A note on sequences witnessing singularity - following Magidor-Sinapova.

Moti Gitik

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Abstract

We address some question raised in Magidor-Sinapova [7] paper.

Suppose $V \subseteq W$, κ is regular in V, but change its cofinality in W. Are there "nice" witnesses for such change? This is a basic question and a lot of work was done around it (probably the most prominent are - Prikry forcing, Jensen and Dodd-Jensen Covering Lemmas, Mitchell Covering Lemmas). Some ZFC results were proved in Dzamonia-Shelah [1] and in [2]. Recently, Magidor and Sinapova [7] studied a supercompact version of it. In this note we address some question raised in this paper.

Let us start with the following:

Theorem 0.1 *Suppose that*

1. $V \subseteq W$.
2. κ *is a regular uncountable cardinal in* V.
3. $\mu > \kappa$ *is a cardinal in* V.
4. *In* V, $2^\mu = \mu^+$.
5. $(\mu^+)^V = \bigcup_{n<\omega} Q_n$, *for some sequence* $\langle Q_n \mid n < \omega \rangle$ *of elements of* $\mathcal{P}_\kappa(\mu^+)$ *of* V.
6. *In* W, $(\mu^+)^V \geq ((2^\omega)^+)^W$.
7. *In* W, $(\mu^{++})^V$ *is a cardinal.*

Let $\langle D_\alpha \mid \alpha < (\mu^{++})^V \rangle \in V$ *be a sequence of clubs in* $\mathcal{P}_\kappa(\mu^+)$ *of* V. *Then there is an increasing sequence* $\langle P_n \mid n < \omega \rangle$ *such that for every* $\alpha < (\mu^{++})^V$, *for all but finitely many* $n < \omega$, $P_n \in D_\alpha$.

Proof. Without loss of generality we can assume that for every $\alpha < \beta < (\mu^{++})^V$ there is $\gamma(\alpha,\beta) < (\mu^+)^V$ such that $D_\beta \cap \{P \in \mathcal{P}_\kappa(\mu^+) \mid \gamma(\alpha,\beta) \in P\} \subseteq D_\alpha$.

Let D be a club $\mathcal{P}_\kappa(\mu^+)$ in V. Then the set $C_D := \{\delta < \mu^+ | D \cap \mathcal{P}_\kappa(\delta)$ is a club $\}$ is a club in μ^+.

Let $\delta < \mu^+$. By the assumption, we have that in V,
$|\{f \mid f : [\delta]^{<\omega} \to [\delta]^{<\kappa}\}| = (\mu^+)^\mu = \mu^+$.
So, there only μ^+ clubs in $\mathcal{P}_\kappa(\delta)$ in V. Also, we have $(\mu^+)^V = \bigcup_{n<\omega} Q_n$, for some sequence $\langle Q_n \mid n < \omega\rangle$ of elements of $\mathcal{P}_\kappa(\mu^+)$ of V. Hence, there is a decreasing sequence of clubs (each of them in V) $\langle E^n_\delta \mid n < \omega\rangle$ of $(\mathcal{P}_\kappa(\delta))^V$, such that
for every $E \subseteq (\mathcal{P}_\kappa(\delta))^V$ a club in V there is $n_E < \omega$ such that for every $n, n_E \le n < \omega$, we have $E^n_\delta \subseteq E$.
Pick, for every $n < \omega$ an element R^n_δ in E^n_δ (take them to be an increasing sequence, as well). Then,
for every $E \subseteq (\mathcal{P}_\kappa(\delta))^V$ a club in V, for every $n, n_E \le n < \omega$, we will have $R^n_\delta \in E$.

By Dzamonia-Shelah [1] or by [2], there is a sequence $\langle \eta_n \mid n < \omega\rangle$ such that for every $H \in \{C_{D_\alpha} \mid \alpha < (\mu^{++})^V\}$ there is n_H such that for every $n, n_H \le n < \omega$, $\eta_n \in H$.

