On the negation of the Singular Cardinals Hypothesis with GCH below

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Abstract

In this paper we attempt to understand the difficulty of getting a model where GCH fails first time at a singular κ and there is an inner model in which κ is a regular cardinal but still 2^κ is big.
In addition, we construct models with $2^\kappa > \kappa^+$ and ultrafilters U over κ such that $U \supseteq Cub_\kappa$ and $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U$.

1 Introduction

There is a tension between the negation of the Singular Cardinals Hypothesis at a given cardinal and the power function below it. A celebrated result of J. Silver [18] states that a singular cardinal of uncountable cofinality cannot be the first that violates GCH.

M. Magidor [14], using extremely sophisticated arguments, showed that this need not be the case with a singular of cofinality ω. Namely, starting with a supercompact cardinal with a huge above, he constructed a model in which $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{n+1}$, for every $n < \omega$.

In the early 80's, H. Woodin came up with a beautiful construction of a model of $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{n+1}$, for every $n < \omega$. The initial assumptions of his construction were optimal, see for example [5], [7].

However, the gap between the singular cardinal and its power in both of the constructions was 2 and not more.

The basic reason for the difficulty was that both arguments based on the Silver-Prikry method of violating SCH, i.e. first a model with a measurable cardinal κ with $2^\kappa > \kappa^+$ was

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constructed and then the Prikry forcing was used to change the cofinality of κ to ω. But having a measurable cardinal κ with $2^\kappa > \kappa^+$ implies that GCH is violated at unboundedly many places below κ. So, the difficult task starts to be to collapse cardinals in order to resurrect GCH below and still keeping $2^\kappa > \kappa^+$.

Later a different method of constructions of models of $\neg$SCH - Extender Based Prikry forcing was introduced in [11]. It allows one to change cofinality of κ and to blow up its power higher simultaneously without adding new bounded subsets to κ.

Following this developments, H. Woodin asked the following natural question:

Assuming that there is no inner model with a strong cardinal, is it possible to have a model M in which $2^{\aleph_\omega} > \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{n+1}$, for every $n < \omega$, and there is an inner model N which satisfies $\kappa = \aleph_\omega$ is a measurable and $2^\kappa \geq (\aleph_{\omega+3})^M$?

A reasonable approach to this question was to use Extender based forcing over κ together with a suitable preparation which say adds many Cohen subsets to ν's below κ, and then, passing into a submodel in which κ is still regular, we combine these Cohens from the preparation together, using Prikry sequences, in order to obtain $2^\kappa-$Cohens over the submodel.

It turned out to be realizable to some degree. Namely, as it was shown in [2], even the Prikry forcing (with a carefully picked $\kappa-$complete ultrafilter) can add κ^+-many mutually generic Cohen subsets to κ over a submodel. However, by [2], neither the original ([11]) nor C. Merimovich ([15]) versions of Extender based Prikry forcings can produce the above type of inner models. Namely, if $\mathcal{P}_E$ denotes the Extender based forcing of [11] and $G \subseteq \mathcal{P}_E$ is generic, then:

For every $A \in V[G] \setminus V, A \subseteq \kappa$, κ changes its cofinality to ω in $V[A]$.

If $\mathbb{P}_E$ denotes the Extender based forcing of [15] and $G \subseteq \mathbb{P}_E$ is generic, then:

For every $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ list of different subsets of κ in $V[G]$, there is $I \subseteq \kappa^{++}, I \in V, |I| = \kappa$ such that κ changes its cofinality to ω in $V[\langle A_\alpha \mid \alpha \in I\rangle]$.

The aim of the first part of the present paper is to use The Mitchell Covering Lemma with some Pcf-arguments to provide further insight into the reasons for the difficulty of getting an inner model in which κ is regular, but still 2^κ is big. In particular, this will provide some progress on Woodin's question.

In the second part, which can be read independently, models with $2^\kappa > \kappa^+$ and ultrafilters U over κ such that $U \supseteq Cub_\kappa$ and $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U$ are constructed. Some questions from T. Benhamou and G. Goldberg [3] are answered. A detailed introduction to this part is presented at the beginning of Section 7.

The dichotomy between non-GCH and GCH, and the resulting tension, is the common thread tying these parts together.

2 Part 1 - Settings and main results

Assume that there is no inner model with a strong cardinal. Let $\mathcal{K}$ denotes the core model. The next two theorems show, in particular, that the Woodin method for restoring GCH below a singular κ with $2^\kappa = \kappa^{++}$, is basically the only possibility.
First we would like to show the following:

Theorem 2.1 *Suppose that in V, $\mathrm{cof}(\kappa) = \omega, 2^\kappa = \kappa^{++}$, GCH holds below κ and there is an inner model $V' \supseteq \mathcal{K}$ in which κ is a regular, but still $2^\kappa \geq \kappa^{++}$.*
Assume that

1. *every $a \subseteq (2^\kappa)^{V'}, |a| < \kappa$ can be covered by a set $b \in V'$ with $|b| \leq \kappa$,*
2. *$V' \models 2^\nu = \nu^+$ for ν's in a club subset of κ in V',*
3. *$(\kappa^{++})^{V'} = \kappa^{++}$.*

Then $|(\tau^+)^{\mathcal{K}}| = \tau$, for unboundedly many cardinals $\tau < \kappa$.

Remark 2.2
1. Note that κ is a measurable in $\mathcal{K}$, and so, by the Mitchell Covering Lemma or just by the weak covering, $(\kappa^+)^{\mathcal{K}} = \kappa^+$.
2. If $(2^\kappa)^{V'} < (\kappa^{+\omega})^{V'}$, then we have the required type of covering, i.e. the condition (1) of the theorem, by standard arguments.
3. If there is no measurable cardinal above κ in $\mathcal{K}$, then again we have the required type of covering, by the Mitchell Covering Lemma.
4. If κ is a measurable cardinal in V', then the required type of covering holds. We assumed that there is no inner model with a strong cardinal, so, by [9], there is no measurable in $\mathcal{K}$ cardinal in the interval $(\kappa, (2^\kappa)^{V'}]$.
5. The actual result will be sharper and it will be possible to relax the condition (2) slightly.

In order to state our further results we will need to define the following form of a strong covering:

Definition 2.3 *Let $V' \subseteq V$, κ be a cardinal in V. Then $Cov(V, V', \kappa^+)$* holds iff:
For every set of ordinals $B \subseteq 2^\kappa$ of cardinality κ^+ there are $I \subseteq B$ of cardinality κ and $I^* \in V', I^* \supseteq I$ such that for some increasing and continuous sequence $\langle M_\nu \mid \nu < \kappa \rangle \in V'$ with $|M_\nu| < \kappa$, for every $\nu < \kappa$, and $I^* \subseteq \bigcup_{\nu<\kappa} M_\nu$, the following holds: for every $\nu < \kappa, |M_\nu \cap I| = |M_\nu \cap I^*|$.

Note that the following density property implies $Cov(V, V', \kappa^+)$:
Every set of ordinals $S' \subseteq 2^\kappa$ of cardinality κ^+ contains a set in V' of cardinality κ.
This property holds, for example, in generic extensions by standard Prikry forcings. We refer to [10] on this subject.

Theorem 2.4 *Suppose that in V, $\mathrm{cof}(\kappa) = \omega, 2^\kappa = \kappa^{++}$, GCH holds below κ and there is an inner model $V' \supseteq \mathcal{K}$ in which κ is a regular, but still $2^\kappa \geq \kappa^{++}$.*
Assume that

1. *every $a \subseteq (2^\kappa)^{V'}, |a| < \kappa$ can be covered by a set $b \in V'$ with $|b| \leq \kappa$,*

2. *$Cov(V, V', \kappa^+)$.*

Then $|(\tau^+)^{\mathcal{K}}| = \tau$, for unboundedly many cardinals $\tau < \kappa$.

The last result relates to the question of Woodin stated in the introduction. Unfortunately it does not provide the full answer due to the assumption (4) which is a strong form of covering.

