Some constructions of ultrafilters over a measurable cardinal.

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Abstract

Some non-normal $\kappa-$complete ultrafilters over a measurable κ with special properties are constructed. Questions by A. Kanamori [4] about infinite Rudin-Frolik sequences, discreteness and products are answered.

1 Introduction.

We present here several constructions of $\kappa-$complete ultrafilters over a measurable cardinal κ and examine their consistency strength. Some questions of Aki Kanamori from [4] are answered.

Section 2 deals with Rudin-Frolik ordering and answers Question 5.11 from [4] about infinite increasing Rudin-Frolik sequences. In Section 3, an example of non-discrete family of ultrafilters is constructed, answering Question 5.12 from [4]. Also the strength of existence of such family is examined. Section 4 deals with products of ultrafilters. A negative answer to Question 5.8 from [4] given.

2 On Rudin-Frolik increasing sequences.

In [4], Aki Kanamori asked if there exists a $\kappa-$ultrafilter with an infinite number of Rudin-Frolik predecessors.

We show that starting with $o(\kappa) = 2$ it is possible.

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Assume GCH. Let

$$\vec{U} = \langle U(\alpha,\beta) \mid (\alpha,\beta) \in \mathrm{dom}(\vec{U}), \alpha \le \kappa, \beta < o^{\vec{U}}(\alpha)\rangle$$

be a coherent sequence such that $o^{\vec{U}}(\kappa) = 2$ and for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) \le 1$. Let

$$A = \{\alpha \mid \exists\beta(\alpha,\beta) \in \mathrm{dom}(\vec{U})\}.$$

Then for every $\alpha \in A$, $o^{\vec{U}}(\alpha) = 1$ and $U(\alpha,0)$ is a normal ultrafilter over α.

We force with Easton support iteration of the Prikry forcings with $U(\alpha,0)$'s (and their extensions), $\alpha \in A$, as in [1] (a better presentation appears in [2]). Let G be a generic. Then for every increasing sequence t of ordinals less than κ, the normal ultrafilter $U(\kappa,1)$ of V extends to a $\kappa-$complete ultrafilter $U(\kappa,1,t)$ in $V[G]$, see [1], p.291.
Denote by b_α the Prikry sequence from G added to α, for every $\alpha \in A$. Then $U(\kappa,1,t)$ concentrates on $\alpha \in A$ for which b_α starts from t, i.e. $b_\alpha \restriction |t| = t$.

Let $\bar{U}(\kappa,0)$ be the canonical extension of $U(\kappa,0)$ to a normal ultrafilter in $V[G]$ defined as in [2] on page 290.
Denote $U(\kappa,1,\langle\rangle)$ by $\bar{U}(\kappa,1)$.

Lemma 2.1 *For every $n, 0 < n < \omega$, $\bar{U}(\kappa,1) = \bar{U}(\kappa,0)^n - \lim\langle U(\kappa,1,t) \mid t \in [\kappa]^n\rangle$.*

Proof. Recall the definition of $U(\kappa,1,t)$, $t \in [\kappa]^m, m < \omega$:
$X \in U(\kappa,1,t)$ iff for some $r \in G, \gamma < \kappa^+, B \in \bar{U}(\kappa,0)$, in $M_{U(\kappa,1)}$ the following holds:

$$r \cup \{\langle t, \underset{\sim}{B}\rangle\} \cup \underset{\sim}{p}_\gamma \Vdash \kappa \in i_{U(\kappa,1)}(\underset{\sim}{X}),$$

where p_γ is the $\gamma-$th element of the canonical master sequence.
In particular, $X \in \bar{U}(\kappa,1)$ iff for some $r \in G, \gamma < \kappa^+, B \in \bar{U}(\kappa,0)$, in $M_{U(\kappa,1)}$ the following holds:

$$r \cup \{\langle\langle\rangle, \underset{\sim}{B}\rangle\} \cup \underset{\sim}{p}_\gamma \Vdash \kappa \in i_{U(\kappa,1)}(\underset{\sim}{X}).$$

Then, for every $t \in [B]^n$, we will have

$$r \cup \{\langle t, \underset{\sim}{B} \setminus \max(t)+1\rangle\} \cup \underset{\sim}{p}_\gamma \Vdash \kappa \in i_{U(\kappa,1)}(\underset{\sim}{X}).$$

So, $X \in U(\kappa,1,t)$. But $[B]^n \in \bar{U}(\kappa,0)^n$, hence $X \in \bar{U}(\kappa,0)^n - \lim\langle U(\kappa,1,t) \mid t \in [\kappa]^n\rangle$.

Hence we showed that $\bar{U}(\kappa,1) \subseteq \bar{U}(\kappa,0)^n - \lim\langle U(\kappa,1,t) \mid t \in [\kappa]^n\rangle$. But this already implies the equality, since both $\bar{U}(\kappa,1)$ and $\bar{U}(\kappa,0)^n - \lim\langle U(\kappa,1,t) \mid t \in [\kappa]^n\rangle$ are ultrafilters. $\square$

Lemma 2.2 *The family $\langle U(\kappa, 1, t) \mid t \in [\kappa]^n \rangle$ is a discrete family of ultrafilters.*

Proof. For each $t \in [\kappa]^n$ set

$$A_t := \{\alpha \in A \mid b_\alpha \restriction n = t\}.$$

Let $t, t' \in [\kappa]^n$ be two different sequences, then, clearly, $A_t \cap A_{t'} = \emptyset$.
$\square$

Recall the following definition:

Definition 2.3 (Frolik and M.E. Rudin) Let U, D be ultrafilters over I. $U \geq_{R-F} D$ iff there is a discrete family $\{E_i \mid i \in I\}$ of ultrafilters over some J such that $U = D-\lim\{E_i \mid i \in I\}$.

So we obtain the following:

Theorem 2.4 *$\bar{U}(\kappa, 1)$ has infinitely many predecessors in the Rudin-Frolik ordering.*

Proof. For every $n, 0 < n < \omega$, use a bijection between $[\kappa]^n$ and κ and transfer $\bar{U}(\kappa, 0)^n$ to κ. The rest follows by Lemmas 2.1, 2.2.
$\square$

Note that for $\kappa-$complete ultrafilters U and D over κ, $U \geq_{R-F} D$ implies $U \geq_{R-K} D$. So, by [5], The existence of a $\kappa-$complete ultrafilter over κ with infinitely many predecessors in the Rudin-Frolik ordering implies by Kanamori [4], that $0^\dagger$ exists. Let us improve this in order to give the exact strength.

Theorem 2.5 *The existence of a $\kappa-$complete ultrafilter over κ with infinitely many predecessors in the Rudin-Frolik ordering implies that $o(\kappa) \geq 2$ in the core model.*

Proof. Note first that for $\kappa-$complete ultrafilters U and D over κ, $U \geq_{R-F} D$ implies $U \geq_{R-K} D$. So, by [5], the existence of a $\kappa-$complete ultrafilter over κ with infinitely many predecessors in the Rudin-Frolik ordering implies that $\exists\lambda o(\lambda) \geq 2$. Let us argue that actually $o(\kappa) \geq 2$ in the core model.
Suppose otherwise. So, $o(\kappa) = 1$. Let $U(\kappa, 0)$ be the unique normal measure over κ in the core model $\mathcal{K}$.
Suppose that, in V, we have a $\kappa-$complete ultrafilter E over κ with infinitely many predecessors in the Rudin-Frolik ordering. Let $\langle E_n \mid n < \omega \rangle$ be a Rudin-Frolik increasing sequence of predecessors of E. Recall that by M.E. Rudin (see [4], 5.5) the predecessors of E are linearly ordered.

Consider $i := i_E \restriction \mathcal{K}$. Then, by [5], it is an iterated ultrapower of $\mathcal{K}$ by its measures. The critical point of i_E is κ, hence $U(\kappa, 0)$ is applied first. Note that $U(\kappa, 0)$ (and its images) can be applied only finitely many times, since M_E is closed under countable (and even κ) sequences of its elements. Denote by k^* the number of such applications.
Let $n \leq \omega$. Similar, consider $i_n := i_{E_n} \restriction \mathcal{K}$. Again, the critical point of i_{E_n} is κ, hence $U(\kappa, 0)$ is applied first. The number of applications of $U(\kappa, 0)$ (and its images)is finite. Denote by k_n the number of such applications.

Now let $n < m < \omega$. We have $E_n <_{R-F} E_m$. Hence, there is a discrete sequence $\langle E_{nm\alpha} \mid \alpha < \kappa \rangle$ of ultrafilters over κ such that

$$E_m = E_n - \lim \langle E_{nm\alpha} \mid \alpha < \kappa \rangle.$$

Then the ultrapower M_{E_m} of V by E_m is $\mathrm{Ult}(M_{E_n}, E'_{nm[id]_{E_n}})$, where $E'_{nm[id]_{E_n}} = i_{E_n}(\langle E_{nm\alpha} \mid \alpha < \kappa \rangle)([id]_{E_n})$ is an ultrafilter over $i_{E_n}(\kappa)$.
Now, in $i_n(\mathcal{K})$, the only normal ultrafilter over $i_{E_n}(\kappa) = i_n(\kappa)$ is $i_n(U(\kappa, 0))$. But this means that i_{E_m} is obtained by more applications of $U(\kappa, 0)$ than i_{E_n}, i.e. $k_n < k_m$.
Similar, $k^* > k_n$, for every $n < \omega$. This means, in particular, that $k^* \geq \omega$, which is impossible. Contradiction.
□

Remark 2.6 Note that the situation with Rudin-Keisler order is different in this respect. Thus, by [3], starting with a measurable κ with $\{o(\kappa) \mid \alpha < \kappa\}$ unbounded in it, it is possible to construct a model with an increasing Rudin-Keisler sequence of the length κ^+.

A similar arguments can be used to produce long increasing Rudin-Frolik sequences. Let us show how to get a sequence of the length $\kappa + 1$[1]
Assume GCH. Let

$$\vec{U} = \langle U(\alpha, \beta) \mid (\alpha, \beta) \in \mathrm{dom}(\vec{U}), \alpha \leq \kappa, \beta < o^{\vec{U}}(\alpha) \rangle$$

be a coherent sequence such that $o^{\vec{U}}(\kappa) = \kappa + 1$ and for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) \leq \kappa$. Let

$$A = \{\alpha \mid \exists \beta (\alpha, \beta) \in \mathrm{dom}(\vec{U}).$$

Then for every $\alpha \in A$, $o^{\vec{U}}(\alpha) \leq \kappa$.

