Some examples of Tukey order on normal ultrafilters

Moti Gitik*

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Abstract

We show consistency from a single measurable of the following statements:

1. there are many normal not Tukey equivalent ultrafilters[1],
2. there are different normal Tukey equivalent normal ultrafilters,
3. there are two normal ultrafilters such that one is strictly above an other in the Tukey order.

1 A model with many normal not Tukey equivalent ultrafilters.

Assume GCH and let κ be a measurable cardinal. Let U be a normal ultrafilter over κ.

Define an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle.$$

Let $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal.
If $\beta < \kappa$ is an inaccessible cardinal then set $Q_\beta = Cohen(\beta)$.

Let G be generic subset of $P_{\kappa+1}$. For every inaccessible $\beta \leq \kappa$, let $f_\beta : \beta \to 2$ denotes the Cohen function added by G. The embedding $j = j_U : V \to M_U = M$ extends to $j^* : V[G] \to M_U[G^*]$ in a standard fashion in $V[G]$.
Set

$$U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}.$$

Then

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[1]Existence of normal Tukey non-equivalent ultrafilters was proved first by T. Benhamow and N. Dobrinen [1]. Here we construct concrete examples of this phenomena.

1. $U^* \supseteq U$,

2. $j_{U^*} = j^*$

3. $M_{U^*} = M_U[G^*]$.

We have $j_U(P) = P_\kappa * P_{(\kappa, j_U(\kappa))}$.

Let us now define other normal ultrafilters, but this time not in $V[G]$ but rather in its generic extension.

Consider the forcing $R = (Cohen(j(\kappa)))^{M[G^* \restriction P_{j(\kappa)}]}$ over $V[G]$.
It is equivalent to $Cohen(\kappa^+)$. So, forcing with it does not add new subsets of κ, and hence U^* remains a normal ultrafilter in the extension.
Let $G(R)$ be a generic subset of R over $V[G]$ with $f_\kappa \in G(R)$. Then j extends to

$$j_{G(R)} : V[G, G(R)] \to M[G^* \restriction P_{j(\kappa)} * G(R) * H^*],$$

where H^* is generated by $j''G(R)$.
Set

$$U^{G(R)} = \{X \subseteq \kappa \mid \kappa \in j_{G(R)}(X)\}.$$

Then

1. $U^{G(R)} \supseteq U$,

2. $j_{U^{G(R)}} = j_{G(R)}$,

3. $M_{U^{G(R)}} = M_U[G^* \restriction P_{j(\kappa)} * G(R) * H^*]$.

The crucial point here is that $G(R)$ is reconstructible from $U^{G(R)}$, and so, $U^{G(R)} \notin V[G]$.

Lemma 1.1 *$\neg(U^{G(R)} \leq_T U^*)$.*

Proof. Suppose otherwise. Then, by Theorem 4.1 of [1], there is $h : [\kappa]^{<\kappa} \to [\kappa]^{<\kappa}$ a function which determines a continuous cofinal map from U^* to $U^{G(R)}$. But note that $h \in V[G]$, since R does not add new subsets to κ. Then, also $U^{G(R)}$ in $V[G]$, which is impossible.
□

Let us use the same idea in order to produce many incompatible normal ultrafilters. Instead of adding a single $G(R)$, we add many mutually generics.

Fix some λ and force with $Cohen(\kappa^+, \lambda)$. Let $\vec{g} = \langle g_\alpha \mid \alpha < \lambda \rangle$ be a generic Cohen functions. Translate them to $\vec{H} = \langle H_\alpha \mid \alpha < \lambda \rangle$ generics for R.

Now, for every $\alpha < \lambda$, define $U_\alpha = U_{H_\alpha}$ as above.
Again, H_α will be reconstructible from U_α over $V[G]$.

Lemma 1.2 *For every $\alpha, \beta < \lambda$, $\neg(U_\alpha \leq_T U_\beta)$.*

Proof. Suppose otherwise. Say, $U_\alpha \leq_T U_\beta$. Then, by Thm.4.1 of [1], there is $h : [\kappa]^{<\kappa} \to [\kappa]^{<\kappa}$ a function which determines a continuous cofinal map from U_β to U_α. But note that $h \in V[G]$, since $Cohen(\kappa^+, \lambda)$ does not add new subsets to κ. Hence, $U_\alpha \in V[G.H_\beta]$. Then, also H_α in $V[G, H_\beta]$, which is impossible.
□

2 Normal Tukey equivalent ultrafilters

Here would like to construct a model with two normal Tukey equivalent ultrafilters.

