\langle X_i \mid i < \kappa\rangle$ a sequence of sets in U. Consider $\pi : \kappa \to \kappa$ such that $[\pi]_U = \kappa$ then:*

$$\Delta^*_{i<\kappa} X_i = \{\nu < \kappa \mid \forall i < \pi(\nu).\ \nu \in X_i\} \in U$$

Proof. Assume otherwise, there the set $E = \{\nu < \kappa \mid \exists i < \pi(\nu).\nu \notin X_i\} \in U$. For every $\nu \in E$ fix $i_\nu < \pi(\nu)$ witnessing $\nu \in E$. The function $f(\nu) = i_\nu$ is below π, in the $<_U$ order.

Since $[\pi]_U = \kappa$, There is $\xi < \kappa$ such that $[f]_U = [C_\xi]_U$. It follows that $E^* = \{\nu \mid i_\nu = \xi\} \in U$. For every $\nu \in E^*$, $\nu \notin X_\xi$, thus, $X_\xi \cap E^* = \emptyset$, contradiction.$\blacksquare$

First we need a generalization of Galvin's theorem (see [7], or [5, Proposition 1.4]):

Proposition 5.15 *Suppose that $2^{<\kappa} = \kappa$ and let F be a normal filter or a p-point ultrafilter over κ . Let $\langle X_i \mid i < \kappa^+\rangle$ be a sequence of sets such that for every $i < \kappa^+$, $X_i \in F$, and let $\langle Z_i \mid i < \kappa^+\rangle$ be any sequence of subsets of κ. Then there is $Y \subseteq \kappa^+$ of cardinality κ, such that*

1. *$\bigcap_{i\in Y} X_i \in F$.*
2. *there is $\alpha \in Y$ such that $[Z_\alpha]^{<\omega} \subseteq \bigcup_{i\in Y\setminus\{\alpha\}} [Z_i]^{<\omega}$*

Proof. (of proposition) For every $\vec{\nu} \in [\kappa]^{<\omega}$, $\alpha < \kappa^+$ and $\xi < \kappa$, let

$$H_{\alpha,\xi,\vec{\nu}} = \{i < \kappa^+ \mid X_i \cap \xi = X_\alpha \cap \xi \wedge \vec{\nu} \in [Z_i]^{<\omega}\}$$

Claim 7 There is $\alpha^* < \kappa^+$ such that for every $\xi < \kappa$ and $\vec{\nu} \in [Z_{\alpha^*}]^{<\omega}$, $|H_{\alpha^*,\xi,\vec{\nu}}| = \kappa^+$

Proof. (of claim) Otherwise, for every $\alpha < \kappa^+$ there is $\xi_\alpha < \kappa$ and $\vec{\nu}_\alpha \in [Z_\alpha]^{<\omega}$ such that $|H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha}| \leq \kappa$. There is $X \subseteq \kappa^+$, $\vec{\nu}^* \in [\kappa]^{<\omega}$ and $\xi^* < \kappa$, such that $|X| = \kappa^+$ and for every

$$\forall \alpha \in X,\ \vec{\nu}_\alpha = \vec{\nu}^* \wedge \xi_\alpha = \xi$$

Since κ is strong limit and $\xi < \kappa$, there are less than κ many possibilities for $X_\alpha \cap \xi^*$. Hence we can shrink X to $X' \subseteq X$ such that $|X'| = \kappa^+$ and find a single set $E^* \subseteq \xi^*$ such that for every $\alpha \in X'$, $X_\alpha \cap \xi^* = E^*$. It follows that for every $\alpha \in X'$:

$$H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha} = H_{\alpha,\xi^*,\vec{\nu}^*} = \{i < \kappa^+ \mid X_i \cap \xi^* = E^* \wedge \vec{\nu}^* \in [Z_i]^{<\omega}\}$$

Hence the set $H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha}$ does not depend on α, which means it is the same for every $\alpha \in X'$. Denote this set by H^*. To see the contradiction, note that for every $\alpha \in X'$, $\alpha \in H_{\alpha,\xi_\alpha,\vec{\nu}_\alpha} = H^*$, thus $X' \subseteq H^*$. Bt then

$$\kappa^+ = |X'| \leq |H^*| \leq \kappa$$

contradiction.$\blacksquare_{claim}$

<u>End of proof of proposition:</u> Let α^* be as in the claim. Let us define $Y \subseteq \kappa^+$ that will witness the lemma. First, enumerate $[Z_{\alpha^*}]^{<\omega}$, $\langle \vec{\nu}_i \mid i < \kappa\rangle$ (Recall that the cardinality of Z_{α^*} is κ by the assumption. Let $\pi : \kappa \to \kappa$ be the function representing κ i.e. $[\pi]_U = \kappa$[4] Since U

[4]If U is normal $\pi = id$.

is p-point or normal, there is a set $X \in U$ such that for every $\alpha < \kappa$ $X \cap \pi^{-1\prime\prime}\alpha$ is bounded in κ. So for every $\alpha < \kappa$, we find $\rho_\alpha > \sup(\pi^{-1\prime\prime}[\alpha+1])$.