[1]Theorem 5.10 of [4] states that this is impossible, however we think that there is a problem in the argument. Namely, on page 346, line 7 - sets depend on β's; this effects the further definition of a function f (line 16). Its unclear how to insure $f(\xi) > f(\xi')$ for most ξ's, and, so f may be constant mod D_0.

We force with Easton support iteration of the Prikry type forcings with extensions of $\langle U(\alpha,\beta) \mid \beta < o^{\vec{U}}(\alpha)\rangle$'s, $\alpha \in A$, as in [1]. Let G be a generic. Then, for every $\alpha \in A$ with $o^{\vec{U}}(\alpha) = 1$ or being a regular uncountable cardinal, Prikry sequence or Magidor sequence of order type $o^{\vec{U}}(\alpha)$ is added by G (more sequences are added, see [1] for detailed descriptions, but be do not need them here). Denote such sequences by b_α.

Let $\bar{U}(\kappa,0)$ be the canonical extension of $U(\kappa,0)$ to a normal ultrafilter in $V[G]$ defined as in [2].

Denote by A' the subset of A which consists of α's with $o^{\vec{U}}(\alpha) = 1$ or being a regular uncountable cardinal.

For every $\delta,\alpha \in A' \cup \{\kappa\}, \delta < \alpha$ we will use an extensions $U(\kappa,\alpha,\langle\rangle)$ and $U(\kappa,\alpha,\langle\delta\rangle)$ of $U(\kappa,\alpha)$. They were defined in [1] as follows:

$X \in U(\kappa,\alpha,\langle\rangle)$ iff for some $r \in G, \gamma < \kappa^+$ and a tree T, in $M_{U(\kappa,1)}$ the following holds:

$$r \cup \{\langle\langle\rangle, \underset{\sim}{T}\rangle\} \cup \underset{\sim}{p}_\gamma \Vdash \kappa \in i_{U(\kappa,1)}(\underset{\sim}{X}),$$

where p_γ is the $\gamma-$th element of the canonical master sequence.

$X \in U(\kappa,\alpha,\langle\delta\rangle)$ iff for some $r \in G, \gamma < \kappa^+$ and a tree T, in $M_{U(\kappa,1)}$ the following holds:

$$r \cup \{\langle\langle\delta\rangle, \underset{\sim}{T}\rangle\} \cup \underset{\sim}{p}_\gamma \Vdash \kappa \in i_{U(\kappa,1)}(\underset{\sim}{X}),$$

where p_γ is the $\gamma-$th element of the canonical master sequence.

Denote further $U(\kappa,\alpha,\langle\rangle)$ by $\bar{U}(\kappa,\alpha)$.

Notice that $U(\kappa,\alpha,\langle\delta\rangle)$ concentrates on ν's with $o^{\vec{U}}(\nu) = \alpha$, $\delta \in b_\nu$ and $b_\nu \cap \delta = b_\delta$.

We have now the following analog of 2.1:

Lemma 2.7 *For every $\alpha \in A'$, $\bar{U}(\kappa,\kappa) = \bar{U}(\kappa,\alpha) - \lim \langle U(\kappa,\kappa,\langle\nu\rangle) \mid o^{\vec{U}}(\nu) = \alpha\rangle$.*

Proof. $X \in \bar{U}(\kappa,\kappa)$ iff for some $r \in G, \gamma < \kappa^+, T$, in $M_{U(\kappa,\kappa)}$ the following holds:

$$r \cup \{\langle\langle\rangle, \underset{\sim}{T}\rangle\} \cup \underset{\sim}{p}_\gamma \Vdash \kappa \in i_{U(\kappa,\kappa)}(\underset{\sim}{X}).$$

Recall that T is a tree consisting of coherent sequences and $Suc_T(\langle\rangle) \in \bar{U}(\kappa,\alpha)$. Then, for every $\nu \in Suc_T(\langle\rangle)$ with $o^{\vec{U}}(\nu) = \alpha$, we will have

$$r \cup \{\langle\langle\nu\rangle, \underset{\sim}{T_{\langle\nu\rangle}}\rangle\} \cup \underset{\sim}{p}_\gamma \Vdash \kappa \in i_{U(\kappa,\kappa)}(\underset{\sim}{X}).$$

So, $X \in U(\kappa,\kappa,\langle\nu\rangle)$. But this holds for $\bar{U}(\kappa,\alpha)-$measure one many ν's, hence $X \in \bar{U}(\kappa,\alpha)-\lim \langle U(\kappa,\kappa,\langle\nu\rangle) \mid o^{\vec{U}}(\nu) = \alpha\rangle$.

Hence we showed that $\bar{U}(\kappa,\kappa) \subseteq \bar{U}(\kappa,\alpha) - \lim \langle U(\kappa,\kappa,\langle\nu\rangle) \mid o^{\vec{U}}(\nu) = \alpha\rangle$. But this already implies the equality, since both $\bar{U}(\kappa,\kappa)$ and $\bar{U}(\kappa,\alpha) - \lim \langle U(\kappa,\kappa,\langle\nu\rangle) \mid o^{\vec{U}}(\nu) = \alpha\rangle$ are ultrafilters.

□

The same argument shows the following:

Lemma 2.8 *For every* $\gamma,\alpha \in A', \alpha < \gamma$, $\bar{U}(\kappa,\gamma) = \bar{U}(\kappa,\alpha) - \lim \langle U(\kappa,\gamma,\langle\nu\rangle) \mid o^{\vec{U}}(\nu) = \alpha\rangle$.

Lemma 2.9 *The family* $\langle U(\kappa,\gamma,\langle\nu\rangle) \mid o^{\vec{U}}(\nu) = \alpha\rangle$ *is a discrete family of ultrafilters, for every* $\gamma,\alpha \in A' \cup \{\kappa\}, \alpha < \gamma$.

Proof. Fix $\gamma,\alpha \in A' \cup \{\kappa\}, \alpha < \gamma$. For each ν with $o^{\vec{U}}(\nu) = \alpha$ set

$$A_\nu := \{\xi \in A' \mid o^{\vec{U}}(\xi) = \gamma, \nu \in b_\gamma \text{ and } b_\gamma \cap \nu = b_\nu\}.$$

Let $\nu,\nu' \in A'$ be two different elements with $o^{\vec{U}}(\nu) = o^{\vec{U}}(\nu') = \alpha$, then, clearly, $A_\nu \cap A_{\nu'} = \emptyset$.

□

So, again as above, we obtain the following:

Theorem 2.10 $\bar{U}(\kappa,\kappa)$ *has* $\kappa-$*many predecessors in the Rudin-Frolik ordering.*

Proof. By Lemmas 2.7, 2.8, the sequence $\langle \bar{U}(\kappa,\gamma) \mid \gamma \in A' \cup \{\kappa\}\rangle$ is R-F-increasing.

□

It follows now that:

Corollary 2.11 *The consistency strength of existence of a* $\kappa-$*complete ultrafilter over* κ *with* $\kappa-$*many predecessors in the Rudin-Frolik ordering is is at least* $\{o(\alpha) \mid \alpha < \kappa\}$ *is unbounded in* κ *and at most* $o(\kappa) = \kappa + 1$.

3 Discrete families of ultrafilters.

Aki Kanamori asked in [4] the following natural question:

If $\{U_\tau \mid \tau < \kappa\}$ *is a family of distinct* $\kappa-$*complete ultrafilters over* κ *and* E *is any* $\kappa-$*complete ultrafilter over* κ*, is there an* $X \in E$ *so that* $\{U_\tau \mid \tau \in X\}$ *is a discrete family?*

We will give a negative answer to this question below.

Let us use the previous construction. We preserve all the notation made there.

Consider the family

$$\{U(\kappa,\kappa,\langle\delta\rangle) \mid \delta,\alpha \in A', \delta < \alpha\}.$$

Lemma 3.1 *The family* $\{U(\kappa,\kappa,\langle\delta\rangle) \mid \delta \in A', \delta < \kappa\}$ *consists of different ultrafilters.*

Proof. Let $U(\kappa,\kappa,\langle\delta\rangle), U(\kappa,\kappa,\langle\delta'\rangle)$ be two different members of the family.
If $o^{\vec{U}}(\delta) = o^{\vec{U}}(\delta')$, then they are different by Lemma 2.9. Suppose that $o^{\vec{U}}(\delta) < o^{\vec{U}}(\delta')$. Then the set

$$\{\nu < \kappa \mid o^{\vec{U}}(\nu) = \nu, \delta \in b_\nu, b_\nu \cap \delta = b_\delta \text{ and } \delta' \notin b_\nu\} \in U(\kappa,\kappa,\langle\delta\rangle) \setminus U(\kappa,\kappa,\langle\delta'\rangle).$$

So we are done.

□

Pick now a $\kappa-$complete (non-principal) ultrafilter D such that the set

$$Z := \{\alpha < \kappa \mid \alpha \text{ is a regular uncountable cardinal }\} \in D.$$

Define now a $\kappa-$complete ultrafilter E over $[\kappa]^2$ as follows:

$$X \in E \text{ iff } \{\alpha \in Z \mid \{\delta < \kappa \mid (\alpha,\delta) \in X\} \in U(\kappa,\alpha,\langle\rangle)\} \in D.$$

I.e. $E = D - \Sigma_\alpha U(\kappa,\alpha,\langle\rangle)$. We can assume that if $(\alpha,\delta) \in X$, for a set $X \in E$, then $o^{\vec{U}}(\delta) = \alpha$, since $U(\kappa,\alpha)$ concentrates on such δ's.
Now, for every pair (α,δ) with $o^{\vec{U}}(\delta) = \alpha$, define $U_{(\alpha,\delta)} = U(\kappa,\kappa,\langle\delta\rangle)$.