For every $\alpha < (\mu^{++})^V$, define a function $f_\alpha : \omega \to \omega \times \omega$.
Pick some $k_0 \ge n_{C_{D_\alpha}}$ and then some $s_0 \ge \max(n_{D_\alpha \cap \mathcal{P}_\kappa(\eta_{k_0})}, k_0)$. Set $f_\alpha(0) = (k_0, s_0)$. Then, $R^{s_0}_{\eta_{k_0}} \in D_\alpha$ and for every $s, s_0 \le s < \omega$, $R^s_{\eta_{k_0}} \in D_\alpha$, as well.
Suppose now that $f_\alpha(n)$ is defined. Define $f_\alpha(n+1)$. Pick first some $k_{n+1} > \max(f_\alpha(n))$ such that $Q_n \in \mathcal{P}_\kappa(\eta_{k_{n+1}})$. Let $s_{n+1} \ge \max(n_{D_\alpha \cap \mathcal{P}_\kappa(\eta_{k_{n+1}})}, \max(f_\alpha(n)))$ be such that $R^{s_{n+1}}_{\eta_{k_{n+1}}} \supseteq R^{f_\alpha(n)}_{\eta_{k_n}} \cup Q_n$.
Set $f_\alpha(n+1) = (k_{n+1}, s_{n+1})$.
Then, $R^{s_{n+1}}_{\eta_{k_{n+1}}} \in D_\alpha$ and for every $s, s_{n+1} \le s < \omega$, $R^s_{\eta_{k_{n+1}}} \in D_\alpha$, as well.

By the assumption $\kappa \ge ((2^\omega)^+)^W$. Hence, there are a stationary $S \subseteq (\mu^{++})^V = (\kappa^+)^W$ and $f : \omega \to \omega \times \omega$ such that $f_\alpha = f$, for every $\alpha \in S$.

Define now an increasing sequence $\langle P_n \mid n < \omega\rangle$ as follows:

$P_n = R^{f(n)_1}_{\eta_{f(n)_0}}$, where $f(n) = (f(n)_0, f(n)_1)$.

Let us argue that the sequence $\langle P_n \mid n < \omega\rangle$ is as desired.
Let first $\alpha < (\mu^{++})^V$ be in S. Consider C_{D_α}. For every $n, n_{C_{D_\alpha}} \le n < \omega$, $\eta_n \in C_{D_\alpha}$.
Now, we have $f = f_\alpha$ and $f_\alpha(0)_0 \ge n_{C_{D_\alpha}}$. Then, $R^s_{\eta_{f(n)_0}} \in D_\alpha$, for every $s, (f(n))_1 \le s < \omega$.
In particular, $P_n \in D_\alpha$ for every n, and we are done.
Let $\alpha < (\mu^{++})^V$ be arbitrary now. Pick $\beta \in S \setminus \alpha$. Then, $P_n \in D_\beta$, for every $n < \omega$. There is $n^*, n(\beta) \le n^* < \omega$ such that $\gamma(\alpha,\beta) \in P_n$, for every $n, n^* \le n < \omega$, since $(\mu^+)^V =$

$\bigcup_{n<\omega} Q_n = \bigcup_{n<\omega} P_n$. Recall that we have $D_\beta \cap \{P \in \mathcal{P}_\kappa(\mu^+) \mid \gamma(\alpha,\beta) \in P\} \subseteq D_\alpha$. Hence, $P_n \in D_\alpha$, for every $n, n^* \leq n < \omega$.
$\square$

The next result has the same proof:

Theorem 0.2 *Suppose that*

1. $V \subseteq W$.
2. κ *is a regular uncountable cardinal in* V.
3. $\mu > \kappa$ *is a regular cardinal in* V.
4. *In* V, $\mu^{<\mu} = \mu$.
5. $\mu = \bigcup_{n<\omega} Q_n$, *for some sequence* $\langle Q_n \mid n < \omega\rangle$ *of elements of* $\mathcal{P}_\kappa(\mu)$ *of* V.
6. $\mu \geq ((2^\omega)^+)^W$.
7. *In* W, $(\mu^+)^V$ *is a cardinal.*

Let $\langle D_\alpha \mid \alpha < (\mu^+)^V\rangle \in V$ *be a sequence of clubs in* $\mathcal{P}_\kappa(\mu)$ *of* V. *Then there is an increasing sequence* $\langle P_n \mid n < \omega\rangle$ *such that for every* $\alpha < (\mu^+)^V$, *for all but finitely many* $n < \omega$, $P_n \in D_\alpha$.