Then by recursion, define β_i for $i < \kappa$. At each step we pick $\beta_i \in H_{\alpha^*,\rho_i+1,\vec{\nu}_i} \setminus \{\beta_j \mid j < i\}$. It is possible to do so since the cardinality of $H_{\alpha^*,\rho_i+1,\vec{\nu}_i} \setminus \{\beta_j \mid j < i\}$ is κ^+, and so far we have defined less than κ^+ many ordinals. Let us prove that $Y = \{\beta_i \mid i < \kappa\} \cup \{\alpha^*\}$ is as wanted. Indeed, by definition, it is clear that $|Y| = \kappa$. Also, if $\vec{\nu} \in [Z_{\alpha^*}]^{<\omega}$, then $\vec{\nu} = \vec{\nu}_i$ for some $i < \kappa$. By definition, $\beta_i \in H_{\alpha^*,\rho_i+1,\vec{\nu}_i}$, hence $\vec{\nu} \in [Z_{\beta_i}]^{<\omega}$, so

$$[Z_{\beta_i}]^{<\omega} \subseteq \bigcup_{x \in Y \setminus \{\alpha^*\}} [Z_x]^{<\omega}$$

Finally, we need to prove that $\bigcap_{i \in Y} X_i \in F$. By proposition 5.14 (or normality),

$$X_{\alpha^*} \cap \Delta^*_{i<\kappa} X_{\beta_i} \in F$$

Let $\zeta \in X_{\alpha^*} \cap \Delta^*_{i<\kappa} X_{\beta_i}$, then for every $i < \pi(\zeta)$, $\zeta \in X_{\beta_i}$. For $i \geq \pi(\zeta)$, by definition of ρ_i, $\zeta < \rho_i$. Since $\beta_i \in H_{\alpha^*,\rho_i+1,\vec{\nu}_i}$

$$X_{\alpha^*} \cap (\rho_i + 1) = X_{\beta_i} \cap (\rho_i + 1)$$

Also, $\zeta \in X_{\alpha^*} \cap (\rho_i+1)$, hence $\zeta \in X_{\beta_i}$. We conclude that $\zeta \in \bigcap_{\alpha \in Y} X_\alpha$. Hence $\bigcap_{\alpha \in Y} X_\alpha \in F$.
■

Now for the main theorem of this section

Theorem 5.16 *Let of $\pi : \mathbb{M}[\vec{U}] \to \mathbb{P}$ be a projection. Let $G \subseteq \mathbb{M}[\vec{U}]$ be V-generic and $H = \pi\bar{[}G]$ be the induced generic for $\mathbb{P}$, then $V[G] \models \mathbb{M}[\vec{U}]/H$ is κ^+-c.c.*

Proof. Assume otherwise, and let $\langle p_i \mid i < \kappa^+\rangle \in V[G]$ be an anthichain in $\mathbb{M}[\vec{U}]/H$. Let $\langle \underset{\sim}{p}_i \mid i < \kappa^+\rangle$ be a sequence of names for them and $r \in G$ such that

$$r \Vdash \langle \underset{\sim}{p}_i \mid i < \kappa^+\rangle \text{ is an antichain in } \mathbb{M}[\vec{U}]/\underset{\sim}{H}$$

Work in V, for every $i < \kappa^+$, let $r \leq r_i \in \mathbb{M}[\vec{U}]$ and $\xi_i \in \mathbb{M}[\vec{U}]$ be such that $r_i \Vdash \underset{\sim}{p}_i = \xi_i$.

Claim 8 $\forall i < \kappa^+ \forall r' \geq r_i \exists q \geq \xi_i \forall q' \geq q \exists r'' \geq r'$ $r'' \Vdash q \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$