Theorem 2.5 *Suppose that in V, $\mathrm{cof}(\kappa) = \omega, 2^\kappa \geq \kappa^{+3}$, GCH holds below κ. Then there is no inner model $V' \supseteq \mathcal{K}$ such that*

1. *κ is regular in V',*

2. *$2^\kappa \geq \kappa^{+3}$,*

3. *every $a \subseteq (2^\kappa)^{V'}, |a| < \kappa$ can be covered by a set $b \in V'$ with $|b| \leq \kappa$,*

4. *$Cov(V, V', \kappa^+)$.*

3 Some general observations

Let us prove several general statements concerning clubs and principal indiscernibles. They are slight generalizations of result by M. Dzamonja, S. Shelah [6] and the author [8] in context of a strong limit cardinal. The following is Proposition 2.1 of [8]:

Proposition 3.1 *Let $V_1 \subseteq V_2$ be two models of ZFC. Let κ be a regular cardinal of V_1 which changes its cofinality to θ in V_2. Suppose that in V_1 there is an almost decreasing (mod nonstationary or equivalently mod bounded) sequence of clubs of κ of length $(\kappa^+)^{V_1}$ so that every club of κ of V_1 almost contains one of the clubs of the sequence. Assume that V_2 satisfies the following:*
(1) $\mathrm{cof}(\kappa^+)^{V_1} \geq ((2^\theta)^+)^{V_2}$ *or* $\mathrm{cof}(\kappa^+)^{V_1} = \theta$*;*
(2) $\kappa > (\theta^+)^{V_2}$.
Then, in V_2, there exists a cofinal in κ sequence $\langle \tau_i \mid i < \theta \rangle$ consisting of ordinals of cofinality $\geq \theta^+$ so that every club of κ in V_1 contains a final segment of $\langle \tau_i \mid i < \theta \rangle$.

The proof of Proposition 3.1 actually gives the following:

Proposition 3.2 *Let $V_1 \subseteq V_2$ be two models of ZFC. Let κ be a strongly inaccessible cardinal of V_1 which changes its cofinality to θ in V_2 but remains a strong limit.*
Suppose that for every set of ordinals a of cardinality $< \kappa$ there is $b \in V_1$ such that $b \supseteq a$ and $|b|^{V_1} \leq \kappa$.
Then, in V_2, for every $\delta < \kappa$ there exists a cofinal in κ sequence $\langle \tau_i \mid i < \theta \rangle$ consisting of ordinals of cofinality $> \delta$ so that every club of κ of V_1 contains a final segment of $\langle \tau_i \mid i < \theta \rangle$.

Proof. We repeat the proof of 2.1 of [8]. In the construction of trees $T(C)$ there, instead of splitting into ω successors allow splitting into $\leq \delta$ successors. Define $(2^\delta)^+$ clubs C_α instead of $(2^{\aleph_0})^+$. We use the covering assumption in order to proceed. Namely, let $\alpha < (2^\delta)^+$ and the sequence of clubs $\langle C_\beta \mid \beta < \alpha \rangle$ was already defined, however it does not need to be in V_1. Let us define C_α. Let $\langle X_i \mid i < \rho \rangle$ be an enumeration of all clubs of κ in V_1. So, for every $\beta < \alpha$ there is $i_\beta < \rho$ such that $C_\beta = X_{i_\beta}$. Consider the set $a = \{i_\beta \mid \beta < \alpha\} \subseteq \rho$. There is $b \in V_1, Z \subseteq \rho, |b|^{V_1} = \kappa$ such that $Z \supseteq Y$. Set $C_\alpha = \Delta_{i \in b} X_i$. Then $C_\alpha \in V_1$ and for every $\beta < \alpha$, C_α is almost included in C_β.
The rest of the argument stays without a change.
□

Now let us show the following:

Proposition 3.3 *Let $V_1 \subseteq V_2$ be two models of ZFC. Let κ be a strongly inaccessible cardinal of V_1 which changes its cofinality to θ in V_2 but remains a strong limit.*
Suppose that for every set of ordinals a of cardinality $< \kappa$ there is $b \in V_1$ such that $b \supseteq a$ and $|b|^{V_1} \leq \kappa$.
Then, in V_2, there exists a cofinal in κ sequence $\langle \tau_i \mid i < \theta \rangle$ such that

1. *every club of κ of V_1 contains a final segment of $\langle \tau_i \mid i < \theta \rangle$,*

2. *the sequence $\langle \operatorname{cof}(\tau_i) \mid i < \theta \rangle$ is cofinal in κ.*

Proof. Suppose otherwise. Using 3.2, for every $\delta < \kappa$ pick a cofinal sequence $\langle \tau_i^\delta \mid i < \theta \rangle$ such that

1. every club of κ of V_1 contains a final segment of $\langle \tau_i \mid i < \theta \rangle$,

2. for every $i < \theta$, $\operatorname{cof}(\tau_i^\delta) > \delta$.

Fix a cofinal in κ sequence $\langle \kappa_i \mid i < \theta \rangle$.
Set $A = \{\tau_i^{\kappa_j} \mid i, j < \theta\}$.
Let $\eta = |A|$. We have $2^\eta < \kappa$, since κ is a strong limit.
Denote by X the set of all subsets A' of A which satisfy the following

1. A' is a cofinal in κ sequence of order type θ,

2. for every $c, d \in A'$, if $c < d$ then $\operatorname{cof}(c) < \operatorname{cof}(d)$,

3. the set $\{\operatorname{cof}(c) \mid c \in A'\}$ is cofinal in κ,

Then for every $x \in X$ there is a club C_x in V_1, such that $x' = x \setminus C_x$ is unbounded in κ. Consider the set $\{C_x \mid x \in X\}$. It can be covered by a set of clubs in V_1 of cardinality κ. Let C be the diagonal intersection of such covering clubs. Then, for every $x \in X$, C is almost contained in C_x.
By the choice of $\tau_i^{\kappa_j}$, there will be $x \in X$ such that $x \subseteq C$. But this impossible, since $x \setminus C_x$ is unbounded in κ and C is almost contained in C_x.
Contradiction.
□

Let us turn now to our context. So, we have $\mathcal{K} \subseteq V' \subseteq V$, κ is regular in $\mathcal{K}, V'$ and singular strong limit in V. Also, we assumed that V' and V satisfy the required covering assumption. Hence, the previous results imply the following:

Proposition 3.4 *Suppose that $\langle \tau_i \mid i < \omega \rangle$ is a cofinal in κ sequence such that every club of κ of $\mathcal{K}$ contains a final segment of $\langle \tau_i \mid i < \omega \rangle$.*
Let N be a covering model and $\langle \tau_i \mid i < \omega \rangle \in N$.
Then a final segment of $\langle \tau_i \mid i < \omega \rangle$ consists of principal indiscernibles of N[1].

[1] An ordinal τ is called a principal indiscernible of N iff for every a finite sequence $\vec{c} \in [\tau]^{<\omega}$, $h^N(\vec{c}) < \tau$, where h^N is the Skolem function for N from $\mathcal{K}$.

Proof. Suppose otherwise. Let $I \subseteq \omega$ be infinite and for every $i \in I$, τ_i is not a principal indiscernible of N. Then there is a finite sequence $\vec{c} \in [\tau_i]^{<\omega}$ such that $h^N(\vec{c}) \geq \tau_i$.
Define $C = \{\nu < \kappa \mid h^{N}{''}[\nu]^{<\omega} \subseteq \nu\}$. It is a club in $\mathcal{K}$. However, $C \cap \{\tau_i \mid i \in I\} = \emptyset$, which is impossible as C is supposed to include a final segment of $\langle \tau_i \mid i < \omega\rangle$. Contradiction.
□

4 Proof of Theorem 2.1

Suppose that such V' exists and $(\tau^+)^{\mathcal{K}} = \tau^+$, for all but boundedly many cardinals $\tau < \kappa$. Let $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ be a sequence of different subsets of κ in V'.

Let us apply Proposition 3.3 to V, V' and find a cofinal in κ sequence $\langle \tau_i \mid i < \omega\rangle$ which satisfies the conclusion of 3.3.
Fix a covering model N^* such that $\langle \tau_i \mid i < \omega\rangle \in N^*$. By Proposition 3.4, a final segment of $\langle \tau_i \mid i < \omega\rangle$ consists of principal indiscernibles of N^*. Denote it by C^*.