Remark 0.3 Note that if $W \supseteq V$, κ is a regular cardinal in V and for some $\mu \geq \kappa$ we have $\mu = \bigcup_{n<\omega} Q_n$, for a sequence $\langle Q_n \mid n < \omega\rangle$ of elements of $\mathcal{P}_\kappa(\mu)$ of V, then all $V-$regular cardinals in the interval $[\kappa, \mu]$ change their cofinality to ω in W.
Thus, first we can assume that the sequence $\langle Q_n \mid n < \omega\rangle$is increasing. Let $\eta \in [\kappa, \mu]$ be a regular cardinal in V. Then $\eta = \bigcup_{n<\omega}(Q_n \cap \eta)$. Set $\eta_n = \sup(Q_n \cap \eta)$, for every $n < \omega$. Then the sequence $\langle \eta_n \mid n < \omega\rangle$ will be cofinal in η.

Proposition 0.4 *Suppose that*

1. $V \subseteq W$,
2. κ *is a regular cardinal in* V,
3. $\mu > \kappa$ *is a cardinal in* V,
4. $\mathrm{cof}^V(\mu) < \kappa$,

5. in V, $\forall \tau < \kappa (\tau^{\mathrm{cof}(\mu)} \le \mu)$,

6. $(\mu^+)^V$ is a cardinal in W.

Then there is a sequence $\langle D_\alpha \mid \alpha < (\mu^+)^V \rangle \in V$ of clubs in $\mathcal{P}_\kappa(\mu)$ of V. such that for any sequence $\langle P_n \mid n < \omega \rangle$ of elements of $(\mathcal{P}_\kappa(\mu))^V$ there is $\alpha < (\mu^+)^V$ such that for infinitely many $n < \omega$, $P_n \notin D_\alpha$.

Proof. Pick in V a set $\mathfrak{a} \subseteq \mu$ of regular cardinals unbounded in μ and of cardinality $\mathrm{cof}(\mu)$ such that $\mathrm{tcf}(\prod \mathfrak{a}, J^{bd}) = \mu^+$, as witnessed by a sequence of functions $\langle f_\xi \mid \xi < \mu^+ \rangle$ in $\prod \mathfrak{a}$.

Consider $\{\mathrm{ran}(f_\xi) \mid \xi < \mu^+\}$. Set $D_\xi = \{P \in \mathcal{P}_\kappa(\mu) \mid P \supseteq \mathrm{ran}(f_\xi)\}$, for every $\xi < \mu^+$. Suppose for a moment that there is a sequence $\langle P_n \mid n < \omega \rangle$ of elements of $\mathcal{P}_\kappa(\mu)$ of V such that for every $\alpha < (\mu^+)^V$, for all but finitely many $n < \omega$, $P_n \in D_\alpha$.

Then there is $i < \omega$ such that $A := \{\xi \mid \mathrm{ran} f_\xi \subseteq P_i\}$ has cardinality μ^+. But $A \in V$ and, in V, $|P_i|^{\mathrm{cof}(\mu)} < \mu^+$, which is impossible. Contradiction.
□

Remark 0.5 Note that the forcing of [3] provides an example of such situation. In the model of [3] we have $\mu = \bigcup_{i<\omega} x_i$, for some $x_i \in (\mathcal{P}_\kappa(\mu))^V$ bounded in μ.
The next proposition shows that it is quite a general phenomena.
Also, this model provides an example of the situation in which μ can be presented as a countable union of members of $(\mathcal{P}_\kappa(\mu))^V$, but there is a sequence $\langle D_\alpha \mid \alpha < (\mu^+)^V \rangle \in V$ of clubs in $\mathcal{P}_\kappa(\mu)$ of V. such that for any sequence $\langle P_n \mid n < \omega \rangle$ of elements of $(\mathcal{P}_\kappa(\mu))^V$ there is $\alpha < (\mu^+)^V$ such that for infinitely many $n < \omega$, $P_n \notin D_\alpha$.

