On tree-like scales

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Abstract

We answer questions on existence of tree-like scales asked by D. Adolf and O. Ben Neria [1].

1 Introduction

L. Pereira [8] introduced and studied an interesting basic notion of a tree-like scales. A model without continuous tree like scale at some product was constructed in [6]. Recently, D. Adolf and O. Ben Neria [1] returned to the subject and proved some very nice results. The purpose of this paper is to answer few questions asked by them.

2 Countable cofinality

L. Pereira showed, starting from a supercompact, that there are products $\prod_{n<\omega} \tau_n$ carring a continuous tree-like scale of length greater than $\sup(\{\tau_n \mid n < \omega\})^+$.
D. Adolf and O. Ben Neria [1] (Question 6) asked whether this is possible from weaker assumptions.
We show that this is indeed the case.

Assume GCH. Let η be a regular cardinal and μ an ordinal.
Define a forcing notion $Q(\eta, \mu)$ for adding μ functions from η to η which are tree-like.

Definition 2.1 $Q(\eta, \mu)$ consists of all functions f such that

1. f is a partial function from $\eta \times \mu \to \eta$ of cardinality less than η,

2. if $(\alpha, \beta), (\alpha, \gamma) \in \mathrm{dom}(f)$ and $f(\alpha, \beta) = f(\alpha, \gamma)$, then for every $\alpha' < \alpha$, $(\alpha', \beta) \in \mathrm{dom}(f)$ iff $(\alpha', \gamma) \in \mathrm{dom}(f)$. Moreover, if $(\alpha', \beta) \in \mathrm{dom}(f)$, then $f(\alpha', \beta) = f(\alpha', \gamma)$.

$Q(\eta,\mu)$ is ordered by the inclusion.

Further we will often identify $f(\alpha,\beta)$ with $f_\beta(\alpha)$.

Lemma 2.2 *$Q(\eta,\mu)$ is $<\eta-$closed forcing notion.*

Lemma 2.3 *$Q(\eta,\mu)$ satisfies η^+-c.c.*

Proof. Let $\{f^i \mid i<\eta^+\}\subseteq Q(\eta,\mu)$. Denote the set $\{\beta<\mu \mid \exists\alpha<\eta((\alpha,\beta)\in \mathrm{dom}(f^i))\}$ by a_i, for every $i<\eta^+$. By shrinking, if necessary, we may assume that the following hold for some $a\subseteq\mu$ and $\alpha^*<\eta$:

1. $\{a_i \mid i<\eta^+\}$ form a $\Delta-$system with the kernel a,
2. for every $i,j<\eta^+$, a_i,a_j are order isomorphic over a.
   Let π_{ij} denotes such isomorphism. Then for every $\beta\in a_i$, $f^i_\beta=f^j_{\pi_{ij}(\beta)}$.

Now, given $i,j<\eta^+$, $f^i\cup f^j$ will be in $Q(\eta,\mu)$ and will be stronger than both f^i and f^j. $\square$

Suppose now that E is a $(\kappa,\kappa^{++})-$extender.[1] We would like to use a slight variation of the Woodin method (see [2]) in order to force $2^\kappa=\kappa^{++}$, preserving measurability and insuring that after a further Prikry forcing there will be a continuous tree like scale.

Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha\le\kappa+1,\beta\le\kappa\rangle$ be the Easton support iteration of $Q(\eta,\eta^{++})$ over inaccessibles $\le\kappa$.

Let G be a generic subset of $P_{\kappa+1}$.

Denote the generic functions added by G over κ by $\langle f^\kappa_\beta \mid \beta<\kappa^{++}\rangle$. Let $G_\kappa=G\cap P_\kappa$. So, $V[G]=V[G_\kappa * \langle f^\kappa_\beta \mid \beta<\kappa^{++}\rangle]$.

Consider $j_E: V\to M_E\simeq \mathrm{Ult}(V,E)$. We would like to extend it to $V[G]$.