Start with some general considerations.

Let U_0 and U_1 be two normal ultrafilters over κ. Consider $W = U_0 \cap U_1$. It is a normal filter over κ.

Suppose that $A_0 \in U_0 \setminus U_1$. Let $A_1 = \kappa \setminus A_0$.
Suppose that A_0 with W generate U_0 and A_1 with W generate U_1.
Note that the fact that A_0 with W generate U_0 does not necessary implies that A_1 with W generate U_1, as will be shown below.
Then,

$$U_i = \{X \subseteq \kappa \mid \exists Y \in W \quad X \subseteq Y \cap A_i\}, i < 2.$$

So, the family

$$\tilde{U}_i = \{Y \cap A_i \mid Y \in W\}$$

is cofinal in U_i, $i < 2$.

Suppose that $U_0 \leq_T U_1$, then, by [2], there is a cofinal monotonic map $H : \tilde{U}_0 \to \tilde{U}_1$. Simplest possibility for such type function would be the function

$$Y \cap A_0 \mapsto Y \cap A_1.$$

However it need not be cofinal.

Lemma 2.1 *Let $\langle B_\alpha \mid \alpha < \kappa^+\rangle$ be a cofinal family in W.*
Suppose that there is $\alpha^* < \kappa^+$ such that for every $X \in W$ there is $Y \in W, Y \cap A_0 \subseteq X \cap A_0$ such that $H(Y \cap A_0) \not\subseteq B_{\alpha^*}$.
Then H is not cofinal.

Proof. For every $X \in W$ we pick $Y_X \in W, Y_X \cap A_0 \subseteq X \cap A_0$ such that $H(Y_X \cap A_0) \nsubseteq B_{\alpha^*}$. Let

$$S = \{Y_X \cap A_0 \mid X \in W\}.$$

Then S is cofinal. However, $H''S$ is not a cofinal subset, as witnessed by B_{α^*}.
□

Hence, in order for H being cofinal the following condition must be satisfied:

$(*)$: *for every $\alpha < \kappa^+$, there is $X_\alpha \in W$, such that for every $Y \in W$, if $Y \cap A_0 \subseteq X_\alpha$, then $H(Y \cap A_0) \subseteq B_\alpha$.*

By replacing X_α with $X_\alpha \cap B_\alpha$, if necessary, we can require in $(*)$ that $X_\alpha \subseteq B_\alpha$.

Suppose that there is $I \subseteq \kappa^+, |I| = \kappa$ such

$$\bigcap_{\alpha \in I} B_\alpha \in U_0 \text{ and } \bigcap_{\alpha \in I} X_\alpha \in U_0.$$

Then $\bigcap_{\alpha \in I} X_\alpha$ contains a set of the form $Y \cap A_0$, for some $Y \in W$. So, $\bigcap_{\alpha \in I} B_\alpha \supseteq H(Y \cap A_0)$. We have $H(Y \cap A_0) \in U_1$. Hence, $\bigcap_{\alpha \in I} B_\alpha \in U_1$. This implies that $\bigcap_{\alpha \in I} B_\alpha \in W$. Hence,

$$\bigcap_{\alpha \in I} B_\alpha \in U_0 \setminus U_1 \text{ implies that } \bigcap_{\alpha \in I} X_\alpha \in W^*,$$

where W^* is the ideal dual to W.

Theorem 2.2 *Suppose there are a cofinal in W family $\{B_\alpha \mid \alpha < \kappa^+\}$ and a family $\{X_\alpha \mid \alpha < \kappa^+\}$ such that*

1. *$X_\alpha \subseteq B_\alpha$, $\alpha < \kappa^+$,*
2. *$X_\alpha \in W$, $\alpha < \kappa^+$,*
3. *if $\beta < \alpha < \kappa^+$, then $B_\beta \cap A_0 \nsubseteq X_\alpha$,*
4. *for every $I \in [\kappa^+]^\kappa$, if $\bigcap_{\alpha \in I} B_\alpha \in U_0 \setminus U_1$, then $\bigcap_{\alpha \in I} X_\alpha \in W^*$.*

Then there is a cofinal map from a cofinal subset $\{B_\alpha \cap A_0 \mid \alpha < \kappa^+\}$ of U_0 to U_1.