Proof. (of claim) Otherwise, there is i and $r' \geq r_i$, such that for every $q \geq \xi_i$, there is $q' \geq q$ such that every $r'' \geq r'$, $r'' \nVdash q' \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$. In particular, the set

$$E = \{q \geq \xi_i \mid \forall r'' \geq r_i . r'' \nVdash q \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}\}$$

is dense above ξ_i. To obtain a contradiction, let G' be any generic for $\mathbb{M}[\vec{U}]$ such that $r' \in G'$. Since $r' \geq r_i \geq r$, $r, r_i \in G'$ and there for $\xi_i = \underset{\sim}{p}_{i,G'} \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}_{G'}$. Denote $H' = \underset{\sim}{H}_{G'}$. Then, there is a V-generic filter G'' for $\mathbb{M}[\vec{U}]$ such that $\xi_i \in G''$ and $\underset{\sim}{H}_{G''} = H'$. By density of E, there is $q \in E \cap G''$ and in particular, $q \in \mathbb{M}[\vec{U}]/H'$. Thus, there is $r' \leq r'' \in G'$ such that $r'' \Vdash q \in \mathbb{M}[\vec{U}]/H'$, contradicting $q \in E$.$\blacksquare_{claim}$

By the claim applied to $r' = r_i$, for every $i < \kappa^+$, there is $q_i \geq \xi_i$ such that

$$(*)_i \qquad \forall q' \geq q_i . \exists r'' \geq r_i . r'' \Vdash q' \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$$

Denote $q_i = \langle t_{i1}, ..., t_{in_i}, \langle \kappa, A(q_i)\rangle\rangle$ and $r_i = \langle s_{i1}, ..., s_{im_i}, \langle \kappa, A(r_i)\rangle\rangle$. Stabilize the sequences $\langle t_{i1}, ..., t_{in_i}\rangle$ and $\langle s_{i1}, ..., s_{im_i}\rangle$ i.e. find $X \subset \kappa^+$ such that $|X| = \kappa^+$ and $\vec{t} = \langle t_1, .., t_n\rangle, \vec{s} = \langle s_1, ..., s_m\rangle$ such that for every $i \in X$

$$\langle t_{i1}, ..., t_{in_i}\rangle = \langle t_1, .., t_n\rangle, \text{ and } \langle s_{i1}, ..., s_{im_i}\rangle = \langle s_1, ..., s_m\rangle$$

This means that for every $i \in X$, $q_i = \vec{t}^\frown\langle \kappa, A(q_i)\rangle$ and $r_i = \vec{s}^\frown\langle \kappa A(r_i)\rangle$. By lemma 5.15, there is $Y \subseteq X$ of cardinality κ, such that

1. $\bigcap_{i \in Y} A(q_i) \in \bigcap_{i<\kappa} U(\kappa, i)$.
2. There is $\alpha^* \in Y$ such that $[A(r_\alpha)]^{<\omega} \subseteq \bigcup_{i \in Y\setminus\{\alpha\}}[A(r_i)]^{<\omega}$

Consider the set $A = \bigcap_{i\in Y} A(q_i)$. For every $i \in Y$, $q_i \leq \vec{t}^\frown\langle \kappa, A\rangle =: q^*$. Consider $\alpha^* \in Y$ which is guaranteed by 5.15. Then there is $r'' \geq r_{\alpha^*}$ such that $r'' \Vdash q^* \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$. Hence there is $\vec{\nu} \in [A(r_{\alpha^*})]^{<\omega}$ such that $r_{\alpha^*}^\frown\vec{\nu} \leq^* r''$. Denote

$$r'' = \langle s_1, ..., s_m, \langle \nu_1, B_1\rangle, ..., \langle \nu_k, B_k\rangle, \langle \kappa, A(r'')\rangle\rangle$$

By the property of α^*, $\vec{\nu} \in \cup_{j \in Y\setminus\{\alpha^*\}}[A(r_j)]^{<\omega}$ and so there is $j \in Y$ such that $\vec{\nu} \in [A(r_j)]^{<\omega}$. Since r_{α^*} and r_j have the same lower part, and $\vec{\nu} \in [A(r_j)]^{<\omega}$, it follows that r'' and r_j are compatible by the condition:

$$r^* = \langle s_1, ..., s_m, \langle \nu_1, B_1 \cap A(r_j)\rangle, ...\langle \nu_k, B_k \cap A(r_j)\rangle, \langle \kappa, A(r_j) \cap A(r'')\rangle\rangle$$

To see the contradiction, note that since $r^* \geq r_{\alpha^*}, r_j$ and r,

$$r^* \Vdash \underset{\sim}{p}_{\alpha^*} = \xi_{\alpha^*}, \underset{\sim}{p}_j = \xi_j \text{ are incompatible in } \mathbb{M}[\vec{U}]/C'$$

But since $r^* \geq r''$,

$$r^* \Vdash q^* \in \mathbb{M}[\vec{U}]/\underset{\sim}{H}$$

Since $q^* \geq q_{\alpha^*} \geq \xi_{\alpha^*}$ and $q^* \geq q_j \geq \xi_j$, then $r^* \Vdash \underset{\sim}{p}_{\alpha^*}, \underset{\sim}{p}_j$ are compatible, contradiction.$\blacksquare$

Note that for $\mathbb{M}[\vec{U}]/C'$ be κ^+-c.c. in $V[C']$, we can use a more abstract and direct argument:

Suppose we have an iteration $P * \underset{\sim}{Q}$ of forcing notions. It is a classical result about the iteration that if for a regular cardinal λ we have

1. P has $\lambda-$c.c.,
2. $\Vdash_P \underset{\sim}{Q}$ has $\lambda - c.c.$,

then $P * \underset{\sim}{Q}$ satisfies $\lambda-$c.c..