Proposition 0.6 *Let $V \subseteq W$, $\kappa \le \mu$ are cardinals in V and $\mu < (\kappa^{+\omega_1})^V$. Assume that all regular cardinals of the interval $[\kappa, \mu]$ change their cofinality to ω in W. Then every $\eta \in [\kappa, \mu]$ can be presented as a union of countably many elements of $(\mathcal{P}_\kappa(\eta))^V$.*

Proof. It is enough to proof the statement for $V-$cardinals η only. Proceed by induction. Suppose that $\eta = \bigcup_{i<\omega} x_i^\eta, x_i^\eta \in (\mathcal{P}_\kappa(\eta))^V$. Turn to $(\eta^+)^V$. Its cofinality in W is ω. Fix a witnessing cofinal sequence $\langle \tau_n \mid n < \omega \rangle$. For every $n < \omega$, let $f_n : \eta \leftrightarrow \tau_n, f_n \in V$.
Set $x_n^{(\eta^+)^V} = \bigcup_{m \le n} f_m{}''(\bigcup_{k \le m} x_k^\eta)$.

If η is a limit cardinal, then by the assumption its cofinality is countable (in V). Let a cofinal sequence $\langle \eta_n \mid n < \omega \rangle \in V$. By induction, for every $n < \omega$ we have $\eta_n = \bigcup_{i<\omega} x_i^{\eta_n}, x_i^{\eta_n} \in (\mathcal{P}_\kappa(\eta))^V$. Now set $x_n^\eta = \bigcup_{m \le n}(x_0^{\eta_m} \cup ... \cup x_m^{\eta_m})$.
□

Actually a bit more general statement is true:

Proposition 0.7 *Let $V \subseteq W$, $\kappa \leq \mu$ be cardinals in V, $\delta \in [\kappa, \mu]$, δ is a union of countably many elements of $(\mathcal{P}_\kappa(\eta))^V$ and $\mu < (\delta^{+\omega_1})^V$. Assume that all regular cardinals of the interval $[\delta, \mu]$ change their cofinality to ω in W. Then every $\eta \in [\kappa, \mu]$ can be presented as a union of countably many elements of $(\mathcal{P}_\kappa(\eta))^V$.*

The proof repeats basically the proof of the previous proposition.

Let us construct a model in which cardinals between κ and μ are collapsed and regular there change cofinality to ω, but μ cannot be presented as a union of countably many members of $(\mathcal{P}_\kappa(\mu))^V$.

Suppose that μ is limit of an increasing sequence $\langle \mu_i \mid i < \omega_1 \rangle$ of measurable cardinals and $\delta > \mu$ is a Woodin cardinal. Force first with the Magidor iteration and add one element Prikry sequence μ_i^* to each μ_i. Then for every $X \subseteq \mu$ in V of cardinality less than κ (or even less then $\bigcup_{n<\omega} \mu_n$), we will have that $X \cap \{\mu_i^* \mid i < \omega_1\}$ is finite.

Collapse now all the cardinals of the intervals (μ_i, μ_{i+1}) and (κ, μ_0). Denote by V_1 such extension of V. Clearly, δ remains Woodin in V_1. Use now the Woodin Stationary Tower forcing (see [5])to change cofinality of κ and each of μ_i's to ω and preserving κ as a cardinal. Let W be such extension of V_1.

Then $\{\mu_i^* \mid i < \omega_1\}$ will witness the desired conclusion between V and W.