Pick, in V, a sequence $\langle N_\alpha \mid \alpha < \kappa^{++}\rangle$ of covering models of a same cardinality below κ with $A_\alpha \in N_\alpha, N^* \subseteq N_\alpha$, for every $\alpha < \kappa^{++}$.
Apply the Mitchell Covering Lemma to N_α.
We will have a Skolem function $h_\alpha \in \mathcal{K}$, $\rho_\alpha < \kappa$ and a sequence of indiscernibles C_α. $N_\alpha \cap \mathcal{K}_\kappa \subseteq h_\alpha{''}(\rho_\alpha; C_\alpha)$. Denote by C^*_α the set of all principal indiscernibles of C_α. It includes an $\omega-$sequence cofinal in κ.

For each $\mu \in C^*_\alpha$, consider $A_\alpha \cap \mu$.
Denote by i^μ_α the index of $A_\alpha \cap \mu$ in a fixed enumeration of $\mathcal{P}(\mu)$ in V'. [2]
Then $i^\mu_\alpha \in N_\alpha$, since $A_\alpha \in N_\alpha$.

By the assumption (2) of the theorem and since C^* is almost contained in every club of κ of V', we can assume the following:

$$(*) \quad \textit{for every } \tau \in C^*, V' \models 2^\tau = \tau^+.$$

Recall that we have $\mathrm{GCH}_{<\kappa}$ in V, but not necessary in V'.
Then, $i^\mu_\alpha < (\mu^+)^{V'}$.
Then there is a finite sequence of indiscernibles $\vec{c}^\mu_\alpha$ below μ such that $i^\mu_\alpha = h_\alpha(\vec{c}^\mu_\alpha, \mu)$.

Now, the number of possibilities for h_α, ρ_α's is κ^+, since $h_\alpha \in \mathcal{K}, \rho_\alpha < \kappa$. Hence, we can find a stationary $S \subseteq \kappa^{++}$ a function h and an ordinal ρ such that for every $\alpha \in S$, $h_\alpha = h$ and $\rho_\alpha = \rho$. By shrinking S, if necessary, we may assume also that C_α, C^*_α's are similar.

[2]Further, in Section 5, a fixed enumeration of $\mathcal{P}(\mu)$ in V will be used instead.

Let I be a subset of S of cardinality κ^+ with $\mathrm{cof}(\sup(I)) = \kappa^+$.

We have $\kappa^+ = (\kappa^+)^{\mathcal{K}} = (\kappa^+)^{V'}$, since κ was regular in $\mathcal{K}$, changed its cofinality in V and so, the Mitchell Covering Lemma applies.
However, in general it is possible that $(\kappa^{++})^{V'}$ is collapsed in V to κ^+, and so, we cannot cover I by a set in V' which cardinality is κ^+ there. This is the reason for the assumption (3) of the theorem.
Find $I^* \in V'$ of cardinality κ^+ which covers I and such that $\sup(I) = \sup(I^*)$.
Let us identify below I with I^*.
Let $\delta = \sup(I)$. Then $\kappa^+ \leq \delta < \kappa^{++}$ and I is unbounded in δ.
Fix a function $\sigma : \kappa^+ \leftrightarrow \delta, \sigma \in V'$.

For every regular in $\mathcal{K}$ cardinal $\nu < \kappa$ define a function $h^\nu : [\nu \cup \{\nu\}]^{<\omega} \to [\nu, \nu^+)$ in $\mathcal{K}$ as follows:
$h^\nu(\langle \xi_1, ..., \xi_n \rangle) = h(\langle \xi_1, ..., \xi_n \rangle)$, if $h(\langle \xi_1, ..., \xi_n \rangle) \in (\nu, \nu^+)$, and
$h^\nu(\langle \xi_1, ..., \xi_n \rangle) = \nu$, otherwise,
for every sequence $\langle \xi_1, ..., \xi_n \rangle \in [\nu \cup \{\nu\}]^{<\omega}$.

Define also a function $\tilde{h}^\nu : [\nu \cup \{\nu\}]^{<\omega} \to \mathcal{P}^{\mathcal{K}}(\nu)$ in $\mathcal{K}$ as follows:
$\tilde{h}^\nu(\langle \xi_1, ..., \xi_n \rangle) =$ the subset of ν in $\mathcal{K}$ which has index $h^\nu(\langle \xi_1, ..., \xi_n \rangle)$ in the canonical enumeration of $\mathcal{K}$ of its subsets of ν,
for every sequence $\langle \xi_1, ..., \xi_n \rangle \in [\nu \cup \{\nu\}]^{<\omega}$.

Finally, let $h^*(\nu)$ be a subset of ν which codes in $\mathcal{K}$ (in a canonical fashion) the function $\tilde{h}^\nu$.
Note that for every $\nu_1 < \nu_2$ in C^*, we will have $h^*(\nu_2) \cap \nu_1 = h^*(\nu_1)$, since h corresponds to an iterated ultrapower (a comparison process between $\mathcal{K}$ and a mouse).
We would like to extend h^* by adding κ to its domain and setting $h^*(\kappa) = \bigcup_{\nu \in C^*} h^*(\nu)$, however such extension need not be in $\mathcal{K}$ anymore. Let us do something slightly more elaborate.
Consider the set $\{U_n \mid n < \omega\}$ of normal measures of extenders of $\mathcal{K}$ to which the members of C^* belong relatively to N^*. Recall that every indiscernible of N^* corresponds to an extender in $\mathcal{K}$ see, for example, Section 4.4 of [16] or first sections of [12], which includes more details. Then this set can be covered in $\mathcal{K}$ by a set of cardinality κ. Denote by $\{U_\tau \mid \tau < \kappa\}$ such a cover, where each of U_τ's is a normal ultrafilter (in $\mathcal{K}$). By normality, there is a partition $\langle Y_\tau \mid \tau < \kappa \rangle$ of κ such that $Y_\tau \in U_\tau$, for every $\tau < \kappa$.
Again, using normality, for every $\tau < \kappa$ there are $Y^*_\tau \subseteq Y_\tau, Y^*_\tau \in U_\tau$ and $Z_\tau \subseteq \kappa$ such that for every $\nu \in Y^*_\tau$, we will have $Z_\tau \cap \nu = h^*(\nu)$.

Define h^{**} by setting $h^{**}(\nu) = \langle Z_\tau \cap \nu \mid \tau < \nu \rangle$, for every $\nu \leq \kappa$.

Let $\nu \in C^*$ and let $\tau_\nu < \kappa$ be such that ν corresponds to U_{τ_ν} as a principal indiscernible, then $\tau_\nu \in h''[\nu]^{<\omega}$, and hence, $\tau_\nu < \nu$. This holds since ν is a principal indiscernible and κ is a limit of such indiscernibles.

Work in V'.

Let $\langle M_\nu \mid \nu < \kappa \rangle$ be an increasing continuous sequence of elementary submodels of H_χ such that

1. $\langle M_\nu \mid \nu \leq \zeta \rangle \in M_{\zeta+1}$,
2. $|M_\nu| < \kappa$,
3. $M_\nu \supseteq \nu$,
4. $h, h^*, h^{**}, \sigma, \langle A_\alpha \mid \alpha \in I \rangle \in M_0$.

Let $C = \{\nu < \kappa \mid M_\nu \cap \kappa = \nu\}$. It is a club in V', since κ is regular there.

Pick $\nu \in C \cap C^*$.

Let $\bar{M}_\nu$ be the transitive collapse of M_ν and π the collapsing function.

Then $\pi(\kappa) = \nu, \pi(A) = A \cap \nu$, for every $A \subseteq \kappa, A \in M_\nu$. Also, $\pi(\langle Z_\tau \mid \tau < \kappa \rangle) = \langle Z_\tau \cap \nu \mid \tau < \nu \rangle$. In particular, $Z_{\tau_\nu} \cap \nu = h^*(\nu) \in \bar{M}_\nu$, since $\tau_\nu < \nu$. Then $\tilde{h}^\nu \in \bar{M}_\nu$, and so, h^ν is in $\bar{M}_\nu$, as well.

Hence, in $\bar{M}_\nu$, the number of subsets of ν indexed (now back in V' and not in $\mathcal{K}$) by ordinals in the range of h^ν is less than $(\nu^+)^{\bar{M}_\nu} = \pi(\kappa^+) < \nu^+$. Let $\nu^* < (\nu^+)^{\bar{M}_\nu}$ be such that

$$\bar{M}_\nu \models \forall X \in \pi(\{A_\alpha \mid \alpha \in I\}) \text{ if the index of } X \text{ appears in } \mathrm{rng}(h^\nu),$$

$$\text{then it is below } \nu^*.$$

Define a function $s \in \prod_{\nu \in C^* \cap C} \nu^+$ by setting $s(\nu) = \nu^*$.