Also, if P has $\lambda-$c.c., $P * \underset{\sim}{Q}$ has $\lambda-$c.c., then $\Vdash_P \underset{\sim}{Q}$ has $\lambda - c.c.$.
Namely, suppose otherwise. Then there are $p \in P$ and a sequence of $P-$names $\langle \underset{\sim}{q}_\alpha \mid \alpha < \lambda \rangle$ such that

$$p \Vdash_P \langle \underset{\sim}{q}_\alpha \mid \alpha < \lambda \rangle \text{ is an antichain in } \underset{\sim}{Q}.$$

Consider now $\{(p, \underset{\sim}{q}_\alpha) \mid \alpha < \lambda\} \subseteq P * \underset{\sim}{Q}$. By $\lambda-$c.c., there are $\alpha, \beta < \lambda, \alpha \neq \beta$ such that $(p, \underset{\sim}{q}_\alpha)$ and $(p, \underset{\sim}{q}_\beta)$ are compatible. Hence, there are $(p', \underset{\sim}{q}') \geq (p, \underset{\sim}{q}_\alpha), (p, \underset{\sim}{q}_\beta)$. But then

$$p' \Vdash_P \underset{\sim}{q}' \text{ is stronger than both } \underset{\sim}{q}_\alpha, \underset{\sim}{q}_\beta,$$

which is impossible, since p' forces that them are members of an antichain.

However, in 5.16, we address a different question:

Suppose that $P * \underset{\sim}{Q}$ satisfies $\lambda-$c.c.. Let $G * H$ be a generic subset of $P * \underset{\sim}{Q}$. Consider the interpretation Q of $\underset{\sim}{Q}$ in $V[G, H]$. Does it satisfies $\lambda-$c.c.?

Clearly, this is not true in general. The simplest P be trivial and Q be the forcing for adding a branch to a Suslin tree. Then, in V^Q, Q will not be c.c.c. anymore.

Our attention in theorem 5.16 is to subforcings and projections of $\mathbb{M}[\vec{U}]$, however the argument given is more general:

Theorem 5.17 *Suppose that $\mathcal{P}$ is either Prikry or Magidor or Magidor-Radin or Radin or Prikry with a p-point ultrafilter forcing and $\underset{\sim}{Q}$ is a projection of $\mathcal{P}$. Let $G(\mathcal{P})$ be a generic subset of $\mathcal{P}$.*
Then, the interpretation of $\underset{\sim}{Q}$ in $V[G(\mathcal{P})]$, satisfies κ^+-c.c. there.

We do not know how to generalize this theorem to wider classes of Prikry type forcing notions.

For example the following may be the first step:

Question 5.18 *Is the result valid for a long enough Magidor iteration of the Prikry forcings?*

The problem is that there is no single complete enough filter here, and so the Galvin Theorem (or its generalization) does not seem to apply.

Question 5.19 *To Which ultrafilters does Galvin's theorem hold?*

One particular example is a fine normal ultrafilter on $P_\kappa(\lambda)$ which is used in the super-compact Prikry forcing (see [4] for the defintiion).

Question 5.20 *Assume that $\lambda^{<\kappa} = \lambda$. Is every quotient forcing of the super compact Prikry forcing also λ^+-c.c. in the generic extension?*

The problem here in generalizing Galvin's theorem to fine normal ultrafilter on $P_\kappa(\lambda)$, is the following:

A set X is bounded in $P_\kappa(\lambda)$ if for some $\xi < \lambda$ there is no $P \in X$ with $\xi \in P$. Such X may be of cardinality λ. However, over κ if X is bounded, then $X \subseteq \xi$ and so $|X| < \kappa$.

In the Galvin's argument, the possibility to stabilize intersections with bounded sets was essential. In the context of $P_\kappa(\lambda)$ such stabilization is just impossible, since if a bounded set has cardinality λ, then there are 2^λ subsets of it, and not $< \kappa$ as in the argument for κ.