Let $E_\kappa=\{X\subseteq\kappa \mid \kappa\in j_E(X)\}$ and $j_{E_\kappa}: V\to M_{E_\kappa}\simeq\mathrm{Ult}(V,E_\kappa)$ be the corresponding elementary embedding.

We will need also $k: M_{E_\kappa}\to M_E$ defined by setting $k([x]_{E_\kappa})=j_E(x)(\kappa)$.

Next, as in [2], we build in $V[G]$ (together with its extension by $<\kappa^+-$closed forcing $\kappa^{++}-$c.c. forcing over $V[G_\kappa,\langle f^\kappa_\beta \mid \beta<(\kappa^{++})^{M_{E_\kappa}}\rangle])$ a generic set H'' over $M_{E_\kappa}[G\restriction(\kappa^{++})^{M_{E_\kappa}}]$.

We require that H'' agrees with $\langle f^\kappa_\beta \mid \beta<(\kappa^{++})^{M_{E_\kappa}}\rangle$ over κ.

In addition chose its part $\langle f''_\beta \mid \beta<j_{E_\kappa}(\kappa^{++})\rangle$ over $j_{E_\kappa}(\kappa)$ to be such that for every $\alpha<(\kappa^{++})^{M_{E_\kappa}}$, $f''_\alpha\restriction\kappa=f^\kappa_\alpha$.

[1] It is possible to start with $o(\kappa)=\kappa^{++}$ and to use [5] instead.

The embedding k is used to move such H'' to H' over $M_E[G]$. Also require that H' agrees with $\langle f^{\kappa}_{\beta} \mid \beta < \kappa^{++}\rangle$ over κ.
Let us denote by $\langle f'_{\beta} \mid \beta < j_E(\kappa^{++})\rangle$ the functions of H' over $j_E(\kappa)$.
By elementarity, then for every $\alpha < \kappa^{++}$, $f'_{\alpha} \restriction \kappa = f^{\kappa}_{\alpha}$.
We would like to change $\langle f'_{\beta} \mid \beta < j_E(\kappa^{++})\rangle$ to $\langle f^{*}_{\beta} \mid \beta < j_E(\kappa^{++})\rangle$ such that for every $\gamma < \kappa^{++}$,

$\aleph$. $f^{*}_{j_E(\gamma)} \restriction \kappa = f^{\kappa}_{\gamma}$,

$\beth$. $f^{*}_{j_E(\gamma)}(\kappa) = \gamma$.

The first condition is just the usual master condition requirement. The second one will be used further to insure that a tree-like continuous scale is constructed.

Similar to the Woodin original construction, in each condition q of
the forcing $Q(j_E(\kappa), j_E(\kappa^{++}))$ only κ many places should be altered and the closure of the forcing allows this.
Note that here changing one of the functions of q may require to make the same change in all other functions in q which agree with the changed one.

Let H^* be such changed H'.

Consider $j^* : V[G] \to M_E[H^*]$. It extends j_E. Define

$$U = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}.$$

It is a normal ultrafilter over κ and actually $j^* = j_U$.
Consider the functions $\langle f^{\kappa}_{\beta} \mid \beta < \kappa^{++}\rangle$.
By elementarity and the condition ($\beth$) above, for every $\beta, \gamma < \kappa^{++}$, we have the following:

1. $\{\nu < \kappa \mid f^{\kappa}_{\beta}(\nu) < \nu^{++}\} \in U$,

2. $\beta < \gamma$ implies that $\{\nu < \kappa \mid f^{\kappa}_{\beta}(\nu) < f^{\kappa}_{\gamma}(\nu)\} \in U$,

3. if β is a limit ordinal of cofinality $< \kappa$ and $\langle \beta_{\xi} \mid \xi < \mathrm{cof}(\beta)\rangle$ is an increasing cofinal in β sequence, then
   $\{\nu < \kappa \mid \langle f^{\kappa}_{\beta_{\xi}} \mid \xi < \mathrm{cof}(\beta)\rangle$ is an increasing cofinal sequence in $f^{\kappa}_{\beta}(\nu)\} \in U$.