By using first collapses over V and then forcing with positive sets it is possible to arrange $V_1, V \subseteq V_1 \subseteq W$, in which $\mu = (\kappa^{+\omega_1})^{V_1}$ and it cannot be presented in W as a union of countably many elements of $(\mathcal{P}_\kappa(\mu))^{V_1}$.

Proposition 0.8 *Let $V \subseteq W$, κ be a cardinal in V, $\delta > \kappa$. Suppose that $(\mathrm{cof}(\delta))^V \geq \kappa$ and $(\mathrm{cof}(\delta))^W > \aleph_0$.*

Then no $\mu \geq \delta$ can be presented as a union of countably many elements of $(\mathcal{P}_\kappa(\mu))^V$.

Proof. Suppose otherwise. Let $\mu = \bigcup_{n<\omega} Q_n$, for some sequence $\langle Q_n \mid n < \omega \rangle$ of elements of $\mathcal{P}_\kappa(\mu)$ of V. Then $\delta = \bigcup_{n<\omega}(Q_n \cap \delta)$. Now, $(\mathrm{cof}(\delta))^V \geq \kappa$ implies that $\sup(Q_n \cap \delta) < \delta$, for every $n < \omega$. Hence, $\langle \sup(Q_n \cap \delta) \mid n < \omega \rangle$ is cofinal in δ, which is impossible, since $(\mathrm{cof}(\delta))^W > \aleph_0$. Contradiction.

□

Remark 0.9 The Namba forcing is a typical example of a situation above. Thus, let $\kappa = \aleph_2, \delta = \aleph_3$. Force with the Namba forcing. Then κ will change its cofinality to ω, δ to ω_1

and both will be collapsed to $\aleph_1$. So, no $\mu \geq \delta$ can be presented as a union of countably many elements of $(\mathcal{P}_\kappa(\mu))^V$.
The Woodin tower forcing P_δ provides other examples of this situation.

Let us give now an application of 0.1.

Theorem 0.10 *Suppose that κ is $\lambda-$strongly compact, $2^\lambda = \lambda^+$ and $\lambda^\omega = \lambda$. Then there is a $Q-$point ultrafilter over $\mathcal{P}_\kappa(\lambda)$, i.e. a fine $\kappa-$complete ultrafilter over $\mathcal{P}_\kappa(\lambda)$ which contains all closed unbounded subsets of $\mathcal{P}_\kappa(\lambda)$.*

Remark 0.11 1.Note that if we allow more strong compactness (say κ is $2^\lambda-$strongly compact), then it is trivial to find a fine $\kappa-$complete ultrafilter over $\mathcal{P}_\kappa(\lambda)$ which contains all clubs on $\mathcal{P}_\kappa(\lambda)$. Just the club filter on $\mathcal{P}_\kappa(\lambda)$ is generated by $\leq 2^\lambda-$many sets, and so it can be extended to a fine $\kappa-$complete ultrafilter over $\mathcal{P}_\kappa(\lambda)$ which contains all clubs on $\mathcal{P}_\kappa(\lambda)$.
2. By classical result of M. Magidor [6], it is possible to have $\lambda-$strongly compact cardinal κ which is the least measurable. In a sense, the theorem shows that some reminiscence of normality always remains.

Proof. Fix a fine $\kappa-$complete ultrafilter U over $\mathcal{P}_\kappa(\lambda)$.
Let $\mathcal{P}$ be the tree Prikry forcing with U. Force with $\mathcal{P}$. Let $G(\mathcal{P}) \subseteq \mathcal{P}$ be generic. Then, by 0.1, in $V[G(\mathcal{P}]$, there is an increasing sequence $\langle P_n \mid n < \omega\rangle$ such that for every club $D \subseteq \mathcal{P}_\kappa(\lambda)$ in V, for all but finitely many $n < \omega$, $P_n \in D$.
Now back in V, we pick a name $\langle \underset{\sim}{P}_n \mid n < \omega\rangle$ such that $(\langle\rangle, [\lambda]^{<\omega})$ forces above. By the properties of $\mathcal{P}$, there is a condition $(\langle\rangle, T) \in \mathcal{P}$ and an increasing sequence $\langle m_n \mid n < \omega\rangle$ of natural numbers such that

- for every $n < \omega$, $t \in T, |t| \geq m_n$ we have $(t, T_t) \parallel \underset{\sim}{P}_n$.