Lemma 3.2 *For every* $X \in E$*, the family* $\{U_\tau \mid \tau \in X\}$ *is not discrete.*

Proof. Let $X \in E$. Suppose that there is a separating sequence $\langle Y_{(\alpha,\delta)} \mid (\alpha,\delta) \in X\rangle$ for $\langle U_{(\alpha,\delta)} \mid (\alpha,\delta) \in X\rangle$. Pick some $\alpha,\alpha' \in \mathrm{dom}(X), \alpha < \alpha'$. Let

$$A_\alpha = \{\delta < \kappa \mid (\alpha,\delta) \in X\}$$

and

$$A_{\alpha'} = \{\delta < \kappa \mid (\alpha',\delta) \in X\}.$$

Then $A_\alpha \in U(\kappa,\alpha,\langle\rangle)$ and $A_{\alpha'} \in U(\kappa,\alpha',\langle\rangle)$. By shrinking X if necessary, assume that $\delta \in A_\alpha$ implies $o^{\vec{U}}(\delta) = \alpha$ and $\delta' \in A_{\alpha'}$ implies $o^{\vec{U}}(\delta') = \alpha'$.
Consider the following set

$$B = \{\nu < \kappa \mid o^{\vec{U}}(\nu) = \nu \text{ and (there are } \delta \in A_\alpha, \delta' \in A_{\alpha'} \text{ such that } \delta < \delta' \text{ and } \delta,\delta' \in b_\nu)\}.$$

Then $B \in U(\kappa,\kappa,\langle\rangle)$. Just take the witnessing tree T_B (as in the definition of $U(\kappa,\kappa,\langle\rangle)$) with the first level

$$A_\alpha \cup A_{\alpha'} \cup (\kappa \setminus (A_\alpha \cup A_{\alpha'}).$$

Then for every $\delta \in A_\alpha$, $B \in U(\kappa, \kappa, \langle \delta \rangle)$. So, $B' := B \cap Y_{(\alpha,\delta)}$ is a subset of B in $U(\kappa, \kappa, \langle \delta \rangle)$. But then an extension of T_B will witness this. In particular there will be $\delta' \in A_{\alpha'}$ such that $B' \in U(\kappa, \kappa, \langle \delta' \rangle)$. This implies that both $Y_{(\alpha,\delta)}$ and $Y_{(\alpha',\delta')}$ are in $U(\kappa, \kappa, \langle \delta' \rangle) = U_{(\alpha,\delta')}$. Hence, $Y_{(\alpha,\delta)} \cap Y_{(\alpha',\delta')} \neq \emptyset$. Contradiction.
□

Now combining Lemmas 3.1, 3.2 we obtain the following:

Theorem 3.3 *In V[G] there are a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa-$complete ultrafilters over κ and a $\kappa-$complete ultrafilter E over κ, so that $\{U_\tau \mid \tau \in X\}$ is a not discrete family for any $X \in E$.*

Corollary 3.4 *The consistency strength of existence a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa-$complete ultrafilters over κ and a $\kappa-$complete ultrafilter E over κ, so that $\{U_\tau \mid \tau \in X\}$ is a not discrete family for any $X \in E$, is at most $o(\kappa) = \kappa + 1$.*

Let us argue now that that $\{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in κ is necessary for this.

Theorem 3.5 *Suppose that there are a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa-$complete ultrafilters over κ and a $\kappa-$complete ultrafilter E over κ, so that $\{U_\tau \mid \tau \in X\}$ is not a discrete family for any $X \in E$. Then $\{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in κ in the Mitchell core model.*

Proof. Suppose otherwise. Let $\{U_\tau \mid \tau < \kappa\}$ be a family of distinct $\kappa-$complete ultrafilters over κ and E be a $\kappa-$complete ultrafilter over κ, so that $\{U_\tau \mid \tau \in X\}$ is a discrete family for any $X \in E$.
Let $\mathcal{K}$ be the Mitchell core model and $o(\kappa) = \eta < \kappa$.
For every $\tau < \kappa$, let j_τ be $i_{U_\tau} \upharpoonright \mathcal{K}$. Then, by [5], j_τ is an iterated ultrapower of $\mathcal{K}$. By [3], the are less than κ possibilities for $j_\tau(\kappa)$. By $\kappa-$completeness of E, we can assume that for every $\tau < \kappa$, $j_\tau(\kappa)$ has a fixed value θ. Denote by Gen_τ the set of generators of j_τ, i.e. the set of ordinals $\nu, \kappa \leq \nu < \theta$ such that for every $n < \omega$, $f : [\kappa]^n \to \kappa, f \in \mathcal{K}$ and $a \in [\nu]^n$, $\nu \neq j_\tau(f)(a)$. Let Gen^*_τ be the subset of Gen_τ consisting of all principle generators of j_τ, i.e. of all $\nu \in Gen_\tau$ such that for every $n < \omega$, $f : [\kappa]^n \to \kappa, f \in \mathcal{K}$ and $a \in [\nu]^n$, $\nu > j_\tau(f)(a)$. Again by [3], the are less than κ possibilities for Gen^*_τ's. So, by $\kappa-$completeness of E,, we can assume that for every $\tau < \kappa$, $Gen^*_\tau = Gen^*$.
Suppose that $\nu \in Gen_\tau$ and ν is not a principle generator. Then there are finite set of generators $b \subseteq \nu$ and $f : [\kappa]^{|b|} \to \kappa, f \in \mathcal{K}$ such that $\nu < j_\tau(f)(b)$.
Set, following W. Mitchell,

$$\alpha(\nu) = \min\{j_\tau(f)(b) \mid b \subseteq \nu \text{ is a finite set of generators},$$

$$f : [\kappa]^{|b|} \to \kappa, f \in \mathcal{K} \text{ and } \nu < j_\tau(f)(b)\}.$$

Let $b_\nu \subseteq \nu$ be the smallest finite set of generators such that for some $f : [\kappa]^{|b_\nu|} \to \kappa$, $f \in \mathcal{K}$, $\alpha(\nu) = j_\tau(f)(b_\nu)$.

Let us call a finite set of generators $a \subseteq Gen_\tau$ *nice* iff for each $\nu \in a$ either ν is a principle generator or it is not and then $b_\nu \subseteq a$.

Consider now $[id]_{U_\tau}$. Find the smallest finite nice set of generators a_τ in Gen_τ such that for some $h_\tau : [\kappa]^{|a_\tau|} \to \kappa, h_\tau \in \mathcal{K}$ we have $[id]_{U_\tau} = j_\tau(h_\tau)(a_\tau)$. We may assume, using $\kappa-$completeness of E, that $a_\tau \cap Gen^*$ has a constant value. Denote it by a^*.

Let us deal first with simpler particular cases.

Suppose first that $a^* = a_\tau$ and it consists only of κ itself, for every $\tau < \kappa$ (or on an $E-$measure one set). Then, for some $\theta < o(\kappa)$, each j_τ is just the ultrapower embedding $i_{U(\kappa,\theta)}$ by a normal measure $U(\kappa, \theta)$ from the sequence of $\mathcal{K}$.

Now the functions h_τ, $\tau < \kappa$ represent ordinals between κ and $i_{U(\kappa,\theta)}(\kappa)$ in this ultrapower. Hence, they are one to one mod $U(\kappa, \theta)$. This means that each U_τ is equivalent to its normal measure as witnessed by h_τ. But such ultrafilters can be easily separated.

Suppose next that $a_\tau = a^* = \{\kappa, \kappa_1\}$, for every $\tau < \kappa$ (or on an $E-$measure one set). Assume that each j_τ is the second ultrapower embedding by a normal measure $U(\kappa, \theta)$ over κ in $\mathcal{K}$, where κ_1 is the image of κ under $i_{U(\kappa,\theta)}(\kappa)$.

Denote $i_{U(\kappa,\theta)}$ by $i_1 : \mathcal{K} \to \mathcal{K}_1$, the ultrapower embedding of $\mathcal{K}_1$ by $i_1(U(\kappa, \theta)$ by $i_{1,2} = i_{i_1(U(\kappa,\theta))} : \mathcal{K}_1 \to \mathcal{K}_2$ and the second ultrapower embedding (the one equal to j_τ's) by $i_2 = i_{1,2} \circ i_1 : \mathcal{K} \to \mathcal{K}_2$. Let $\kappa_2 = i_2(\kappa)$. Then we have $[id]_{U_\tau} = i_2(h_\tau)(\kappa, \kappa_1) \in [\kappa_1, \kappa_2)$, for every $\tau < \kappa$.

Let us deal first with different mod $U(\kappa, \theta)^2$ functions among h_τ's. So, let $Z \subseteq \kappa$ be a set of such functions, i.e. for every $\tau \neq \tau'$ in Z, $h_\tau \neq h_{\tau'}$ mod $U(\kappa, \theta)^2$.

Our prime interest will be in $\langle \text{rng}(h_\tau) \mid \tau \in Z\rangle$. We will argue that there is a set $C \in U(\kappa, \theta)^2$ such that $\langle h_\tau{}''[C \setminus \tau + 1]^2 \mid \tau \in Z\rangle$ is a disjoint family, which in turn will witness that the family $\langle U_\tau \mid \tau \in Z\rangle$ is discrete.

Let $\tau \in Z$ and $\beta < \kappa$. Define $h^\beta_\tau : \beta \to \kappa \setminus \beta$ by setting $h^\beta_\tau(\alpha) = h_\tau(\alpha, \beta)$.

Consider $i_1(\langle h^\beta_\tau \mid \beta < \kappa\rangle)(\kappa) : \kappa \to \kappa_1 \setminus \kappa$. Denote it by h'_τ.

Suppose for a moment that for some $\tau, \tau' \in Z, \tau \neq \tau'$, $h'_\tau = h'_{\tau'}$ mod $U(\kappa, \theta)$. Then there is a set $H \in U(\kappa, \theta)$ such that

$$\{\beta < \kappa \mid h^\beta_\tau \restriction H \cap \beta = h^\beta_{\tau'} \restriction H \cap \beta\} \in U(\kappa, \theta).$$

But then

$$H \subseteq \{\alpha < \kappa \mid \{\beta < \kappa \mid h_\tau(\alpha, \beta) = h_{\tau'}(\alpha, \beta)\} \in U(\kappa, \theta)\}$$

. Hence,

$$\{\alpha < \kappa \mid \{\beta < \kappa \mid h_\tau(\alpha, \beta) = h_{\tau'}(\alpha, \beta)\} \in U(\kappa, \theta).$$

Which is impossible.
Hence, $\tau, \tau' \in Z, \tau \neq \tau'$ implies $h'_\tau \neq h'_{\tau'}$ mod $U(\kappa, \theta)$.
Now, using normality of $U(\kappa, \theta)$ and covering by a set in $\mathcal{K}$ of cardinality κ, it is easy to find $A \in U(\kappa, \theta)$ such that $\tau, \tau' \in Z, \tau < \tau'$ implies

$$\mathrm{rng}(h'_\tau \upharpoonright A \setminus \tau') \cap \mathrm{rng}(h'_{\tau'} \upharpoonright A \setminus \tau') = \emptyset.$$