The last property insures the continuity.

Let now $h : \kappa \to \kappa$ be such that for every inaccessible $\nu < \kappa$, $h(\nu) < \nu^{++}$. Consider $\tau = j^*(h)(\kappa)$. Then $\tau < \kappa^{++}$. Recall that we have $j^*(f^{\kappa}_{\tau+1})(\kappa) = \tau + 1$, so $h < f^{\kappa}_{\tau+1}$ mod U.

Let us force with the Prikry forcing with U. Let $\langle \kappa_n \mid n < \omega \rangle$ be the corresponding Prikry sequence.
Define a scale $\langle t_\beta \mid \beta < \kappa^{++} \rangle$ in $\prod_{n<\omega} \kappa_n^{++}$.
Just set $t_\beta(n) = f_\beta^\kappa(\kappa_n)$, if $f_\beta^\kappa(\kappa_m) < \kappa_m^{++}$, for every $m \geq n$ and let it be 0 otherwise.

The properties above insure that such $\langle t_\beta \mid \beta < \kappa^{++} \rangle$ will be a tree-like continuous scale.

3 Uncountable cofinality

D. Adolf and O. Ben Neria [1], Question 3, asked about existence of continuous tree-like scales for uncountable cofinality.
We will show that it is possible to have such scales even with $\neg$SCH.

Let $\vec{E} = \langle E(\mu, \tau) \mid \mu \leq \kappa, \tau < o^{\vec{E}}(\mu) \rangle$ be a coherent sequence of $(\mu, \mu^{++})-$extenders and for some regular length $\delta < \kappa$, $o^{\vec{E}}(\kappa) = \delta$.

We use the forcing of the previous section to blow up powers of $\mu \leq \kappa$ to μ^{++} and extending $\vec{E}$ to such generic extension. An additional Cohen function is forced to pin out starting points for the additional forcings over μ^+'s. The result will be a coherent sequence of measures $\vec{U}$. We force finally with the Magidor forcing with $\vec{U}$ and change the cofinality of κ to δ.

4 Non-existence of tree-like scales for uncountable cofinality

A model with no continuous tree-like scales in a product $\prod_{n<\omega} \eta_n$ for some sequence of regular cardinals $\langle \eta_n \mid n < \omega \rangle$ was consructed in [6]. The SCH breaks at $\bigcup_{n<\omega} \eta_n$ in this model.
In [1] a different model was constructed. SCH holds there and the initial assumptions are optimal.

D. Adolf and O. Ben Neria [1] asked whether it is possible to show non-existence for uncountable cofinality.

First note that the method of [6] can be applied to the Merimovich Extender Based Magidor forcing, replacing the Extender Based Prikry forcing in [6]. In the resulting model SCH breaks down.

O. Ben Neria asked if it is possible to keep SCH and to use weaker assumptions.
We will show that it is possible. The argument is heavily based on those of [1].

Let us deal for simplicity with cofinality ω_1. The general case is similar.
Our initial assumption will be the following:

there is a cardinal λ *of cofinality* ω_1 *such that the set* $\{o(\mu) \mid \mu < \lambda\}$ *is unbounded in* λ.

By [1], Theorem 5, this assumption seems to be optimal.

