Two Stationary Sets with Different Gaps of the Power Function

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Abstract

Starting with a strong cardinal a model with a cardinal κ of cofinality $\aleph_1$ such that both sets $\{\alpha < \kappa \mid 2^\alpha = \alpha^{++}\}$ and $\{\alpha < \kappa \mid 2^\alpha = \alpha^{+++}\}$ are stationary is constructed.

0 Introduction

The classical theorem of Silver states that if κ is a singular cardinal of uncountable cofinality and $2^\kappa > \kappa^+$, then the set $\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$ contains a club. But what if $2^\kappa = \kappa^+$, can both sets

$$\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$$

and

$$\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$$

be stationary?

The question is still open. The purpose of the present paper is to construct a model with a cardinal κ of cofinality $\aleph_1$ such that both sets $\{\alpha < \kappa \mid 2^\alpha = \alpha^{++}\}$ and $\{\alpha < \kappa \mid 2^\alpha = \alpha^{+++}\}$ are stationary. We start from a regular cardinal κ having a coherent sequence of $(\kappa, \kappa^{+\omega+3})-$extenders of the length $\aleph_1$. A variation of the extender based Magidor forcing (see [5], [?]) is used to change its cofinality to $\aleph_1$ blowing up powers of cardinals over the generic Magidor sequence below. The point will be to arrange a different behaviour on stationary sets. For this a short extenders forcing will be used.

1 Preliminary Settings

1.1 Cofinal sequences and stationary set

Let us attach to every δ, $0<\delta<\omega_1$, a successor ordinal $\delta^*<\delta$ so that for every successor ordinal $\tau<\omega_1$ the set of δ's with $\tau=\delta^*$ is stationary. Clearly, the set

$$\begin{aligned} C &= \{\mu<\omega_1 \mid \mu \text{ is limit ordinal and for every successor } \tau<\mu \\ &\qquad \text{the set of } \delta\text{'s below } \mu \text{ with } \delta^*=\tau \text{ is unbounded in } \mu\} \end{aligned}$$

is closed unbounded.

Let S be a subset of $\lim(\lim(C))$. It can be nonstationary, but in the interesting cases S will be stationary, costationary. For every $\mu\in C$ fix a cofinal sequence $\langle\mu'_n|n<\omega\rangle$. Let $\mu\in C$. We define another cofinal sequence $\langle\mu_n|n<\omega\rangle$ as follows:

$$\mu_0=0,\quad \mu_{n+1}=\min\{\delta<\mu\mid \delta \text{ limit},\quad \delta\notin S,\quad \delta^*=\mu_n \quad\text{and}\quad \delta\geq\mu'_{n+1}\}+1\ .$$

The advantage of using such improved cofinal sequences is that once we have μ_n then $\langle\mu_k|k\leq n\rangle$ is uniquely determined without need in μ. Thus $\mu_{n-1}=(\mu_n-1)^*$, $\mu_{n-2}=(\mu_{n-1}-1)^*$, etc.

1.2 The extenders sequence

We assume that the ground model satisfies GCH and has a coherent sequence

$$\vec{E}=\langle E(\alpha,\beta)\mid \alpha\leq\kappa,\quad \alpha\in\operatorname{dom}\vec{E},\quad \beta<\omega_1\rangle$$

such that for every $\alpha\leq\kappa$ and $\beta<\omega_1$ the following hold:

(a) $E(\alpha,\beta)$ is an $(\alpha,\alpha^{+\omega+3})$-extender over α.

(b) (coherence)

$$j_{E(\alpha,\beta)}(\vec{E})\restriction(\alpha+1)=\langle E(\alpha',\beta')\mid(\alpha'<\alpha)\quad\text{or}\quad(\alpha'=\alpha\quad\text{and}\quad\beta'<\beta)\rangle\ ,$$

where $j_{E(\alpha,\beta)}:V\to M\simeq\mathrm{Ult}(V,E(\alpha,\beta))$ is the elementary embedding by $E(\alpha,\beta)$

(c) there are disjoint subsets $\langle E_{\alpha,i}\mid i<\omega_1\rangle$ of α such that $E_{\alpha,\beta}$ belongs to the normal measure $E(\alpha,\beta)(\alpha)$ of $E(\alpha,\beta)$ and for every $\gamma\leq\alpha$, $\gamma\in\operatorname{dom}\vec{E}$, $i<\omega_1$

$$E_{\alpha i}\cap\gamma=E_{\gamma i}\ ,$$

where for $\tau<\alpha^{+\omega+3}$ the τ-th measure $E(\alpha,\beta)(\tau)$ of $E(\alpha,\beta)$ is the set $\{X\subseteq\alpha\mid\tau\in j_{E(\alpha,\beta)}(X)\}$.

(d) $\vec{E} \restriction (\alpha, \beta) = j_{E(\alpha,\beta)}(f)(\alpha)$ for some $f \in {}^{\alpha}V_{\alpha}$, i.e., essentially it depends on the normal measure $E(\alpha, \beta)(\alpha)$, where

$$\vec{E} \restriction (\alpha, \beta) = \langle \vec{E}(\alpha', \beta') \mid (\alpha' < \alpha) \quad \text{or} \quad (\alpha' = \alpha \quad \text{and} \quad \beta' < \beta) \rangle \ .$$

1.3 Types

Previously Short Extenders forcings were used in context of a singular cardinal κ which is a limit of cardinals κ_n's carrying extenders and types were defined over κ_n's (say $\kappa_n^{+k_n+2}$ with sequences $\langle k_n \mid n < \omega \rangle$ converging to infinity). Here κ is a regular cardinal, but we will work around $\kappa^{+\omega}$ and will use $\kappa^{+n}, n < \omega$ as a replacement.

Let χ be a regular cardinal large enough (thus $\kappa^{+\omega+4}$ will do it). For $k \leq \omega$ we consider a structure

$$\mathfrak{A}_k = \langle H(\chi^{+k}), \in, \vec{E}, \text{ the enumeration of } [\kappa^{+\omega+3}]^{\leq\kappa}, \text{ and of } [\kappa^{+\omega+2}]^{\leq\kappa},$$

$$\langle \chi^{+n} \mid n \leq k \rangle, \kappa, 0, 1, \ldots, \alpha \ldots \mid \alpha < \kappa^{+k} \rangle$$

in an appropriate language which we denote $\mathcal{L}_k$.

For an ordinal $\xi < \chi$ (usually ξ will be below $\kappa^{+\omega+3}$) we denote by $tp_k(\xi)$ the $\mathcal{L}_k$-type realized by ξ in $\mathfrak{A}_k$.

Let $\mathcal{L}'_k$ be the language obtained from $\mathcal{L}_k$ by adding a new constant c'. For $\delta < \chi$ let $\mathfrak{a}_{k,\delta}$ be the $\mathfrak{L}'_k$-structure obtained from $\mathfrak{a}_k$ by interpreting c' as δ. The type $tp_k(\delta, \xi)$ is the $\mathcal{L}'_k$-type realized by ξ in $\mathfrak{A}_{k,\delta}$. Further, we shall identify types with ordinals corresponding to them in some fixed well ordering of $\mathcal{P}(\kappa^{+k})$.

Definition 1.1 Let $k \leq \omega$ and $\beta < \kappa^{+\omega+3}$ (or $\beta < \kappa^{+\omega+2}$). β is called k-good iff

1. for every $\gamma < \beta$ $tp_k(\gamma, \beta)$ is realized unboundedly often below $\kappa^{+\omega+3}$ (or respectively, $\kappa^{+\omega+2}$);

2. for every bounded $a \subseteq \beta$ of cardinality $\leq \kappa$ there is $\alpha < \beta$ corresponding to a in the enumeration of $[\kappa^{+\omega+3}]^{\leq\kappa}$ (or respectively $[\kappa^{+\omega+2}]^{\leq\kappa}$).

The next two lemmas are proved in [1].

Lemma 1.2 *The set $\{\beta < \kappa^{+\omega+i} \mid \beta$ is ω-good$\}$ contains a club, for every $i < 2$.*

Lemma 1.3 *Let $0 < k \leq \omega$ and β be k-good. Then there are arbitrarily large $k-1$-good ordinals below β.*

1.4 The Preparation Forcing $\mathcal{P}'$

The relevant preparation forcing $\mathcal{P}'$ here will be just $\mathcal{P}'$ of Gap 3 with κ replaced by $\kappa^{+\omega}$ (and then κ^+ by $\kappa^{+\omega+1}$, κ^{++} by $\kappa^{+\omega+2}$, κ^{+3} by $\kappa^{+\omega+3}$). So, models of cardinality $\kappa^{+\omega+1}$ and those of cardinality $\kappa^{+\omega+2}$ (ordinals) will be the only one used. This way the final forcing will satisfy only $\kappa^{+\omega+2}$–c.c. and the cardinals of the interval $[\kappa^{++}, \kappa^{+\omega+1}]$ will be collapsed. But that is what we actually desire. Thus $(\kappa^{+\omega+2})^V$ will be turned into κ^{++} and $(\kappa^{+\omega+3})^V$ into κ^{+3}.

In order to force with $\mathcal{P}'$ and preserve the desired strongness– the corresponded preparation should be made below κ.

2 The Main Forcing

The next definition combines the Extender Based Magidor forcing with a certain short extenders forcing.