Theorem 5.21 *Let $W \models ZFC$ and $T \subseteq \mathbb{P}$ be any W-generic filter and let λ be a regular cardinal in $W[T]$. Assume $\mathbb{P}$ is λ-c.c. in $W[T]$. Then in $W[T]$ there are no fresh subsets of λ with respect to W.*

Remark 1 *Note that it is crucial that $\mathbb{P}$ is λ-c.c. in the generic extension, otherwise there are trivial examples which contradict this. Namely, The forcing which Adds a branch through a Suslin tree, is c.c.c., but the branch added is a fresh subset of ω_1.*

Proof. Toward a contradiction, assume that $A \in W[T] \setminus W$ is fresh subset of λ. Let $\underset{\sim}{A}$ be a name for A in $\mathbb{P}$. For every $\alpha < \lambda$ define in W

$$X_\alpha = \{B \subseteq \alpha \mid ||\underset{\sim}{A} \cap \alpha = B|| \neq 0\}$$

where the truth value is taken in $RO(\mathbb{P})$- the complete boolean algebra of regular open sets for $\mathbb{P}$. Different B's in X_α yeild incompatible conditions of $\mathbb{P}$ and we have λ-c.c by assumption, thus (even in $W[T]$)

$$\forall \alpha < \lambda \ |X_\alpha| < \lambda$$

For every $B \in X_\alpha$ define $b(B) = ||\underset{\sim}{A} \cap \alpha = B||$. Assume that $B' \in X_\beta$ and $\alpha \leq \beta$ then $B = B' \cap \alpha \in X_\alpha$. Moreover $b(B') \leq_B b(B)$ (we Switch to boolean algebra notation $p \leq_B q$ means p extends q). Note that for such B, B' if $b(B') <_B b(B)$, then there is

$$0 < p \leq_B (b(B) \setminus b(B')) \leq_B b(B)$$

Therefore

$$p \cap b(B') \leq_B (b(B) \setminus b(B')) \cap b(B') = 0$$

meaning $p \perp b(B')$. Work in $W[T]$, denote $A_\alpha = A \cap \alpha$. By freshness

$$\forall \alpha < \lambda \; A_\alpha \in W$$

thus $A_\alpha \in X_\alpha$. Consider the $\leq_B$-non-increasing sequence $\langle b(A_\alpha) \mid \alpha < \lambda \rangle$. If there exists some $\gamma^* < \lambda$ on which the sequence stabilizes, define

$$A' = \bigcup \{B \subseteq \lambda \mid \exists \alpha \; b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha = B\} \in W$$

Claim that $A' = A$, notice that if B, B', α, α' are such that

$$b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha = B, \quad b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha' = B'$$

With out loss of generality, $\alpha \leq \alpha'$ then we must have $B' \cap \alpha = B$ otherwise, the non zero condition $b(A_{\gamma^*})$ would force contradictory information. Consequently, for every $\xi < \lambda$ there exists $\xi < \gamma < \lambda$ such that $b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \gamma = A \cap \gamma$, hence $A' \cap \gamma = A \cap \gamma$. This is a contradiction to $A \notin M$. We conclude that the sequence $\langle b(A_\alpha) \mid \alpha < \lambda \rangle$ does not stabilize. By regularity of λ, there exists a subsequence $\langle b(A_{i_\alpha}) \mid \alpha < \lambda \rangle$ which is strictly decreasing. Use the observation we made to find $p_\alpha \leq_B b(A_{i_\alpha})$ such that $p_\alpha \perp b(A_{i_{\alpha+1}})$. Since $b(A_{i_\alpha})$ are decreasing, for any $\beta > \alpha$ $p_\alpha \perp b(A_{i_\beta})$ thus $p_\alpha \perp p_\beta$. This shows that $\langle p_\alpha \mid \alpha < \lambda \rangle \in W[T]$ is an antichain of size λ which contradiction.
$\blacksquare$

Theorem 5.22 $A \in V[C']$ *and* $V[A] = V[C']$.

Proof. Otherwise A would have been a fresh subset of κ^+ with respect to the models $V[C'] \subseteq V[C'][G]$ which is a generic extension of the κ^+-c.c. forcing $\mathbb{M}[\vec{U}]/C'$ in $V[C'][G]$ contradiction the last theorem.$\blacksquare$

Following theorem 5.21, we state here another result related to fresh subsets in Prikry-type models.

Sets of ordinals above κ^+: By induction on $\sup(A) = \lambda > \kappa^+$. It suffices to assume that λ is a cardinal.

