Negation of the Singular Cardinals Hypothesis with GCH below

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Abstract

The purpose of this paper is to provide an attempt to understand the difficulty of getting a model where GCH breaks first time at a singular κ and there is an inner model in which κ is a regular cardinal but still with 2^κ big.

1 Introduction

There is a tension between the negation of the Singular Cardinals Hypothesis the power function below it. A celebrated result of J. Silver [14] states that a singular cardinal of uncountable cofinality cannot be the first that violates GCH.

M. Magidor [11], using extremely sophisticated arguments, showed that this need not be the case with a singular of cofinality ω. Namely, starting with a supercompact cardinal with a huge above, he constructed a model in which $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{n+1}$, for every $n < \omega$.

In early 80-th, Hugh Woodin came up with a beautiful construction of a model of $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{n+1}$, for every $n < \omega$. The initial assumptions of his construction were optimal.

However, the gap between the singular cardinal and its power in both of the constructions was 2 and not more.

The basic reason for the difficulty was that both arguments based on the Silver-Prikry method of violating SCH, i.e. first a model with a measurable cardinal κ with $2^\kappa > \kappa^+$ was constructed and then the Prikry forcing was used to change the cofinality of κ to ω. But having a measurable cardinal κ with $2^\kappa > \kappa^+$ implies that GCH is violated at unboundedly

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many places below κ. So, a hard tusk starts to be to collapse cardinals in order to resurrect GCH below and still keeping $2^\kappa > \kappa^+$.

Later a different method of constructions of models of $\neg$SCH - Extender Based Prikry forcing was introduced in [9]. It allows to change cofinality of κ and to blow up its power higher simultaneously without adding new bounded subsets to κ.

Following this developments, H. Woodin asked the following natural question:

Assuming that there is no inner model with a strong cardinal, is it possible to have a model M in which $2^{\aleph_\omega} > \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{n+1}$, for every $n < \omega$, and there is an inner model N such that $\kappa = \aleph_\omega$ is a measurable and $2^\kappa \geq (\aleph_{\omega+3})^M$?

A reasonable approach to this question was to use Extender based forcing over κ together with a suitable preparation which say adds many Cohen subsets to ν's below κ, and then, passing into a submodel in which κ is still regular, we combine this Cohen's from the preparation together, using Prikry sequences, in order to obtain $2^\kappa-$Cohens over the submodel.

It turned out to be realizable to some degree. Namely, as it was shown in [2], even the Prikry forcing (with carefully picked $\kappa-$complete ultrafilter) can add κ^+-many mutually generic Cohen subsets to κ over a submodel. However, by [2], neither the original ([9]) nor C. Merimovich ([12]) versions of Extender based Prikry forcings cannot produce the above type of inner models. Namely, if $\mathcal{P}_E$ denotes the Extender based forcing of [9] and $G \subseteq \mathcal{P}_E$ is generic, then:
For every $A \in V[G] \setminus V, A \subseteq \kappa$, κ changes its cofinality to ω in $V[A]$.
If $\mathbb{P}_E$ denotes the Extender based forcing of [12] and $G \subseteq \mathbb{P}_E$ is generic, then:
For every $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ list of different subsets of κ in $V[G]$, there is $I \subseteq \kappa^{++}, I \in V, |I| = \kappa$ such that κ changes its cofinality to ω in $V[\langle A_\alpha \mid \alpha \in I\rangle]$.

The aim of the present paper is to use The Mitchell Covering Lemma with some Pcf-arguments in order through some more light on the reasons of the difficulty to have an inner model in which κ is regular, but still 2^κ is big. In particular, this will provide some progress on the question of Woodin.

2 Settings and main results

Assume $\neg 0^\P$. Let $\mathcal{K}$ denotes the core model.
First we would like to show the following:

Theorem 2.1 *Suppose that in V, $\mathrm{cof}(\kappa) = \omega, 2^\kappa = \kappa^{++}$, GCH holds below κ and there is an inner model $V' \supseteq \mathcal{K}$ in which κ is a regular, but still $2^\kappa \geq \kappa^{++}$.*
Assume that

1. *every $a \subseteq (2^\kappa)^{V'}, |a| < \kappa$ can be covered by a set $b \in V'$ with $|b| \leq \kappa$,*
2. *$V' \models 2^\nu = \nu^+$ for ν's in a club subset of κ in V',*
3. *$(\kappa^{++})^{V'} = \kappa^{++}$.*

Then $|(\tau^+)^{\mathcal{K}}| = \tau$, for unboundedly many cardinals $\tau < \kappa$.

Remark 2.2
1. Note that κ is a measurable in $\mathcal{K}$, and so, by the Mitchell Covering Lemma, $(\kappa^+)^{\mathcal{K}} = \kappa^+$.
2. If $(2^\kappa)^{V'} < \kappa^{+\omega}$, then we have the required type of covering by standard arguments.
3. If there is no measurable cardinal above κ in $\mathcal{K}$, then again we have the required type of covering, by the Mitchell Covering Lemma.
4. If κ is a measurable cardinal in V', then the required type of covering holds. $\neg 0^\P$ is assumed, so, by [7], there is no measurable in $\mathcal{K}$ cardinal in the interval $(\kappa, (2^\kappa)^{V'}]$.

In order to state our further results we will need to define the following form of a strong covering:

Definition 2.3 *Let $V' \subseteq V$, κ be a cardinal in V. Then $Cov(V, V', \kappa^+)$* holds iff:
For every set of ordinals $B \subseteq 2^\kappa$ of cardinality κ^+ there are $I \subseteq B$ of cardinality κ and $I^* \in V', I^* \supseteq I$ such that for some increasing and continuous sequence $\langle M_\nu \mid \nu < \kappa \rangle \in V'$ with $|M_\nu| < \kappa$, for every $\nu < \kappa$, and $I^* \subseteq \bigcup_{\nu<\kappa} M_\nu$, the following holds: for every $\nu < \kappa, |M_\nu \cap I| = |M_\nu \cap I^*|$.