The forcing will change the cofinality of κ to ω_1 by adding to it a Magidor sequence $\langle \kappa_\tau \mid \tau < \omega_1 \rangle$. For each $\tau < \omega_1$, we will have $2^{\kappa_\tau} \geq (\kappa_\tau^{+\omega+2})^V$. This will be due to the fact that extenders $E(\kappa_\tau, \beta), \beta < \tau$ has length above $\kappa_\tau^{+\omega+2}$ and nothing special (like new assignments and equivalence relations $\longleftrightarrow$) will be done for measures with indexes below $\kappa_\tau^{+\omega+2}$. So the correspondence (assignment) will be here the natural one:

$\xi < \kappa_\tau^{+\omega+2}$ will correspond the ξ–th measure of the extenders over κ_τ.

If $\tau \in S$, then we would like to have $2^{\kappa_\tau} = (\kappa_\tau^{+\omega+3})^V$, and $2^{\kappa_\tau} = (\kappa_\tau^{+\omega+2})^V$, whenever $\tau \notin S$.

A new essential point in the present construction will be as follows. Let $\tau \in S$. Suppose that κ_τ is determined and it is a limit of κ_ν's with $\nu \notin S$. We would like that $\{\kappa_\nu^{+\omega+2} \mid \nu \notin S\}$ correspond to $\kappa_\tau^{+\omega+3}$, i.e. $\{\kappa_\nu^{+\omega+2} \mid \nu \notin S\} \subseteq \mathfrak{b}_{\kappa_\tau^{+\omega+3}}$, where $\mathfrak{b}_{\kappa_\tau^{+\omega+3}}$ is the pcf–generator. This already requires drops in cofinality. Thus we need some sequence to correspond to $\kappa_\tau^{+\omega+2}$. We will use $\langle \kappa_{\tau_n}^{+\omega+2} \mid n < \omega \rangle$ for this purpose, where $\langle \tau_n \mid n < \omega \rangle$ is a cofinal in τ sequence consisting of successor ordinals that was reserved in advance. So not all of the set $\{\kappa_\nu^{+\omega+2} \mid \nu \notin S\}$ will be a part of $\mathfrak{b}_{\kappa_\tau^{+\omega+3}}$, rather only

$$\{\kappa_\nu^{+\omega+2} \mid \nu \notin S\} \setminus \{\kappa_{\tau_n}^{+\omega+2} \mid n < \omega\}.$$

Suppose also that for every $n < \omega$ we have $\eta_n \notin S, \tau_n < \eta_n < \tau_{n+1}$ such that $\kappa_{\tau_n}^{+\omega+2}$ is connected with $\kappa_{\eta_n}^{+\omega+2}$ (i.e. basically $\kappa_{\tau_n}^{+\omega+2} \in \mathfrak{b}_{\kappa_{\eta_n}^{+\omega+2}}$). Note that since there are $\aleph_1$ many τ's and both $S, \omega_1 \setminus S$ are stationary, situations like above always must occur. Let $n < \omega$. We have that $\kappa_{\eta_n}^{+\omega+2}$ is connected to $\kappa_\tau^{+\omega+3}$, $\kappa_{\tau_n}^{+\omega+2}$ is connected to $\kappa_{\eta_n}^{+\omega+2}$ and $\kappa_{\tau_n}^{+\omega+2}$ is connected

to $\kappa_\tau^{+\omega+2}$. This looks like a problem and in the actual setting has to do with the chain condition ($\kappa_\tau^{+\omega+3}$–c.c. instead of the desired $\kappa_\tau^{+\omega+2}$–c.c.). A way to overcome this difficulty will be as follows. Once we have two conditions such that in the first some $\alpha < \kappa_\tau^{+\omega+3}$ corresponds to a measure $\alpha^* < \kappa_\tau^{+\omega+2}$ of $E(\kappa_\tau, \eta_n)$ and α^* corresponds to a measure γ^* of $E(\kappa_\tau, \tau_n)$ via the assignment of $E(\kappa_\tau, \eta_n)$ (i.e. on a set of measure one for a maximal measure of the condition for $E(\kappa_\tau, \eta_n)$). In addition some $\gamma < \kappa_\tau^{+\omega+2}$ corresponds to the measure γ^* of $E(\kappa_\tau, \tau_n)$. Suppose the second condition is the same, but instead of α we have some different ordinal β, say $\alpha < \beta$, but $\alpha^* = \beta^*$ and the rest is the same.
We need to be able to put such two conditions into one stronger than both of them in order to verify the chain condition. Say we like to extend the first condition. So, as usual, we find some $\mu > \alpha^*$ which realize the same type over the common part and attach it to β instead of α^* (which corresponds to β in the second condition). The problem with this is that α^* corresponds to γ^*, but μ does not.
But let us do the following: attach μ to γ^* as well. So, in a sense, we loose a one to one correspondence of the assignment function for $\kappa_{\eta_n}^{+\omega+2}$. In order to compensate this, let us require that the α^*-th and μ-th sequences to κ_{η_n} differ all the time above κ_{τ_n}.

The above will be implemented as follows. A non direct extensions of a condition which determine the value of κ_τ will be allowed to identify α^* and μ as above, but then on a set of measure one for $\underset{\sim}{\kappa_\tau}$ (i.e. with different choices of κ_τ) they will be kept different.
Now, since a non-direct extensions can be made only at finitely many places (i.e. a condition decides only finitely many κ_τ's) the generic omega sequences corresponding from κ_{η_n} for α^* and μ will be eventually different.
We do not require that assignments between $\kappa_\tau^{+\omega+2}$ and measures of $E(\kappa_\tau, \eta_n)$ and between measures $E(\kappa_\tau, \tau_n)$ and $E(\kappa_\tau, \eta_n)$ are identity. i.e. $\gamma < \kappa_\tau^{+\omega+2}$ corresponds to γ-th measure of $E(\kappa_\tau, \eta_n)$, and γ-th measure of $E(\kappa_\tau, \tau_n)$ corresponds to γ-th measure of $E(\kappa_\tau, \eta_n)$.
Actually, it cannot be the identity once the above was implemented. The assignment functions however will be identity on $\kappa_\tau^{+\omega+1}$.

Let $\tau < \eta < \omega_1$. Denote the connection function between levels η and τ by $a_{\eta\tau}$. There are five possibilities.

1. $\tau, \eta \notin S$.
   Then we connect between $\kappa_\eta^{+\omega+2}$ and $\kappa_\tau^{+\omega+2}$.

2. $\tau, \eta \in S$.
   This implies that τ is not one of η_n's.
   We connect $\kappa_\eta^{+\omega+3}$ to both $\kappa_\tau^{+\omega+2}$ and to $\kappa_\tau^{+\omega+3}$.

3. $\eta \notin S, \tau \in S$.
   Then we connect $\kappa_\eta^{+\omega+2}$ to both $\kappa_\tau^{+\omega+2}$ and to $\kappa_\tau^{+\omega+3}$.

4. $\eta \in S, \tau \notin S$ and $\tau = \eta_n$ for some $n < \omega$.
   Then we connect $\kappa_\eta^{+\omega+2}$ to $\kappa_\tau^{+\omega+2}$.

5. $\eta \in S, \tau \notin S$ and τ is not one of η_n's.
   Then we connect $\kappa_\eta^{+\omega+3}$ to $\kappa_\tau^{+\omega+2}$. Dropping in cofinality is used to deal with this case.

In all the cases $\kappa_\eta^{+\omega+1}$ is connected to $\kappa_\tau^{+\omega+1}$ and the connection between them is just the identity.

Let us explain more the cofinalities drops that will occur here. The complication is due to the fact that the cofinality from ω_1 many places may drop to a same value. Thus, for example, τ_0 may be the first element of the fixed ω-sequence for ω_1 many ordinals $\alpha < \omega_1$. So, we will have drops from $\kappa_\alpha^{+\omega+2}$ to $\kappa_{\tau_0}^{+\omega+2}$ for ω_1- many α's. This disturbs completeness of the forcing (at least direct extensions in it).
Let us deal with this as follows (the explanation continues with τ_0, but τ_0 may be viewed as an arbitrary point of the fixed ω-sequence to arbitrary $\tau \in S$). Let us assume that $\tau' =: \tau_0+1$ is not in S and is not a member of any of this fixed ω-sequences. Then we would like to add $\kappa_{\tau'}^{+\omega+2}$-many ω-sequences to $\kappa_{\tau'}$. Now require that once κ_{τ_0} is determined, i.e. a non-direct extension of a condition was made over level τ_0, then the same was done over τ' and $\kappa_{\tau'}$ is determined as well. $\kappa_{\tau'}$ is linked with the rest ω_1 many cardinals (not yet determined) $\underset{\sim}{\kappa}_\beta$'s. We will now "copy" more or less from $\kappa_{\tau'}$ to $\underset{\sim}{\kappa}_\beta$'s.
Let us clarify one subtle point in this context (i.e. once there are many drops to a single cofinality). Suppose for simplicity that $\omega \in S$, but $n \notin S$, for each $n < \omega$. Suppose also that $\langle 2n \mid n < \omega\rangle$ is the fixed sequence for ω. In a generic extension (and we are interested only in its part below κ_ω) let f_i be the generic ω-sequence for $i < \kappa_\omega^{+\omega+3}$. Then $f_i \in \prod_{n<\omega} \kappa_{2n+1}^{+\omega+2}$, for $i < \kappa_\omega^{+\omega+3}$. In addition, if $\mathrm{cof}(i) = \kappa_\omega^{+\omega+2}$, then $\mathrm{cof}(f_i(n)) = \kappa_{2n}^{+\omega+1}$, because of dropping in cofinality.
Let $\langle g_j \mid j < \kappa_\omega^{+\omega+2}\rangle$ be generic ω-sequences in $\prod_{n<\omega} \kappa_{2n}^{+\omega+2}$, i.e. corresponding to $\kappa_\omega^{+\omega+2}$. Let $V_1 = V[\langle g_j \mid j < \kappa_\omega^{+\omega+2}\rangle]$.
Now suppose the following:

(*) for every $i < \kappa_\omega^{+\omega+3}$ of cofinality $\kappa_\omega^{+\omega+2}$ (or just for stationary many of such i's) there is a sequence $\langle s(i,n) \mid n < \omega\rangle \in V_1$ such that for all but finitely many n's $s(i,n)$ is bounded in i, but $\{f_j(n) < f_i(n) \mid j \in s(i,n)\}$ is unbounded in $f_i(n)$.