We assume GCH and let λ be the least cardinal of cofinality ω_1 such that for a coherent sequence $\vec{U}$ of ultrafilters the set $\{o^{\vec{U}}(\mu) \mid \mu < \lambda\}$ is unbounded in λ.[2]

Define now an ω_1-sequence of cardinals below λ Let λ_0^0 be a measurable cardinal below λ. Pick $\lambda_0^1 < \lambda$ to be a measurable cardinal above λ_0^0 with $o^{\vec{U}}(\lambda_0^1) = \lambda_0^0$. Set $\lambda_0 = (\lambda_0^1)^+$. Continue by induction on $i < \omega_1$. Let $\lambda_i^0 < \lambda$ to be a measurable cardinal above $\bigcup_{j<i} \lambda_j$ with $o^{\vec{U}}(\lambda_i^0) =$ the first measurable above $\bigcup_{j<i} \lambda_j$. Pick $\lambda_i^1 < \lambda$ to be a measurable cardinal above λ_i^0 with $o^{\vec{U}}(\lambda_i^1) = \lambda_i^0$. Set $\lambda_i = (\lambda_i^1)^+$.

As in [1] we iterate now the forcing of [3] to turn the Mitchell order into the Rudin-Keisler and this way to generate extenders E_i^j over λ_i^j, $i < \omega_1, j < 2$. Denote the length of E_i^j by κ_i^j. So, $\kappa_i^1 = \lambda_i^0$.

Next step will be to iterate Prikry type forcings $\langle Q_i, \leq, \leq^* \rangle, i < \omega_1$ defined below. We use the Magidor ($\leq^*$ −full support) iteration for this, see [4].

Each Q_i will be a simplified variant of the short extender forcing similar to the one used in [1]. It will have only two levels instead of ω.
Namely E_i^0 is used at the first level and E_i^1 at the second. Cohen functions at both levels act from λ_i, and so, their parts inside conditions have cardinality $< \lambda_i$.

Definition 4.1 Q_i consists of conditions q_i such that either

1. $q_i = \langle f_i^0, f_i^1 \rangle$, where
   (a) f_i^0 is partial function from λ_i to λ_i^0 with $\mathrm{dom}(f_i^0) \in \lambda_i$ a successor ordinal,
   (b) f_i^1 is partial function from λ_i to λ_i^1 with $\mathrm{dom}(f_i^1) \in \lambda_i$ a successor ordinal,
   (c) $\mathrm{dom}(f_i^0) = \mathrm{dom}(f_i^1)$,
   (d) for every limit $\alpha \in \mathrm{dom}(f_i^0)$, if $\omega_1 < \mathrm{cof}(\alpha) < \kappa_i^0$, then there is a club C in α such that for every $\beta < \gamma, \beta, \gamma \in C$, we have $f_i^0(\beta) < f_i^0(\gamma)$, $f_i^0(\alpha) = \bigcup_{\beta \in C} f_i^0(\beta)$ and $f_i^1(\beta) < f_i^1(\gamma)$,$f_i^0(\alpha) = \bigcup_{\beta \in C} f_i^0(\beta)$.
   This condition will be used in order to insure that the final generis scale will be a very good scale.

[2]It is possible to do a similar construction with λ which is not the first. The only change would be that the sequence defined below should be made cofinal in λ.

Or

2. $q_i = \langle f_i^0, \langle a_i^1, A_i^1, f_i^1 \rangle \rangle$, where

   (a) f_i^0 is partial function from λ_i to λ_i^0 with $\mathrm{dom}(f_i^0) \in \lambda_i$ a successor ordinal,

   (b) for every limit $\alpha \in \mathrm{dom}(f_i^0)$, if $\omega_1 < \mathrm{cof}(\alpha) < \kappa_i^0$, then there is a club C in α such that for every $\beta < \gamma, \beta, \gamma \in C$, we have $f_i^0(\beta) < f_i^0(\gamma)$ and $f_i^0(\alpha) = \bigcup_{\beta \in C} f_i^0(\beta)$.