Proof. Our settings imply that the family $\{B_\alpha \cap A_i \mid \alpha < \kappa^+\}$ is cofinal in $U_i, i < 2$.

Define H on $\{B_\alpha \cap A_0 \mid \alpha < \kappa^+\}$ by induction.

Suppose that for every $\beta < \alpha$, $H(B_\beta \cap A_0)$ is defined. Define $H(B_\alpha \cap A_0)$. Consider

$I_\alpha = \{\beta < \alpha \mid B_\alpha \cap A_0 \subseteq X_\beta\}$. Then, $\bigcap_{\beta \in I_\alpha} X_\beta \supseteq B_\alpha \cap A_0$, and so it is not in W^*. Hence, $\bigcap_{\beta \in I_\alpha} B_\beta \in U_0 \cap U_1 = W$. Set

$$H(B_\alpha \cap A_0) = A_1 \cap \bigcap_{\beta \in I_\alpha} B_\beta.$$

Let S be a cofinal subfamily of $\{B_\alpha \cap A_0 \mid \alpha < \kappa^+\}$. Suppose that its image is not cofinal. Then there is $\beta < \kappa^+$ such that $B_\beta \cap A_1 \not\supseteq H(B_\alpha \cap A_0)$, for every α with $B_\alpha \cap A_0 \in S$.

Claim 1 *There is $\alpha, \beta < \alpha < \kappa^+$ such that $B_\alpha \cap A_0 \in S$ and $B_\alpha \cap A_0 \subseteq X_\beta$.*

Proof. Assume that $|\{\gamma \leq \beta \mid B_\gamma \cap A_0 \in S\}| = \kappa$. Otherwise, just take the intersection of all of them, and then proccedd as below. Let $\langle B_{\gamma_i} \cap A_0 \mid i < \kappa\rangle$ be an enumeration of the set $\{B_\gamma \cap A_0 \in S \mid \gamma \leq \beta\}$. Set $C = \Delta_{i<\kappa} B_{\gamma_i} \cap A_0$. Then $C \subseteq^* B_\gamma \cap A_0$, for every $\gamma \leq \beta, B_\gamma \cap A_0 \in S$. By normality, $C \in U_0$. Split C into two disjoint sets C_0, C_1. Then one of them, say C_0, in U_0. So, $C_0 \subseteq^* B_\gamma \cap A_0$ and $|B_\gamma \cap A_0 \setminus C_0| = \kappa$, for every γ as above. Set $C^* = C_0 \cap X_\beta$. Pick $\alpha < \kappa^+$ such that $B_\alpha \cap A_0 \subseteq C^*$ and $B_\alpha \cap A_0 \in S$. Then $\alpha > \beta$ and we are done.
□ of the claim.

Pick $\alpha, \beta < \alpha < \kappa^+$ such that $B_\alpha \cap A_0 \in S$ and $B_\alpha \cap A_0 \subseteq X_\beta$. Then $\beta \in I_\alpha$. So, by the definition of H,

$$H(B_\alpha \cap A_0) = A_1 \cap \bigcap_{\gamma \in I_\alpha} B_\gamma \subseteq A_1 \cap B_\beta.$$

Contradiction.
□

2.1 Forcing construction

We have U is a normal ultrafilter in V and $\langle A^\alpha \mid \alpha < \kappa^+\rangle$ is a generating family.

We would like to add generically sequences $\langle B_\alpha \mid \alpha < \kappa^+\rangle$, $\langle X_\alpha \mid \alpha < \kappa^+\rangle$, $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ such that

1. each of this three sequences is strictly $\subset^*$ −decreasing,
2. $A_\alpha \subseteq^* X_\alpha \subseteq B_\alpha$, for every $\alpha < \kappa^+$,
3. if for some $\nu < \kappa$, $\nu \in X_\alpha \cap A_\beta$, then $A_\beta \setminus \nu \subseteq X_\alpha$, for every $\alpha < \beta < \kappa^+$.

Let us describe the forcing. It will be the Easton support iteration needed in order to preserve measurability. But let us first deal with a single component.

Let λ be an inaccessible cardinal. We will define Q_λ.