<u>case1:</u> $cf^{V[G]}(\lambda) > \kappa$, the arguments for κ^+ works.

case2: $cf^{V[G]}(\lambda) \leq \kappa$ and since κ is singular in $V[G]$ then $cf^{V[G]}(\lambda) < \kappa$. Since $\mathbb{M}[\vec{U}]$ satisfies $\kappa^+ - c.c.$ we must have that $\nu := cf^V(\lambda) \leq \kappa$. Fix $\langle \gamma_i | \ i < \nu \rangle \in V$ cofinal in λ. Work in $V[A]$, for every $i < \nu$ find $d_i \subseteq \kappa$ such that $V[d_i] = V[A \cap \gamma_i]$. By induction, there exists $C^* \subseteq C_G$ such that $V[\langle d_i \mid i < \nu \rangle] = V[C^*]$, therefore

1. $\forall i < \nu \ A \cap \gamma_i \in V[C^*]$
2. $C^* \in V[A]$

Work in $V[C^*]$, for $i < \nu$ fix $\langle X_{i,\delta} \mid \delta < 2^{\gamma_i} \rangle = P(\gamma_i)$ then we can code $A \cap \gamma_i$ with some δ_i such that $X_{i,\delta_i} = A \cap \gamma_i$. By the previous metheods, we can find $C'' \subseteq C_G$ such that $V[C''] = V[\langle \delta_i \mid i < \nu \rangle]$ finally we can find $C' \subseteq C_G$ such that $V[C'] = V[C^*, C'']$, it follows that $V[A] = V[C']$
$\blacksquare$

Let us conclude another resault about fresh subsets in Prikry, Magidor, Magidor-Radin forcings.

Theorem 5.23 *Assume that $o^{\vec{U}}(\kappa) < \kappa^+$ and let $G \subseteq \mathbb{M}[\vec{U}]$ be V-generic. If $A \in V[G]$ is fresh subset with respect to V, then $cf^{V[G]}(\sup(A)) = \omega$*

Proof. By induction on κ. Let A be a fresh subset, then if $A \in V[C_G \cap \alpha]$ for some $\alpha < \kappa$, we are done. Assume that $A \notin V[C_G \cap \alpha]$, in particular $\sup(A) \geq \kappa$. Let us start with $\sup(A) = \kappa$. Toward a contradiction assume that $\lambda := cf^{V[G]}(\kappa) > \omega$, since we assume that $o^{\vec{U}}(\kappa) < \kappa^+$, then $\omega < \lambda < \kappa$. Also find Let $\langle c_\alpha \mid \alpha < \lambda \rangle$ be a cofinal continuous subsequence of C_G such that $c_0 > \lambda$. Let $\langle \underset{\sim}{c}'_\alpha \mid \alpha < \lambda \rangle$ be a sequence of names for it. Also let $\underset{\sim}{A}$ a name for A.

Let $p \in G \restriction (\lambda, \kappa)$ be such that

$$p \Vdash \underset{\sim}{A} \text{ is fresh} \wedge \langle \underset{\sim}{c}'_\alpha \mid \alpha < \lambda \rangle \text{ is a cofinal continuous subsequence of } C_G$$

As in 3.4, for every $i < \lambda$ find a condition $p \leq^* p^{(i)}$ such that if there is $\vec{\alpha} \in [\kappa]^{<\omega}$ and $A(\vec{\alpha}) \subseteq \max(\vec{\alpha})$ such that

$$p^{(i)\frown}\vec{\alpha} \Vdash \underset{\sim}{A} \cap \max(\vec{\alpha}) = A(\vec{\alpha}) \wedge \max(\vec{\alpha}) = \underset{\sim}{c}'_i$$

then there is a $\vec{U}$-fat tree of extensions of $p^{(i)}$, T_i, with $\vec{\alpha} \in mb(T_i)$, such that

$$\forall t \in mb(T_i). \ \exists A(t) \subseteq \max(t). \ p^{(i)\frown}t \Vdash \underset{\sim}{A} \cap \max(t) = A(t) \wedge \underset{\sim}{c}'_i = \max(t)$$

To see that there is such $\vec{\alpha}$, find any $\vec{\alpha}$ and q such that $p^{(i)\frown}\vec{\alpha} \leq^* q$ and $q \Vdash \max(\vec{\alpha}) = \underset{\sim}{c}'_i$ and then above $\max(\vec{\alpha})$ there is enough closure to decide $\underset{\sim}{A} \cap \max(\vec{\alpha})$. Hence there is an

$q\restriction(\max(\vec{\alpha}),\kappa)\leq^* q_{\max(\vec{\alpha})<}$ and $q\restriction\max(\vec{\alpha})\leq q_{\leq\max(\vec{\alpha})}$ such that $\langle q_{\leq\max(\vec{\alpha})},q_{\max(\vec{\alpha})<}\rangle$ is as wanted.