   (c) $\langle a_i^1, A_i^1, f_i^1 \rangle$ as in the short extender forcing with E_i^1,
   in addition we require that

      i. $\mathrm{dom}(f_i^0) = \mathrm{dom}(a_i^1) \cup \mathrm{dom}(f_i^1)$,

      ii. if $\alpha \in \mathrm{dom}(a_i^1)$ is a non-limit point of $\mathrm{dom}(a_i^1)$, then either it is a successor ordinal or it is a limit ordinal of cofinality $\geq \kappa_i^0$,

      iii. for every limit $\alpha \in \mathrm{dom}(f_i^1)$, if $\omega_1 < \mathrm{cof}(\alpha) < \kappa_i^0$, then there is a club C in α[3] such that for every $\beta < \gamma, \beta, \gamma \in C$, we have $f_i^1(\beta) < f_i^1(\gamma)$ and $f_i^0(\alpha) = \bigcup_{\beta \in C} f_i^0(\beta)$.

Or

3. $q_i = \langle \langle a_i^0, A_i^0, f_i^0 \rangle, \langle a_i^1, A_i^1, f_i^1 \rangle \rangle$, where

   (a) $\langle \langle a_i^0, A_i^0, f_i^0 \rangle, \langle a_i^1, A_i^1, f_i^1 \rangle \rangle$ as in the short extender forcing with E_i^0, E_i^1,
   in addition we require that

      i. $\mathrm{dom}(a_i^0) \cup \mathrm{dom}(f_i^0) = \mathrm{dom}(a_i^1) \cup \mathrm{dom}(f_i^1)$,

      ii. if $\alpha \in \mathrm{dom}(a_i^j)$ is a non-limit point of $\mathrm{dom}(a_i^j)$, then either it is a successor ordinal or it is a limit ordinal of cofinality $\geq \kappa_i^0$, for any $j < 2$,

      iii. for every $j < 2$ and for every limit $\alpha \in \mathrm{dom}(f_i^j)$, if $\omega_1 < \mathrm{cof}(\alpha) < \kappa_i^0$, then there is a club C in α such that for every $\beta < \gamma, \beta, \gamma \in C$, we have $f_i^j(\beta) < f_i^j(\gamma)$ and $f_i^j(\alpha) = \bigcup_{\beta \in C} f_i^j(\beta)$.

This version of the forcing shares all the usual properties, only the closure should be replaced by the strategic closure with Good Player playing at limit stages.

Let P denotes such an iteration of Q_i's. We assume that for every $q = \langle q_i \mid i < \omega_1 \rangle$ for all but finitely many i's $\ell(q_i) = 0$.
Let $G \subseteq P$ be a generic.

[3] Note that $\mathrm{dom}(a_i^1)$ is a closed set. So $\mathrm{dom}(a_i^1)$ must be bounded in α.

Denote by τ_i^j the largest indiscernible, for every block $i < \omega_1$ and level $j < 2$. Let $\langle t_{i\beta} \mid \beta < \lambda^i \rangle$ be the corresponding generic functions in $\tau_i^0 \times \tau_i^1$.

Fix a very good scale $\langle h_\alpha \mid \alpha < \lambda^+ \rangle$ in $\prod_{i<\omega_1} \lambda_i / bounded$.[4]

Set $\mathfrak{b} = \{\lambda_i \mid i < \omega_1\}$. We will often will not distinguish between $\prod_{i<\omega_1} \lambda_i$ and $\prod \mathfrak{b}$, as well as between i and λ_i.

Set $\mathfrak{a} = \mathfrak{b} \cup \{\tau_i^j \mid i < \omega_1, j < 2\}$.

Now let us extend $\langle h_\alpha \mid \alpha < \lambda^+ \rangle$ to a continuous scale $\langle g_\alpha \mid \alpha < \lambda^+ \rangle$ in $\prod \mathfrak{a}/bounded$ using t_i^j's.

Proceed as follows.

Set $g_\alpha \restriction \mathfrak{b} = h_\alpha$ and $g_\alpha(\tau_i^j) = t_{ih_\alpha(\lambda_i)}(j)$, for every $i < \omega_1, j < 2$.