Note that the following density property implies $Cov(V, V', \kappa^+)$:
Every set of ordinals $S' \subseteq 2^\kappa$ of cardinality κ^+ contains a set in V' of cardinality κ.
We refer to [8] on this subject.

The next theorem shows, in particular, that the Woodin method for restoring GCH below a singular κ with $2^\kappa = \kappa^{++}$, is basically the only possible.

Theorem 2.4 *Suppose that in V, $\mathrm{cof}(\kappa) = \omega, 2^\kappa = \kappa^{++}$, GCH holds below κ and there is an inner model $V' \supseteq \mathcal{K}$ in which κ is a regular, but still $2^\kappa \geq \kappa^{++}$.*
Assume that

1. *every* $a \subseteq (2^\kappa)^{V'}, |a| < \kappa$ *can be covered by a set* $b \in V'$ *with* $|b| \leq \kappa$,

2. $Cov(V, V', \kappa^+)$.

Then $|(\tau^+)^{\mathcal{K}}| = \tau$, *for unboundedly many cardinals* $\tau < \kappa$.

The last result relates to the question of Woodin stated in the introduction. Unfortunately it does not provide the full answer due to the assumption (4) on a strong form of covering.

Theorem 2.5 *Suppose that in* V, $\text{cof}(\kappa) = \omega, 2^\kappa \geq \kappa^{+3}$, *GCH holds below* κ. *Then there is no inner model* $V' \supseteq \mathcal{K}$ *such that*

1. κ *is regular in* V',

2. $2^\kappa \geq \kappa^{+3}$,

3. *every* $a \subseteq (2^\kappa)^{V'}, |a| < \kappa$ *can be covered by a set* $b \in V'$ *with* $|b| \leq \kappa$,

4. $Cov(V, V', \kappa^+)$.

3 Some general observations

Let us prove several general statements concerning clubs and principle indiscernibles. They are a kind of slight generalizations of result by M. Dzamonja, S. Shelah [4] and the author [6] in context of a srong limit cardinal. The following is Proposition 2.1 of [6]:

Proposition 3.1 *Let* $V_1 \subseteq V_2$ *be two models of ZFC. Let* κ *be a regular cardinal of* V_1 *which changes its cofinality to* θ *in* V_2. *Suppose that in* V_1 *there is an almost decreasing (mod nonstationary or equivalently mod bounded) sequence of clubs of* κ *of length* $(\kappa^+)^{V_1}$ *so that every club of* κ *of* V_1 *almost contains one of the clubs of the sequence. Assume that* V_2 *satisfies the following:*
(1) $\text{cof}(\kappa^+)^{V_1} \geq (\theta)^+$ *or* $\text{cof}(\kappa^+)^{V_1} = \theta$;
(2) $\kappa > \theta^+$.
Then, in V_2, *there exists a cofinal in* κ *sequence* $\langle \tau_i \mid i < \theta \rangle$ *consisting of ordinals of cofinality* $> \theta^+$ *so that every club of* κ *of* V_1 *contains a final segment of* $\langle \tau_i \mid i < \theta \rangle$.

The proof of it actually gives the following:

Proposition 3.2 *Let $V_1 \subseteq V_2$ be two models of ZFC. Let κ be a strongly inaccessible cardinal of V_1 which changes its cofinality to θ in V_2 but remains a strong limit.*
Suppose that every set of ordinals a of cardinality $< \kappa$ there is $b \in V_1$ such that $b \supseteq a$ and $|b|^{V_1} \leq \kappa$.
Then, in V_2, for every $\delta < \kappa$ there exists a cofinal in κ sequence $\langle \tau_i \mid i < \theta \rangle$ consisting of ordinals of cofinality $> \delta$ so that every club of κ of V_1 contains a final segment of $\langle \tau_i \mid i < \theta \rangle$.

Proof. We repeat the proof of 2.1 of [6]. In the construction of trees $T(C)$ their, instead of splitting into ω allow splittings into $\leq \delta$. Define $(2^\delta)^+$ clubs C_α instead of $(2^{\aleph_0})^+$. We use the covering assumption in order to proceed. Namely, let $\alpha < (2^\delta)^+$ and the sequence of clubs $\langle C_\beta \mid \beta < \alpha \rangle$ was already defined, however it need not be in V_1. Define C_α. Let $\langle X_i \mid i < \rho \rangle$ be an enumeration of all clubs of κ in V_1. So, for every $\beta < \alpha$ there is $i_\beta < \rho$ such that $C_\beta = X_i$. Consider the set $a = \{i_\beta \mid \beta < \alpha\} \subseteq \rho$. There is $b \in V_1, Z \subseteq \rho, |b|^{V_1} = \kappa$ such that $Z \supseteq Y$. Set $C_\alpha = \Delta_{i \in b} X_i$. Then $C_\alpha \in V_1$ and for every $\beta < \alpha$, C_α is almost included in C_β. The rest of the argument stays without a change.
□

Now let us show the following:

Proposition 3.3 *Let $V_1 \subseteq V_2$ be two models of ZFC. Let κ be a strongly inaccessible cardinal of V_1 which changes its cofinality to θ in V_2 but remains a strong limit.*
Suppose that every set of ordinals a of cardinality $< \kappa$ there is $b \in V_1$ such that $b \supseteq a$ and $|b|^{V_1} \leq \kappa$.
Then, in V_2, there exists a cofinal in κ sequence $\langle \tau_i \mid i < \theta \rangle$ so that

1. *every club of κ of V_1 contains a final segment of $\langle \tau_i \mid i < \theta \rangle$,*
2. *the sequence $\langle \operatorname{cof}(\tau_i) \mid i < \theta \rangle$ is cofinal in κ.*

Proof. Suppose otherwise. Using 3.2, for every $\delta < \kappa$ pick a cofinal sequence $\langle \tau_i^\delta \mid i < \theta \rangle$ such that

1. every club of κ of V_1 contains a final segment of $\langle \tau_i \mid i < \theta \rangle$,
2. for every $i < \theta$, $\operatorname{cof}(\tau_i^\delta) > \delta$.