Let $\langle f_\xi \mid \xi < \kappa^+ \rangle$ be canonical functions in $\prod_{\xi < \kappa} \xi^+$, in $\mathcal{K}$.

Lemma 4.1 *There is $\eta < \kappa^+$ such that $f_\eta \restriction C^* \cap C$ dominates s mod finite.*

Proof. Pick a covering model N with $s, C^* \cap C \in N$. We may assume that each $\nu \in C^* \cap C$ is a principal indiscernible of N, by dropping finitely many points if necessary. By the assumption, $\nu^+ = (\nu^+)^{\mathcal{K}}$, hence there are no indiscernibles in the interval $(\nu, \nu^+]$.

Define a function $g \in \prod_{\gamma < \kappa} \gamma^+$ as follows:

$$g(\gamma) = \sup(h^{N''}\gamma) \cap \gamma^+.$$

Then $g \in \mathcal{K}$, since h^N is in $\mathcal{K}$. In addition, $g(\nu) > s(\nu)$, for every $\nu \in \mathrm{dom}(s)$, since $s \in N$ and there are no indiscernibles in the interval $(\nu, \nu^+]$.
Now find $\eta < \kappa^+$ such that f_η dominates g.
□

Pick $\eta < \kappa^+$ such that the canonical function $f_\eta \restriction C^* \cap C$ dominates s.

Let $\langle R_\nu \mid \nu < \kappa \rangle$ be an increasing continuous sequence of elementary submodels of H_χ such that

1. $\langle R_\nu \mid \nu \leq \zeta \rangle \in R_{\zeta+1}$,
2. $|R_\nu| < \kappa$,
3. $M_\nu \subseteq R_\nu$,
4. $\eta \in R_0$.

Let $E = \{\nu \in C \mid R_\nu \cap \kappa = \nu\}$.
Pick a typical $\nu \in E \cap C^*$ which is a principal indiscernible and $\nu^* < f_\eta(\nu)$.
Let $\bar{R}_\nu$ be the transitive collapse of R_ν and φ the collapsing function.
Then $\varphi(\kappa) = \nu, \varphi(A) = A \cap \nu$, for every $A \subseteq \kappa, A \in R_\nu$. Let M'_ν be $\varphi[M_\nu]$. Then $M'_\nu \preceq \bar{R}_\nu$. Also, $\bar{M}_\nu$ is the transitive collapse of M'_ν. Let $\psi : M'_\nu \leftrightarrow \bar{M}_\nu$ be the collapsing function. Note that $M'_\nu \cap \nu = \bar{M}_\nu \cap \nu = \bar{R}_\nu \cap \nu = \nu$. So, $M'_\nu \cap \varphi(\kappa^+)$ is an ordinal, and hence, $\psi \restriction M'_\nu \cap \varphi(\kappa^+)$ is the identity. In particular, $\psi(\nu^*) = \nu^*$.
So, by elementarity,

$$\bar{R}_\nu \models \forall X \in \varphi(\{A_\alpha \mid \alpha \in I\}) \text{ if the index of } X \text{ appears in } \mathrm{rng}(h^\nu),$$

$$\text{then it is below } \nu^*.$$

Let us argue that. $\varphi(\eta) = f_\eta(\nu)$.

It's likely the general proof below about canonical functions and transitive collapses is already known.

Claim 1 $\varphi(\eta) = f_\eta(\nu)$.

Proof. Let us recall one of the definitions of canonical functions $\langle f_\xi \mid \xi < \kappa^+ \rangle$ in $\prod_{\xi<\kappa} \xi^+$, in $\mathcal{K}$.
We proceed by induction on ξ. If $\xi = \xi' + 1$, then let $f_\xi(\tau) = f_{\xi'}(\tau) + 1$, for every $\tau < \kappa$.

Suppose now that $\xi < \kappa^+$ is a limit ordinal. Pick the least club $\{\xi_i \mid i < \mathrm{cof}(\xi)\}$. Set $f_\xi(\tau) = \bigcup_{i<\tau} f_{\xi_i}(\tau)$, for every $\tau < \kappa$.

Let now $\langle R_\nu \mid \nu < \kappa \rangle$ be as above or just any other increasing continuous sequence of elementary submodels of H_χ with $\eta \in R_0$.
Set $R_\kappa = \bigcup_{\nu<\kappa} R_\nu$.
Let us show by induction on $\xi \in R_\kappa \cap \kappa^+$ that for a club C_ξ of ν's, $f_\xi(\nu) = \varphi(\xi)$.
Let us deal with a limit ξ of cofinality κ. Other cases are immediate. Let $\{\xi_i \mid i < \kappa\}$ be the least club in ξ. By elementarity it is in R_κ.
By induction, for every $i < \kappa$, there is a club C_{ξ_i} in κ such that for every $\tau \in C_{\xi_i}$, $f_{\xi_i}(\tau) = \varphi_\tau(\xi_i)$, where φ_τ denotes the transitive collapse of R_τ. Set $C_\xi = \Delta_{i<\kappa} C_{\xi_i}$. Let $\tau \in C_\xi$. By the definition, $f_\xi(\tau) = \bigcup_{i<\tau} f_{\xi_i}(\tau)$. But, for every $i < \tau, \tau \in C_{\xi_i}$, and hence, by induction, $f_{\xi_i}(\tau) = \varphi_\tau(\xi_i)$. Also, $\varphi_\tau(\{\xi_i \mid i < \kappa\}) = \{\varphi_\tau(\xi_i) \mid i < \tau = \varphi_\tau(\kappa)\}$. By elementarity, then $\varphi_\tau(\eta) = \bigcup \varphi_\tau(\{\xi_i \mid i < \kappa\}) = \bigcup\{\varphi_\tau(\xi_i) \mid i < \tau\}$, and we are done.
$\square$ of the claim.

We have, $\nu^* < f_\eta(\nu)$. Hence, the index i^ν_η of $A_{\sigma(\eta)} \cap \nu$ in the enumeration of subsets of ν will not appear in the range of h^ν. This is impossible, since $h = h_{\sigma(\eta)}$ and there is a finite sequence of indiscernibles $\vec{c}^{\,\nu}_{\sigma(\eta)}$ below ν such that $i^\nu_{\sigma(\eta)} = h_{\sigma(\eta)}(\vec{c}^{\,\nu}_{\sigma(\eta)}, \nu)$.

Remark 4.2 *Note that the standard Extender Based Prikry forcing over $\mathcal{K}$ satisfies the conditions (2) and (3) of the theorem. So, the argument above shows that there is no intermediate model in which κ is regular and $2^\kappa > \kappa^+$. However, this is under the assumption that there is no inner model with a strong cardinal, in contrast to [2].*

5 Proof of Theorem 2.4

Let us show how to modify the previous argument in order to replace the assumptions that $V' \models 2^\tau = \tau^+$, *for every* $\tau \in C^*$ and that $\kappa^{++} = (\kappa^{++})^{V'}$ by a strong form of covering.

Let $\langle A_\alpha \mid \alpha < \kappa^{++} \rangle$, $\langle N_\alpha \mid \alpha < \kappa^{++} \rangle$, h, S be as in Section 4.

For every $\alpha < \kappa^{++}, \tau \in C^*$, let i^τ_α be the index of $A_\alpha \cap \tau$ in a fixed enumeration of $\mathcal{P}(\tau)$, but now in V. Consider the function $\tau \mapsto i^\tau_\alpha$. Denote it by g_α. By $\mathrm{GCH}_{<\kappa}$, $g_\alpha \in \prod_{\tau \in C^*} \tau^+$. There is a finite increasing sequence of indiscernibles $\vec{c}^{\,\tau}_\alpha \in [i^\tau_\alpha + 1]^{<\omega}$ such that $g_\alpha(\tau) = i^\tau_\alpha = h(\vec{c}^{\,\tau}_\alpha)$.