Lemma 4.2 *$\langle g_\alpha \mid \alpha < \lambda^+ \rangle$ is a scale in $\prod \mathfrak{a}/bounded$.*

Proof. Let $f \in \prod \mathfrak{a}$. Work in V. Let $\underset{\sim}{f}$ be a name of f and p a condition. Then, by standard arguments (Prikry condition), there is $q = \langle\langle q_i^0, q_i^1 \rangle \mid i < \omega_1 \rangle \geq^* p$ and a function $r \in \prod \mathfrak{b}$ such that for every $i < \omega_1$ the following hold:

1. $q \Vdash \underset{\sim}{f}(\lambda_i) < r(\lambda_i)$,
2. for every $\nu_0 \in A^{q_i^0}$, $q^\frown \nu_0 \parallel \underset{\sim}{f}(\underset{\sim}{\tau}_i^0)$,
3. for every $\langle \nu_0, \nu_1 \rangle \in A^{q_i^0} \times A^{q_i^1}$, $q^\frown \langle \nu_0, \nu_1 \rangle \parallel \underset{\sim}{f}(\underset{\sim}{\tau}_i^1)$.

Pick $\alpha < \lambda^+$ such that $h_\alpha > r$ mod bounded and in addition $h_\alpha(\lambda_i) > \sup(\mathrm{dom}(a_i^0(q_i^0) \cup \mathrm{dom}(a_i^1(q_i^1) \cup \mathrm{dom}(f_i^0(q_i^0) \cup \mathrm{dom}(f_i^1(q_i^1)$. Suppose, for simplification of the notation, that everywhere.

Now, we extend q to $s = \langle\langle s_i^0, s_i^1 \rangle \mid i < \omega_1 \rangle \geq^* q$ such that for every $i < \omega_1$,

1. $mc(\mathrm{dom}(a_i^0(s)) = mc(\mathrm{dom}(a_i^1(s))$,
2. $h_\alpha(\lambda_i) \in \mathrm{dom}(a_i^0(s))$,
3. for every $\nu_0 \in A^{s_i^0}$, $s^\frown \nu_0 \Vdash \underset{\sim}{f}(\underset{\sim}{\tau}_i^0) < \nu_0$,
4. for every $\langle \nu_0, \nu_1 \rangle \in A^{s_i^0} \times A^{s_i^1}$, $s^\frown \langle \nu_0, \nu_1 \rangle \Vdash \underset{\sim}{f}(\underset{\sim}{\tau}_i^1) < \nu_1$.

[4]It may be a need to force fist $\square_{\lambda^+}$ in order to get such a scale. Also note that any new function in this product is dominated by one from V.

Find $\beta, \alpha \leq \beta < \lambda^+$ such that h_β dominates the function $\lambda_i \mapsto mc(\mathrm{dom}(a_i^0(s)))$. Then s will force that g_β will dominate f.

$\square$

Lemma 4.3 *$\langle g_\alpha \mid \alpha < \lambda^+\rangle$ is a very good scale in $\prod \mathfrak{a}$/bounded.*

Proof. Definition 4.1 insures this. Just intersect all relevant clubs.

$\square$

Let us show the following:[5]

Lemma 4.4 *Let $F : \lambda \to \lambda$ be a function in $V[G]$ such that for every $i < \omega_1$, $F \restriction \lambda_i^1 : \lambda_i^1 \to \lambda_i^0$. Let $p \in G$ be a condition which forces this. Assume for simplicity that $\ell(p_i) = 0$, for every $i < \omega_1$.*
Then there are $p^* \geq^* p$, a function f and, every $i < \omega_1$, for $\xi < \lambda_i^1$ a maximal antichain $Z_{i\xi}$ in $P \restriction i$ above $p^* \restriction i$ such that the following holds:
if $z \in Z_{i\xi}, \langle \nu_0, \nu_1\rangle \in A_i^0(p^*) \times A_i^1(p^*)$, then $z^\frown(p_i^\frown\langle\nu_0,\nu_1\rangle)^\frown p^* \setminus i \Vdash \underset{\sim}{F}(\xi) = f(\xi, z, \nu_0, \nu_1)$.