This statement is true in $\mathcal{K}_1$, hence by elementarity,

$$\{\beta < \kappa \mid \mathrm{rng}(h^\beta_\tau \upharpoonright (A \cap \beta) \setminus \tau') \cap \mathrm{rng}(h^\beta_{\tau'} \upharpoonright (A \cap \beta) \setminus \tau') = \emptyset\} \in U(\kappa, \theta).$$

Fix $\tau \in Z$. Let $\tau' \in Z$ be different from τ. Set

$$B^{\tau'}_\tau = \{\beta < \kappa \mid \mathrm{rng}(h^\beta_\tau \upharpoonright (A \cap \beta) \setminus \tau') \cap \mathrm{rng}(h^\beta_{\tau'} \upharpoonright (A \cap \beta) \setminus \tau') = \emptyset\},$$

if $\tau < \tau'$ and

$$B^{\tau'}_\tau = \{\beta < \kappa \mid \mathrm{rng}(h^\beta_\tau \upharpoonright (A \cap \beta) \setminus \tau) \cap \mathrm{rng}(h^\beta_{\tau'} \upharpoonright (A \cap \beta) \setminus \tau) = \emptyset\},$$

if $\tau' < \tau$. Then $B^{\tau'}_\tau \in U(\kappa, \theta)$. The set

$$E_\tau = \{\beta < \kappa \mid \forall \alpha < \beta' < \beta (h_\tau(\alpha, \beta') < \beta)\} \in U(\kappa, \theta).$$

Set $C_\tau = (A \setminus \tau) \cap E_\tau \cap \Delta_{\tau' \in Z, \tau' \neq \tau} B^{\tau'}_\tau$. Then for every $\alpha, \alpha', \beta \in C_\tau$ with $\alpha, \alpha' < \beta$, $\alpha \neq \alpha'$ we have

$$(*) h_\tau(\alpha, \beta) \neq h_{\tau'}(\alpha', \beta),$$

once $\tau' \in Z, \tau' \neq \tau$ and $\tau' < \beta$.

Suppose now $\tau, \tau' \in Z, \tau \neq \tau'$, $(\alpha, \beta), (\alpha', \beta') \in [C_\tau]^2 \cap [C_{\tau'}]^2$. Assume for a moment that $h_\tau(\alpha, \beta) = h_{\tau'}(\alpha', \beta')$.
Note first that $\beta = \beta'$, since $h_\tau(\alpha, \beta) \geq \beta$, $h_{\tau'}(\alpha', \beta') \geq \beta'$ and $\beta, \beta' \in E_\tau \cap E_{\tau'}$. But then

$$h_\tau(\alpha, \beta) \neq h_{\tau'}(\alpha', \beta),$$

by the previous paragraph.

Finally let $C = \Delta_{\tau \in Z} C_\tau$. The sequence $\langle h_\tau{}''[C \setminus \tau + 1]^2 \mid \tau \in Z\rangle$ will be as desired. Thus let $\tau < \tau', \tau, \tau' \in Z$ and $(\alpha, \beta) \in [C \setminus \tau + 1]^2, (\alpha', \beta') \in [C \setminus \tau' + 1]^2$. If $\beta \leq \tau'$, then

$h_\tau(\alpha, \beta) < \beta' \leq h_{\tau'}(\alpha', \beta')$, since $\beta' \in C \setminus \tau' + 1$, and so, $\beta' \in C_\tau \subseteq E_\tau$. If $\beta > \tau'$, then $\beta \in C_{\tau'}$. So, $\beta \neq \beta'$, say $\beta > \beta'$ will imply

$$\beta' \leq h_{\tau'}(\alpha', \beta') < \beta \leq h_\tau(\alpha, \beta).$$

Suppose that $\beta = \beta'$. But $\beta > \tau'$, hence by (*) above $h_\tau(\alpha, \beta) \neq h_{\tau'}(\alpha', \beta)$.

Let us deal now with ultrafilters from the sequence $\langle U_\tau \mid \tau < \kappa \rangle$ such that the ordinals $[id]_{U_\tau}$'s are the same and of the form $i_2(h)(\kappa, \kappa_1)$, for some $h : [\kappa]^2 \to \kappa, h \in \mathcal{K}$. Assume for simplicity that every $\tau < \kappa$ is like this.

Denote $h_* U(\kappa, \theta)^2$ by $\mathcal{V}$. We have then that for every $X \subseteq \kappa, X \in \mathcal{K}$,

$$X \in \mathcal{V} \Leftrightarrow i_2(h)(\kappa, \kappa_1) \in i_2(X) \Leftrightarrow [id]_{U_\tau} \in i_2(X) \Leftrightarrow [id]_{U_\tau} \in i_{U_\tau}(X) \Leftrightarrow X \in U_\tau.$$

So, $U_\tau \supseteq \mathcal{V}$, for every $\tau < \kappa$.

Let $\pi : \kappa \to \kappa, \pi \in \mathcal{K}$ be a projection of $\mathcal{V}$ to the normal ultrafilter $Rudin - Keisler$ below $\mathcal{V}$, i.e. to $U(\kappa, \theta)$. Assume that $\mathcal{V}$ is Rudin-Keisler equivalent to $U(\kappa, \theta)^2$. The case $\mathcal{V} =_{R-K} U(\kappa, \theta)$ is similar and no other possibility can occur here. So,

$$\kappa = [\pi]_{\mathcal{V}} = i_2(\pi)([id]_{\mathcal{V}}) = i_2(\pi)(h(\kappa, \kappa_1)) = i_2(\pi)([id]_{U_\tau}),$$

for every $\tau < \kappa$. Which means that for every $\tau < \kappa$, π is a projection of U_τ to its normal measure.

Now the conclusion follows by the following likely known lemma.

Lemma 3.6 *Let $\langle E_\alpha \mid \alpha < \kappa \rangle$ be a family of pairwise different $\kappa-$complete ultrafilters over κ which have the same projection to their least normal measures. Then the family is discrete.*

Proof. Denote by π this common projection.
Let $\alpha < \kappa$. For every $\beta < \kappa, \beta \neq \alpha$, pick $A_\alpha^\beta \in E_\alpha \setminus E_\beta$. Let

$$B_\alpha = \{\nu < \kappa \mid \pi(\nu) > \alpha\}.$$

Then $B_\alpha \in E_\alpha$, since $\pi_* E_\alpha$ is not principal ultrafilter. Set

$$A_\alpha = \Delta^*_{\beta < \kappa, \beta \neq \alpha} A_\alpha^\beta = \{\nu < \kappa \mid \forall \beta < \pi(\nu)(\beta \neq \alpha \to \nu \in A_\alpha^\beta)\}.$$

Then $A_\alpha \in E_\alpha$. Let

$$A_\alpha^* = A_\alpha \cap B_\alpha \cap \bigcap_{\beta < \alpha} (\kappa \setminus A_\beta^\alpha).$$

Clearly, $A^*_\alpha \in E_\alpha$.

Let us argue that the sets $\langle A^*_\alpha \mid \alpha < \kappa \rangle$ are pairwise disjoint. So, let $\alpha < \alpha' < \kappa$. Suppose that $\nu \in A^*_\alpha \cap A^*_{\alpha'}$. Then $\nu \in B_{\alpha'}$, and hence, $\pi(\nu) > \alpha' > \alpha$. But then, $\nu \in A_\alpha$ implies that $\nu \in A^{\alpha'}_\alpha$, which is impossible since $\nu \in A^*_{\alpha'} \subseteq \kappa \setminus A^{\alpha'}_\alpha$.

□

Let us turn now to the general case. So, we have for each $\tau < \kappa$, the smallest finite nice set of generators a_τ in Gen_τ and $h_\tau : [\kappa]^{|a_\tau|} \to \kappa, h_\tau \in \mathcal{K}$ such that $[id]_{U_\tau} = j_\tau(h_\tau)(a_\tau)$. Also, $i_{U_\tau} \restriction \mathcal{K} = j_\tau$ is an iterated ultrapower of $\mathcal{K}$ by its measures.

If $a_\tau = a^*$ or just a_τ's are the same, for most (mod E)τ's, then the previous arguments apply without much changes. Suppose that this does not happen, i.e. for an $E-$measure one set of τ, $a_\tau \neq a^*$. Assume that this is true for every $\tau < \kappa$ and also that $|a_\tau| = |a_{\tau'}|$, for every $\tau, \tau' < \kappa$.

Then for every $\tau < \kappa$, let $\langle \mu_{\tau,k} \mid k < m \rangle$ be an increasing enumeration of $Gen_\tau \cap (a_\tau \setminus a^*)$. Then $\alpha(\mu_{\tau,0}) > \mu_{\tau,0}$. By the definition of $\alpha(\mu_{\tau,0})$, we have $b_{\mu_{\tau,0}} \subseteq a^* \cap \mu_{\tau,0}$ and $f_{\mu_{\tau,0}} \in \mathcal{K}$ such that

$$j_\tau(f_{\mu_{\tau,0}})(b_{\mu_{\tau,0}}) = \alpha(\mu_{\tau,0}).$$

Similar, for each $k, 0 < k < m$, $\alpha(\mu_{\tau,k}) > \mu_{\tau,k}$ and there are $b_{\mu_{\tau,k}} \subseteq a_\tau \cap \mu_{\tau,k}$ and $f_{\mu_{\tau,k}} \in \mathcal{K}$ such that

$$j_\tau(f_{\mu_{\tau,k}})(b_{\mu_{\tau,k}}) = \alpha(\mu_{\tau,k}).$$

Note if $\mu_{\tau,k} < \mu_{\tau,k'}$ and no generator of j_τ seats in between, then $\alpha(\mu_{\tau,k}) \geq \alpha(\mu_{\tau,k'})$. Also note that if δ is of a form $\alpha(\mu_{\tau,k})$, for some $\tau < \kappa$, then the number of generators with this δ bounded in κ, since the set $\{o(\eta) \mid \eta < \kappa\}$ is bounded in κ.

Using the $\kappa-$completeness of E, we can assume that all a_τ's are generated in the same fashion over a^* with respect to the order and number and order of applications of the $\alpha(-), b_-$. Stating this more precisely the structures

$$\mathcal{A}_\tau = \langle a_\tau, <, a^*, \alpha(-), b_-, ... \rangle$$

are isomorphic over a^*.

Let us deal with the following partial case, in the general one mainly the notation are more complicated.