Fix a cofinal in κ sequence $\langle \kappa_i \mid i < \theta \rangle$.
Set $A = \{\tau_i^{\kappa_j} \mid i, j < \theta\}$.
Let $\eta = |A|$. We have $2^\eta < \kappa$, since κ is a strong limit.
Denote by X the set of all subsets A' of A which satisfy the following

1. A' is a cofinal in κ sequence of order type θ,

2. for every $c, d \in A'$, if $c < d$ then $\mathrm{cof}(c) < \mathrm{cof}(d)$,

3. the set $\{\mathrm{cof}(c) \mid c \in A'\}$ is cofinal in κ,

Then for every $x \in X$ there is a club C_x in V_1, such that $x' = x \setminus C_x$ is unbounded in κ. Consider the set $\{C_x \mid x \in X\}$. It can be covered by a set of clubs in V_1 of cardinality κ. Let C be the diagonal intersection of such covering clubs. Then, for every $x \in X$, C is almost contained in C_x.
By the choice of $\tau_i^{\kappa_j}$, there will be $x \in X$ such that $x \subseteq C$. But this impossible, since $x \setminus C_x$ is unbounded in κ and C is almost contained in C_x.
Contradiction.
□

Turn now to our context. So, we have $\mathcal{K} \subseteq V' \subseteq V$, κ is regular in $\mathcal{K}, V'$ and singular strong limit in V. Also, we assumed that V' and V satisfy the required covering assumption. Hence, the previous results imply the following:

Proposition 3.4 *Suppose that $\langle \tau_i \mid i < \omega \rangle$ is a cofinal in κ sequence such that every club of κ of $\mathcal{K}$ contains a final segment of $\langle \tau_i \mid i < \omega \rangle$.*
Let N be a covering model and $\langle \tau_i \mid i < \omega \rangle \in N$.
Then a final segment of $\langle \tau_i \mid i < \omega \rangle$ consists of principle indiscernibles of N.

Proof. Suppose otherwise. Let $I \subseteq \omega$ be infinite and for every $i \in I$, τ_i is not a principle indiscernible of N. Then there is a finite sequence $\vec{c} \in [\tau_i]^{<\omega}$ such that $h^N(\vec{c}) \geq \tau_i$.
Define $C = \{\nu < \kappa \mid h^{N''}[\nu]^{<\omega} \subseteq \nu\}$. It is a club in $\mathcal{K}$. However, $C \cap \{\tau_i \mid i \in I\} = \emptyset$, which is impossible since C is supposed to include a final segment of $\langle \tau_i \mid i < \omega \rangle$. Contradiction.
□

4 Proof of Theorem 2.1

Suppose that such V' exists and $(\tau^+)^{\mathcal{K}} = \tau^+$, for all but boundedly many cardinals $\tau < \kappa$. Let $\langle A_\alpha \mid \alpha < \kappa^{++} \rangle$ be a sequence of different subsets of κ in V'.

Pick a sequence $\langle N_\alpha \mid \alpha < \kappa^{++} \rangle$ of covering models of a same cardinality cardinality below κ with $A_\alpha \in N_\alpha$, for every $\alpha < \kappa^{++}$.
Apply the Mitchell Covering Lemma to N_α.
We will have a Skolem function $h_\alpha \in \mathcal{K}$, $\rho_\alpha < \kappa$ and the sequence of indiscernibles C_α.

Denote by C_α^* the set of all principle indiscernibles of C_α. It includes an $\omega-$sequence cofinal in κ.

For each $\mu \in C_\alpha^*$, consider $A_\alpha \cap \mu$.
Denote by i_α^μ the index of $A_\alpha \cap \mu$ in a fixed enumeration of $\mathcal{P}(\mu)$ in V'.
Further, in Section 5, a fixed enumeration of $\mathcal{P}(\mu)$ in V will be used instead.
Then $i_\alpha^\mu \in N_\alpha$, since $A_\alpha \in N_\alpha$.

Let us apply Proposition 3.3 to V, V' and find a cofinal in κ sequence $\langle \tau_i \mid i < \omega \rangle$ which satisfies the conclusion of 3.3.
Fix a covering model N^* such that $\langle \tau_i \mid i < \omega \rangle \in N^*$. Set $C^* = N^* \cap \{\tau_i \mid i < \omega\}$. By Proposition 3.4, C^* is cofinal in κ.
Without loss of generality we can assume that each N_α includes N^*.

By the assumption (2) of the theorem and since C^* is almost contained in every club of κ of V', we can assume the following:

$$(*) \quad V' \models 2^\tau = \tau^+, \textit{ for every } \tau \in C^*.$$

Recall that we have $\mathrm{GCH}_{<\kappa}$ in V, but not necessary in V'.
Then, $i_\alpha^\mu < (\mu^+)^{V'}$.
Then there is a finite sequence of indiscernibles $\vec{c}_\alpha^\mu$ below μ such that $i_\alpha^\mu = h_\alpha(\vec{c}_\alpha^\mu, \mu)$.