Note that there are no indiscernibles in the interval $(\tau, \tau^+]$, since we are assuming that $(\tau^+)^{\mathcal{K}} = \tau^+$, for every $\tau \in C^*$. Hence,

$$\mathrm{tcf}(\prod_{\tau \in C^*} \tau^+, <_{J^{bd}_\kappa}) = \kappa^+,$$

as every function in this product will be bounded by the restriction of a function in $\mathcal{K}$ to C^*, see Lemma 4.1 for this type of argument.
Let $\vec{f} = \langle f_\alpha \mid \alpha < \kappa^+ \rangle$ be a witnessing scale. We have $\kappa^{++}-$many g_α's, so there are $S' \subseteq S, |S'| = |S| = \kappa^{++}$ and $\alpha^* < \kappa^+$ such that f_{α^*} dominates each $g_\alpha, \alpha \in S'$. By shrinking S' if necessary, we can assume that there is $\gamma^* \in C^*$ such that for every $\gamma \in C^* \setminus \gamma^*$, $f_{\alpha^*}(\gamma) > g_\alpha(\gamma)$. Assume for simplicity that $\gamma^* = \min C^*$.

For every $\tau \in C^*$ we fix a function

$$e_\tau : f_{\alpha^*}(\tau) \leftrightarrow |\tau|.$$

For every $\alpha \in S'$, define a function $s_\alpha \in \prod_{\tau \in C^*} |\tau|$ by setting

$$s_\alpha(\tau) = e_\tau(g_\alpha(\tau)).$$

Let us consider few cases.

Case 1 *There is $\delta < \kappa$ such that for an unbounded $C' \subseteq C^*$, the following holds: $c \in C' \Rightarrow \mathrm{cof}(|c|) < \delta$.*

Then, using $\mathrm{GCH}_{<\kappa}$, it is easy to find some $g \in \prod_{\tau \in C'} |\tau|$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, g dominates s_α. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$.
$\square$ of Case 1.

Suppose now that there is no such δ. Then the set

$$\{\mathrm{cof}(|c|) \mid c \in C'\}$$

is unbounded in κ. By shrinking C^* if necessary, we can assume that the sequence

$$\langle \mathrm{cof}(|c|) \mid c \in C^* \rangle$$

is strictly increasing. Consider then $\mathrm{pcf}(\{|c| \mid c \in C^*\}) \setminus \kappa$. It is a subset of the set $\{\kappa^+, \kappa^{++}\}$, since $2^\kappa = \kappa^{++}$ and κ is a strong limit.

Case 2 *There is $C' \subseteq C^*$ such that* $\mathrm{tcf}(\prod_{c \in C'} |c|, <_{J^{bd}_\kappa}) = \kappa^+$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^+ \rangle$ be a witnessing scale.
Then there are $\xi^* < \kappa^+$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, p_{ξ^*} dominates $s_\alpha \restriction C'$. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.
$\square$ of Case 2.

Case 3 *There is $C' \subseteq C^*$ such that* $\mathrm{tcf}(\prod_{c \in C'} |c|, <_{J^{bd}_\kappa}) = \kappa^{++}$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^{++} \rangle$ be a witnessing scale. Take S'' to be a subset of S' of cardinality κ^+. Then there will be $\xi^* < \kappa^{++}$ such that for every $\alpha \in S''$, p_{ξ^*} dominates s_α. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.
$\square$ of Case 3.

So, in either case we are able to find a function g which dominates subsets of S' of cardinality κ^{++} or κ^+.

We showed the following crucial property:

($\aleph$) *There are an unbounded $E \subseteq C^*$ and $B \subseteq S, |B| = \kappa^+$ such that for every $\tau \in E$, the set*

$$\{A_\alpha \cap \tau \mid \alpha \in B\}$$

has cardinality strictly less than $|\tau|$.

This holds since the corresponding set

$$\{i^\tau_\alpha \mid \alpha \in B\}$$

has cardinality less than $|\tau|$.

Now we would like to ran the argument with an elementary chain based on the strong form of covering $Cov(V, V', \kappa^+)$ defined 2.3.

Recall that $Cov(V, V', \kappa^+)$ denotes the following strong covering property:
For every set of ordinals $B \subseteq 2^\kappa$ of cardinality κ^+ there are $I \subseteq B$ of cardinality κ and $I^* \in V', I^* \supseteq I$ such that for some increasing and continuous sequence $\langle M_\nu \mid \nu < \kappa \rangle \in V'$ with $|M_\nu| < \kappa$, for every $\nu < \kappa$, and $I^* \subseteq \bigcup_{\nu<\kappa} M_\nu$, the following holds: for every $\nu < \kappa, |M_\nu \cap I| = |M_\nu \cap I^*|$.

The next lemma will provide the desired contradiction and complete the proof of the theorem.

Lemma 5.1 *It is impossible to have simultaneously both $Cov(V, V', \kappa^+)$ and ($\aleph$).*

Proof. Suppose that both $Cov(V, V', \kappa^+)$ and ($\aleph$) hold.
Apply $Cov(V, V', \kappa^+)$ to B given by ($\aleph$). Let $I, I^*, \langle M_\nu \mid \nu < \kappa \rangle$ be witnessing sets.

Work in V'. Pick $\langle R_\nu \mid \nu < \kappa \rangle$ to be an increasing continuous sequence of elementary submodels of H_χ, with χ large enough, such that

1. $\langle R_\nu \mid \nu \leq \zeta \rangle \in R_{\zeta+1}$,
2. $|R_\nu| < \kappa$,

3. $M_\nu \subseteq R_\nu$,

4. $I^*, \langle A_\alpha \mid \alpha \in I^* \rangle \in R_0$.

Let $\langle i^*_\nu \mid \nu < \kappa \rangle$ be an enumeration of I^* in $V' \cap R_0$.

Set $X = \{\nu < \kappa \mid R_\nu \cap \kappa = \nu$ and $M_\nu \cap I^* = \{i^*_\zeta \mid \zeta < \nu\}\}$.

Clearly, X is in V' and it is a closed unbounded subset of κ. Then, X contains a final segment of E, where $E \subseteq C^*$ is from ($\aleph$). Pick $\eta \in E \cap X$.

Then, $R_\eta \cap \eta = \eta$. By elementarity, $R_\eta \cap I^* = \{i^*_\nu \mid \nu < \eta\}$. So, $R_\eta \cap I^* = M_\eta \cap I^*$. Hence, in V,

$$|\eta| = |R_\eta \cap I^*| = |M_\eta \cap I^*| = |M_\eta \cap I| = |R_\eta \cap I|.$$

For every $\alpha \in I^* \cap R_\eta$, $A_\alpha \in R_\eta$. In particular, for every $\alpha \in I \cap R_\eta$, $A_\alpha \in R_\eta$. By elementarity,

$$R_\eta \models \forall \alpha, \beta \in I^* (\alpha \neq \beta \to A_\alpha \neq A_\beta).$$

We have $R_\eta \cap \kappa = \eta$, hence $A_\alpha \cap \eta \neq A_\beta \cap \eta$, for every $\alpha, \beta < (\kappa^{++})^V \cap R_\eta, \alpha \neq \beta$. In particular, for every $\alpha, \beta \in R_\eta \cap I, \alpha \neq \beta$, $A_\alpha \cap \eta \neq A_\beta \cap \eta$. So,

$$|\{A_\alpha \cap \eta \mid \alpha \in I\}| \geq |R_\eta \cap I| = |\eta|.$$

But $I \subseteq B$, $\eta \in E$, hence the set

$$\{A_\alpha \cap \eta \mid \alpha \in I\}$$

has cardinality strictly less than $|\eta|$, by ($\aleph$). It is impossible. Contradiction.

□

This completes the proof of Theorem 2.4.

6 Proof of Theorem 2.5

We deal now with a possibility that successors of principal indiscernibles are collapsed.

Assume here that $2^\kappa = \kappa^{+3}$. If $2^\kappa > \kappa^{+3}$, then we just collapse 2^κ to κ^{+3}. Suppose that there is $V', \mathcal{K} \subseteq V' \subseteq V$ such that

1. κ is a regular in V',

2. $(2^\kappa)^{V'} \geq \kappa^{+3}$.

Let $\langle A_\alpha \mid \alpha < \kappa^{+3}\rangle$ be a sequence in V' of $\kappa^{+3}-$subsets of κ. Keep the notation of the previous sections and define $C^*, \langle N_\alpha \mid \alpha < \kappa^{+3}\rangle, h, S \subseteq \kappa^{+3}$ as before.