Assume that there is a set $Z \subseteq \kappa$ of cardinality κ such that for some $a^{**} \subseteq a^*$, for every $\tau \in Z$ there is $\mu_\tau \in a_\tau \setminus \max(a^{**})$ such that

1. $\alpha(\mu_\tau) = j_\tau(f_{\mu_\tau})(a^{**})$,

2. $\mu_\tau \leq [id]_{U_\tau} < \alpha(\mu_\tau)$,

3. if $\tau \neq \tau'$ are in Z, then $\alpha(\mu_\tau) \neq \alpha(\mu_{\tau'})$.

Note that once $\alpha(\mu_\tau)$ is fixed, the number of possible $\mu_{\tau'}$'s with $\alpha(\mu_\tau) = \alpha(\mu_{\tau'})$ is below κ, since $\{o(\xi) \mid \xi < \kappa\}$ is bounded in κ. So the condition 3 above is not really very restrictive.

Note also that if $\tau \neq \tau'$ are in Z, then $\mu_\tau < \mu_{\tau'}$ implies $\alpha(\mu_\tau) < \mu_{\tau'}$ and $\mu_\tau > \mu_{\tau'}$ implies $\alpha(\mu_{\tau'}) < \mu_\tau$. Since $\mu_\tau, \mu_{\tau'}$ are generators (indiscernibles) corresponding to different measurables $\alpha(\mu_\tau), \alpha(\mu_{\tau'})$ and this measurables depend (were generated by) on a^{**} only.

Now we would like to use the arguments similar to the previous considered case and split not only $\alpha(\mu_\tau)$'s but rather the intervals they generate.
First note that the set

$$\{\alpha(\mu_{\tau'}) \mid \tau' \in Z \text{ and } \mu_{\tau'} < \mu_\tau\}$$

is bounded below μ_τ, due to the cofinality considerations. So we can pick some $\alpha^-(\mu_\tau)$ of a form $j_\tau(f^-_{\mu_\tau})(a^{**})$ in the interval $(\sup(\{\alpha(\mu_{\tau'}) \mid \tau' \in Z \text{ and } \mu_{\tau'} < \mu_\tau\}), \mu_\tau)$.
Let

$$\mathcal{U} = \{X \subseteq [\kappa]^{|a^{**}|} \mid X \in \mathcal{K}, a^{**} \in j_\tau(X)\}.$$

Then it is a $\kappa-$complete ultrafilter over $[\kappa]^{|a^{**}|}$ in $\mathcal{K}$ which is a product of finitely many normal measures over κ.
Our aim will be to find a set $C \subseteq [\kappa]^{|a^{**}|}$ in $\mathcal{K}$ such that

1. $a^{**} \in j_\tau(C)$, for all $\tau \in Z$,

2. the intervals $[f^-_{\mu_\tau}(\vec{\nu}), f_{\mu_\tau}(\vec{\nu})], [f^-_{\mu_{\tau'}}(\vec{\nu}'), f_{\mu_{\tau'}}(\vec{\nu}')]$ are disjoint whenever $\tau \neq \tau'$ are in Z and $\vec{\nu} \in C, \min(\vec{\nu}) > \tau, \vec{\nu}' \in C, \min(\vec{\nu}') > \tau'$.

Denote $\max(a^{**})$ by β and $a^{**} \setminus \{\beta\}$ by $\vec{\alpha}$.
Let $U(\kappa,\theta)$ be the last measure of $\mathcal{U}$, i.e. $\mathcal{U} = (\mathcal{U} \restriction [\kappa]^{|a^{**}|-1}) \times U(\kappa,\theta)$.
Let $\tau \in Z$ and $\beta < \kappa$. Define $g^\beta_\tau : \beta \to \kappa \setminus \beta$ by setting $g^\beta_\tau(\vec{\alpha}) = f_{\mu_\tau}(\vec{\alpha}, \beta)$ and $g^{-\beta}_\tau : \beta \to \kappa \setminus \beta$ by setting $g^{-\beta}_\tau(\vec{\alpha}) = f^-_{\mu_\tau}(\vec{\alpha}, \beta)$.
Consider

$$i_{U(\kappa,\theta)}(\langle g^\beta_\tau \mid \beta < \kappa\rangle)(\kappa) : [\kappa]^{|a^{**}|-1} \to i_{U(\kappa,\theta)}(\kappa) \setminus \kappa.$$

Denote it by g'_τ. Similar let

$$i_{U(\kappa,\theta)}(\langle g^{-\beta}_\tau \mid \beta < \kappa\rangle)(\kappa) : [\kappa]^{|a^{**}|-1} \to i_{U(\kappa,\theta)}(\kappa) \setminus \kappa.$$

Denote it by $g_\tau^{-'}$.
Suppose for a moment that for some $\tau, \tau' \in Z, \tau \neq \tau'$, $g_\tau^{-'} < g_{\tau'}^{-'} \leq g_\tau' \bmod \mathcal{U} \restriction [\kappa]^{|a^{**}|-1}$. Then there is a set $H \in \mathcal{U} \restriction [\kappa]^{|a^{**}|-1}$ such that for every $\vec{\alpha} \in H$, the set

$$\{\beta < \kappa \mid g_\tau^{-\beta}(\vec{\alpha}) < g_{\tau'}^{-\beta}(\vec{\alpha}) \leq g_\tau^\beta(\vec{\alpha})\} \in U(\kappa, \theta).$$

But then

$$H \subseteq \{\vec{\alpha} \in [\kappa]^{|a^{**}|-1} \mid \{\beta < \kappa \mid g_\tau^-(\vec{\alpha}, \beta) < g_{\tau'}^-(\vec{\alpha}, \beta) \leq g_\tau(\overrightarrow{\alpha, \beta})\} \in U(\kappa, \theta)\}.$$

Hence,

$$\{\vec{\alpha} \in [\kappa]^{|a^{**}|-1} \mid \{\beta < \kappa \mid g_\tau^-(\vec{\alpha}, \beta) < g_{\tau'}^-(\vec{\alpha}, \beta) \leq g_\tau(\overrightarrow{\alpha, \beta})\} \in U(\kappa, \theta)\} \in \mathcal{U} \restriction [\kappa]^{|a^{**}|-1}.$$

Which is impossible.
Hence, $\tau, \tau' \in Z, \tau \neq \tau'$ implies $\neg(g_\tau^{-'} < g_{\tau'}^{-'} \leq g_\tau') \bmod \mathcal{U} \restriction [\kappa]^{|a^{**}|-1}$. Which means, by switching between τ and τ' is necessary, that $g_\tau' < g_{\tau'}^{-'} \bmod \mathcal{U} \restriction [\kappa]^{|a^{**}|-1}$ or $g_{\tau'}' < g_\tau^{-'} \bmod \mathcal{U} \restriction [\kappa]^{|a^{**}|-1}$.
Now, using induction, normality of components of $\mathcal{U} \restriction [\kappa]^{|a^{**}|-1}$ and covering the set $\{\{g_\tau^{-'}, g_\tau'\} \mid \tau \in Z\}$ by a set in $\mathcal{K}$ of cardinality κ, if necessary, we can find $A \in \mathcal{U} \restriction [\kappa]^{|a^{**}|-1}$ such that $\tau, \tau' \in Z, \tau \neq \tau'$ implies that for every $\vec{\nu}, \vec{\nu}' \in A$ with $\min(\vec{\nu}) > \tau, \min(\vec{\nu}') > \tau'$ the intervals

$$[g_\tau^{-'}(\vec{\nu}), g_\tau'(\vec{\nu})], [g_{\tau'}^{-'}(\vec{\nu}'), g_{\tau'}'(\vec{\nu}')] \text{ are disjoint .}$$

Thus, we can assume that the functions $g_\tau^{-'}, g_\tau'$ are not constant, just otherwise the set of relevant generators can be reduced to a smaller one.
Split into two cases according to the supremums of the ranges.

Case 1. Same supremum.

So assume for simplification of notation that for every $\tau \in Z$ the ranges of the functions $g_\tau^{-'}, g_\tau'$ have the same supremum χ. Then χ has cofinality κ, and let $\langle \chi_\gamma \mid \gamma < \kappa \rangle$ be a cofinal sequence.
Now we proceed similar to what was done in the beginning with h_τ, only an induction on size of a^{**} should be used.

Case 1. Different supremums.

Then we deal with this different supremums and split them. This will provide the desired conclusion also for $g_\tau^{-'}, g_\tau'$'s.

Now, the statement that for every $\vec{\nu}, \vec{\nu}' \in A$ with $\min(\vec{\nu}) > \tau, \min(\vec{\nu}') > \tau'$ the intervals

$$[g_\tau^{-'}(\vec{\nu}), g_\tau'(\vec{\nu})], [g_{\tau'}^{-'}(\vec{\nu}'), g_{\tau'}'(\vec{\nu}')] \text{ are disjoint ,}$$

is true in $\mathcal{K}_1$, hence by elementarity,

$$\{\beta < \kappa \mid \forall \vec{\nu}, \vec{\nu}' \in A \cap [\beta]^{|a^{**}|-1}(\min(\vec{\nu}) > \tau \wedge \min(\vec{\nu}') > \tau' \to$$

$$[g_\tau^{-\beta}(\vec{\nu}), g_\tau^\beta(\vec{\nu})] \cap [g_{\tau'}^{-\beta}(\vec{\nu}'), g_{\tau'}^\beta(\vec{\nu}')] = \emptyset)\} \in U(\kappa, \theta).$$

Fix $\tau \in Z$. Let $\tau' \in Z$ be different from τ. Set

$$B_\tau^{\tau'} = \{\beta < \kappa \mid \forall \vec{\nu}, \vec{\nu}' \in A \cap [\beta \setminus \tau']^{|a^{**}|-1}([g_\tau^{-\beta}(\vec{\nu}), g_\tau^\beta(\vec{\nu})] \cap [g_{\tau'}^{-\beta}(\vec{\nu}'), g_{\tau'}^\beta(\vec{\nu}')] = \emptyset)\},$$

if $\tau < \tau'$ and

$$B_\tau^{\tau'} = \{\beta < \kappa \mid \forall \vec{\nu}, \vec{\nu}' \in A \cap [\beta \setminus \tau]^{|a^{**}|-1}([g_\tau^{-\beta}(\vec{\nu}), g_\tau^\beta(\vec{\nu})] \cap [g_{\tau'}^{-\beta}(\vec{\nu}'), g_{\tau'}^\beta(\vec{\nu}')] = \emptyset)\},$$

if $\tau' < \tau$. Then $B_\tau^{\tau'} \in U(\kappa, \theta)$. The set

$$E_\tau = \{\beta < \kappa \mid \forall \vec{\alpha} < \beta' < \beta(g_\tau(\vec{\alpha}, \beta') < \beta)\} \in U(\kappa, \theta).$$

Set $C_\tau = E_\tau \cap \Delta_{\tau' \in Z, \tau' \neq \tau} B_\tau^{\tau'}$. Then for every $\vec{\alpha}, \vec{\alpha}' \in (A \setminus \tau), \beta \in C_\tau$ with $\alpha, \alpha' < \beta$, $\alpha \neq \alpha'$ we have

$$(**)[g_\tau^-(\vec{\alpha}, \beta), g_\tau(\vec{\alpha}, \beta)] \cap [g_{\tau'}^-(\vec{\alpha}', \beta), g_{\tau'}(\vec{\alpha}', \beta)] = \emptyset$$

once $\tau' \in Z, \tau' \neq \tau$ and $\tau' < \beta$.