Now, the number of possibilities for h_α, ρ_α's is κ^+, since $h_\alpha \in \mathcal{K}, \rho_\alpha < \kappa$. Hence, we can find a stationary $S \subseteq \kappa^{++}$ a function h and an ordinal ρ such that for every $\alpha \in S$, $h_\alpha = h$ and $\rho_\alpha = \rho$. By shrinking S, if necessary, we may assume also that C_α, C_α^*'s are similar.

Let I be a subset of S of cardinality κ^+ with $\mathrm{cof}(\sup(I)) = \kappa^+$.

We have $\kappa^+ = (\kappa^+)^{\mathcal{K}} = (\kappa^+)^{V'}$, since κ was regular in $\mathcal{K}$, changed its cofinality in V and so, the Mitchell Covering Lemma applies.
However, in general it is possible that $(\kappa^{++})^{V'}$ is collapsed in V to κ^+, and so, we cannot cover I by a set in V' which cardinality there is κ^+. This is the reason for the assumption (3) of the theorem.
Find $I^* \in \mathcal{K}$ of cardinality κ^+ which covers I and such that $\sup(I) = \sup(I^*)$.
Let us identify below I with I^*.
Denote this supremum by δ. Then $\kappa^+ \leq \delta < \kappa^{++}$ and I is unbounded in δ.
Fix in $\mathcal{K}$ a function $\sigma : \kappa^+ \leftrightarrow \delta$.

Let U be the normal ultrafilter of $\mathcal{K}$ which concentrates on non-measurable cardinals.
Consider $j_U(h)(\kappa) = [\nu \mapsto h \restriction [\nu \cup \{\nu\}]^{<\omega}]_U$, where we restrict $h \restriction [\nu \cup \{\nu\}]^{<\omega}$ only to values in ν^+, say setting all the rest to be 0. It follows that, in $\mathcal{K}$, $j_U(h)(\kappa) : [\kappa \cup \{\kappa\}]^{<\omega} \to \kappa^+$.

Work in V'.
Let $\langle M_\nu \mid \nu < \kappa \rangle$ be an increasing continuous sequence of elementary submodels of H_χ such that

1. $\langle M_\nu \mid \nu \leq \zeta \rangle \in M_{\zeta+1}$,
2. $|M_\nu| < \kappa$,
3. $M_\nu \supseteq \nu$,
4. $h, j_U(h)(\kappa), \sigma, \langle A_\alpha \mid \alpha \in I \rangle \in M_0$.

Let $C = \{\nu < \kappa \mid M_\nu \cap \kappa = \nu\}$. It is a club in V', since κ is regular there.

Pick a typical $\nu \in C \cap C^*$.
Let $\bar{M}_\nu$ be the transitive collapse of M_ν and π the collapsing function.
Then $\pi(\kappa) = \nu, \pi(A) = A \cap \nu$, for every $A \subseteq \kappa, A \in M_\nu$.
Now, in $\bar{M}_\nu$, the number of subsets of ν indexed by ordinals in the range of $h \restriction [\nu \cup \{\nu\}]^{<\omega}$ is less than $(\nu^+)^{\bar{M}_\nu} = \pi(\kappa^+) < \nu^+$. Let $\nu^* < (\nu^+)^{\bar{M}_\nu}$ be such that

$$\bar{M}_\nu \models \forall \xi (\nu^* < \xi < \nu^+ \to \text{ the index of } \pi(A_{\sigma(\pi^{-1}(\xi))}) = A_{\sigma(\pi^{-1}(\xi))} \cap \nu$$

$$\text{does not appears in the range of } h \restriction [\nu \cup \{\nu\}]^{<\omega}).$$

Define a function $s \in \prod_{\nu \in C^* \cap C} \nu^+$ by setting $s(\nu) = \nu^*$.
Let $\langle f_\xi \mid \xi < \kappa^+ \rangle$ be canonical functions in $\prod_{\xi < \kappa} \xi^+$, in $\mathcal{K}$.

Lemma 4.1 *There is $\eta < \kappa^+$ such that $f_\eta \restriction C^* \cap C$ dominates s mod finite.*

Proof. Pick a covering model N with $s, C^* \cap C \in N$. We may that each $\nu \in C^* \cap C$ is a principle indiscernible of N, by dropping finitely many points if necessary. By the assumption, $\nu^+ = (\nu^+)^{\mathcal{K}}$, hence there is no indiscernibles in the interval $(\nu, \nu^+]$.
Define a function $g \in \prod_{\gamma < \kappa} \gamma^+$ as follows:

$$g(\gamma) = \sup(h^{N\prime\prime}\gamma) \cap \gamma^+.$$

Then $g \in \mathcal{K}$, since h^N is in $\mathcal{K}$. In addition, $g(\nu) > s(\nu)$, for every $\nu \in \text{dom}(s)$, since $s \in N$ and is no indiscernibles in the interval $(\nu, \nu^+]$.
Now find $\eta < \kappa^+$ such that f_η dominates g.
□

Pick $\eta < \kappa^+$ such that the canonical function $f_\eta \upharpoonright C^* \cap C$ which dominates s.

Let $\langle R_\nu \mid \nu < \kappa \rangle$ be an increasing continuous sequence of elementary submodels of H_χ such that

1. $\langle R_\nu \mid \nu \leq \zeta \rangle \in R_{\zeta+1}$,
2. $|R_\nu| < \kappa$,
3. $M_\nu \subseteq R_\nu$,
4. $h, j_U(h)(\kappa), \sigma, \langle A_\alpha \mid \alpha \in I \rangle, \eta \in R_0$.