Start with $Cohen(\lambda)$.

Next force with a forcing R_λ. It consists of triples (t_1, t_2, t_3) of functions from some $\delta < \lambda$ to $Cohen(\lambda)$ such that

1. $t_1(\nu) \leq t_2(\nu) \leq t_3(\nu)$ as conditions in the Cohen forcing, for every $\nu < \delta$.

The purpose is to add B_α at the first coordinate, X_α at the second and A_α at the third. We repeat the process λ^+-many times in order to add such sets for every $\alpha < \lambda^+$.
Thus set $Q^0_\lambda = Cohen(\lambda) \times R_\lambda$. Let $i \leq \lambda^+$ and suppose that for every $i' < i$, $Q^{i'}_\lambda$ is defined. Define Q^i_λ.
If i is limit, then Q^i_λ will the iteration of $Q^{i'}_\lambda$'s with $< \lambda-$support.
Suppose that $i = i' + 1$. Then let Q^i_λ be the set of all triples (t_1, t_2, t_3) of functions from some $\delta < \lambda$ to $Q^{i'}_\lambda$ such that

1. $t_1(\nu) \leq t_2(\nu) \leq t_3(\nu)$, for every $\nu < \delta$.
2. for every $\mu \leq \delta, i_0 < i_1 \leq i'$, if $t_{2i_0}(\mu), t_{3i_1}(\mu) \in G(Q^{i'}_\lambda)$, then for every $\nu \in \mathrm{dom}(t_{2i_0}) \setminus \mu$, $t_{2i_0}(\nu) \leq t_{3i_1}(\nu)$.

Set $Q_\lambda = Q^{\lambda^+}_\lambda$.
The order on Q_λ is defined in the usual fashion with a small addition.

Let $p = \langle p_i \mid i < \lambda^+\rangle, q = \langle q_i \mid i < \lambda^+\rangle \in Q_\lambda$. Set $p \geq q$ iff

1. $p_i \geq q_i$, for every $i < \lambda^+$,
2. if $i' < i < \lambda^+$, $\nu \leq \mathrm{dom}(q_{i'})$, then $p_{i2} \restriction \nu$ is of the form $q_{i''2}$, for some $i'' \leq i'$, with probably a bounded change.

The following lemma follows from the definition:

Lemma 2.3 *Q_λ is $\lambda-$closed forcing which satisfies λ^+-c.c.*

Now, we iterate such Q_λ's below a measurable cardinal κ.
Namely, let $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle$ be the Easton support iteration, where Q_β is trivial unless β is an inaccessible and this case Q_β is defined in V^{P_β} as above.

Let G be a generic subset of $P_{\kappa+1}$.
Denote by $\langle f_{\kappa\alpha i} \mid \alpha < \kappa^+, i \in \{1, 2, 3\}\rangle$ the triples of generic functions added by $G \restriction Q_\kappa$.

Lemma 2.4 *κ remains measurable in $V[G]$.*

Proof. Start with an normal ultrapower embedding $j : V \to M$. We would like to extend it to $j^* : V[G] \to M[G^*]$.

Define a master conditions sequence $\langle r_\alpha \mid \alpha < \kappa^+ \rangle$, in order to do this.

Fix, in V, an elementary chain $\langle N_i \mid i < \kappa^+ \rangle, N_i \in M$ such that $|N_i| = \kappa, N_i \supseteq \kappa$, $\langle N_{i'} \mid i' < i \rangle \in N_i$ and $\bigcup_{i<\kappa^+}(N_i \cap j(\kappa^+)) = j(\kappa^+)$. Fix a list of all dense open sets $\langle D_i \mid i < \kappa^+ \rangle$ of $P_{j(\kappa)+1}/G$. We can assume that each $D_i \in N_{i+1}$. Denote $\sup(N_i \cap j(\kappa))$ by η_i.

First, consider $G \restriction Q_\kappa$. It belongs to $M[G]$. Extend to a condition of the hight η_0. Do this as follows.

For every $\alpha < \kappa^+$ with $j(\alpha) \in N_0$, we arrange

- $j(f_{j(\kappa),j(\alpha),1}(\kappa), j(f_{j(\kappa),j(\alpha),2}(\kappa), j(f_{j(\kappa),j(\alpha),3}(\kappa)$ to be in G^*,
- $X_{j(\alpha)} \cap \eta_0 \setminus \kappa + 1 = \emptyset$,
- $X_{j(\alpha)} \cap \eta_0 = X_\beta \eta_0$, for some $\beta < \kappa^+$.