Work in V_1. Consider a function $i \mapsto \sup(\cup_{n<\omega} s(i,n))$. Find a stationary $S \subseteq (\kappa_\omega^{+\omega+3})^V$

on which it takes a constant value δ. Now, in V_1, we still have $\delta^\kappa = \kappa^\omega = (\kappa_\omega^{+\omega+2})^V < (\kappa_\omega^{+\omega+3})^V$. Hence, there is $S_1 \subseteq S$ stationary such that for every $i, i' \in S_1$ we have $\langle s(i,n) \mid n < \omega\rangle = \langle s(i',n) \mid n < \omega\rangle$.
Let now $i < i'$ be in S_1. Then in the full extension (*) implies that for all but finitely many n's we have $f_i(n) = f_{i'}(n)$. Which is impossible and hence (*) must fail.
The meaning of this is that in this type of a cofinality dropping it is impossible to relay completely (i.e. once arranging assignment functions) on points of a drop corresponding to smaller cofinality. This is a reason of taking $\kappa_{\tau'}$ and copping from it to relevant $\underset{\sim}{\kappa}_\beta$'s as was described above.

Let us explain the process of copping and assignment functions that allow it. For each $\alpha < \beta < \omega_1$, we will have the assignment function $a_{\beta\alpha}$ which will be as usual an isomorphism between suitable structures and $f_{\beta\alpha}$ that is comes from a Cohen part of a condition. As before the both components will be put together once a non-direct extension which decides $\kappa_\alpha, \kappa_\beta$ was taken. The new element in the present context will be a commutativity.
Thus suppose that we have in addition $\gamma, \beta < \gamma < \omega_1$. We require that

1. $a_{\gamma\alpha} = a_{\beta\alpha} \circ a_{\gamma\beta}$,

2. $f_{\gamma\alpha} = f_{\beta\alpha} \circ f_{\gamma\beta}$

once non of κ_i's is decided. Domains of f_{ij}'s are sets of pairs which have the first coordinates corresponding to potential choices of $\underset{\sim}{\kappa}_i$'s.
Suppose now that a non-direct extension was made and as a result κ_α was decided. Then $a_{\gamma\alpha}$ and $a_{\beta\alpha}$ are incorporated into new $f_{\gamma\alpha}$ and $f_{\beta\alpha}$ respectively in the usual fashion. Now $a_{\gamma\beta}$, $f_{\gamma\beta}$ together with $f_{\beta\alpha}$ should give $f_{\gamma\alpha}$.
Assume that we have in addition $\alpha', \alpha < \alpha' < \beta$ such that a dropping occurs from γ to α and α' at the same level, i.e. for some $n < \omega$, we have $\gamma_n < \alpha, \alpha' < \gamma_{n+1}$, where $\langle \gamma_n \mid \gamma < \omega\rangle$ is the fixed sequence for γ. Assume also that $\underset{\sim}{\kappa}_{\alpha'}$ is not decided yet. Then we use $a_{\gamma\beta}$,$f_{\gamma\beta}$ to pick elements of the level β, i.e. over $\underset{\sim}{\kappa}_\beta$, and then $a_{\beta\alpha'}$ to move them down to the level α'. Note that it is possible that $\beta_k < \alpha' < \beta_{k+1}$, for some $k < \omega$, and $\underset{\sim}{\kappa}_{\beta_k}$ is not decided yet.
Once κ_β is decided, then the argument showing $\kappa_\beta^{+\omega+2}$–c.c. of the forcing up to κ_β turns now the usual form.

Let us turn to formal definitions. First let us define pure conditions.

Definition 2.1 (Pure conditions) The set $\bar{\mathcal{P}}$ consists of sequences of the form
$\langle\langle \xi, p^\xi\rangle \mid \xi \in s\rangle, \langle A_\alpha \mid \alpha < \omega_1\rangle, \langle a_{\gamma\beta}, f_{\gamma\beta} \mid \beta < \gamma \leq \omega_1\rangle, \langle a^\delta_{\beta\alpha}, f^\delta_{\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa\rangle\rangle$
satisfying the following conditions:

1. $s \subseteq \kappa^{+\omega+2}$ such that

   (a) $|s| \leq \kappa$,

   (b) $\max(s)$ exists and it is above every other element of s in the order of each of the extenders $E(\kappa, \xi), \xi < \omega_1$.

2. p^ξ is a finite increasing sequence of ordinals below κ,

3. $A_\alpha \in E(\kappa, \alpha)(\max(s))$,

4. $a_{\gamma\beta}$ is an assignment function (an isomorphism between suitable structures) over κ. Let us state the particular properties $a_{\gamma\beta}$ related to the fixed stationary set S.

   (a) If $\beta, \gamma \notin S$, then $a_{\gamma\beta}$ is an isomorphism between a generic suitable structure over $\kappa^{+\omega+2}$ and a suitable structure over $\kappa^{+\omega+2}$.
   This eventually will connect $\kappa_\gamma^{+\omega+2}$ with $\kappa_\beta^{+\omega+2}$, where $\kappa_\gamma, \kappa_\beta$ are γ-th and β-th elements of a generic Magidor sequence.

   (b) $\beta, \gamma \in S$, then $a_{\gamma\beta}$ is an isomorphism between a generic suitable structure over $\kappa^{+\omega+3}$ and a suitable structure over $\kappa^{+\omega+3}$, but so that ordinals correspond to models of cardinality $\underset{\sim}{\kappa}_{\gamma_n}^{+\omega+2}$ and $\kappa^{+\omega+2}$ drops down to $\underset{\sim}{\kappa}_{\gamma_n}^{+\omega+2}$ where γ_n is the maximal member $\leq \beta$ of the fixed cofinal in γ sequence $\langle \gamma_n \mid n < \omega \rangle$.
   This way we would like to connect $\kappa_\gamma^{+\omega+3}$ with both $\kappa_\beta^{+\omega+3}$ and $\kappa_\beta^{+\omega+2}$, in addition $\kappa_\gamma^{+\omega+2}$ will drop to $\kappa_{\gamma_n}^{+\omega+2}$.
   Note, that due to the dropping, the cardinality of $a_{\gamma\beta}$ should be κ_{γ_n}, once we have κ_{γ_n} is decided. It is bad (Prikry condition) to keep the cardinality of $a_{\gamma\beta}$ below κ and then to choose κ_{γ_n} above it. Instead let us allow the cardinality of $a_{\gamma\beta}$ to be a name which depends on the choice of κ_{γ_n}. Recall that in cofinality drops $\mathrm{rng}(a_{\gamma\beta})$ depends on the choice of a point from A_{γ_n}. So, here also the domain is a name, but both are always subsets of the support of the condition s.

   (c) If $\gamma \notin S$ and $\beta \in S$, then $a_{\gamma\beta}$ is an isomorphism between a generic suitable structure over $\kappa^{+\omega+2}$ and a suitable structure over $\kappa^{+\omega+3}$ so that ordinals correspond to models of cardinality $\kappa^{+\omega+1}$.
   This way we would like to connect $\kappa_\gamma^{+\omega+2}$ with both $\kappa_\beta^{+\omega+3}$ and $\kappa_\beta^{+\omega+2}$. Thus, if $\xi \in \mathrm{dom}(a_{\gamma\beta}) \cap On$ then $a_{\gamma\beta}(\xi)$ will be a model of the size $\kappa^{+\omega+1}$. $a_{\gamma\beta}(\xi) \cap \kappa^{+\omega+2}$ will be an ordinal. We have here a kind of splitting where ξ corresponds to both an ordinal $a_{\gamma\beta}(\xi) \cap \kappa^{+\omega+2}$ for $\omega + 2$ and a model $a_{\gamma\beta}(\xi)$ for $\omega + 3$. No drops are needed in this case.

(d) If $\gamma \in S$ and $\beta = \gamma_n$, for some $n < \omega$ (and, in particular, $\beta \notin S$), then $a_{\gamma\beta}$ is an isomorphism between a generic suitable structure over $\kappa^{+\omega+2}$ and a suitable structure over $\kappa^{+\omega+2}$.
This eventually will connect $\kappa_\gamma^{+\omega+2}$ with $\kappa_\beta^{+\omega+2}$.

(e) If $\gamma \in S, \beta \notin S$ and β is not one of γ_n's, then $a_{\gamma\beta}$ is an isomorphism between a generic suitable structure over $\kappa^{+\omega+3}$ and a suitable structure over $\kappa^{+\omega+2}$, but so that $\kappa^{+\omega+2}$ drops down to the maximal $\gamma_n \leq \beta$, where $\langle \gamma_n \mid n < \omega \rangle$ is the fixed cofinal in γ sequence.
This way $\kappa_\gamma^{+\omega+3}$ will be connected with $\kappa_\beta^{+\omega+2}$. In addition $\kappa_\gamma^{+\omega+2}$ will drop to $\kappa_{\gamma_n}^{+\omega+2}$.

5. $a^\delta_{\beta\alpha}$ is an assignment function (an isomorphism between suitable structures) over δ.
It satisfies the conditions 4a–4e above only with κ replaced by δ.