The basic idea will be to explore the selection between three available cardinalities $\kappa^+, \kappa^{++}, \kappa^{+3}$ for collections of subsets of κ in V' against only two related cofinalities of products of the form $\prod_{\tau \in C^*} \tau^+$ and $\prod C^*$.

For every $\alpha < \kappa^{+3}, \tau \in C^*$, let i^τ_α be the index of $A_\alpha \cap \tau$ in a fixed enumeration of $\mathcal{P}(\tau)$ in V. Consider the function $\tau \mapsto i^\tau_\alpha$. Denote it by g_α. By $\mathrm{GCH}_{<\kappa}$, $g_\alpha \in \prod_{\tau \in C^*} \tau^+$. There is a finite increasing sequence of indiscernibles $\vec{c}^{\,\tau}_\alpha \in [i^\tau_\alpha + 1]^{<\omega}$ such that $g_\alpha(\tau) = i^\tau_\alpha = h(\vec{c}^{\,\tau}_\alpha)$.

Denote the set $\{\tau^+ \mid \tau \in C^*\}$ by C^{*+}.
Consider $\mathrm{pcf}(C^{*+}) \setminus \kappa$. It is a subset of the set $\{\kappa^+, \kappa^{++}, \kappa^{+3}\}$. Let $C^{*+} = C^{*+}_1 \cup C^{*+}_2 \cup C^{*+}_3$ be a splitting of C^{*+} into sets which are generators of $\kappa^+, \kappa^{++}, \kappa^{+3}$ respectively. It is possible that some of them are empty. Let us consider a few cases.

Case 1 $C^{*+}_3 \neq \emptyset$.

Then $\mathrm{tcf}(\prod C^{*+}_3, <_{J^{bd}_\kappa}) = \kappa^{+3}$.
Let $\vec{f} = \langle f_\alpha \mid \alpha < \kappa^{+3}\rangle$ be a witnessing scale. Let $\alpha \in S$. Define a function t_α on C^{*+}_3 by setting

$$t_\alpha(\tau^+) = g_\alpha(\tau) = i^\tau_\alpha.$$

Then $t_\alpha \in \prod C^{*+}_3$, and so it is bounded by a function from the scale $\vec{f}$.
Take any $S' \subseteq S$ of cardinality κ^{++}. There will be $\alpha^* < \kappa^{+3}$ such that f_{α^*} dominates each $t_\alpha, \alpha \in S'$. By shrinking S' if necessary, we can assume that there is $\gamma^* \in C^{*+}_3$ such that for every $\gamma \in C^{*+}_3 \setminus \gamma^*$, $f_{\alpha^*}(\gamma) > t_\alpha(\gamma)$. Assume for simplicity that $\gamma^* = \min C^{*+}_3$.
Set

$$C'_3 = \{\tau \in C^* \mid \tau^+ \in C^{*+}_3\}.$$

For every $\tau \in C'_3$ we fix a function

$$e_\tau : f_{\alpha^*}(\tau^+) \leftrightarrow |\tau|.$$

For every $\alpha \in S'$, define a function $s_\alpha \in \prod_{\tau \in C'_3} |\tau|$ by setting

$$s_\alpha(\tau) = e_\tau(t_\alpha(\tau^+)).$$

Subcase 1.1 *There is $\delta < \kappa$ such that for an unbounded $C' \subseteq C'_3$, the following holds:* $c \in C' \Rightarrow \mathrm{cof}(|c|) < \delta$.[3]

Then, using $\mathrm{GCH}_{<\kappa}$, it is easy to find some $g \in \prod_{\tau \in C'} |\tau|$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, g dominates s_α. By shrinking a bit more if necessary, we can assume

[3]It is parallel to Case 1 from the previous section.

that the domination takes place from the same point for every $\alpha \in S''$.
$\square$ of Subcase 1.1.

Suppose now that there is no such δ. Then the set

$$\{\mathrm{cof}(|c|) \mid c \in C'\}$$

is unbounded in κ. By shrinking C' if necessary, we can assume that the sequence

$$\langle \mathrm{cof}(|c|) \mid c \in C' \rangle$$

is strictly increasing. Consider then $\mathrm{pcf}(\{|c| \mid c \in C'\}) \setminus \kappa$. Again, it is a subset of the set $\{\kappa^+, \kappa^{++}, \kappa^{+3}\}$.

Subcase 1.2 *There is* $C'' \subseteq C'$ *such that* $\mathrm{tcf}(\prod_{c \in C''} |c|, <_{J_\kappa^{bd}}) = \kappa^+$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^+ \rangle$ be a witnessing scale.
Then there are $\xi^* < \kappa^+$ and $S'' \subseteq S', |S''| = |S'|$ such that for every $\alpha \in S''$, p_{ξ^*} dominates $s_\alpha \restriction C''$. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.
$\square$ of Subcase 1.2.

Subcase 1.3 *There is* $C'' \subseteq C'$ *such that* $\mathrm{tcf}(\prod_{c \in C''} |c|, <_{J_\kappa^{bd}}) = \kappa^{+3}$.

Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^{+3} \rangle$ be a witnessing scale. Then, $|S'| = \kappa^{++}$ implies that there is $\xi^* < \kappa^{+3}$ such that for every $\alpha \in S''$, p_{ξ^*} dominates $s_\alpha \restriction C''$. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.
$\square$ of Subcase 1.3.

Suppose now that Subcases 1.2,1.3 fail. Then $\mathrm{tcf}(\prod_{c \in C'} |c|, <_{J_\kappa^{bd}}) = \kappa^{++}$.
Let $\vec{p} = \langle p_\xi \mid \xi < \kappa^{++} \rangle$ be a witnessing scale. Take S'' to be a subset of S' of cardinality κ^+. Then there will be $\xi^* < \kappa^{++}$ such that for every $\alpha \in S''$, p_{ξ^*} dominates s_α. By shrinking a bit more if necessary, we can assume that the domination takes place from the same point for every $\alpha \in S''$. Set $g = p_{\xi^*}$.

So, in either case we are able to find a function g which dominates subsets of S' of cardinality κ^{++} or κ^+.

Case 2 $C_2^{*+} \neq \emptyset$.

Case 3 $C_1^{*+} \neq \emptyset$.

The treatment of Cases 2,3 is completely similar to those of Case 1.

We showed the following crucial property:

(ℵ) *There are an unbounded $E \subseteq C^*$ and $B \subseteq S, |B| = \kappa^+$ such that for every $\tau \in E$, the set*

$$\{A_\alpha \cap \tau \mid \alpha \in B\}$$

has cardinality strictly less than $|\tau|$.

This holds since the corresponding set

$$\{i^\tau_\alpha \mid \alpha \in B\}$$

has cardinality strictly less than $|\tau|$.

Recall that the strong covering property $Cov(V, V', \kappa^+)$ is assumed, so, we can apply Lemma 5.1 in order to derive the contradiction.
This completes the proof of Theorem 2.5.

7 Part 2 - Introduction

Let κ be a measurable cardinal and $2^\kappa > \kappa^+$. Then, by a classical result of D. Scott, for unboundedly many $\nu < \kappa$, $2^\nu > \nu^+$. Moreover, if U is a normal ultrafilter over κ, then

$$\{\nu < \kappa \mid 2^\nu > \nu^+\} \in U.$$

But what will happen if we drop the normality assumption? It is easy to construct a model in which $2^\kappa > \kappa^+$ and for some $\kappa-$complete ultrafilter U over κ,

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U.$$

Just start with a sufficiently large cardinal κ, say supercompact or 2-strong. Assume GCH. Force with an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle,$$

where $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal. For an inaccessible $\beta \leq \kappa$, let Q_β be the Cohen forcing $Cohen(\beta, \beta^{++})$ which adds $\beta^{++}-$many subsets to β. Let $G \subseteq P_{\kappa+1}$ be a generic. κ will remain a measurable in $V[G]$ and $2^\kappa = \kappa^{++}$. Let W be a normal ultrafilter over κ. Consider its ultrapower embedding $j_W : V[G] \to M_W$. Note that in $V[G]$ for every successor cardinal $\nu < \kappa$, $2^\nu = \nu^+$. Set $\delta = (\kappa^+)^{M_W}$. Then, by elementarity, $M_W \models 2^\delta = \delta^+$. Define

$$U = \{X \subseteq \kappa \mid \delta \in j_W(X)\}.$$

Then

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U.$$

In this example U is Rudin-Keisler equivalent to a normal ultrafilter and concentrate on a non-stationary set.
But what will happen if we require that U contains all closed unbounded subsets of κ?