Continue further in a standard fashion. Insure that the generic functions such that $j^*(f_{\beta 1})(\kappa) = r_\alpha$, for every $\alpha < \kappa^+$, for some $\beta \geq \alpha$.

□

Recall that Q_κ starts with the Cohen forcing $Cohen(\kappa)$. Let $f_\kappa : \kappa \to 2$ be the generic Cohen function added by $G \restriction Q_\kappa$.

We have sets

$$A_k = \{\nu < \kappa \mid f_\kappa(\nu) = k\}, k < 2.$$

Define a normal filter W over κ in which both A_0 and A_1 are positive. In order to do this just repeat the argument of the previous lemma, define a master condition sequence $\langle r_i \mid i < \kappa^+ \rangle$, but put into W only sets $Z \subseteq \kappa$ such that there are $p \in G$ and $i < \kappa^+$ such that $(p, r_i \setminus \{(\kappa_1, \kappa, 0), (\kappa_1, \kappa, 1)\})$ forces that $\kappa \in j(\underset{\sim}{Z})$.

We use $G \restriction Q_\kappa$ to define families $\langle B'_\alpha \mid \alpha < \kappa^+ \rangle, \langle B_\alpha \mid \alpha < \kappa^+ \rangle, \langle A_\alpha \mid \alpha < \kappa^+ \rangle$ and $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$ inside W, again removing from each r_α the value of $f_{\kappa_1}(\kappa)$, i.e. $\{(\kappa_1, \kappa, 0), (\kappa_1, \kappa, 1)\})$.

Set

$$B'_\alpha = \{\nu < \kappa \mid f_{\alpha 1}(\nu) \in G\}, X_\alpha = \{\nu < \kappa \mid f_{\alpha 2}(\nu) \in G\}, A_\alpha = \{\nu < \kappa \mid f_{\alpha 3}(\nu) \in G\},$$

for every $\alpha < \kappa^+$. Now, using $\langle B'_\alpha \mid \alpha < \kappa^+\rangle$, we define a $\subset$ −cofinal in U^G family $\langle B_\alpha \mid \alpha < \kappa^+\rangle$. First intersect B'_α with a generator A^α of U. Then build a $\subseteq^*$ −decreasing sequence by making bounded changes and shrinking inside the set $(B'_\alpha \cap A^\alpha) \setminus X_\alpha$.

The next density lemma will be crucial.

Lemma 2.5 *The generic extension $V[G]$ satisfies the following:*

For every $\eta < \kappa^+$, there is $\nu(\eta) < \kappa$ and there are arbitrary large inaccessible cardinals $\rho \in A_\eta, |A_\eta \cap \rho| = \rho$ such that for every $\alpha < \eta$ either

1. *$X_\alpha \supseteq A_\eta \setminus \nu(\eta)$,*
   or

2. *$X_\alpha \cap \rho$ is disjoint from A_η.*
   In addition

3. *$f_\kappa \restriction [\nu(\eta), \rho) \cap A_\eta$ has constant value 0.*

4. *For every $A \in U$, we can find ρ as above such that $\rho \in A$ and $|A \cap A_\eta \cap \rho| = \rho$.*

Proof. Work in $V[G \restriction \kappa]$ and use a density argument.
Thus, let $q \in Q_\kappa$ be an arbitrary condition. We will extend it to a condition p which forces the conclusion of the lemma.

Let $\delta(q) < \kappa$ be the hight of coordinates of q. Set $\nu(\eta) = \delta(q)$.
By extending q if necessary, we can assume that η already appears in q as the third coordinate of some coordinate of q.
Let $\rho < \kappa$ be an arbitrary large inaccessible such that $V_\rho[G \restriction \rho] \preceq V_\kappa[G \restriction \kappa]$. Define p to be an extension of q such that

1. $A_\eta \cap \rho$ is unbounded in ρ, i.e. we extend the third coordinate for η such that unboundedly many times below ρ, members of $G \restriction \rho$ are chosen.