Let $E = \{\nu \in C \mid R_\nu \cap \kappa = \nu\}$.
Pick a typical $\nu \in E \cap C^*$ which is a principal indiscernible and $\nu^* < f_\eta(\nu)$.
Let $\bar{R}_\nu$ be the transitive collapse of R_ν and φ the collapsing function.
Then $\varphi(\kappa) = \nu, \varphi(A) = A \cap \nu$, for every $A \subseteq \kappa, A \in R_\nu$. Let M'_ν be $\varphi[M_\nu]$. Then $M'_\nu \preceq \bar{R}_\nu$. Also, $\bar{M}_\nu$ is the transitive collapse of M'_ν. Let $\psi : M'_\nu \leftrightarrow \bar{M}_\nu$ be the collapsing function. Note that $M'_\nu \cap \nu = \bar{M}_\nu \cap \nu = \bar{R}_\nu \cap \nu = \nu$. So, $M'_\nu \cap \varphi(\kappa^+)$ is an ordinal, and hence, $\psi \upharpoonright M'_\nu \cap \varphi(\kappa^+)$ is the identity. In particular, $\psi(\nu^*) = \nu^*$.

In addition, $\varphi(\eta) = f_\eta(\nu)$.
Also, $\varphi(j_U(h)(\kappa)) = h \upharpoonright [\nu \cup \{\nu\}]^{<\omega}$.

We have, $\nu^* < f_\eta(\nu)$. Hence, the index i^ν_η of $A_{\sigma(\eta)} \cap \nu$ in the enumeration of subsets of ν will not appear in the range of $h \upharpoonright [\nu \cup \{\nu\}]^{<\omega}$. This is impossible, since $h = h_{\sigma(\eta)}$ and there is a finite sequence of indiscernibles $\vec{c}^{\,\nu}_{\sigma(\eta)}$ below ν such that $i^\nu_{\sigma(\eta)} = h_{\sigma(\eta)}(\vec{c}^{\,\nu}_{\sigma(\eta)}, \nu)$.

Remark 4.2 *Note that the standard Extender Based Prikry forcing over $\mathcal{K}$ satisfies the conditions (2) and (3) of the theorem. So, the argument above shows that there is no intermediate model in which κ is regular and $2^\kappa > \kappa^+$. However, this is under the assumption that there is no inner model with a strong cardinal, in contrast to [2].*

5 Proof of Theorem 2.4

Let us show how to modify the previous argument in order to replace the assumptions (*) and (**) by a strong form of covering, i.e., we do not require that $V' \models 2^\tau = \tau^+$, *for every* $\tau \in C^*$ and that $\kappa^{++} = (\kappa^{++})^{V'}$.

For every $\alpha < \kappa^{++}, \tau \in C^*$, let i^τ_α be the index of $A_\alpha \cap \tau$ in a fixed enumeration of $\mathcal{P}(\tau)$, but now in V. Consider the function $\tau \mapsto i^\tau_\alpha$. Denote it by g_α. By $\mathrm{GCH}_{<\kappa}$, $g_\alpha \in \prod_{\tau \in C^*} \tau^+$. There

is a finite increasing sequence of indiscernibles $\vec{c}^{\tau}_{\alpha} \in [i^{\tau}_{\alpha}+1]^{<\omega}$ such that $g_{\alpha}(\tau) = i^{\tau}_{\alpha} = h(\vec{c}^{\tau}_{\alpha})$. Actually, $\vec{c}^{\tau}_{\alpha} \in [\tau+1]^{<\omega}$, since we are assuming that
$(\tau^+)^{\mathcal{K}} = \tau^+$, and so, there are no indiscernibles in the interval $(\tau, \tau^+]$. Denote by n^{τ}_{α} the length of $\vec{c}^{\tau}_{\alpha}$. By similarity of models N_{α}, we can assume that n^{τ}_{α} does not depend on α. Let n^{τ} be such value.
Replace $\vec{c}^{\tau}_{\alpha}$ by $c^{\tau}_{\alpha n^{\tau}} + c^{\tau}_{\alpha n^{\tau}-1} + ... + c^{\tau}_{\alpha 0}$, if $n^{\tau} > 0$. Let h' be the corresponding replacement of h, i.e., set $h'(\xi_n + \xi_{n-1} + ... + \xi_0) = h(\langle \xi_0, ..., \xi_n \rangle)$.

Note that there are no indiscernibles in the interval $(\tau, \tau^+]$, since we are assuming that $(\tau^+)^{\mathcal{K}} = \tau^+$, for every $\tau \in C^*$. Hence,

$$\mathrm{tcf}(\prod_{\tau \in C^*} \tau^+, <_{J^{bd}_{\kappa}}) = \kappa^+.$$

Just every function in this product will be bounded by the restriction of a function in $\mathcal{K}$ to C^*.
Let $\vec{f} = \langle f_{\alpha} \mid \alpha < \kappa^{++} \rangle$ be a witnessing scale. We have $\kappa^{++}-$many g_{α}'s, so there are $S' \subseteq S, |S'| = |S| = \kappa^{++}$ and $\alpha^* < \kappa^{++}$ such that f_{α^*} dominates each $g_{\alpha}, \alpha \in S'$. By shrinking S' if necessary, we can assume that there is $\gamma^* \in C^*$ such that for every $\gamma \in C^* \setminus \gamma^*$, $f_{\alpha^*}(\gamma) > t_{\alpha}(\gamma)$. Assume for simplicity that $\gamma^* = \min C^*$.