6. $f_{\gamma\beta}$ is a partial function of cardinality at most κ from $\kappa^{+\omega+3}$ to κ,

7. $f^\delta_{\gamma\beta}$ is a partial function of cardinality at most δ from $\delta^{+\omega+3}$ (or from $\delta^{+\omega+2}$) to δ,

8. (Disjointness of domains)

   - $\mathrm{dom}(a_{\gamma\beta}) \cap \mathrm{dom}(f_{\gamma\beta}) = \emptyset$, for every $\beta < \gamma < \omega_1$.
   - $\mathrm{dom}(a^\delta_{\beta\alpha}) \cap \mathrm{dom}(f^\delta_{\beta\alpha}) = \emptyset$, for every $\alpha < \beta < o^{\vec{E}}(\delta)$.

   Let us state now the conditions which deal with a commutativity.

9. $a_{\gamma\alpha} = a_{\gamma\beta} \circ a_{\beta\alpha}$, for each $\alpha < \beta < \gamma < \omega_1$,

10. $a^\delta_{\gamma\alpha} = a^\delta_{\gamma\beta} \circ a^\delta_{\beta\alpha}$, for each $\alpha < \beta < \gamma < o^{\vec{E}}(\delta)$.
Note that only $\mathrm{dom}(a_{\omega_1\beta}) \subseteq s$. If $\gamma < \omega_1$ and $\gamma \in S$, then $\mathrm{dom}(a_{\gamma\beta}) \subseteq \kappa^{+\omega+3}$ which may be larger than s. Our main interest will in the type $\mathrm{dom}(a_{\gamma\beta})$ realizes and further the equivalence relation $(\longleftrightarrow)$ will identify conditions accordingly.

11. If $\nu \in \mathrm{dom}(a_{\gamma\beta})$ and $\mathrm{cof}(\nu) = \kappa^+$, then $\mathrm{cof}(a_{\gamma\beta}(\nu) = \kappa^+$.
Note that such ν's will be usually non–limit points, since there will be drops in cofinalities.

    The next conditions deal with compatibility.

12. (Compatibility) For each $\beta < \gamma < \omega_1$ and $\eta \in A_\gamma$ we require that

$$a^\eta_{\gamma\beta} \bigcup \{\langle \bar{\xi}, \bar{\rho} \rangle \mid \text{ there are } \langle \xi, \rho \rangle \in a_{\gamma\beta} \text{ such that } \eta \text{ is permitted to both}$$

$$p^\xi, p^\rho, \pi_{\max(s),\xi}(\eta) = \bar{\xi} \text{ and } \pi_{\max(s),\rho}(\eta) = \bar{\rho}\}$$

is a function and it is an order preserving (or even an isomorphism between suitable structures.
This means that the copy of $a_{\gamma\beta}$ over η is compatible with the "local" function $a^\eta_{\gamma\beta}$.

13. For every $\alpha < \beta < \gamma < \omega_1$ and $\xi \in \mathrm{dom}(f_{\gamma\alpha})$ the following hold:

   (a) $\xi \in \mathrm{dom}(f_{\gamma\beta})$,

   (b) if $\nu = f_{\gamma\alpha}(\xi)$ and $\delta = f_{\gamma\beta}(\xi)$, then $o(\delta^0) > \alpha$ implies $f^{\delta^0}_{o(\delta^0)\alpha}(\delta) = \nu$.

14. For every $\gamma < \omega_1$, $\xi \in \bigcup_{\beta<\gamma} \mathrm{dom}(a_{\gamma\beta})$ and $k < \omega$
the following set is a final segment in γ:

$$\{\beta < \gamma \mid a_{\gamma\beta}(\xi) \text{ is defined and } k \text{ good}\}.$$

Let us deal now with the following situation. Suppose that $\gamma \in S$, $\alpha < \gamma$, $\alpha \notin \{\gamma_i \mid i < \omega\}$ and $n < \omega$ is a maximal such that $\gamma_n < \alpha$. Then $\underset{\sim}{\kappa}^{+\omega+2}_\gamma$ is connected with $\underset{\sim}{\kappa}^{+\omega+2}_{\gamma_n}$ and $\underset{\sim}{\kappa}^{+\omega+3}_\gamma$ with $\underset{\sim}{\kappa}^{+\omega+2}_\alpha$, and if $\alpha \in S$, then also with $\underset{\sim}{\kappa}^{+\omega+3}_\alpha$. On the other hand $\underset{\sim}{\kappa}^{+\omega+2}_\alpha$ is connected with $\underset{\sim}{\kappa}^{+\omega+2}_{\gamma_n}$, if $\alpha \notin S$ and $\underset{\sim}{\kappa}^{+\omega+3}_\alpha$ is connected with $\underset{\sim}{\kappa}^{+\omega+3}_{\gamma_n}$, if $\alpha \in S$, but $\gamma_n \notin \{\alpha_i \mid i < \omega\}$. We need to break down this connections between α and γ_n in order to keep the thing working. Namely the $\kappa^{+\omega+2}_\gamma$ chain condition of the forcing at the level γ is effected. It is easy to deal with a single γ_n (or with $\{\gamma_i \mid i \leq n\}$). We can just identify images at the level γ_n of different ordinals (or models) from the level α. The situation starts to be more involved once instead of a single γ we have ω–many with the corresponding γ_n's unbounded in α. This always occurs due to stationarity of S. The problem in this case is that if we identify too much, then there will not be enough ω–sequences to make 2^{κ_α} big.
Let us state now conditions that allow to identify certain ordinals (models), but still keep many sequences different.

15. Suppose $\delta \in A_\gamma$, for some $\gamma \in S$. Then $o^{\vec{E}}(\delta^0) = \gamma$. Consider $a^\delta_{\gamma\alpha}, a^\delta_{\gamma\gamma_n}$, where $\alpha < \gamma$ and $n < \omega$ is the maximal such that $\gamma_n < \alpha$.
We have a connection $a^\delta_{\alpha\gamma_n}$ between $\underset{\sim}{\kappa}_\alpha$ and $\underset{\sim}{\kappa}_{\gamma_n}$. Let us disturb it. We will allow to

identify some values. There is $\bar{a}^\delta_{\alpha\gamma_n} \subseteq \mathrm{dom}(a^\delta_{\alpha\gamma_n})$ on which $a^\delta_{\alpha\gamma_n}$ is order preserving (or isomorphism) but the rest of $\mathrm{dom}(a^\delta_{\alpha\gamma_n})$ is mapped into the image of $\bar{a}^\delta_{\alpha\gamma_n}$ by $a^\delta_{\alpha\gamma_n}$.
It will be allowed to change $\bar{a}^\delta_{\alpha\gamma_n}$ (once extending a condition) and to pass to a different set on which the order is preserved.

Now suppose that instead of a single γ_n (or even bounded many ones) α is a limit of ordinals ξ_i which members of the ω–sequences for ordinals $\xi \geq \gamma$.
Then for each such such ξ_i we will have a set $\bar{a}^\delta_{\alpha\xi_i} \subseteq \mathrm{dom}(a^\delta_{\alpha\xi_i})$ on which $a^\delta_{\alpha\xi_i}$ is order preserving (or isomorphism). Require the following:

(a) if ν, μ are in the support over $\underset{\sim}{\kappa}_\alpha$ and $\nu \neq \mu$, then the set

$$\{\xi_i < \alpha \mid a^\delta_{\alpha\xi_i}(\nu) = a^\delta_{\alpha\xi_i}(\mu)\}$$

is bounded in α;

(b) (Minimality) if ν, μ are in the support over $\underset{\sim}{\kappa}_\alpha$, $\nu \in \bar{a}^\delta_{\alpha\xi_i}$ and $a^\delta_{\alpha\xi_i}(\nu) = a^\delta_{\alpha\xi_i}(\mu)$, then $\mu \geq \nu$.

Let us explain how the above condition allows to run the chain condition argument. We will need to prove $\kappa_\gamma^{+\omega+2}$–c.c. of the forcing up to the level γ. As usual a Δ–system is formed and at the final stage of the argument we will need to put together two conditions with indexes from it. The problem here is that the kernel of the Δ–system includes the parts of both conditions over $\kappa_\gamma^{+\omega+2}$. The assignment functions then will move this common part to κ_{γ_n}. So over the level α we will have different things that should correspond to the same one over the level γ_n. Suppose for simplicity that $\alpha \notin S$. Then (over κ_α) we will have the kernel x (the image under $a^\delta_{\gamma\alpha}$'s) the rest of the first condition (again only over κ_α) y and the second z above it. We need to identify the images of y and z over the level γ_n, but still keep them different or even one above the other co-boundedly often. Denote by $\bar{y}, \bar{z}$ the parts on which we have the order preservation. Let $\nu = \min(\bar{y} \setminus x)$ and $\mu = \min(\bar{z} \setminus z)$. The sequences for ν and for μ behave the same way (i.e. equal, less, bigger) relatively the sequences for members of x, as parts of the Δ–system. Densely often above the level γ and hence in the sequence of conditions above the level γ used to determine the members of the antichain over γ, both ν and μ appear. So, their sequence from the level α differ on a co-bounded subset. This means that a final segment of both of them differs from any of the sequences of the kernel x as well as one from an other. In particular this means that $\nu = \min(y)$

and $\mu = \min(z)$, by the minimality item above. Now we can combine the conditions together.

Define a direct extension order $\leq^*$ on $\bar{\mathcal{P}}$ in the obvious fashion as follows.