Suppose now $\tau, \tau' \in Z, \tau \neq \tau'$, $\vec{\alpha}, \vec{\alpha}' \in (A \setminus \tau) \cap (A \setminus \tau'), \beta \in C_\tau, \beta' \in C_{\tau'}$. Assume for a moment that

$$[g_\tau^-(\vec{\alpha}, \beta), g_\tau(\vec{\alpha}, \beta)] \cap [g_{\tau'}^-(\vec{\alpha}', \beta'), g_{\tau'}(\vec{\alpha}', \beta')] \neq \emptyset$$

Note first that $\beta = \beta'$, since $\beta \leq g_\tau^-(\vec{\alpha}, \beta) \leq g_\tau(\alpha, \beta)$, $\beta' \leq g_{\tau'}^-(\vec{\alpha}', \beta') \leq g_{\tau'}(\alpha', \beta')$ and $\beta, \beta' \in E_\tau \cap E_{\tau'}$. But then

$$[g_\tau^-(\vec{\alpha}, \beta), g_\tau(\vec{\alpha}, \beta)] \cap [g_{\tau'}^-(\vec{\alpha}', \beta'), g_{\tau'}(\vec{\alpha}', \beta')] \neq \emptyset,$$

by the previous paragraph.

Finally let $\tilde{C} = \Delta_{\tau \in Z} C_\tau$ and

$$C = \{(\vec{\alpha}, \beta) \mid \vec{\alpha} \in A, \beta \in \tilde{C} \text{ and } \beta > \max(\vec{\alpha})\}.$$

Such C will be as desired. Thus let $\tau < \tau', \tau, \tau' \in Z$ and $(\vec{\alpha}, \beta) \in C \setminus \tau + 1, (\vec{\alpha}', \beta') \in C \setminus \tau' + 1$. If $\beta \leq \tau'$, then $g_\tau(\alpha, \beta) < \beta' \leq g_{\tau'}^-(\alpha', \beta')$, since $(\vec{\alpha}', \beta') \in C \setminus \tau' + 1$, and so, $\beta' \in C_\tau \subseteq E_\tau$. If $\beta > \tau'$, then $\beta \in C_{\tau'}$. So, $\beta \neq \beta'$, say $\beta > \beta'$ will imply

$$\beta' \leq g_{\tau'}(\vec{\alpha}', \beta') < \beta \leq g_\tau^-(\vec{\alpha}, \beta).$$

Suppose that $\beta = \beta'$. But $\beta > \tau'$, hence by (**) above

$$[g^-_\tau(\vec{\alpha}, \beta), g_\tau(\vec{\alpha}, \beta)] \cap [g^-_{\tau'}(\vec{\alpha}', \beta), g_{\tau'}(\vec{\alpha}', \beta)] = \emptyset.$$

□

4 Products of ultrafilters.

In [4], Aki Kanamori asked the following question (Question 5.8 there):

If $\mathcal{U}$ and $\mathcal{V}$ are $\kappa-$complete ultrafilters over κ such that $\mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{V} \times \mathcal{U}$, is there a $\mathcal{W}$ and integers n and m so that $\mathcal{U} \simeq \mathcal{W}^n$ and $\mathcal{V} \simeq \mathcal{W}^m$?

Solovay gave an affirmative answer once "$\mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{V} \times \mathcal{U}$" is replaced by "$\mathcal{U} \times \mathcal{V} \simeq \mathcal{V} \times \mathcal{U}$", and Kanamori once $\mathcal{U}$ is a $p-$point, see [4] 5.7, 5.9.

We would like to show that the negative answer is consistent assuming $o(\kappa) = \kappa$. Two examples will be produced. The following will be shown:

Theorem 4.1 *Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa-$complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over κ such that*

1. *$\mathcal{V} >_{R-K} \mathcal{U}$,*
2. *$\mathcal{V} \times \mathcal{U} >_{R-K} \mathcal{U} \times \mathcal{V}$.*

Theorem 4.2 *Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa-$complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over κ such that*

1. *$\mathcal{V}$ is a normal measure,*
2. *$\mathcal{V}$ is the projection of $\mathcal{U}$ to its least normal measure,*
3. *$\mathcal{V} \times \mathcal{U} >_{R-K} \mathcal{U} \times \mathcal{V}$.*

Proof of the first theorem.

Let us keep the notation of the previous section.

So, we have $\kappa-$complete ultrafilters $U(\kappa, \alpha, t), \alpha < \kappa, t \in [\kappa]^{<\omega}$ which extend $U(\kappa, \alpha)$'s. Denote $U(\kappa, \alpha, \langle\rangle)$ by $\bar{U}(\kappa, \alpha)$.

Let $f : \kappa \to \kappa$. Define

$$U_f = \{X \subseteq \kappa \mid \{\alpha < \kappa \mid X \in \bar{U}(\kappa, f(\alpha))\} \in \bar{U}(\kappa, 0)\},$$

i.e.

$$U_f = \bar{U}(\kappa, 0) - \lim_{\alpha<\kappa} \bar{U}(\kappa, f(\alpha)).$$

Then U_f is a $\kappa-$complete ultrafilter over κ.

It is noted in [3], that if $f \leq g \mod \bar{U}(\kappa, 0)$, then $U_f \leq_{R-K} U_g$.

Our prime interest will be in $f = id$ and $g = id + 1$.

Set $\mathcal{U} = U_{id}$ and $\mathcal{V} = U_{id+1}$.

We would like to argue that $\mathcal{U} \times \mathcal{V} <_{R-K} \mathcal{V} \times \mathcal{U}$.

Note that neither $\mathcal{U}$ nor $\mathcal{V}$ are of the form $\mathcal{W}^n$, for $n > 1$, since the only ultrafilters Rudin-Keisler below $\mathcal{U}$ are $\bar{U}(\kappa, \alpha), \alpha < \kappa$ and their finite powers, those below $\mathcal{V}$ are $\bar{U}(\kappa, \alpha), \alpha < \kappa$, $\mathcal{U}$ and their finite powers. Just examine the ultrapowers by $\mathcal{U}$ nor $\mathcal{V}$.

In particular, $\mathcal{V} \neq \mathcal{U}^n, n < \omega$.

Suppose that $B \in \mathcal{U} \times \mathcal{V}$. Then

$$\{\mu < \kappa \mid \{\xi < \kappa \mid (\mu, \xi) \in B\} \in \mathcal{V}\} \in \mathcal{U}.$$

Denote

$$A = \{\mu < \kappa \mid \{\xi < \kappa \mid (\mu, \xi) \in B\} \in \mathcal{V}\}$$

and for each $\mu < \kappa$, let

$$A_\mu = \{\xi < \kappa \mid (\mu, \xi) \in B\}.$$

Recall that

$$\mathcal{U} = \bar{U}(\kappa, 0) - \lim \langle \bar{U}(\kappa, \alpha) \mid \alpha < \kappa \rangle.$$

Hence, there is $Z \in \bar{U}(\kappa, 0)$ such that for every $\alpha \in Z$, $A \in \bar{U}(\kappa, \alpha)$.

Similar,

$$\mathcal{V} = \bar{U}(\kappa, 0) - \lim \langle \bar{U}(\kappa, \alpha + 1) \mid \alpha < \kappa \rangle.$$

Hence, for every $\mu \in A$, there is $Y_\mu \in \bar{U}(\kappa, 0)$ such that for every $\alpha \in Y_\mu$, $A_\mu \in \bar{U}(\kappa, \alpha + 1)$.

Set

$$X = Z \cap \Delta_{\mu \in A} Y_\mu.$$

Then $X \in \bar{U}(\kappa, 0)$ and for every $\alpha \in X$ we have

$$A \in \bar{U}(\kappa, \alpha) \text{ and } \forall \mu \in A \cap \alpha (A_\mu \in \bar{U}(\kappa, \alpha + 1)).$$

Then, by elementarity, in $M_{\mathcal{V}}$, for every $\alpha \in i_{\mathcal{V}}(X)$,

$$i_{\mathcal{V}}(A) \in \bar{U}(i_{\mathcal{V}}(\kappa), \alpha) \text{ and } \forall \mu \in i_{\mathcal{V}}(A) \cap \alpha (A'_\mu \in \bar{U}(i_{\mathcal{V}}(\kappa), \alpha + 1)),$$

where $i_{\mathcal{V}}(\langle A_\mu \mid \mu < \kappa\rangle) = \langle A'_\mu \mid \mu < i_{\mathcal{V}}(\kappa)\rangle$.