For every $\tau \in C^*$ we fix a function

$$e_{\tau} : f_{\alpha^*}(\tau) \leftrightarrow |\tau|.$$

For every $\alpha \in S'$, define a function $s_{\alpha} \in \prod_{\tau \in C^*} |\tau|$ by setting

$$s_{\alpha}(\tau) = e_{\tau}(g_{\alpha}(\tau)).$$

Let us consider few cases.

Case 1 *There is $\delta < \kappa$ such that for an unbounded $C' \subseteq C^*$, the following holds: $c \in C' \Rightarrow \mathrm{cof}(|c|) < \delta$.*

Then, using $\mathrm{GCH}_{<\kappa}$, it is easy to find some $g \in \prod_{\tau \in C'} |\tau|$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, g dominates s_{α}. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$.
$\square$ of Case 1.

Suppose now that there is no such δ. Then the set

$$\{\mathrm{cof}(|c|) \mid c \in C'\}$$

is unbounded in κ. By shrinking C^* if necessary, we can assume that the sequence

$$\langle \operatorname{cof}(|c|) \mid c \in C^* \rangle$$

is strictly increasing. Consider then $\operatorname{pcf}(\{|c| \mid c \in C^*\}) \setminus \kappa$. It is a subset of the set $\{\kappa^+, \kappa^{++}\}$, since $2^\kappa = \kappa^{++}$ and κ is a strong limit.

Case 2 *There is $C' \subseteq C^*$ such that* $\operatorname{tcf}(\prod_{c \in C'} |c|, <_{J^{bd}_\kappa}) = \kappa^+$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^+ \rangle$ be a witnessing scale.

Then there are $\xi^* < \kappa^+$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, p_{ξ^*} dominates $s_\alpha \restriction C'$. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.

$\square$ of Case 2.

Case 3 *There is $C' \subseteq C^*$ such that* $\operatorname{tcf}(\prod_{c \in C'} |c|, <_{J^{bd}_\kappa}) = \kappa^{++}$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^{++} \rangle$ be a witnessing scale. Take S'' to be a subset of S' of cardinality κ^+. Then there will be $\xi^* < \kappa^{++}$ such that for every $\alpha \in S''$, p_{ξ^*} dominates s_α. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.

$\square$ of Case 3.

So, in either case we are able to find a function g which dominates subsets of S' of cardinality κ^{++} or κ^+.

We showed the following crucial property:

($\aleph$) *There are an unbounded $E \subseteq C^*$ and $B \subseteq S, |B| = \kappa^+$ such that for every $\tau \in E$, the set*

$$\{A_\alpha \cap \tau \mid \alpha \in B\}$$

has cardinality less than $|\tau|$.

This holds since the corresponding set

$$\{i^\tau_\alpha \mid \alpha \in B\}$$

has cardinality less than $|\tau|$.

Now let us run the argument with an elementary chain and use the strong form of covering $Cov(V, V', \kappa^+)$ defined 2.3.

Apply it to B which was constructed above, i.e. from ($\aleph$).

Let $I, I^*, \langle M_\nu \mid \nu < \kappa \rangle$ be a witnessing sets.

Work in V'. Pick $\langle R_\nu \mid \nu < \kappa \rangle$ to be an increasing continuous sequence of elementary submodels of H_χ, with χ large enough, such that

1. $\langle R_\nu \mid \nu \leq \zeta \rangle \in R_{\zeta+1}$,

2. $|R_\nu| < \kappa$,

3. $M_\nu \subseteq R_\nu$,

4. $I^*, \langle A_\alpha \mid \alpha \in I^* \rangle \in R_0$.

Let $\langle i^*_\nu \mid \nu < \kappa \rangle$ be an enumeration of I^* in $V' \cap R_0$.

Set $X = \{\nu < \kappa \mid R_\nu \cap \kappa = \nu \text{ and } M_\nu \cap I^* = \{i^*_\zeta \mid \zeta < \nu\}\}$.

Clearly, X is in V' and it is a closed unbounded subset of κ. Then, X contains a final segment of E, where $E \subseteq C^*$ is from ($\aleph$). Pick $\eta \in E \cap X$.

Then, $R_\eta \cap \eta = \eta$. By elementarity, $R_\eta \cap I^* = \{i^*_\nu \mid \nu < \eta\}$. So, $R_\eta \cap I^* = M_\eta \cap I^*$. Hence, in V,

$$|\eta| = |R_\eta \cap I^*| = |M_\eta \cap I^*| = |M_\eta \cap I| = |R_\eta \cap I|.$$

For every $\alpha \in I^* \cap R_\eta$, $A_\alpha \in R_\eta$. In particular, for every $\alpha \in I \cap R_\eta$, $A_\alpha \in R_\eta$. By elementarity,

$$R_\eta \models \forall \alpha, \beta \in I^* (\alpha \neq \beta \rightarrow A_\alpha \neq A_\beta).$$

We have $R_\eta \cap \kappa = \eta$, hence $A_\alpha \cap \eta \neq A_\beta \cap \eta$, for every $\alpha, \beta < (\kappa^{+3})^V \cap R_\eta, \alpha \neq \beta$. In particular, for every $\alpha, \beta \in R_\eta \cap I, \alpha \neq \beta$, $A_\alpha \cap \eta \neq A_\beta \cap \eta$. So,

$$|\{A_\alpha \cap \eta \mid \alpha \in I\}| \geq |R_\eta \cap I| = |\eta|.$$

But $I \subseteq B$, $\eta \in E$, hence the set

$$\{A_\alpha \cap \eta \mid \alpha \in I\}$$

has cardinality less than $|\eta|$, by ($\aleph$). It is impossible. Contradiction.
This completes the proof of Theorem 2.4.

6 Proof of Theorem 2.5

We deal now with a possibility that successors of principle indiscernibles are collapsed.