Definition 2.2 (Direct extension order) Let $p = \langle\langle \xi, p^\xi\rangle \mid \xi \in s\rangle, \langle A_\alpha \mid \alpha < \omega_1\rangle, \langle a_{\gamma\beta}, f_{\gamma\beta} \mid \beta < \gamma \leq \omega_1\rangle, \langle a^\delta_{\beta\alpha}, f^\delta_{\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa\rangle\rangle$ and $q = \langle\langle \xi, q^\xi\rangle \mid \xi \in t\rangle, \langle B_\alpha \mid \alpha < \omega_1\rangle, \langle b_{\gamma\beta}, g_{\gamma\beta} \mid \beta < \gamma \leq \omega_1\rangle, \langle b^\delta_{\beta\alpha}, g^\delta_{\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa\rangle\rangle$ be in $\bar{\mathcal{P}}$. Set $q \leq^* p$ iff

1. $s \supseteq t$,
2. $p^\xi = q^\xi$,for every $\xi \in t$,
3. $\pi''_{\alpha,\max(s),\max(t)} A_\alpha \subseteq B_\alpha$, for every $\alpha < \omega_1$,
4. $a_{\gamma\beta} \supseteq b_{\gamma\beta}$, for every $\beta < \gamma < \omega_1\rangle$,
5. $f_{\gamma\beta} \supseteq g_{\gamma\beta}$, for every $\beta < \gamma < \omega_1\rangle$,
6. $a^\delta_{\beta\alpha} \subseteq b^\delta_{\beta\alpha}$, for every $\alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa$,
7. $f^\delta_{\beta\alpha} \subseteq g^\delta_{\beta\alpha}$, for every $\alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa$.

Let us define now an extension of a pure condition by one element.

Definition 2.3 (One element extension) Let $p = \langle\langle \xi, p^\xi\rangle \mid \xi \in s\rangle, \langle A_\alpha \mid \alpha < \omega_1\rangle, \langle a_{\gamma\beta}, f_{\gamma\beta} \mid \beta < \gamma \leq \omega_1\rangle, \langle a^\delta_{\beta\alpha}, f^\delta_{\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa\rangle\rangle \in \bar{\mathcal{P}}$, $\beta < \omega_1$ and $\eta \in A_\beta$. Define the extension of p by η, $p^\frown\eta$. It consists of two parts the upper part p^{up} and the lower part p^{lo}, where

$$p^{up} = \langle\langle \xi, p^\xi\rangle \mid \xi \in s \text{ and } \eta \text{ not permitted for } p^\xi\rangle, \langle A_\alpha \setminus \eta \mid o^{\vec{E}}(\eta) < \alpha < \omega_1\rangle,$$

$$\langle a_{\gamma\beta}, f_{\gamma\beta} \mid o^{\vec{E}}(\eta) < \beta < \gamma \leq \omega_1\rangle, \langle a^\delta_{\beta\alpha}, f^\delta_{\beta\alpha} \mid o^{\vec{E}}(\eta) < \alpha < \beta \leq o^{\vec{E}}(\delta), \eta < \delta < \kappa\rangle,$$

$$\langle f_{\gamma o^{\vec{E}}(\eta)} \mid o^{\vec{E}}(\eta) < \gamma < \omega_1\rangle, \langle f^\delta_{\gamma o^{\vec{E}}(\eta)} \mid o^{\vec{E}}(\eta) < \gamma \leq o^{\vec{E}}(\delta)\rangle, \langle \bar{f}_{\gamma o^{\vec{E}}(\eta)} \mid o^{\vec{E}}(\eta) < \gamma \leq \omega_1\rangle\rangle,$$

$$p^{lo} = \langle\langle \pi_{\max(s),\xi}(\eta), p^\xi\rangle \mid \xi \in s \text{ and } \eta \text{ is permitted for } p^\xi\rangle, \langle A_\alpha \cap \eta \mid \alpha < o^{\vec{E}}(\eta)\rangle,$$

$$\langle \bar{a}^\eta_{\gamma\beta}, f^\eta_{\gamma\beta} \mid \beta < \gamma \leq o^{\vec{E}}(\eta)\rangle, \langle a^\delta_{\beta\alpha}, f^\delta_{\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \eta^0\rangle\rangle,$$

where

1. $\bar{f}_{\gamma o^{\vec{E}}(\eta)}$ is a combination of $f_{\gamma o^{\vec{E}}(\eta)}$ with $a_{\gamma o^{\vec{E}}(\eta)}$, i.e.

$$\bar{f}_{\gamma o^{\vec{E}}(\eta)} = f_{\gamma o^{\vec{E}}(\eta)} \restriction \{\xi \mid (f_{\gamma o^{\vec{E}}(\eta)}(\xi))^0 = \eta^0\} \cup$$

$$\{\langle \xi, \pi_{\max(s), a_{\gamma o^{\vec{E}}(\eta)}(\xi)}(\eta)\rangle \mid \xi \in \mathrm{dom}(a_{\gamma o^{\vec{E}}(\eta)}), \eta \text{ is permitted for } p^{\xi}\}.$$

   Note that $a_{\gamma o^{\vec{E}}(\eta)}$ may be a name according to Definition 2.1(4b). So in such a case also $\bar{f}_{\gamma o^{\vec{E}}(\eta)}$ will be a name.
   It is important that all connections between the upper part (including η) and the lower part go through the level η. This way a completeness of the upper part regains.

   Note also that the following situation may occur:
   the connection between levels γ and α (i.e. $a_{\gamma\alpha}$) used models of greater cardinality than those between $o^{\vec{E}}(\eta)$ and α , for some $\alpha < o^{\vec{E}}(\eta)$. If this happen, then we still base the connection between $o^{\vec{E}}(\eta)$ and α (i.e. $\bar{a}^{\eta}_{o^{\vec{E}}(\eta)\alpha}$) on smaller models, but require that the largest of bigger size belongs to one of the smaller size.

2. $\bar{a}^{\eta}_{\gamma\beta} = a^{\eta}_{\gamma\beta}$, unless $\gamma = o^{\vec{E}}(\eta)$. If $\gamma = o^{\vec{E}}(\eta)$, then $\bar{a}^{\eta}_{\gamma\beta}$ is the combination of $a^{\eta}_{\gamma\beta}$ with the copy of $a_{\gamma\beta}$ over η, i.e.

$$\bar{a}^{\eta}_{\gamma\beta} = a^{\eta}_{\gamma\beta} \bigcup \{\langle \bar{\xi}, \bar{\rho}\rangle \mid \text{ there are } \langle \xi, \rho\rangle \in a_{\gamma\beta} \text{ such that } \eta \text{ is permitted to both}$$

$$p^{\xi}, p^{\rho}, \pi_{\max(s),\xi}(\eta) = \bar{\xi} \text{ and } \pi_{\max(s),\rho}(\eta) = \bar{\rho}\}.$$

   Note that such defined $\bar{a}^{\eta}_{\gamma\beta}$ is an order preserving function by 2.1 (12).
   The connection $a_{\epsilon\beta}$, for ϵ's above $o^{\vec{E}}(\eta)$, is replaced now by the composition of $f^{\eta}_{\epsilon, o^{\vec{E}}(\eta)}$ (or its further extensions) with $\bar{a}^{\eta}_{o^{\vec{E}}(\eta),\beta}$.

The definition above takes care of a problem of completeness. Thus the following situation always occurs:
some $\delta < \omega_1$ is a dropping level for $\aleph_1$–many levels of a pure condition p.
In this case the direct extension order $\leq^*$ will be at most κ_δ–closed. But once a one element non-direct extension was made using $\eta^0 > \kappa_\delta$, the following will happen:
the connection functions $a_{\beta\delta}$ split to parts up to η and below η, by 2.3(1). The parts above η are η^0–closed. Note that connections to the level of η not one to one anymore, since a's are replaced by f's.
Not only connections to the level of η stop to be one to one, but in addition also connections to levels which drop to the level of η or below. Let us explain this point in more details. Thus suppose that we have levels $\gamma > \beta, \gamma \in S$ above the level of η, for some $k < \omega$, $\gamma_{k+1} > \beta$

but $\beta > \gamma_k$ and $\gamma_k \leq$ the level of η. In this situation we have a drop in cofinality from β to γ_k. The cardinality of $a_{\gamma\beta}$ should be at most η (actually it should be less than κ_{γ_k} once this cardinal is decided). So we should loose completeness due to the size of $a_{\gamma\beta}$ for each $\beta, \gamma_k < \beta < \gamma_{k+1}$. Also there may be $\aleph_1$ many γ's with γ_m below the level η.
The way to overcome this difficulty will be to replace connection functions $a_{\gamma\beta}$ (for γ, β which are like this) by a combination of it with the function $u_{\gamma\beta}$ which mention η. We give up the order preservation here.
An additional refinement is needed, since $\kappa_\gamma^{+\omega+2}$–c.c. of the forcing up to the level γ may be effected otherwise. Namely, running the Δ–system argument there will be a need to deal with the following situation:
two different ordinals (or models) $\xi_1 < \xi_2$ at the level γ in two conditions that we like to combine and to attach to them different ordinals $\rho_1 < \rho_2$ at the level β. Say presently the same ρ corresponds to both. Usually we pick some ρ' similar enough to ρ and extend conditions by adding the missing ξ and sending it to ρ'. But in the present situation for each choice of κ_{γ_k} there may be some ζ in the common part which $u_{\gamma\beta}$ moves (in both conditions) to ρ. This makes impossible to move from ρ to ρ' and so to combine such conditions.
Let us define $u_{\gamma\beta}$ more carefully. Thus instead of relying on γ_k let us move to $\gamma_k + 1$. It is not in S and it is not of the form δ_n for any $\delta \in S$. So, $\gamma_k + 1$ is the first level above γ_k which drops to γ_k. We require that if a non-direct extension was made at the level γ_k, then such extension made at the level $\gamma_k + 1$ as well. Now, $u_{\gamma\beta}$ will keep information about connection to the level $\gamma_k + 1$ instead those to γ_k. The advantage in the chain condition argument will be that in the situation described above, we first arrange compatibility at the level $\gamma_k + 1$, i.e. find some similar τ, τ' at this level and make the assignment function a_{γ,γ_k+1} to move ξ_1, ξ_2 to τ, τ'. Now we can pick ρ' and move ρ, ρ' also to τ, τ' but using a_{β,γ_k+1}. The problem that we had above (using γ_k instead of $\gamma_k + 1$) does not occur now, since we managed to get different values at the level $\gamma_k + 1$. This was impossible with γ_k due to different cofinalities of levels β and γ_k. Note that cofinalities of levels β and γ_k are the same.