The answer is that it is still possible.

We will present several constructions. The first one will be an easy forcing extension from a supercompact cardinal. The second will be from a strong cardinal and with some additional properties. However, the gap between κ and 2^κ which we can reach using the second construction is only 2.
It turns out that the situation here is surprisingly similar to those of the previous sections. The third construction extends the second to larger gaps, but uses a supercompactness.

8 The first construction

Assume GCH and let κ be a $\kappa^{++}-$supercompact cardinal.
Fix a normal ultrafilter W over $\mathcal{P}_\kappa(\kappa^{++})$. Let $j_W : V \to M_W$ be the corresponding elementary embedding. Then ${}^{\kappa^{++}}M_W \subseteq M_W$.

As above, we force with an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa\rangle,$$

where $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal. For an inaccessible $\beta \leq \kappa$, let Q_β be the Cohen forcing $Cohen(\beta, \beta^{++})$ which adds $\beta^{++}-$many subsets to β. Let $G \subseteq P_{\kappa+1}$ be a generic. κ will remain a $\kappa^{++}-$supercompact cardinal in $V[G]$ and $2^\kappa = \kappa^{++}$. Moreover W extends to a normal ultrafilter W^* and j_W extends to $j_{W^*} : V[G] \to M_{W^*} = M_W[G^*]$. Still we have ${}^{\kappa^{++}}M_{W^*} \subseteq M_{W^*}$.
Let $\langle C_\alpha \mid \alpha < \kappa^{++}\rangle$ be an enumeration of all clubs of κ in $V[G]$. Then, the set $Y = \{j_{W^*}(C_\alpha) \mid \alpha < \kappa^{++}\}$ is in M_{W^*} and consists of less than $j_W(\kappa)-$many clubs. Hence, $C = \bigcap Y$ is a club of $j_W(\kappa)$ in M_{W^*}, see for example [5], 12.6,7.

Note that in $V[G]$ for every singular cardinal $\nu < \kappa$, $2^\nu = \nu^+$.
Working in M_{W^*}, pick a singular cardinal $\delta \in C$ of cofinality ω. Then, by elementarity, $M_{W^*} \models 2^\delta = \delta^+$. Define

$$U = \{X \subseteq \kappa \mid \delta \in j_{W^*}(X)\}.$$

Then

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U.$$

Moreover, by the choice of δ, we will have that $U \supseteq Cub_\kappa$.

In addition, let us observe the following:

Let

$$U_\kappa = \{X \subseteq \kappa \mid \kappa \in j_U(X)\}.$$

It is the projection of U to the least normal below it in the Rudin-Keisler ordering. Let $k : M_{U_\kappa} \to M_U$ be the corresponding embedding. Then $\mathrm{crit}(k) < [id]_U$, since $M_U \models \mathrm{cof}([id]_U) = \omega$.

9 Construction from the optimal assumptions

Assume GCH and suppose that a cardinal κ is $2-$strong. Let E be a witnessing extender. Instead, we can start from $o(\kappa) = \kappa^{++}$ which is optimal and, using [7], arrange the same type of a situation.

Define an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle.$$

Let $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal.

If $\beta < \kappa$ is an inaccessible cardinal then set $Q_\beta = Q^0_\beta * Q^1_\beta$, where Q^0_β is an atomic forcing which consists of two incompatible elements 0 and 1.

If the generic for Q^0_β is 0, then Q^1_β is the Cohen forcing for adding β^{++} subsets to β , $Cohen(\beta, \beta^{++})$.

Otherwise, i.e. if the generic for Q^0_β is 1, then $Q^1_\beta = Cohen(\beta, \beta^{++}) * Col(\beta, \beta^+)$.

Set $Q_\kappa = Cohen(\kappa, (\kappa^{++})^{M_{E(\kappa)}}) * (Cohen(\kappa^+, \kappa^{++}) \times Cohen(\kappa, [(\kappa^{++})^{M_{E(\kappa)}}, \kappa^{++}))$.

Let G be generic subset of $P_{\kappa+1}$. Then, in $V[G]$, $2^\kappa = \kappa^{++}$.

By Woodin's arguments, see [5], κ will remain a measurable in $V[G]$ and the embedding $j_E : V \to M_E$ extends to $j^* : V[G] \to M_E[G^*]$. We take here the generic for Q^0_κ to be 0.

Set

$$U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}.$$

Then

1. $U^* \supseteq U$,

2. $j_{U^*} = j^*$

3. $M_{U^*} = M_E[G^*]$.

We have

$$j_E(P) = P_\kappa * Cohen(\kappa, \kappa^{++}) * P_{(\kappa, j_E(\kappa))} * Cohen(j_E(\kappa), (j_E(\kappa)^{++})^{M_{j_E(E(\kappa))}})$$
$$*(Cohen(j_E(\kappa^+), j_E(\kappa^{++})) \times Cohen(j_E(\kappa), [(j_E(\kappa)^{++})^{M_{j_E(E(\kappa))}}, j_E(\kappa^{++}))).$$

Consider now $U^* \times U^*$. We have that $j_{U^* \times U^*}$ extends $j_{E\times E} = j_{j_E(E)} \circ j_E$. Denote $j_E(\kappa)$ by κ_1 and $j_{E\times E}(\kappa) = j_{j_E(E)}(\kappa_1)$ by κ_2. Then $j_{j_E(E)} : M_E \to M_{E\times E}$ and κ_1 is its critical point.

Apply $j_{E\times E}$ to $P_{\kappa+1}$. In $M_{E\times E}$,

$$P_{\kappa_2+1} = P_\kappa * \underset{\sim}{Q}_\kappa * P_{(\kappa,\kappa_1)} * \underset{\sim}{Q}_{\kappa_1} * P_{(\kappa_1,\kappa_2)} * \underset{\sim}{Q}_{\kappa_2}.$$

We have, in $M_{E\times E}[G^* \cap P_{\kappa_1}]$, $Q_{\kappa_1} = Q^0_{\kappa_1} * Q^1_{\kappa_1}$. Let us take a generic for $Q^0_{\kappa_1}$ to be 1. Then $Q^1_{\kappa_1}$ will be $Cohen(\kappa_1, \kappa_1^{++}) * Col(\kappa_1, \kappa_1^+)$.[4]

Still $j_{j_E(E)}$ extends. Let $k : M_E[G^*] \to M_{E\times E}[G^{**}]$ be such extension. We will have also $j^{**} : V[G] \to M_{E\times E}[G^{**}]$ such that $j^{**} = k \circ j^*$.

Define

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j^{**}(X)\}.$$

Then W will be a $\kappa-$complete ultrafilter over κ which extends Cub_κ. Since for every club $C \subseteq \kappa$, $j^*(C)$ is a club in κ_1 in $M_E[G^*]$, and so, $\kappa_1 \in k(j^*(C)) = j^{**}(C)$.

Now, in $M_{E\times E}[G^{**}]$, $2^{\kappa_1} = \kappa_1^+$, since $Col(\kappa_1, \kappa_1^+)$ was applied to restore GCH over κ_1. Hence,

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W,$$

and we are done.

In addition, $U^* \leq_{R-K} W$, and if $k : M_{U^*} \to M_W$ is a corresponding embedding, then $\mathrm{crit}(k) = \kappa_1$.

Remark 9.1 *Note that as in the Woodin construction, the above gives a gap two and not more. We will see the reasons for such phenomenon in the next section.*

[4] Note that

$$Cohen(\kappa_1, \kappa_1^{++}) * Col(\kappa_1, \kappa_1^+) = Cohen(\kappa_1, \kappa_1^{++}) \times Col(\kappa_1, \kappa_1^+) = Col(\kappa_1, \kappa_1^+) \times Cohen(\kappa_1, \kappa_1^{++}).$$

It is possible to find a generic for the corresponding collapse over the ultrapower $M_{E(\kappa)}$, by the normal measure of E already in $V[G]$, then we force as Woodin does (see [5]) in order to obtain a generic for the corresponding Cohen part, and then, move such generics to κ_1 in M_E.

10 On the consistency strength

Suppose that $W \supseteq Cub_\kappa$, $2^\kappa = \kappa^{++}$ and $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$.