Proof. The proof is standard. The only point is that after $\langle \nu_0, \nu_1\rangle$ are added the forcing Q_i is $< \lambda_i = (\lambda_i^1)^+ -$strategically closed, and so, we can accumulate all the decisions made into a single condition.

$\square$

We would like to use the following weakening of ABSP of [1].

Definition 4.5 ABSP$'$ with respect to $\prod \mathfrak{a}$ asserts that there exists a club $C \subseteq \mathcal{P}_\lambda(H_\theta)$ of structures $N \prec H_\theta$ so that for every internally approachable $N \in C$ there exists $\mu < \omega_1$ such that for every $F \in N$, $F : \lambda \to \lambda$ and every $i, \mu \leq i < \omega_1$, if $F(\chi_N(\tau_i^1)) < \tau_i^0$, then $F(\chi_N(\tau_i^1)) < \chi_N(\tau_i^0)$.

The proof of Lemma 15 ((3(ii)) of [1] gives the following:

Lemma 4.6 *ABSP$'$ implies that there is now continuous tree-like scale in $\prod \mathfrak{a}$.*

Similar to Theorem 29 of [1], let us show:

Lemma 4.7 *ABSP$'$ with respect to $\prod \mathfrak{a}$ holds in $V[G]$.*

[5]It is a weak form of Lemma 23 of [1] adapted to the present situation.

Proof. Suppose otherwise. Then there is a stationary $S \subseteq \mathcal{P}_\lambda(H_\theta)$ of internally approachable elementary substructures such that for every $N \in S$ and $\mu < \omega_1$ there is a function $F^N_\mu : \lambda \to \lambda$ and $i^N_\mu, \mu \le i^N_\mu < \omega_1$ which satisfy

$$\chi_N(\tau^0_{i^N_\mu}) \le F(\chi_N(\tau^1_{i^N_\mu})) < \tau^0_{i^N_\mu}.$$

As in [1], then there is $S^* \subseteq \lambda^+$ stationary and two fixed sequences $\langle F_\mu \mid \mu < \omega_1\rangle, \langle i_\mu \mid \mu < \omega_1\rangle$, such that for every $\delta \in S^*$ there exists $N \in S$ so that

1. $\delta = \chi_N(\lambda^+)$,
2. $\langle F_\mu \mid \mu < \omega_1\rangle = \langle F^N_\mu \mid \mu < \omega_1\rangle$,
3. $\langle i_\mu \mid \mu < \omega_1\rangle = \langle i^N_\mu \mid \mu < \omega_1\rangle$.

For each $\delta \in S^*$ there are $m_\delta < \omega_1$ and $N \in S$ such that for every $\mu, m_\delta \le \mu < \omega_1, j < 2$, $g_\delta(\tau^j_\mu) = \chi_N(\tau^j_\mu)$.
So,

$$g_\delta(\tau^0_{i_\mu}) \le F_\mu(g_\delta(\tau^1_{i_\mu})) < \tau^0_{i_\mu}.$$

Now, in V, let p be a condition which forces the above. Apply Lemma 4.4 repeatedly to functions $\langle \underset{\sim}{F}_\mu \mid \mu < \omega_1\rangle$ and get a single $p' \ge^* p$ and functions $\langle f_\mu \mid \mu < \omega_1\rangle$ which satisfy the conclusion of the lemma.
Also, by shrinking if necessary, we can assume that p' decides $\langle i_\mu \mid \mu < \omega_1\rangle$.
Pick now $\alpha < \lambda^+$ such that for all but boundedly many $i < \omega_1$, $h_\alpha(i) > \mathrm{dom}(a^1_i(p')) \cup \mathrm{dom}(f^1_i(p'))$. Suppose for simplisity that this holds for every i. Extend p' to $p^* \ge^* p'$ such that $h_\alpha(i) = mc(\mathrm{dom}(a^1_i(p^*)))$, for every $i \ge \ell(p^*)$.