Inside a pure condition:

1. $u_{\gamma\beta} \subseteq \kappa^2$.
   Note that we do not require that $u_{\gamma\beta}$ is a function. It is needed in order to prove $\kappa_\gamma^{+\omega+2}$–c.c. of the final forcing below γ. A single value may prevent a possibility of putting together equivalent condition.

2. $|u_{\gamma\beta}| < \kappa$,

3. $\mathrm{dom}(u_{\gamma\beta}) \subseteq \mathrm{dom}(f_{\gamma\beta})$,

4. if for some $\tilde{\xi} \in \mathrm{dom}(u_{\gamma\beta})$ we have $|u_{\gamma\beta}(\tilde{\xi})| > 1$, then

   (a) $\mathrm{rng}(a_{\gamma\beta}) \supseteq u_{\gamma\beta}(\tilde{\xi})$,
   or

   (b) for some $\xi \in \mathrm{dom}(a_{\gamma\beta})$, $u_{\gamma\beta}(\tilde{\xi}) \in a_{\gamma\beta}(\xi)$ and $u_{\gamma\beta}(\tilde{\xi})$ is simply definable inside $a_{\gamma\beta}(\xi)$ (say using ordinals from $\underset{\sim}{\kappa}{}^{+}_{\beta}$).

Suppose now that κ_{γ_k+1} was decided. We combine then $a_{\gamma\beta}$ and $u_{\gamma\beta}$ into a new $u_{\gamma\beta}$ as follows.
Let $\tilde{\xi} \in \mathrm{dom}(u_{\gamma\beta})$.
Case 1. *For some* $\xi \in \mathrm{dom}(a_{\gamma\beta})$ $a_{\gamma\beta}(\xi) = u_{\gamma\beta}(\tilde{\xi})$ *or* $a_{\gamma\beta}(\xi) \in u_{\gamma\beta}(\tilde{\xi})$.
If there is such ξ with $f_{\gamma\gamma_k+1}(\xi) = f_{\gamma\gamma_k+1}(\tilde{\xi})$, then we leave $\tilde{\xi}$ in the domain of the new $u_{\gamma\beta}$ and leave only $a_{\gamma\beta}(\xi)$ as its unique image. Otherwise $\tilde{\xi}$ is removed.
Case 2. *Not Case 1, but there is* $\xi \in \mathrm{dom}(a_{\gamma\beta})$,$u_{\gamma\beta}(\tilde{\xi}) \in a_{\gamma\beta}(\xi)$ *and* $u_{\gamma\beta}(\tilde{\xi})$ *is simply definable inside* $a_{\gamma\beta}(\xi)$.
If there is such ξ with $f_{\gamma\gamma_k+1}(\tilde{\xi})$ simply definable inside $f_{\gamma\gamma_k+1}(\xi)$, then we leave $\tilde{\xi}$ in the domain of the new $u_{\gamma\beta}$ and leave only the one inside $a_{\gamma\beta}(\xi)$ as its unique image. Otherwise $\tilde{\xi}$ is removed.
Case 3. *Not Case 1, Case 2.*
Then we keep $u_{\gamma\beta}(\tilde{\xi})$ as it is.

The idea here is that once a non–direct extension over $\gamma_k + 1$ was made– we more or less copy the connection between levels $\gamma, \gamma_k + 1$ to those between γ, β. So the function $u_{\gamma\beta}$, which replaces $a_{\gamma\beta}$ now, is not order preserving. For each $\tilde{\xi}$ in its domain we have $f_{\gamma\gamma_k+1}(\xi) = f_{\beta\gamma_k+1}(u_{\gamma\beta}(\tilde{\xi}))$ which is viewed as a non–direct extension over $\gamma_k + 1$ responsible for $u_{\gamma\beta}(\tilde{\xi})$.
Cases 1–3 above describe how the irrelevant (for the choices made over $\gamma_k + 1$–level) information is removed.

It may be that a non-direct extension was made at some level $\alpha, \gamma_{k+1} < \alpha < \beta$.

Denote the set of one element extensions of elements of $\bar{\mathcal{P}}$ by $\bar{\mathcal{P}}_1$. Extend $\leq^*$ to $\bar{\mathcal{P}}_1$ in the obvious fashion. Repeat Definition 2.3 and define $\bar{\mathcal{P}}_2$ to be the set of one element extensions of elements of $\bar{\mathcal{P}}_1$, etc.
Finally set

$$\mathcal{P} = \bigcup_{n<\omega} \bar{\mathcal{P}}_n,$$

where $\bar{\mathcal{P}}_0 = \bar{\mathcal{P}}$. Extend $\leq^*$ to $\bar{\mathcal{P}}_n$'s and $\mathcal{P}$ in the obvious fashion.

Definition 2.4 (Order) Let $p, q \in \mathcal{P}$. Set $p \geq q$ iff there exists a finite sequence $\langle r_k \mid k < n < \omega \rangle$ of elements of $\mathcal{P}$ such that

1. $r_0 = q$,
2. $r_{n-1} = p$,
3. $r_k \leq^* r_{k+1}$ or r_{k+1} is a one element extension of r_k, for every $k < n - 1$.

Let $p \in \mathcal{P}$ and for some $\alpha < \omega_1$ the value of κ_α is decided. Then p splits naturally into two parts – the part $p_{\leq \alpha}$ from the level α down and the part $p_{>\alpha}$ above the level α.

Definition 2.5 Let $\alpha < \omega_1$.

1. Set $\mathcal{P}_{\leq \alpha}$ to be the set of all $p_{\leq \alpha}$ with $p \in \mathcal{P}$ which decides κ_α.
2. Set $\mathcal{P}_{>\alpha}$ to be the set of all $p_{>\alpha}$ with $p \in \mathcal{P}$ which decides κ_α.

Lemma 2.6 *Let $\alpha < \omega_1$. Then $\langle \mathcal{P}_{>\alpha}, \leq^* \rangle$ is κ_α^+–closed forcing.*

Proof. The proof follows from the way conditions split–namely 2.3(1).
□

Lemma 2.7 *(Prikry condition)*
$\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.

Proof. Given Lemma 2.6 – standard arguments apply.
□

3 The Main Forcing Order

Define a partial order $\longrightarrow$ on $\mathcal{P}$ such that $\langle \mathcal{P}, \rightarrow \rangle$ will be nice subforcing of $\langle \mathcal{P}, \leq \rangle$ and $\langle \mathcal{P}_{\leq \alpha}, \rightarrow \rangle$ will satisfy $\kappa_\alpha^{+\omega+2}$–c.c., for every $\alpha < \omega_1$.

We start with a definition of equivalences $\longleftrightarrow$.

Definition 3.1 (Equivalence of pure conditions) Let $\eta < \omega_1$,

$p = \langle\langle \xi, p^\xi\rangle \mid \xi \in s\rangle, \langle A_\alpha \mid \alpha < \eta\rangle, \langle a_{\gamma\beta}, f_{\gamma\beta} \mid \beta < \gamma \leq \eta\rangle, \langle a^\delta_{\beta\alpha}, f^\delta_{\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa_\eta\rangle\rangle$ and

$q = \langle\langle \xi, q^\xi\rangle \mid \xi \in t\rangle, \langle B_\alpha \mid \alpha < \eta\rangle, \langle b_{\gamma\beta}, g_{\gamma\beta} \mid \beta < \gamma \leq \eta\rangle, \langle b^\delta_{\beta\alpha}, g^\delta_{\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa_\eta\rangle\rangle$

be in $\mathcal{P}_{\leq\eta}$.
Set $p \longleftrightarrow_\eta q$ iff the following hold:

1. $s = t$,
2. $A_\alpha = B_\alpha$ for each $\alpha < \eta$,
3. $p^\xi = q^\xi$ for every $\xi \in s$,
4. $f_{\gamma\beta} = g_{\gamma\beta}$, for every $\beta < \gamma \leq \eta$,
5. $f^\delta_{\beta\alpha} = g^\delta_{\beta\alpha}$, for every $\alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa_\eta$,
6. $a^\delta_{\beta\alpha} = b^\delta_{\beta\alpha}$, for every $\alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa_\eta$,
7. $a_{\gamma\beta} = b_{\gamma\beta}$, $\beta < \gamma \leq \eta$,
8. $\mathrm{dom}(a_{\eta\gamma}) = \mathrm{dom}(b_{\eta\gamma})$, for every $\gamma < \eta$,
9. $\mathrm{rng}(a_{\eta\gamma})$ and $\mathrm{rng}(b_{\eta\gamma})$ realize the same k-type, for every $k < \omega$, for a final segment of γ's below η. Moreover they always (for each $\gamma < \eta$) realize the same 4-type.