Assume here that $2^\kappa = \kappa^{+3}$. If $2^\kappa > \kappa^{+3}$, then we just collapse 2^κ to κ^{+3}. Suppose that there is $V', \mathcal{K} \subseteq V' \subseteq V$ such that

1. κ is a regular in V',

2. $(2^\kappa)^{V'} \geq \kappa^{+3}$.

Let $\langle A_\alpha \mid \alpha < \kappa^{+3}\rangle$ be a sequence in V' of $\kappa^{+3}-$subsets of κ. Keep the notation of the previous sections and define $C^*, \langle N_\alpha \mid \alpha < \kappa^{+3}\rangle, h, S \subseteq \kappa^{+3}$ as before.

The basic idea will be to explore the choice between three available cardinalities $\kappa^+, \kappa^{++}, \kappa^{+3}$ for collections of subsets of κ in V' against only two related cofinalities of products of the form $\prod_{\tau \in C^*} \tau^+$ and $\prod C^*$.

For every $\alpha < \kappa^{+3}, \tau \in C^*$, let i^τ_α be the index of $A_\alpha \cap \tau$ in a fixed enumeration of $\mathcal{P}(\tau)$ in V. Consider the function $\tau \mapsto i^\tau_\alpha$. Denote it by g_α. By GCH$_{<\kappa}$, $g_\alpha \in \prod_{\tau \in C^*} \tau^+$. There is a finite increasing sequence of indiscernibles $\vec{c}^{\,\tau}_\alpha \in [i^\tau_\alpha + 1]^{<\omega}$ such that $g_\alpha(\tau) = i^\tau_\alpha = h(\vec{c}^{\,\tau}_\alpha)$. Denote by n^τ_α the length of $\vec{c}^{\,\tau}_\alpha$. By similarity of models N_α, we can assume that n^τ_α does not depend on α. Let n^τ be such value.
Replace $\vec{c}^{\,\tau}_\alpha$ by $c^\tau_{\alpha n^\tau} + c^\tau_{\alpha n^\tau - 1} + ... + c^\tau_{\alpha 0}$, if $n^\tau > 0$. Let h' be the corresponding replacement of h, i.e., set $h'(\xi_n + \xi_{n-1} + ... + \xi_0) = h(\langle \xi_0, ..., \xi_n\rangle)$.

Denote the set $\{\tau^+ \mid \tau \in C^*\}$ by C^{*+}.
Consider pcf$(C^{*+}) \setminus \kappa$. It is a subset of the set $\{\kappa^+, \kappa^{++}, \kappa^{+3}\}$. Let $C^{*+} = C^{*+}_1 \cup C^{*+}_2 \cup C^{*+}_3$ be a splitting of C^{*+} into sets which are generators of $\kappa^+, \kappa^{++}, \kappa^{+3}$ respectively. It is possible that some of them are empty. Let us consider few cases.

Case 1 $C^{*+}_3 \neq \emptyset$.

Then tcf$(\prod C^{*+}_3, <_{J^{bd}_\kappa}) = \kappa^{+3}$.
Let $\vec{f} = \langle f_\alpha \mid \alpha < \kappa^{+3}\rangle$ be a witnessing scale. Let $\alpha \in S$. Define a function t_α on C^{*+}_3 by setting

$$t_\alpha(\tau^+) = h'(c^\tau_{\alpha n^\tau} + c^\tau_{\alpha n^\tau - 1} + ... + c^\tau_{\alpha 0}).$$

Then $t_\alpha \in \prod C^{*+}_3$, and so it is bounded by a function from the scale $\vec{f}$.
Take any $S' \subseteq S$ of cardinality κ^{++}. There will be $\alpha^* < \kappa^{+3}$ such that f_{α^*} dominates each $t_\alpha, \alpha \in S'$. By shrinking S' if necessary, we can assume that there is $\gamma^* \in C^{*+}_3$ such that for every $\gamma \in C^{*+}_3 \setminus \gamma^*$, $f_{\alpha^*}(\gamma) > t_\alpha(\gamma)$. Assume for simplicity that $\gamma^* = \min C^{*+}_3$.
Set

$$C'_3 = \{\tau \in C^* \mid \tau^+ \in C^{*+}_3\}.$$

For every $\tau \in C'_3$ we fix a function

$$e_\tau : f_{\alpha^*}(\tau^+) \leftrightarrow |\tau|.$$

For every $\alpha \in S'$, define a function $s_\alpha \in \prod_{\tau \in C'_3} |\tau|$ by setting

$$s_\alpha(\tau) = e_\tau(t_\alpha(\tau^+)).$$

Subcase 1.1 *There is $\delta < \kappa$ such that for an unbounded $C' \subseteq C'_3$, the following holds:* $c \in C' \Rightarrow \mathrm{cof}(|c|) < \delta$.

Then, using $\mathrm{GCH}_{<\kappa}$, it is easy to find some $g \in \prod_{\tau \in C'} |\tau|$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, g dominates s_α. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$.
$\square$ of Subcase 1.1.

Suppose now that there is no such δ. Then the set

$$\{\mathrm{cof}(|c|) \mid c \in C'\}$$

is unbounded in κ. By shrinking C' if necessary, we can assume that the sequence

$$\langle\, \mathrm{cof}(|c|) \mid c \in C' \rangle$$

is strictly increasing. Consider then $\mathrm{pcf}(\{|c| \mid c \in C'\}) \setminus \kappa$. Again, it is a subset of the set $\{\kappa^+, \kappa^{++}, \kappa^{+3}\}$.