Also, for every club C in $\mathcal{P}_\kappa(\lambda)$ there are $n_C < \omega$ and a tree T_C such that

1. $(\langle\rangle, T_C) \geq^* (\langle\rangle, T)$,

2. for every $t \in T_C$ with $|t| \geq m_{n_C}$ we have $(t, (T_C)_t \Vdash \underset{\sim}{P}_{n_C} \in \check{C}$.

Set $[\alpha] = \{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in P\}$, for every $\alpha < \lambda$.
There is $n^* < \omega$ and $A \subseteq \lambda$ of cardinality λ such that for every $\alpha \in A$ we have $n_{[\alpha]} = n^*$.
Pick an enumeration $\langle C_\gamma \mid \gamma < \lambda^+\rangle$ of clubs of $\mathcal{P}_\kappa(A)$, so that for every $\beta < \gamma < \lambda^+$, there is $\delta(\beta, \gamma) \in A$ such that for every $Q \in C_\gamma$, if $\delta(\beta, \gamma) \in Q$, then $Q \in C_\beta$

Let for $C \subseteq \mathcal{P}_\kappa(A)$, C^λ denotes the set $\{P \in \mathcal{P}_\kappa(\lambda) \mid P \cap A \in C\}$. Then, by Menas (see [4], 8.27), if $C \subseteq \mathcal{P}_\kappa(A)$ is a club then C^λ is a club in $\mathcal{P}_\kappa(\lambda)$.
Find a stationary $S \subseteq \lambda^+$ and $n^{**} \geq n^*$ such that for every $\gamma \in S$, $n_{C^\lambda_\gamma} = n^{**}$.

Consider $U^{m_{n^{**}}}$ (i.e. the product U with itself $m_{n^{**}}-$many times). Then for every condition $(\langle\rangle, R) \in \mathcal{P}$, $R \restriction m_{n^{**}} \in U^{m_{n^{**}}}$.
Define a projection map $F : [\mathcal{P}_\kappa(\lambda)]^{m_{n^{**}}} \to \mathcal{P}_\kappa(\lambda)$ as follows:

$$F(t) = \begin{cases} \emptyset, & \text{if } t \notin T; \\ P, & \text{if } t \in T \text{ and } (t, T_t) \Vdash \underset{\sim}{P}_{n^{**}} = \check{P} \ . \end{cases}$$

Set

$$\mathcal{V} = \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid F^{-1\prime\prime} X \in U^{m_{n^{**}}}\}.$$