Let $q \ge p^*$ be such that for some $\delta > \alpha$, $q \Vdash \delta \in \underset{\sim}{S}^*$. By taking further direct extension of q if necessary, we can assume that q already decides m_δ.
There is some $\gamma < \omega_1$ such that $h_\delta(\beta) > h_\alpha(\beta)$, for every $\beta, \gamma \le \beta < \omega_1$.
Again, by taking further direct extension of q if necessary, we can assume that starting with some $\eta, \max(m_\delta, \gamma) < \eta < \omega_1$, we have $\ell(q_i) = 0$ and $h_\alpha(i), h_\delta(i) \in \mathrm{dom}(a(q^0_i))$.
Finally, consider $i_\eta \ge \eta$.
Recall that we have

$$g_\delta(\tau^0_{i_\eta}) \le F_\eta(g_\delta(\tau^1_{i_\eta})) < \tau^0_{i_\eta},$$

in $V[G]$ and, in V, for every $\xi < \lambda^1_{i\eta}$ there is a maximal antichain $Z_{i_\eta\xi} \subseteq P \restriction i_\eta$ above $p^* \restriction i$ such that the following holds:

$$\text{if } z \in Z_{i_\eta \xi}, \langle \nu_0, \nu_1 \rangle \in A^0_{i_\eta}(p^*) \times A^1_{i_\eta}(p^*),$$

$$\text{then } z^\frown (p_{i_\eta}{}^\frown \langle \nu_0, \nu_1 \rangle)^\frown p^* \setminus i_\eta \Vdash \underset{\sim}{F}(\xi) = f(\xi, z, \nu_0, \nu_1).$$

Pick now $\zeta \in A^1_{i_\eta}(q)$. Let ξ be its projection to $h_\delta(i_\eta)$, i.e. $\xi = \pi_{mc(a^1_{i_\eta}(q))h_\delta(i_\eta)}(\zeta)$ and let $\nu_1 \in A^1_{i_\eta}(p^*)$ be the projection of ξ to $h_\alpha(i_\eta)$.
Then, for every $z \in Z_{i_\eta \xi}, \nu_0 \in A^0_{i_\eta}(p^*)$,

$$z^\frown (p_{i_\eta}{}^\frown \langle \nu_0, \nu_1 \rangle)^\frown p^* \setminus i_\eta \Vdash \underset{\sim}{F}(\xi) = f(\xi, z, \nu_0, \nu_1).$$

By shrinking $A^0_{i_\eta}(p^*)$ if necessary, we can assume that the element z is constant, since $|Z_{i_\eta \xi}|$ is relatively small.
Define a function $s : A^0_{i_\eta}(p^*) \to \lambda^0_{i_\eta}$ by setting

$$s(\nu_0) = f(\xi, z, \nu_0, \nu_1).$$

Now, $a^0_{i_\eta}(q)(h_\alpha(i_\eta)) < a^0_{i_\eta}(q)(h_\delta(i_\eta))$ and both are generators of the extender $E^0_{i_\eta}$, hence $j_{E^0_{i_\eta}}(s)(h_\alpha(i_\eta)) < h_\delta(i_\eta)$.
Pick now any $\rho \in A^0_{i_\eta}(q)$. Let $\nu_0 \in A^0_{i_\eta}(p^*)$ be its projection to $h_\alpha(i_\eta)$.
But then

$$p \leq^* p^* \leq z^\frown (q_{i_\eta}{}^\frown \langle \rho, \xi \rangle)^\frown q \setminus i_\eta \Vdash \underset{\sim}{F}(\xi) = f(\xi, z, \nu_0, \nu_1) = s(\nu_0) < \nu_1 = \underset{\sim}{g}_\delta(\tau^0_{i_\eta})$$

$$\text{and } \xi = \underset{\sim}{g}_\delta(\tau^1_{i_\eta}).$$

Contradiction.
□

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