Subcase 1.2 *There is $C'' \subseteq C'$ such that* $\mathrm{tcf}(\prod_{c \in C''} |c|, <_{J^{bd}_\kappa}) = \kappa^+$ *or* $\mathrm{tcf}(\prod_{c \in C''} |c|, <_{J^{bd}_\kappa}) = \kappa^{+3}$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^+ \rangle$ be a witnessing scale.
Then there are $\xi^* < \kappa^+$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, p_{ξ^*} dominates $s_\alpha \restriction C''$. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.
$\square$ of Subcase 1.2.

Subcase 1.3 *There is $C'' \subseteq C'$ such that* $\mathrm{tcf}(\prod_{c \in C''} |c|, <_{J^{bd}_\kappa}) = \kappa^{+3}$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^{+3} \rangle$ be a witnessing scale. Then, $|S'| = \kappa^{++}$ implies that there is $\xi^* < \kappa^{+3}$ such that for every $\alpha \in S''$, p_{ξ^*} dominates $s_\alpha \restriction C''$. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.
$\square$ of Subcase 1.3.

Suppose now that Subcases 1.2,1.3 fail. Then $\mathrm{tcf}(\prod_{c \in C'} |c|, <_{J^{bd}_\kappa}) = \kappa^{++}$.
Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^{++} \rangle$ be a witnessing scale. Take S'' to be a subset of S' of cardinality κ^+. Then there will be $\xi^* < \kappa^{++}$ such that for every $\alpha \in S''$, p_{ξ^*} dominates s_α. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.

So, in either case we are able to find a function g which dominates subsets of S' of cardinality κ^{++} or κ^+.

Case 2 $C_2^{*+} \neq \emptyset$.

Case 3 $C_1^{*+} \neq \emptyset$.

The treatment of Cases 2,3 is completely similar to those of Case 1.

We showed the following crucial property:

($\aleph$) *There are an unbounded $E \subseteq C^*$ and $B \subseteq S, |B| = \kappa^+$ such that for every $\tau \in E$, the set*

$$\{A_\alpha \cap \tau \mid \alpha \in B\}$$

has cardinality less than $|\tau|$.

This holds since the corresponding set

$$\{i_\alpha^\tau \mid \alpha \in B\}$$

has cardinality less than $|\tau|$.

Now let us run the argument with an elementary chain and use the strong form of covering for κ^+ (2.3).

Recall that $Cov(V, V', \kappa^+)$ denotes the following strong covering property:
For every set of ordinals $B \subseteq 2^\kappa$ of cardinality κ^+ there are $I \subseteq B$ of cardinality κ and $I^* \in V', I^* \supseteq I$ such that for some increasing and continuous sequence $\langle M_\nu \mid \nu < \kappa\rangle \in V'$ with $|M_\nu| < \kappa$, for every $\nu < \kappa$, and $I^* \subseteq \bigcup_{\nu<\kappa} M_\nu$, the following holds: for every $\nu < \kappa, |M_\nu \cap I| = |M_\nu \cap I^*|$.

We assume $Cov(V, V', \kappa^+)$. Apply it to B which was constructed above, i.e. from ($\aleph$).

Let $I, I^*, \langle M_\nu \mid \nu < \kappa\rangle$ be a witnessing sets.

Work in V'. Pick $\langle R_\nu \mid \nu < \kappa\rangle$ to be an increasing continuous sequence of elementary submodels of H_χ, with χ large enough, such that

1. $\langle R_\nu \mid \nu \leq \zeta\rangle \in R_{\zeta+1}$,
2. $|R_\nu| < \kappa$,
3. $M_\nu \subseteq R_\nu$,
4. $I^*, \langle A_\alpha \mid \alpha \in I^*\rangle \in R_0$.

Let $\langle i_\nu^* \mid \nu < \kappa\rangle$ be an enumeration of I^* in $V' \cap R_0$.

Set $X = \{\nu < \kappa \mid R_\nu \cap \kappa = \nu$ and $M_\nu \cap I^* = \{i_\zeta^* \mid \zeta < \nu\}\}$.

Clearly, X is in V' and it is a closed unbounded subset of κ. Then, X contains a final segment of E, where $E \subseteq C^*$ is from ($\aleph$). Pick $\eta \in E \cap X$.

Then, $R_\eta \cap \eta = \eta$. By elementarity, $R_\eta \cap I^* = \{i^*_\nu \mid \nu < \eta\}$. So, $R_\eta \cap I^* = M_\eta \cap I^*$. Hence, in V,

$$|\eta| = |R_\eta \cap I^*| = |M_\eta \cap I^*| = |M_\eta \cap I| = |R_\eta \cap I|.$$

For every $\alpha \in I^* \cap R_\eta$, $A_\alpha \in R_\eta$. In particular, for every $\alpha \in I \cap R_\eta$, $A_\alpha \in R_\eta$. By elementarity,

$$R_\eta \models \forall \alpha, \beta \in I^* (\alpha \neq \beta \to A_\alpha \neq A_\beta).$$

We have $R_\eta \cap \kappa = \eta$, hence $A_\alpha \cap \eta \neq A_\beta \cap \eta$, for every $\alpha, \beta < (\kappa^{+3})^V \cap R_\eta, \alpha \neq \beta$. In particular, for every $\alpha, \beta \in R_\eta \cap I, \alpha \neq \beta$, $A_\alpha \cap \eta \neq A_\beta \cap \eta$. So,

$$|\{A_\alpha \cap \eta \mid \alpha \in I\}| \geq |R_\eta \cap I| = |\eta|.$$

But $I \subseteq B$, $\eta \in E$, hence the set

$$\{A_\alpha \cap \eta \mid \alpha \in I\}$$

has cardinality less than $|\eta|$, by $(\aleph)$. It is impossible. Contradiction.
This completes the proof of Theorem 2.5.

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