Let $\rho^{\mathcal{U}}$ denotes $[id]_{\mathcal{U}}$. Then $\rho^{\mathcal{U}} \in i_{\mathcal{U}}(A)$. We have a natural embedding $\sigma : M_{\mathcal{U}} \to M_{\mathcal{V}}$ and it does not move $\rho^{\mathcal{U}}$, since its critical point is $i_{\mathcal{U}}(\kappa)$.
Then,

$$\rho^{\mathcal{U}} = \sigma(\rho^{\mathcal{U}}) \in \sigma(i_{\mathcal{U}}(A)) = i_{\mathcal{V}}(A).$$

Note that generators of $\bar{U}(\kappa, 0)$ appear unboundedly many times below $\rho_{\mathcal{V}} > \rho_{\mathcal{U}}$. Let α^* be, say, the least generator such generator above $\rho^{\mathcal{U}}$.
Then $\alpha^* \in i_{\mathcal{V}}(X) \setminus \rho^{\mathcal{U}} + 1$. So,

$$\forall \mu \in i_{\mathcal{V}}(A) \cap \alpha^*(A'_\mu \in \bar{U}(i_{\mathcal{V}}(\kappa), \alpha^* + 1)).$$

Now, $\bar{U}(i_{\mathcal{V}}(\kappa), \alpha^* + 1)) <_{R-K} U(i_{\mathcal{V}}(\kappa), id) = i_{\mathcal{V}}(\mathcal{U})$. Let η represents a corresponding projection function in the ultrapower of $M_{\mathcal{V}}$ by $i_{\mathcal{V}}(\mathcal{U})$.
Then for all $\mu \in i_{\mathcal{V}}(A) \cap \alpha^*$, $\eta \in i_{i_{\mathcal{V}}(\mathcal{U})}(A'_\mu)$.
Hence,

$$\eta \in i_{i_{\mathcal{V}}(\mathcal{U})}(A'_{\rho^{\mathcal{U}}}).$$

So,

$$(\rho^{\mathcal{U}}, \eta) \in i_{i_{\mathcal{V}}(\mathcal{U})}(B).$$

We are done, since then

$$\{E \subseteq [\kappa]^2 \mid (\rho^{\mathcal{U}}, \eta) \in i_{i_{\mathcal{V}}(\mathcal{U})}(E)\} \supseteq \mathcal{U} \times \mathcal{V},$$

but $\mathcal{U} \times \mathcal{V}$ is an ultrafilter, so

$$\{E \subseteq [\kappa]^2 \mid (\rho^{\mathcal{U}}, \eta) \in i_{i_{\mathcal{V}}(\mathcal{U})}(E)\} = \mathcal{U} \times \mathcal{V},$$

which means that

$$\mathcal{U} \times \mathcal{V} <_{R-K} \mathcal{V} \times \mathcal{U}.$$

□

The second theorem can be deduced from the first, but let us give a direct argument.
Proof of the second theorem.
Let us show now that $\bar{U}(\kappa, 0) \times \mathcal{U} >_{R-K} \mathcal{U} \times \bar{U}(\kappa, 0)$.
Note that $\bar{U}(\kappa, 0)$ is normal. By Kanamori [4], it is impossible to have $\mathcal{V} \times \mathcal{U} >_{R-K} \mathcal{U} \times \mathcal{V}$ once $\mathcal{U}$ is normal or even a $P-$point.

We have

$$\mathcal{U} = \bar{U}(\kappa, 0) - \lim \langle \bar{U}(\kappa, \alpha) \mid \alpha < \kappa \rangle.$$

So, the ultrapower with $\mathcal{U}$ is obtained as follows. First $\bar{U}(\kappa, 0)$ is applied. We have

$$i_{\bar{U}(\kappa,0)} : V \to M_{\bar{U}(\kappa,0)}.$$

Next $\bar{U}(i_{\bar{U}(\kappa,0)}(\kappa), \kappa)$ is applied over $M_{\bar{U}(\kappa,0)}$. We have

$$i_{\bar{U}(i_{\bar{U}(\kappa,0)}(\kappa),\kappa)} : M_{\bar{U}(\kappa,0)} \to M_{\bar{U}(i_{\bar{U}(\kappa,0)}(\kappa),\kappa)}.$$

The composition is the ultrapower embedding by $\mathcal{U}$, i.e.

$$i_{\mathcal{U}} = i_{\bar{U}(i_{\bar{U}(\kappa,0)}(\kappa),\kappa)} \circ i_{\bar{U}(\kappa,0)} : V \to M_{\mathcal{U}} = M_{\bar{U}(i_{\bar{U}(\kappa,0)}(\kappa),\kappa)}.$$

Consider $\bar{U}(\kappa, 0) \times \mathcal{U}$.
So, we have $i_{\bar{U}(\kappa,0)} : V \to M_{\bar{U}(\kappa,0)}$ followed by $i_{\bar{U}(\kappa,0)}(\mathcal{U}) = U(i_{\bar{U}(\kappa,0)}(\kappa), id)$. The application of $U(i_{\bar{U}(\kappa,0)}(\kappa), id)$ to $M_{\bar{U}(\kappa,0)}$ has the similar description to the one above.
Namely, $i_{\bar{U}(\kappa,0)}(\bar{U}(\kappa, 0))$ is used first followed by

$$\bar{U}(i_{i_{\bar{U}(\kappa,0)}(\bar{U}(\kappa,0))}(i_{\bar{U}(\kappa,0)}(\kappa)), i_{\bar{U}(\kappa,0)}(\kappa)).$$

In order to simplify the notation, let us denote $i_{\bar{U}(\kappa,0)}$ by i_1, $M_{\bar{U}(\kappa,0)}$ by M_1, $i_{\bar{U}(\kappa,0)}(\kappa)$ by κ_1, the second ultrapower of $\bar{U}(\kappa, 0)$ by M_2 and the image of κ_1 there by κ_2.
Then $i_{\bar{U}(\kappa,0)\times\mathcal{U}} : V \to M_{\bar{U}(\kappa,0)\times\mathcal{U}}$ is $i_1 : V \to M_1$ followed by $i_{\bar{U}(\kappa_1,0)} : M_1 \to M_2$ and then by $i_{\bar{U}(\kappa_2,\kappa_1)} : M_2 \to M_{\bar{U}(\kappa,0)\times\mathcal{U}}$.
Note that in M_2, we have $\bar{U}(\kappa_2, \kappa_1) >_{R-K} \bar{U}(\kappa_2, \kappa)$ and even
$\bar{U}(\kappa_2, \kappa_1) >_{R-K} \bar{U}(\kappa_2, \kappa) \times \bar{U}(\kappa_2, 0)$.
Pick (η, ρ) which represents a corresponding projection function in the ultrapower of M_2 by $\bar{U}(\kappa_2, \kappa_1)$.
Let us argue that

$$\{E \subseteq [\kappa]^2 \mid (\eta, \rho) \in i_{\bar{U}(\kappa,0)\times\mathcal{U}}(E)\} \supseteq \mathcal{U} \times \bar{U}(\kappa, 0).$$

Let $A \in \mathcal{U}$, then

$$[id]_{\bar{U}(\kappa_1,\kappa)} \in i_{\mathcal{U}}(A) = i_{\bar{U}(\kappa_1,\kappa)}(i_1(A)).$$

Then, in M_1,

$$i_1(A) \in \bar{U}(\kappa_1, \kappa).$$

Apply the second ultrapower embedding $i_{\bar{U}(\kappa_1,0)}$ to it. Note that its critical point is $\kappa_1 > \kappa$. Then,

$$i_2(A) = i_{\bar{U}(\kappa_1,0)}(i_1(A)) \in i_{\bar{U}(\kappa_1,0)}(\bar{U}(\kappa_1,\kappa)) = \bar{U}(\kappa_2,\kappa).$$

Next apply $i_{\bar{U}(\kappa_2,\kappa_1)} : M_2 \to M_{\bar{U}(\kappa,0)\times\mathcal{U}}$. So, by the choice of η,

$$\eta \in i_{\bar{U}(\kappa,0)\times\mathcal{U}}(A) = i_{\bar{U}(\kappa_2,\kappa_1)}(i_2(A)).$$

Suppose now that $B \in \mathcal{U} \times \bar{U}(\kappa,0)$. Set

$$A := \{\mu < \kappa \mid \{\xi < \kappa \mid (\mu,\xi) \in B\} \in \bar{U}(\kappa,0)\}.$$

Then $A \in \mathcal{U}$ and for every $\mu \in A$ the set

$$A_\mu := \{\xi < \kappa \mid (\mu,\xi) \in B\} \in \bar{U}(\kappa,0).$$

Apply i_2. Then, in M_2,

$$\forall \mu \in i_2(A)(A_\mu \in \bar{U}(\kappa_2,0)).$$

But, by above, we have

$$i_2(A) \in \bar{U}(\kappa_2,\kappa),$$

hence,

$$i_2(B) \in \bar{U}(\kappa_2,\kappa) \times \bar{U}(\kappa_2,0).$$

So,

$$(\eta,\rho) \in i_{\bar{U}(\kappa,0)\times\mathcal{U}}(B),$$

and we are done.

□

Let us address now the strength issue.

Theorem 4.3 *Suppose that there is no inner model in which κ is a measurable with $\{o(\alpha) \mid \alpha < \kappa\}$ unbounded in it. Then for any two $\kappa-$complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over κ, if $\mathcal{V}\times\mathcal{U} \geq_{R-K} \mathcal{U}\times\mathcal{V}$, then there is an integer n such that $\mathcal{V} =_{R-K} \mathcal{U}^n$.*

Proof. Suppose that there is no inner model in which κ is a measurable with $\{o(\alpha) \mid \alpha < \kappa\}$ unbounded in it. Then the separation holds and there are no κ non-Rudin-Keisler equivalent ultrafilters which are Rudin-Keisler below some $\kappa-$complete ultrafilter.

Let $\mathcal{U}$ and $\mathcal{V}$ be two $\kappa-$complete ultrafilters over κ and $\mathcal{V} \times \mathcal{U} \geq_{R-K} \mathcal{U} \times \mathcal{V}$. Let $(\rho, \eta) \in [i_{\mathcal{V}\times\mathcal{U}}(\kappa)]^2$ generates $\mathcal{U} \times \mathcal{V}$, i.e.

$$\mathcal{U} \times \mathcal{V} = \{X \subseteq [\kappa]^2 \mid (\rho, \eta) \in i_{\mathcal{V}\times\mathcal{U}}(X)\}.$$

Clearly, then $\eta > i_{\mathcal{V}}(\kappa)$. Consider in $M_{\mathcal{V}}$ an ultrafilter $\mathcal{W}$ defined by η, i.e.

$$\mathcal{W} := \{Z \subseteq i_{\mathcal{V}}(\kappa) \mid \eta \in i_{i_{\mathcal{V}}(\mathcal{U})}(Z)\}.$$

Clearly, $\mathcal{W} \leq_{R-K} i_{\mathcal{V}}(\mathcal{U})$. Find a sequence of ultrafilters $\langle \mathcal{W}_\alpha \mid \alpha < \kappa\rangle$ which represents $\mathcal{W}$ in the ultrapower by $\mathcal{V}$, i.e.

$$i_{\mathcal{V}}(\langle \mathcal{W}_\alpha \mid \alpha < \kappa\rangle)([id]_{\mathcal{V}}) = \mathcal{W}.$$

So, for most (mod $\mathcal{V}$) α's, $\mathcal{W}_\alpha \leq_{R-K} \mathcal{U}$.
Note that

$$\mathcal{V} = \mathcal{V} - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa\rangle.$$