By recursion, define A^i_s for $s\in T_i\setminus mb(T_i)$. Let $s\in Lev_{ht(T_i)-1}(T_i)$, then we can shrink $\mathrm{Succ}_{T_i}(s)$ and find A^i_s such that for every $\alpha\in\mathrm{Succ}_{T_i}(s)$, $A(s^\frown\alpha)=A^i_s\cap\alpha$. Note that if $n_i\notin I_i$, then $A^i_s=A(s^\frown\alpha)$ for every $\alpha\in\mathrm{Succ}_{T_i}(s)$.

Generally, take $s\in T_i$ and assume that for every α is $\mathrm{Succ}_{T_i}(s)$, $A^i_{s^\frown\alpha}$ is defined. We can find a single A^i_s and shrink $\mathrm{Succ}_{T_i}(s)$ such that for every $\alpha\in\mathrm{Succ}_{T_i}(s)$, $A^i_{s^\frown\alpha}\cap\alpha=A^i_s\cap\alpha$.

Now we move to $V[A]$, for every i, define recursively ρ^i_k for $k\leq n_i:=ht(T_i)$. Let $\rho^i_0=\min(A\Delta A^i_{\langle\rangle})+1$.

$$\rho^i_{k+1}=\sup(\min(A\Delta A^i_{\langle\delta_1,...,\delta_k\rangle})+1\mid\delta_1<\rho^i_0,...,<\delta_k<\rho^i_k))$$

Let $\vec{c}_i\in mb(T_i)$ such that $p^{(*)\frown}\vec{c}_i\in G$, let us argue that for every $k\leq n_i$, $\rho^i_k>(\vec{c}_i)_k$.

By construction of the tree T_i, $A\cap c_i=A^i_{\vec{c}_i}\cap c_i$. now for every $j\leq n_i$,

$$A\cap(\vec{c}_i)_j=A^i_{\langle(\vec{c}_i)_0,...,(\vec{c}_i)_{j-1}\rangle}\cap(\vec{c}_i)_j$$

In particular, $A\cap(\vec{c}_i)_0=A_{\langle\rangle}\cap(\vec{c}_i)_0$. Since $A\cap\rho^i_0\neq A_{\langle\rangle}\cap\rho^i_0$, it follows that $(\vec{c}_i)_0<\rho^i_0$. Assume that $(\vec{c}_i)_j<\rho^i_j$ for every $j\leq k$. Since $A^i_{\langle(\vec{c}_i)_0,...,(\vec{c}_i)_k\rangle}\cap(\vec{c}_i)_{k+1}=A\cap(\vec{c}_i)_{k+1}$, then

$$(\vec{c}_i)_{k+1}<\min(A^i_{\langle(\vec{c}_i)_0,...,(\vec{c}_i)_k\rangle}\Delta A)\leq\rho^i_{k+1}$$

It remains to see that $\rho^i_k<\kappa$. Again by induction on k, $\rho^i_0<\kappa$ since $A\neq A_{\langle\rangle}$, as $A_{\langle\rangle}\in V[C_G\cap\lambda]$ but $A\notin V[C_G\cap\lambda]$.

Toward a contradiction assume that $\rho^i_{k+1}=\kappa$. Back to $V[C_G\cap\lambda]$, consider the collection

$$\{A^i_{\langle\alpha_0,...,\alpha_k\rangle}\mid\alpha_0<\rho^i_0,...,\alpha_k<\rho^i_k\}$$

Then for every $\gamma<\kappa$ pick any distinct $\vec{\alpha}_1,\vec{\alpha}_2$ such that $A^i_{\vec{\alpha}_1}\neq A^i_{\vec{\alpha}_2}$, but $A^i_{\vec{\alpha}_1}\cap\gamma=A^i_{\vec{\alpha}_2}\cap\gamma$. To see that there are such $\vec{\alpha}_1,\alpha_2$, by assumption that $\rho^i_{k+1}=\kappa$ there is $\vec{\alpha}_1$ such that $\eta_1:=\min(A\Delta A^i_{\vec{\alpha}_1})>\gamma$, hence $A^i_{\vec{\alpha}_1}\cap\gamma=A\cap\gamma$. Let $\vec{\alpha}_2$ be such that $\min(A\Delta A^i_{\vec{\alpha}_2})>\eta_1$. In particular, $A^i_{\vec{\alpha}_1}\neq A^i_{\vec{\alpha}_2}$, but $A^i_{\vec{\alpha}_1}\cap\gamma=A\cap\gamma=A^i_{\vec{\alpha}_2}\cap\gamma$. Since this is all in $V[C_G\cap\lambda]$, where κ is still measurable, then we can find unboundedly many γ's with the same $\vec{\alpha}_1,\vec{\alpha}_2$, which is clearly a contradiction.

So we found a sequence $\langle\rho^i_{n_i}\mid i<\lambda\rangle\in V[A]$ such that $\rho^i_{n_i}>c_i$. Let Z be the closure of $\{\rho_i\mid i<\lambda\}$. Since $\lambda>\omega$, there is some a limit $\alpha<\lambda$ such that $c_\alpha<\kappa$ is a limit point of Z.