Lemma 0.12 *For every $\alpha \in A$, $[\alpha] \in \mathcal{V}$.*

Proof. Suppose otherwise. Then $X := \{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \notin P\} \in \mathcal{V}$. Set $Y = F^{-1\prime\prime}X$. Then $Y \in U^{m_{n^{**}}}$. Recall that we have a tree $T_{[\alpha]}$ such that for every $t \in T_{[\alpha]}$ with $|t| \geq m_{n_{[\alpha]}}$ we have $(t, (T_{[\alpha]})_t \Vdash \underset{\sim}{P}_{n_{[\alpha]}} \in \check{[\alpha]}$. Which means that $(t, (T_{[\alpha]})_t \Vdash \check{\alpha} \in \underset{\sim}{P}_{n_{[\alpha]}}$. The sequence $\langle P_n \mid n < \omega \rangle$ is forced to be increasing, hence, for every $n, n_{[\alpha]} \leq n < \omega$, $(t, (T_{[\alpha]})_t \Vdash \check{\alpha} \in \underset{\sim}{P}_n$, and so, $(t, (T_{[\alpha]})_t \Vdash \underset{\sim}{P}_n \in \check{[\alpha]}$.
Now, let us shrink the tree $T_{[\alpha]}$ to a tree T' by replacing $T_{[\alpha]} \restriction m_{n^{**}}$ with $T_{[\alpha]} \restriction m_{n^{**}} \cap Y$. Note that both members of this intersection are in $U^{m_{n^{**}}}$. Hence, $(\langle\rangle, T')$ will be a condition in $\mathcal{P}$ and will be stronger than $\langle\rangle, T_{[\alpha]}$.
Pick some $t \in T_{[\alpha]}$ with $|t| = m_{n^{**}}$.
Let $F(t) = P$, for some P. t belongs to Y, hence $\alpha \notin P$. However, we have $(t, T_t) \Vdash \underset{\sim}{P}_{n^{**}} = \check{P}$. Hence a stronger condition (t, T'_t) forces the same. Recall that $n^{**} \geq n^* = n_{[\alpha]}$. So, $(t, (T_{[\alpha]})_t \Vdash \check{\alpha} \in \underset{\sim}{P}_{n^{**}}$. Then, also, $(t, T'_t) \Vdash \check{\alpha} \in \underset{\sim}{P}_{n^{**}}$. But then, α must belong to P which is impossible. Contradiction.
$\square$ of the lemma.

The next lemma is similar.

Lemma 0.13 *For every $\gamma \in S$, $C^\lambda_\gamma \in \mathcal{V}$.*

Consider now

$$\mathcal{V}^* = \{X \restriction A \mid X \in \mathcal{V}\},$$

where $X \restriction A = \{P \cap A \mid P \in X\}$.

Lemma 0.14 *$\mathcal{V}^*$ is a fine $\kappa-$complete ultrafilter over $\mathcal{P}_\kappa(A)$ which includes all club subsets of $\mathcal{P}_\kappa(A)$.*

Proof. For every $\alpha \in A$, $[\alpha] \in \mathcal{V}$, by Lemma 0.12. Then $[\alpha] \upharpoonright A = \{P \cap A \mid \alpha \in P\} \in \mathcal{V}^*$. But $\{P \cap A \mid \alpha \in P\} = \{Q \in \mathcal{P}_\kappa(A) \mid \alpha \in Q\}$. So, $\mathcal{V}^*$ is fine.

Let now $C \subseteq \mathcal{P}_\kappa(A)$ be a club. We like to show that $C \in \mathcal{V}^*$. Then there are $\gamma \in S$ and $\delta \in A$ such that for every $Q \in C_\gamma$, if $\delta \in Q$, then $Q \in C$. So, $C_\gamma \cap [\delta] \upharpoonright A \subseteq C$. Hence, it is enough to show that $C_\gamma \in \mathcal{V}^*$. But this follows from Lemma 0.13.
$\square$ of the lemma.

Now it is easy to finish the proof of the theorem. Just pick an injection $\sigma : A \longleftrightarrow \lambda$ and move using it $\mathcal{V}^*$ from $\mathcal{P}_\kappa(A)$ to $\mathcal{P}_\kappa(\lambda)$. Namely let $\mathcal{V}^{**}$ be defined as follows:
$X \in \mathcal{V}^{**}$ iff $\sigma^{-1}X \in \mathcal{V}^*$, where $\sigma^{-1}X = \{\sigma^{-1}{}''P \mid P \in X\}$.
$\square$

Let us conclude with the following:

Conjecture. Suppose that

1. $V \subseteq W$ models of ZFC with same ordinals,
2. κ is a regular cardinal in V,
3. $\mathrm{cof}(\kappa) = \omega$ in W,
4. $\aleph_1^V = \aleph_1^W$,
5. V, W agree about a final segment of cardinals.

Then there is a subclass V' of V which is a model of ZFC, agree with V about a final segment of cardinals, and there is a sequence witnessing singularity of κ (in W) which is generic over V' for either Namba, Woodin tower or Prikry type forcing.

References

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