Namely,

$$X \in \mathcal{V} \Leftrightarrow \eta \in i_{\mathcal{V}\times\mathcal{U}}(X) \Leftrightarrow i_{\mathcal{V}}(X) \in \mathcal{W}$$

$$\Leftrightarrow \{\alpha < \kappa \mid X \in \mathcal{W}_\alpha\} \in \mathcal{V} \Leftrightarrow X \in \mathcal{V} - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa\rangle.$$

The sequence $\langle \mathcal{W}_\alpha \mid \alpha < \kappa\rangle$ may contain same ultrafilters, but among them must be κ different. Just otherwise, mod $\mathcal{V}$ they will be the same. Let $\mathcal{W}'$ be this ultrafilter. Then, $\mathcal{V} = \mathcal{V} - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa\rangle$, implies $\mathcal{V} = \mathcal{W}'$. So, $\mathcal{V} \leq_{R-K} \mathcal{U}$.
Now, if $\rho < i_{\mathcal{V}}(\kappa)$, then $\mathcal{U} \leq_{R-K} \mathcal{V}$. Hence, $\mathcal{U} =_{R-K} \mathcal{V}$, which is impossible.
Assume for a while that $\rho < i_{\mathcal{V}}(\kappa)$.
Still among this different $\mathcal{W}_\alpha$'s may be many which are Rudin-Keisler equivalent.
If the number of the equivalence classes has cardinality κ then we are done. Suppose otherwise. Then there is $\mathcal{W}'$ such that $\mathcal{W}_\alpha =_{R-K} \mathcal{W}'$, for almost every α mod $\mathcal{V}$.
Set $\alpha \sim \beta$ iff $\mathcal{W}_\alpha = \mathcal{W}_\beta$. Let $t : \kappa \to \kappa$ be a function which picks exactly one ultrafilter in such equivalence classes.
Set $\mathcal{V}' = t_*\mathcal{V}$. Then

$$\mathcal{V} = \mathcal{V}' - \lim \langle \mathcal{W}_\alpha \mid \alpha < \kappa\rangle.$$

Now, using the separation property, the ultrapower by $\mathcal{V}$ is the ultrapower by $\mathcal{V}'$ followed by $\mathcal{W}_{[id]_{\mathcal{V}'}}$.
But $\mathcal{W}_{[id]_{\mathcal{V}'}} =_{R-K} i_{\mathcal{V}'}(\mathcal{W}')$, so its ultrapower is the same as those by $i_{\mathcal{V}'}(\mathcal{W}')$. This means

that the iterated ultrapower is just $\mathcal{V}' \times \mathcal{W}'$.
So, $\mathcal{V}' \times \mathcal{W}' =_{R-K} \mathcal{V}$.
Then

$$\mathcal{V} \leq_{R-K} \mathcal{V}' \times \mathcal{U} \text{ and } \mathcal{V}' <_{R-K} \mathcal{V}.$$

Following Kanamori [4],5.9, we would like to argue that $\mathcal{U} \times \mathcal{V}' \leq_{R-K} \mathcal{V}$ and then to apply induction to

$$\mathcal{U} \times \mathcal{V}' \leq_{R-K} \mathcal{V}' \times \mathcal{U}.$$

I.e. there will be $n < \omega$ such that $\mathcal{V}' =_{R-K} \mathcal{U}^n$, and then

$$\mathcal{U} \times \mathcal{V}' \leq_{R-K} \mathcal{V} \leq_{R-K} \mathcal{V}' \times \mathcal{U}$$

will imply that $\mathcal{V} =_{R-K} \mathcal{U}^{n+1}$. Denote $[t]_{\mathcal{V}}$ by ρ'. By Kanamori [4],5.4, it is enough to show that for any not constant mod $\mathcal{V}$ function $g : \kappa \to \kappa$,

$$\rho < i_{\mathcal{V}\times\mathcal{U}}(g)(\rho').$$

Also, Kanamori [4],5.4, we know that for any not constant mod $\mathcal{V}$ function $g : \kappa \to \kappa$,

$$\rho < i_{\mathcal{V}\times\mathcal{U}}(g)(\eta).$$

So it will be enough to show that there is $s : \kappa \to \kappa$ such that

$$\rho' = i_{\mathcal{V}\times\mathcal{U}}(s)(\eta).$$

Define such s by using the separation property $\mathcal{W}_\alpha$'s relatively to $\mathcal{V}'$.
Thus let

$$\langle A_\alpha \mid \alpha \in B\rangle$$

be a disjoint family of sets, $B \in \mathcal{V}'$ such that each $A_\alpha \in \mathcal{W}_\alpha$. Consider

$$\langle A'_\alpha \mid \alpha \in i_{\mathcal{V}\times\mathcal{U}}(B)\rangle = i_{\mathcal{V}\times\mathcal{U}}(\langle A_\alpha \mid \alpha \in B\rangle).$$

Then $\eta \in A'_{\rho'}$, since η generates $W_{\rho'}$ in $M_{\mathcal{V}}$.
So, define $s : \kappa \to \kappa$ by setting

$$s(\mu) = \min(\{\alpha \mid \mu \in A_\alpha\}).$$

Suppose now that $\rho \geq i_{\mathcal{V}}(\kappa)$. Then, as above, replacing η by (ρ, η), we will have in $M_{\mathcal{V}}$ an ultrafilter W defined by (ρ, η), i.e.

$$W := \{Z \subseteq [i_{\mathcal{V}}(\kappa)]^2 \mid (\rho, \eta) \in i_{i_{\mathcal{V}}(\mathcal{U})}(Z)\}.$$

Clearly, $W \leq_{R-K} i_{\mathcal{V}}(\mathcal{U})$. Find a sequence of ultrafilters $\langle W_\alpha \mid \alpha < \kappa\rangle$ which represents W in the ultrapower by $\mathcal{V}$, i.e.

$$i_{\mathcal{V}}(\langle W_\alpha \mid \alpha < \kappa\rangle)([id]_{\mathcal{V}}) = W.$$

So, for most (mod $\mathcal{V}$) α's, $W_\alpha \leq_{R-K} \mathcal{U}$.
Note that

$$\mathcal{U} \times \mathcal{V} = \mathcal{V} - \lim \langle W_\alpha \mid \alpha < \kappa\rangle.$$

Namely,

$$X \in \mathcal{U} \times \mathcal{V} \Leftrightarrow (\rho, \eta) \in i_{\mathcal{V}\times\mathcal{U}}(X) \Leftrightarrow i_{\mathcal{V}}(X) \in W$$

$$\Leftrightarrow \{\alpha < \kappa \mid X \in W_\alpha\} \in \mathcal{V} \Leftrightarrow X \in \mathcal{V} - \lim \langle W_\alpha \mid \alpha < \kappa\rangle.$$

The sequence $\langle W_\alpha \mid \alpha < \kappa\rangle$ may contain same ultrafilters, but among them must be κ different. Just otherwise, mod $\mathcal{V}$ they will be the same. Let W' be this ultrafilter. Then, $\mathcal{U} \times \mathcal{V} = \mathcal{V} - \lim \langle W_\alpha \mid \alpha < \kappa\rangle$, implies $\mathcal{U} \times \mathcal{V} = W'$. So, $\mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{U}$, which is impossible.
Still among this different W_α's may be many which are Rudin-Keisler equivalent.
If the number of the equivalence classes has cardinality κ then we are done. Suppose otherwise. Then there is W' such that $W_\alpha =_{R-K} W'$, for almost every α mod $\mathcal{V}$.
Set $\alpha \sim \beta$ iff $W_\alpha = W_\beta$. Let $t : \kappa \to \kappa$ be a function which picks exactly one ultrafilter in such equivalence classes.
Set $\mathcal{V}' = t_*\mathcal{V}$. Then

$$\mathcal{U} \times \mathcal{V} = \mathcal{V}' - \lim \langle W_\alpha \mid \alpha < \kappa\rangle.$$

Now, using the separation property, the ultrapower by $\mathcal{U} \times \mathcal{V}$ is the ultrapower by $\mathcal{V}'$ followed by $W_{[id]_{\mathcal{V}'}}$.
But $W_{[id]_{\mathcal{V}'}} =_{R-K} i_{\mathcal{V}'}(W')$, so its ultrapower is the same as those by $i_{\mathcal{V}'}(W')$. This means that the iterated ultrapower is just $\mathcal{V}' \times W'$.
So, $\mathcal{V}' \times W' =_{R-K} \mathcal{U} \times \mathcal{V}$. Then by Kanamori [4] (5.6), at least one of the following three possibilities must holds:

1. $W' =_{R-K} \mathcal{V}$ and $\mathcal{V}' =_{R-K} \mathcal{U}$;
2. there is a $\kappa-$complete ultrafilter F, such that $\mathcal{V}' =_{R-K} \mathcal{U} \times F$ and $\mathcal{V} =_{R-K} F \times W'$;
3. there is a $\kappa-$complete ultrafilter G such that $\mathcal{U} =_{R-K} \mathcal{V}' \times G$ and $W' =_{R-K} G \times \mathcal{V}$.

Suppose for a moment that the first possibility occurs. Then

$$\mathcal{U} \geq_{R-K} \mathcal{W}' =_{R-K} \mathcal{V} \geq_{R-K} \mathcal{V}' =_{R-K} \mathcal{U}.$$

So, $\mathcal{U} =_{R-K} \mathcal{V}$, and then $\mathcal{U} \times \mathcal{V} =_{R-K} \mathcal{V} \times \mathcal{U}$, which is impossible.

Suppose now that the second possibility occurs.
Then $\mathcal{V} \geq_{R-K} \mathcal{V}'$ and $W' \leq_{R-K} U$ imply

$$\mathcal{U} \times F \leq_{R-K} F \times W' \leq_{R-K} F \times \mathcal{U}.$$

But, also (2) implies that $\mathcal{V} >_{R-K} F$. So, we can apply the induction to

$$\mathcal{U} \times F \leq_{R-K} F \times \mathcal{U}.$$

Consider now the third possibility.
Then $\mathcal{U} \geq_{R-K} W'$ and $\mathcal{V} \geq_{R-K} \mathcal{V}'$ imply

$$\mathcal{V} \times G \geq_{R-K} \mathcal{V}' \times G \geq_{R-K} G \times \mathcal{V}.$$

But, also (3) implies that $\mathcal{U} >_{R-K} G$. So, we can apply the induction to

$$\mathcal{V} \times G \geq_{R-K} G \times \mathcal{V}.$$

$\square$

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