2. Now new $\gamma < \kappa^+$ are added to the support of q, i.e., $\mathrm{supp}(p) = \mathrm{supp}(q)$.

3. For every $\alpha \in \mathrm{supp}(q) \cap \eta$, if $X_\alpha \cap \delta(q)$ does not contain a final segment of $A_\eta \cap \delta(q)$, then in p, $X_\alpha \cap [\delta(q), \rho) = \emptyset$.
   This means that $q_{2\alpha}(\nu)$ is picked outside of $G \restriction \rho$, for every $\nu \in [\delta(q), \rho)$.

4. Define $f_\kappa \restriction \rho$ in order to satisfy the third condition of the statement of the lemma.

The desired conclusion follows by the definition of the order on Q_κ. Namely, if $p \in G \upharpoonright Q_\kappa$, then for any $\alpha < \eta$, $X_\alpha \cap \rho = X_\beta \cap \rho$, for some $\beta \in supp(p) = \mathrm{supp}(q)$, and $X_\beta \cap \rho$ is fine with respect to A_η.
$\square$

It follows now:

Lemma 2.6 *The following holds in $V[G]$:*

For every $\eta < \kappa^+$, there is $\nu(\eta) < \kappa$ such that for every $\beta < \eta$ and $A \in U$, if $X_\beta \supseteq A_\eta \cap A \cap A_0$, then $X_\beta \supseteq A_\eta \cap A \setminus \nu(\eta)$.

We define U_0 by adding A_0 to W and U_1 by adding A_1 to W.
By 2.2, $U_0 \equiv_T U_1$.

3 Two normal ultrafilters such that one is Tukey strictly below another

We will combine the ideas of the previous sections.
Let U be a normal ultrafilter in V. Force with the forcing $P_{\kappa+1}$ of the previous section. Let $G \subseteq P_{\kappa+1}$ be a generic, let $U_0, U_1 \supseteq U$ be a normal ultrafilters in $V[G]$ defined in Section 2 and let $\langle B_\alpha \mid \alpha < \kappa^+\rangle, \langle X_\alpha \mid \alpha < \kappa^+\rangle, \langle A_\alpha \mid \alpha < \kappa^+\rangle$ be the sequences defined there. Only, replace $Cohen(\alpha)$ by $Cohen(\alpha,\alpha)$. Let $\langle r_i \mid i < \kappa^+\rangle$ the master condition sequence used in Section 2 and assume that the condition $\{(\kappa_1, \kappa, 0)\}$ is inside, i.e. this sequence defines U_0.

Define R as in Section 1 in $V[G]$ and let $G(R)$ be its generic subset of it over $V[G]$. Let, as in Section 1, $U^{G(R)}$ be the corresponding extension of U.
Consider $U_0 \times U^{G(R)}$. Then $U_0 <_T U_0 \times U^{G(R)}$, just the projection to the first coordinate will witness $U_0 \leq_T U_0 \times U^{G(R)}$, and, by the arguments of Section 1, $U_0 \not\equiv_T U_0 \times U^{G(R)}$.
However, $U_0 \times U^{G(R)}$ is not normal. Let us fix this.

For every $i < \kappa^+$, set
$$A^0_i = \{\nu < \kappa \mid h_{r_i}(\nu) \in G\},$$
where $h_{r_i} : \kappa \to \kappa$ is a function which represents r_i in M_1. Then $\langle A^0_i \mid i < \kappa^+\rangle$ generates U_0 mod U.

Consider $j_{U_0 \times U^{G(R)}} : V[G, G(R)] \to M_{U_0 \times U^{G(R)}}$. The ground model of $M_{U_0 \times U^{G(R)}}$ is M_2 - the second ultrapower by U. Denote by κ_2 the corresponding image of κ.
Change two values of the Cohen function of this double ultrapower:
$f_{\kappa_2}(\kappa)$ to κ_1 and $f_{\kappa_2}(\kappa_1)$ to κ. The generic over M_2 set is obtained by first using $\langle r_i \mid i < \kappa^+\rangle$

for the forcing up to $\kappa_1 + 1$ and then the image of $G(R)$.
Let $\langle s_i \mid i < \kappa^+\rangle$ be the corresponding master condition sequence such that $s_i \restriction \kappa_1 + 1 = r_i, i < \kappa^+$.