Definition 3.2 (Equivalence) Let $\eta < \omega_1$ and $p, q \in \mathcal{P}_{\leq\eta}$.
Set $p \longleftrightarrow_\eta q$ iff the following hold

1. the sets of κ_γ's for $\gamma < \eta$ determined by p and by q are the same.
   Denote this set by X and let $Y = \{\gamma < \eta \mid \kappa_\gamma \in X\}$. Clearly both X and Y are finite.
2. If $Y = \emptyset$, then $p \longleftrightarrow_\eta q$ as in Definition 3.1.
3. If $Y \neq \emptyset$, then
   (a) $p_{\leq\min(Y)} \longleftrightarrow_{\min(Y)} q_{\leq\min(Y)}$,
   (b) $p_{>\max(Y)} \longleftrightarrow_\eta q_{>\max(Y)}$,
   (c) $(p_{\leq\gamma})_{>\beta} \longleftrightarrow_\gamma (q_{\leq\gamma})_{>\beta}$, for any two successive elements $\beta < \gamma$ of Y.

Definition 3.3 (Main order of $\mathcal{P}_{\leq\eta}$) Let $\eta < \omega_1$ and $p, q \in \mathcal{P}_{\leq\eta}$.
Set $p \rightarrow_\eta q$ iff there exists a finite sequence $\langle r_i \mid i \leq k \rangle$ of elements of $\mathcal{P}_{\leq\eta}$ such that

1. $p = r_0$,
2. $q = r_k$,
3. for every $i < k$ either
   - $r_i \leq r_{i+1}$
     or
   - $r_i \longleftrightarrow_\eta r_{i+1}$.

Definition 3.4 (Main order) Let $p, q \in \mathcal{P}$. We set $p \rightarrow q$ iff either

1. $p \leq q$
   or
2. there is $\eta < \omega_1$ such that κ_η is determined the same way in both p, q and the following hold:
   (a) $p_{>\eta} \leq q_{>\eta}$.
   It means that nothing new, not taken into account by "$\leq$", happen above level η.
   (b) $p_{\leq\eta} \rightarrow_\eta q_{\leq\eta}$.

The next lemma insures that $\langle \mathcal{P}, \rightarrow \rangle$ is a nice subforcing of $\langle \mathcal{P}, \leq \rangle$, i.e. every dense open set in $\langle \mathcal{P}, \rightarrow \rangle$ generates such a set in $\langle \mathcal{P}, \leq \rangle$. The proof is similar to the corresponding lemma of [**?**, Sec. 5].

Lemma 3.5 *Suppose that $p \rightarrow q \leq q'$ then there is $p' \geq p$ such that $q' \rightarrow p'$, where $p, q, q', p' \in \mathcal{P}$.*

Lemma 3.6 *The following hold in $V^{\langle \mathcal{P}, \rightarrow \rangle}$, for every limit $\eta < \omega_1$:*

- $2^{\kappa_\eta} \geq (\kappa_\eta^{+\omega+2})^V$,
- *if $i \in S$, then $2^{\kappa_\eta} \geq (\kappa_\eta^{+\omega+3})^V$.*

Note that actually all the cardinals in the interval $[\kappa_\eta^{++}, \kappa_\eta^{+\omega+1}]$ are collapsed to κ_η^{+} which itself is preserved as the successor of a singular.

Lemma 3.7 *Let $\eta < \omega_1$. If $\eta \notin S$ then, in $V^{\langle \mathcal{P}, \to \rangle}$, $2^{\kappa_\eta} \leq (\kappa_\eta^{+\omega+2})^V$.*

Proof. Just the forcing splits into $\mathcal{P}_{>\eta}$, which is κ_η^+ closed, and into $\mathcal{P}_{\leq\eta}$. Formally the cardinality of $\mathcal{P}_{\leq\eta}$ is $\kappa_\eta^{+\omega+3}$, but actually the forcing $\langle \mathcal{P}_{\leq\eta}, \to_\eta \rangle$ produces η–sequences indexed only by $\kappa_\eta^{+\omega+2}$, since $\eta \notin S$. The formal argument is given below.

Let $M \prec H(\chi)$, for χ big enough, containing all the relevant information such that

- $|M| = \kappa_\eta^{+\omega+2}$,
- $M \cap \kappa_\eta^{+\omega+3}$ is an ordinal,
- M is closed under $\kappa_\eta^{+\omega+1}$ sequences of its elements.

Let $p \in \mathcal{P}_{\leq\eta}$. Then, using elementarity there will be a condition $p^M \in M \cap \mathcal{P}_{\leq\eta}$ such that $p \longleftrightarrow_\eta p^M$ and even $p \restriction \kappa_i^{+\omega+2} = p^M \restriction \kappa_i^{+\omega+2}$. This means that $\langle \mathcal{P}_{\leq\eta}, \to_\eta \rangle$ and $\langle \mathcal{P}_{\leq\eta} \cap M, \to_\eta \rangle$ are just the same from forcing point of view. But $|\mathcal{P}_{\leq\eta} \cap M| = \kappa_\eta^{+\omega+2}$.
$\square$

Our next task will be to show $\kappa_\eta^{+\omega+2}$-c.c. of the forcing $\langle \mathcal{P}_{\leq\eta}, \to_\eta \rangle$ in $V^{\mathcal{P}'}$ for each $\eta < \omega_1$.

Lemma 3.8 *(Chain condition lemma) Suppose that $\eta < \omega_1$ and the value of κ_η is decided. Then, in $V^{\mathcal{P}'}$, $\langle \mathcal{P}_{\leq\eta}, \to_\eta \rangle$ satisfies $\kappa_\eta^{+\omega+2}$ – c.c.*

Proof. Work in $V^{\mathcal{P}'}$.
Let us deal with $\eta \notin S$. If $\eta \in S$, then the argument is very similar but with ordinals replaced by models see [4].

If $\eta = \eta' + 1$, then we decide $\kappa_{\eta'}$. The forcing $\langle \mathcal{P}_\eta, \to_\eta \rangle$ will be then consists of two parts $\mathcal{P}_{\eta'}$ and a part isomorphic to the Cohen forcing for adding subsets to κ_η^+. As in Lemma 3.7, $\mathcal{P}_{\eta'}$ is equivalent to a forcing of small cardinality, and so we are done.

Assume now that η is a limit ordinal.

Let $\langle p_\tau \mid \tau < \kappa_\eta^{+\omega+2} \rangle$ be a sequence of elements of $\mathcal{P}_{\leq\eta}$ and

$$p_\tau = \langle \langle \langle \xi, p_\tau^\xi \rangle \mid \xi \in s_\tau \rangle, \langle A_{\tau\alpha} \mid \alpha < \eta \rangle, \langle a_{\tau\gamma\beta}, f_{\tau\gamma\beta} \mid \beta < \gamma \leq \eta \rangle, \langle a^\delta_{\tau\beta\alpha}, f^\delta_{\tau\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa_\eta \rangle \rangle$$

We may assume without loss of generality that p_τ's are of this form, since the number of possibilities for low parts is small and so they may be assumed to be the same and then just ignored since then the incompatibility if occurs will be due to the upper parts of the conditions.

Shrinking and replacing by $\longleftrightarrow_\eta$ if necessary, we can assume that the following hold:

1. $A_{\tau\alpha} = A_{\tau'\alpha}$ for each $\tau, \tau' < \kappa_\eta^{+\omega+2}$ and $\alpha < \eta$,

2. $\langle s_\tau \mid \tau < \kappa_\eta^{+\omega+2} \rangle$ form a Δ-system with a kernel s,

3. $s_\tau, s_{\tau'}$ are order isomorphic over s for every $\tau, \tau' < \kappa_\eta^{+\omega+2}$,

4. $p_\tau^\xi = p_{\tau'}^{\sigma_{\tau\tau'}(\xi)}$, where $\sigma_{\tau\tau'}$ is the order isomorphism between s_τ and $s_{\tau'}$,

5. $s \supseteq s_\xi \cap \kappa_\eta^{+\omega+1}$ for every $\xi < \kappa_\eta^{+\omega+2}$,

6. $\min(s_\xi \setminus s) \geq \xi$,

7. $\langle a^\delta_{\tau\beta\alpha}, f^\delta_{\tau\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa_\eta \rangle = \langle a^\delta_{\tau'\beta\alpha}, f^\delta_{\tau'\beta\alpha} \mid \alpha < \beta \leq o^{\vec{E}}(\delta), \delta < \kappa_\eta \rangle$, for every $\tau, \tau' < \kappa_\eta^{+\omega+2}$,

8. $f_{\tau\gamma\beta}$ and $f_{\tau'\gamma\beta}$ are compatible for every $\beta < \gamma \leq \eta, \tau, \tau' < \kappa_\eta^{+\omega+2}$,

9. $a_{\tau\gamma\beta} = a_{\tau'\gamma\beta}$, for every $\beta < \gamma < \eta, \tau, \tau' < \kappa_\eta^{+\omega+2}$.
   Note that in order to insure this we may need to pass to conditions that are $\longleftrightarrow_\eta$ equivalent to the original ones.

10. $\mathrm{rng}(a_{\tau\eta\beta}) = \mathrm{rng}(a_{\tau'\eta\beta})$, for every $\beta < \eta, \tau, \tau' < \kappa_\eta^{+\omega+2}$.
    As in the previous item, the above may require to pass to $\longleftrightarrow_\eta$ equivalent conditions.