To see the contradiction, note that on one hand, $A\cap c_\alpha\in V$, and therefore the set $Z\cap c_\alpha$ is defined in V, on the other hand, $c_\alpha>\lambda$ is measurable in V, and $|Z\cap c_\alpha|=\lambda$, contradiction.

For general A, if $\lambda := cf^V(\sup(A)) \leq \kappa$ then there is a fresh set $X \subseteq \lambda$ such that $V[A] = V[X]$. To see this, pick in V a cofinal sequence $\langle \eta_i \mid i < \lambda \rangle$ in $\sup(A)$. Then By κ^+-c.c, there is $F \in V$, such that

1. $Dom(F) = \lambda$.
2. For every $i < \lambda$, $|F(i)| = \kappa$.
3. $A \cap \eta_i \in F(i)$.

For each $i < \lambda$, find in V, an enumeration $\langle x^i_j \mid j < \kappa \rangle$ of $F(i)$, such that for every $W \in F(i)$, $\{j < \kappa \mid x^i_j = W\}$ is unbounded in κ.

Move to $V[A]$, inductively define $\langle \gamma_i \mid i < \lambda \rangle$ increasing such that $x^i_{\gamma_i} = A \cap \eta_i$.

Set $\gamma_0 = \min(j \mid x^0_j = A \cap \eta_0)$. Assume that γ_i was defined for every $i \leq k < \lambda$, define $\gamma_{k+1} = \min(j > \gamma_k \mid x^{k+1}_j = A \cap \eta_{k+1})$. Note that at limit stage δ, the sequence $\langle \gamma_i \mid i < \delta \rangle$ is definable using only the enumeration and $A \cap \eta_\delta$ which is all available in V. hence $\gamma'_\delta = \sup(\gamma_i \mid i < \delta) < \kappa$ and we define $\gamma_\delta = \min(j > \gamma'_\delta \mid x^\delta_j = A \cap \eta_\delta)$.

Let $X = \{\gamma_i \mid i < \lambda\} \subseteq \kappa$. Since $\langle \gamma_i \mid i < \lambda \rangle$ is increasing, $cf^{V[G]}(\sup(X)) = cf^{V[G]}(\lambda)$, $V[A] = V[X]$ and X is fresh. It follows by the proof for subsets of κ that $cf^{V[G]}(X) = \omega$, hence $cf^{V[G]}(\sup(A)) = \omega$.

Finally, if $\lambda \geq \kappa^+$, by theorems 5.17,5.21 there cannot be a fresh subset of with $cf(\sup(A)) \geq \kappa^+$.$\blacksquare$

6 Open problems

Let us conclude with some related open problems:

Distinguishing from the case where $o^{\vec{U}}(\kappa) < \kappa$, we do not have here a classification of the subforcings of $\mathbb{M}[\vec{U}]$.

Question 6.1 *What are the subforcings of* $\mathbb{M}[\vec{U}]$*?*

Using theorem 1.1, it remains to consider models of the form $V[C']$ for some $C' \subseteq C_G$, and try to classify the forcings which generates these models.

Our conjecture is the following

Conjecture 6.2 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic filter, where $o^{\vec{U}}(\kappa) = \kappa$. If $V \subseteq M \subseteq V[G]$ is a transitive ZFC model, then either it is a finite iteration of Magidor like forcings as in [3], or there is a tree $T \subseteq [\kappa]^{<\omega}$ in V such that $ht(T) = \omega$ and for every $t \in T$ and every $\alpha \in \mathrm{Succ}_T(t)$, there is a name $\underset{\sim}{\mathbb{M}[\vec{U}]}^*_{t^\frown\alpha}$ for a Magidor-like forcing, such that if H is V-generic filter for the forcing adding a branch through the tree T along with the forcings $\underset{\sim}{\mathbb{M}[\vec{U}]}^*_{t^\frown\alpha}$ corresponding to the branch, then $M = V[H]$.*

Question 6.3 *Suppose that $o^{\vec{U}}(\kappa) = \kappa^+$. Is still every set of ordinals in the extension equivalent to a subsequence of a generic sequence?*

Note that the situation here is more involved since κ stays regular in $V[G]$ and there is no way to identify the measure associated to a member of the generic C_G.

Question 6.4 *The same as 2, but with $o^{\vec{U}}(\kappa) \geq \kappa^+$.*

Question 6.5 *What can we say about other Prikry type forcing notions ?*

Probably the simplest would be to deal a long enough Magidor iteration of the Prikry forcings and to analyze its subforcings.

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