Denote by F the resulting normal ultrafilter over κ.
For every $i < \kappa^+$, set

$$A_i^1 = \{\nu < \kappa \mid h_{s_i}(\nu, f_\kappa(\nu)) \in G\},$$

where $h_{s_i} : [\kappa]^2 \to \kappa$ is a function which represents s_i in M_2. Then $\langle A_i^1 \mid i < \kappa^+\rangle$ generates F mod U.

We claim that $F \geq_T U_0$. Note that as above $U_0 \not\geq_T F$, and so, this will imply that $F \not\equiv_T U_0$.
It will be a bit more convenient to replace F by its two dimensional version F' defined by setting

$$Z \subseteq [\kappa]^2 \text{ is in } F' \text{ iff } (\kappa, \kappa_1) \in j_F(X).$$

Then we will have

$$Z \in F' \text{ iff}$$

$$\exists p \in G \quad \exists \alpha < \kappa^+ \quad p^\frown r_\alpha^\frown s_\alpha \setminus (\kappa_1 + 1 \cup \{(\kappa_2, \kappa, \kappa_1)\})^\frown \{(\kappa_2, \kappa, \kappa_1)\} \Vdash (\kappa, \kappa_1) \in j(\underset{\sim}{X}).$$

Pick functions $h_1^\alpha, h_2^\alpha : \kappa \to V_{\kappa+1}$ which represent r_α and $s_\alpha \setminus (\kappa_1 + 1 \cup \{(\kappa_2, \kappa, \kappa_1)\})$
Set

$$E_k^\alpha = \{\nu < \kappa \mid h_k^\alpha(\nu) \in G\}, k = 1, 2.$$

Then the sequence

$$\langle E_0^\alpha \times \kappa \cap \kappa \times E_1^\alpha \cap \{(\nu, f_\kappa(\nu)) \mid \nu < \kappa\} \mid \alpha < \kappa^+\rangle$$

generates F' mod U^2.

We have a $\subseteq^*$ $-$generating family $\langle A^\alpha \mid \alpha < \kappa^+\rangle$ for U in V.

Then, $\langle A^\alpha \cap B_\alpha \mid \alpha < \kappa^+\rangle$ will generate W and $\langle A^\alpha \cap B_\alpha \cap A^0 \mid \alpha < \kappa^+\rangle$ will be cofinal in U_0. We can use this sets to replace E_0^α's. Then the sequence

$$\langle A^\alpha \cap B_\alpha \times \kappa \cap \kappa \times (E_1^\alpha \cap A^\alpha) \cap \{(\nu, f_\kappa(\nu)) \mid \nu < \kappa\} \mid \alpha < \kappa^+\rangle$$

will be cofinal in F'.

Define H on this set by induction.
Suppose that for every $\beta < \alpha$, $H(A^\beta \cap B_\beta \times \kappa \cap \kappa \times (E_1^\beta \cap A^\beta) \cap \{(\nu, f_\kappa(\nu)) \mid \nu < \kappa\})$ is defined. Define $H(A^\alpha \cap B_\alpha \times \kappa \cap \kappa \times (E_1^\alpha \cap A^\alpha) \cap \{(\nu, f_\kappa(\nu)) \mid \nu < \kappa\})$. Consider

$$I_\alpha = \{\beta < \alpha \mid A^\alpha \cap B_\alpha \cap \{\nu < \kappa \mid f_\kappa(\nu) \in E_1^\alpha \cap A^\alpha\} \subseteq X_\beta\}.$$

Then,

$$\bigcap_{\beta \in I_\alpha} X_\beta \supseteq A^\alpha \cap B_\alpha \cap \{\nu < \kappa \mid f_\kappa(\nu) \in E_1^\alpha \cap A^\alpha\} \subseteq X_\beta\}.$$

Set

$$H(A^\alpha \cap B_\alpha \times \kappa \cap \kappa \times (E_1^\alpha \cap A^\alpha) \cap \{(\nu, f_\kappa(\nu)) \mid \nu < \kappa\}) = A_0 \cap \bigcap_{\beta \in I_\alpha} (A^\beta \cap B_\beta).$$

The argument similar to the one of the previous section shows that such H is as desired, i.e., it witnesses $U_0 \leq_T F'$.

References

[1] T. Benhamou, N. Dobrinen, COFINAL TYPES OF ULTRAFILTERS OVER MEASURABLE CARDINALS

[2] N. Dobrinen, S. Todorcevic, Tukey types of ultrafilters,