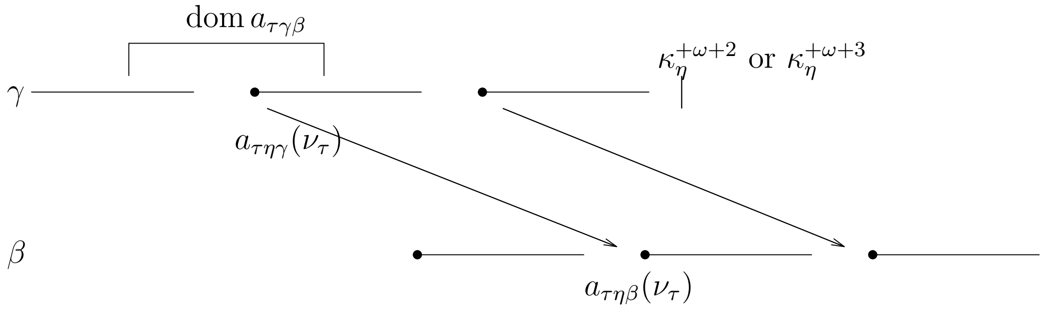
Extending, if necessary, we may assume that for every τ if $\nu \in s_\tau$ and $\mathrm{cof}(\nu) \leq \kappa_\eta$ then a closed cofinal sequence witnessing $\mathrm{cof}(\nu)$ is contained in s_τ. This implies $\mathrm{cof}(\min(s_\tau \setminus s)) \geq \kappa_\eta^+$. Let $\nu_\tau = \min(s_\tau \setminus s)$ for $\tau < \kappa_\eta^{+\omega+2}$. Consider $\langle a_{\tau\eta\beta}(\nu_\tau) \mid \beta < \gamma \rangle$. By Definition 2.1(14), for every $k < \omega$, $a_{\tau\eta\beta}(\nu_\tau)$ is k-good, for a final segment of $\beta < \eta$.
Shrinking if necessary, we may assume that this final segments are the same for all τ's and k's. Fix an increasing sequence $\langle k_i \mid i < \omega \rangle$ which converges to infinity and $k_0 \geq 4$. Make a non-direct extension and freeze (i.e. make it independent of τ's) the initial segment consisting of β's which are not k_0-good.

Let now $\tau' < \tau < \kappa_\eta^{+\omega+2}$. We would like to show compatibility of p_τ and $p_{\tau'}$ in $\langle \mathcal{P}_{\leq\eta}, \rightarrow_\eta \rangle$. We proceed similar to [1, 2.20].
Let $\ell < \omega$ and γ be from the final segment of k_ℓ. Consider $a_{\tau\eta\gamma}(\nu_\tau)$. It is k_ℓ–good, of cofinality κ_η^+ and, once κ_γ is determined, it will correspond to an ordinal of cofinality κ_γ^+, by Definition 2.1(11)). Consider $k_\ell - 1$–type that $(\mathrm{rng}(a_{\tau\eta\gamma}) \setminus a_{\tau\eta\gamma}(\nu_\tau)) \cup (\bigcup_{\beta<\gamma} \mathrm{dom}(a_{\tau\gamma\beta}) \setminus a_{\tau\eta\gamma}(\nu_\tau))$ realizes over $(\mathrm{rng}(a_{\tau\eta\gamma}) \cap a_{\tau\eta\gamma}(\nu_\tau)) \cup (\bigcup_{\beta<\gamma} \mathrm{dom}(a_{\tau\gamma\beta}) \cap a_{\tau\eta\gamma}(\nu_\tau))$. Note that the last set is bounded in $a_{\tau\eta\gamma}(\nu_\tau)$ since its size is $\leq \kappa_\eta$. Realize this type below $a_{\tau\eta\gamma}(\nu_\tau)$ over

$(\mathrm{rng}(a_{\tau\eta\gamma}) \cap a_{\tau\eta\gamma}(\nu_\tau)) \cup (\bigcup_{\beta<\gamma} \mathrm{dom}(a_{\tau\gamma\beta}) \cap a_{\tau\eta\gamma}(\nu_\tau))$. Let t_γ denotes the result. Now we change $a_{\tau'\eta\gamma}$ in the obvious fashion by sending the part that was above $a_{\tau\eta\gamma}(\nu_\tau)$ into t_γ. This allows to combine $a_{\tau\eta\gamma}$ with such changed $a_{\tau'\eta\gamma}$.

The above takes care of assignment functions from the level η. What remains is to change $a_{\tau'\gamma\beta}$, for $\beta < \gamma < \eta$, according to the commutativity requirements (Definition 2.1(9)). It can be done easily using the fact that $a_{\tau\eta\beta}(\nu_\tau) = a_{\tau\gamma\beta}(a_{\tau\eta\gamma}(\nu_\tau))$, for every $\beta < \gamma < \eta$.

See the diagram:



□

Lemma 3.9 *Let $\eta < \omega_1$. Every cardinal of V of the form $\kappa_\eta^{+n}, 2 \leq n \leq \omega + 1$ is collapsed to $\kappa_\eta^+ = (\kappa_\eta^+)^V$ in $V^{\langle \mathcal{P}_\eta, \to_\eta \rangle}$.*

Proof. Just size of assignment functions is κ_η over η and $\to_\eta$ does not effect things below $\kappa_\eta^{+\omega+1}$.
□

4 Collapsing Successors of Singulars

In this section we describe how using supercompacts to collapse κ_i^+'s one can obtain a model satisfying

(1) $2^{\kappa_i} = \kappa_i^{+3}$, if $i \in S$.

(2) $2^{\kappa_i} = \kappa_i^{++}$, if $i \notin S$.

(3) $(\kappa^+)^V < \kappa^+$.

The construction repeats the previous one, but instead of using the usual extender sequence, we shall use here a $\mathcal{P}_\kappa(\kappa^+)$ extender sequence of the length $\kappa^{+\omega+3}$. Let us define such a sequence. Assume that κ is $\kappa^{+\omega+3}$ – supercompact. Let $j : V \longrightarrow M$ be a witnessing embedding. Define from j a $\mathcal{P}_\kappa(\kappa^+)$ – extender sequence $\langle E_\tau \mid \tau < \kappa^{+\omega+3}\rangle$ of the length $\kappa^{+\omega+3}$ as follows: for every $X \subseteq \mathcal{P}_\kappa(\kappa^+) \times V_\kappa$

$$X \in E_\tau \ \text{ iff } \ \langle j''\kappa^+, \tau\rangle \in j(X)\ .$$

Let $N_\tau = Ult(V, E_\tau)$, $N = Ult(V, \langle E_\tau \mid \tau < \kappa^{+\omega+3}\rangle)$ and

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "i_\tau"'] \arrow[dr, "i"] & M \\
N_\tau \arrow[r, "k_\tau"'] & N \arrow[u, "k"']
\end{tikzcd}$$

be the corresponding diagram with embeddings defined in the usual way.

Lemma 4.1 *$i_\tau''\kappa^+ \in N_\tau$, ${}^{\kappa^+}N_\tau \subseteq N_\tau$, $i''\kappa^+ \in N$, ${}^{\kappa^+}N \subseteq N$, $crit(k) = \kappa^{+3}$, $H_{\kappa^{+\omega+3}} = (H_{\kappa^{+\omega+3}})^N$.*

Proof. Just note that $i_\tau''\kappa^+$ is represented by the function $(P, \alpha) \mapsto P$. $\square$

The extender based Prikry forcing with such extender $\langle E_\tau \mid \tau < \kappa^{+\omega+3}\rangle$ will blow up the power of κ to $\kappa^{+\omega+3}$ but also will collapse κ^+ to κ changing its cofinality to ω, due to the $\mathcal{P}_\kappa(\kappa^+)$ – supercompact ingredient of the extender.

Here we will use a version of Magidor extended based forcing defined in previous sections, with only change to $\mathcal{P}_\kappa(\kappa^+)$ – extenders, but the supports now will be of cardinality κ^+, due to number of possible choices from a supercompact measure over $\mathcal{P}_\kappa(\kappa^+)$, and not κ, as before. [1]

Thus we assume that

$$\vec{E} = \langle E(\alpha, \beta) \mid \alpha \leq \kappa, \alpha \in \mathrm{dom}\vec{E}\ ,\ \beta < \omega_1\rangle$$

is a coherent sequence satisfying condition (a) – (c) of $\vec{E}$ of Section 1. Only in (a) we require here that $E(\alpha, \beta)$ is a $(\mathcal{P}_\alpha(\alpha^+), \alpha^{+\omega+3})$ extender, i.e one of the type considered above. Also, $E(\alpha, \beta)(\tau)$ will be now the set

$$\{X \subseteq \mathcal{P}_\alpha(\alpha^+) \times \alpha \mid (j_{E(\alpha,\beta)}''(\alpha^+), \tau) \in j_{E(\alpha,\beta)}(X)\}\ .$$

The rest of the construction is without changes. The supercompact part of the forcing will change cofinality of each $(\kappa_i^+)^V (i < \omega_1)$ to ω by adding to it a cofinal sequence of order type i.

[1] Actually this is the point that prevents collapse of $\kappa^{+\omega+1}$ to κ, and it is collapsed rather to κ^{++}, which will be the new κ^+.

5 Concluding Remarks

5.1 Down to $\aleph_{\omega_1}$

Combining the present construction with the techniques for collapsing cardinals of Merimovich [5] it is possible to turn κ into $\aleph_{\omega_1}$. For $i < \omega_1$, we start collapses from $\kappa_i^{+\omega+5}$ and insure by this that they will depend only on the normal measure of the extender $E(\kappa, i)$. This way the equivalence relation $\leftrightarrow$ will not effect them.

5.2 Other Stationary Sets

Recall that S was a subset of a club. Outside of a club we are basically free. Only, as in 5.1, for each $i < \omega_1$ we need to start changes above $\kappa_i^{+\omega+5}$ in order to make the final thing work.

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