Assume that there is no inner model with a strong cardinal. Let $\mathcal{K}$ be the core model. By W. Mitchell [16] or by R. Schindler [17], $j_W \restriction \mathcal{K}$ is an iterated ultrapower of $\mathcal{K}$.

An ordinal $\alpha < j_W(\kappa)$ is called *a generator* of j_W, if there is no $n < \omega$ and $f : [\kappa]^n \to \kappa$ such that for some $\alpha_1 < ... < \alpha_n < \alpha$, $j_W(f)(\alpha_1, ..., \alpha_n) = \alpha$.

Let us call an ordinal $\alpha < j_W(\kappa)$ *a principal generator* of j_W, if there is no $n < \omega$ and $f : [\kappa]^n \to \kappa$ such that for some $\alpha_1 < ... < \alpha_n < \alpha$, $j_W(f)(\alpha_1, ..., \alpha_n) \geq \alpha$.

Then, κ and $[id]_W$ are principal generators, but probably there are more.

Denote by P the set of all generators of j_W.

Consider

$$\{j_W(f)(\alpha_1, ..., \alpha_n) \mid n < \omega, \alpha_1 < ... < \alpha_n, \alpha_1, ..., \alpha_n \in P \cap [id]_W, f : [\kappa]^n \to V\}.$$

Let M be the transitive collapse of it and let $j : V \to M$ be the corresponding elementary embedding.

Define $k : M \to M_W$ by setting $k(j(f)(\alpha_1, ..., \alpha_n)) = j_W(f)(\alpha_1, ..., \alpha_n)$.

Then $j_W = k \circ j$ and $[id]_W = j(\kappa)$ will be the critical point of k.

Note that in the construction of the previous section, we have $P = \{\kappa, [id]_W\}$ and M is the ultrapower by the normal ultrafilter generated by κ.

Denote $[id]_W$ by κ_1 and $j_W(\kappa)$ by κ_2.

We have then that $M \models 2^{\kappa_1} = \kappa_1^{++}$, by the elementarity of j, and $M_W \models 2^{\kappa_1} = \kappa_1^+$, since $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$.

Lemma 10.1 *$(\mathcal{P}(\kappa_1))^M \subseteq (\mathcal{P}(\kappa_1))^{M_W}$, and so, $(\kappa_1^+)^M \leq (\kappa_1^+)^{M_W}$.*

Proof. Let $A \subseteq \kappa_1, A \in M$. Then $k(A) \in M_W$. We have $\kappa_1 = \text{crit}(k)$, so $k(A) \cap \kappa_1 = A$. Hence, $A \in M_W$.

□

Assume now that $(\kappa_1^+)^M$ ***is not collapsed in*** M_W***.***

Let $\vec{A} = \langle A_\alpha \mid \alpha < \kappa^{++} \rangle$ be an enumeration of $\mathcal{P}(\kappa)$. Consider the images of $\vec{A}$ under j and j_W. Let $\vec{A}^1 = j(\vec{A}) = \langle A^1_\beta \mid \beta < (\kappa_1^{++})^M \rangle$ and $\vec{A}^2 = j_W(\vec{A}) = \langle A^2_\gamma \mid \gamma < (\kappa_2^{++})^{M_W} \rangle$.

Lemma 10.2 *If $\vec{A}^1 \in M_W$, then $((\kappa_1)^{++})^M$ is collapsed to $(\kappa_1^+)^{M_W}$ in M_W.*

Proof. This follows from $M_W \models 2^{\kappa_1} = \kappa_1^+$.

□

Lemma 10.3 *$((\kappa_1)^{++})^M$ cannot be collapsed to $(\kappa_1^+)^{M_W}$.*

Proof. By the anti-large cardinal assumption made, we have

$$\mathrm{cof}(((\kappa_1)^+)^{M_W}) = \kappa^+ \text{ and } \mathrm{cof}(((\kappa_1)^{++})^M) = \kappa^{++},$$

since the pointwise images of κ^+ and of κ^{++} are unbounded in $((\kappa_1)^+)^{M_W}$ and in $((\kappa_1)^{++})^M$ respectively.

If the cofinality of $((\kappa_1)^{++})^M$ is changed to something below, in M_W, then $((\kappa_1)^{++})^M$ will be measurable in $\mathcal{K}^M$ and $M \models 2^{\kappa_1} = \kappa_1^{++}$, so, by [12], there must be an extender in $\mathcal{K}^M$ of a measurable length. Such possibility is ruled out as well by the anti-large cardinal assumption made.
□

Remark 10.4 Similar, under the same anti-large cardinal assumptions, if $2^\kappa > \kappa^{++}$, then there will be no room for such type of collapses.

The following property was introduced by T. Benhamou and G. Goldberg [3]:

Definition 10.5 Let U be a $\kappa-$complete ultrafilter over κ. $\diamondsuit^-_{par}(U)$ holds iff there are $A \in M_U$, λ and a family $\{X_i \mid i < 2^\kappa\}$ of 2^κ different subsets of κ such that

1. $\{j_U(X_i) \cap \lambda \mid i < 2^\kappa\} \subseteq A$,
2. there is no function $f : \kappa \to \kappa$ such that $j_U(f)(|A|^{M_U}) \geq \lambda$.

Theorem 10.6 *Assume that there is no inner model with a strong cardinal.*
Suppose that $W \supseteq Cub_\kappa$, $2^\kappa = \kappa^{++}$ and $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$.
Then every one of the following properties is impossible:
(ℵ) $(\kappa_1^+)^M$ is not collapsed in M_W and $\vec{A}^1 \in M_W$,
(ב) $(\mathcal{P}(\kappa_1))^{M_W} \subseteq (\mathcal{P}(\kappa_1))^M$,
(ג) $\neg\diamondsuit^-_{par}(W)$,
(ד) W has the Galvin property.

Proof. If (ℵ) holds, then we can use 10.2 and 10.3 to derive a contradiction.

Suppose that (ב) holds.

Let $\langle B_\xi \mid \xi < \kappa_1^+\rangle$ be an enumeration of all subsets of κ_1 in M_W. Remember that $j''\kappa^+$ is unbounded in κ_1^+ of M which is the same as κ_1^+ of M_W, by Lemma 10.1 and (ב). So, there

will be $S \subseteq \kappa^{++}, |S| = \kappa^{++}$ and $\eta < \kappa_1^+$ such that every $A^1_\alpha, \alpha \in j''S$ appears below η in the enumeration $\langle B_\xi \mid \xi < \kappa_1^+ \rangle$ of subsets of κ_1 in M_W. Fix, in M_W, a function $e_\eta : \kappa_1 \leftrightarrow \eta_1$. Then there will be $\mu < \kappa_1$ and $S' \subseteq S$, of cardinality κ^{++}, if $\mathrm{cof}(\kappa_1) = \kappa^+$ (which usually the case), or of κ^+, otherwise such that $e_\eta{}''\mu$ includes the indexes of all $A^1_\alpha, \alpha \in j''S'$.
Now, using ($\sqsupset$), it is not hard to find $X \in M, |X| < \kappa_1$ such that $A^1_{j(\beta)} \in X$, for every $\beta \in S'$. Then, there is $Y \in M, |Y| < \kappa_1$ such that $Y \supseteq j''S'$, which is impossible. Contradiction.
Note that the family $\{B_\xi \mid \xi \in e_\eta{}''\mu\}$ will be a cover in M_W of $\{A^1_\alpha \mid \alpha \in j''S'\}$. This witnesses $\diamondsuit^-_{par}(W)$, which is impossible by ($\beth$). Contradiction.
By [3], the Galvin property implies $\neg\diamondsuit^-_{par}(W)$. So we are done.
□

11 Construction - gaps above 2

We obtain gaps larger than 2 by using supercompactness, as was done in Section 8. The key difference in this construction is that $k \restriction \kappa_1 = id$, where k, as in Sections 8, 9 is the connecting embedding from the normal ultrapower.

Let us deal with a gap 3. The general case uses the same idea.

Assume GCH and let κ be a $\kappa^{+3}-$supercompact cardinal. Fix a normal ultrafilter U over $\mathcal{P}_\kappa(\kappa^{+3})$ witnessing this. Let $j_U : V \to M_U \simeq V^{\mathcal{P}_\kappa(\kappa^{+3})}/U$ be the correspondent elementary embedding. Denote j_U by j and M_U by M. Let $\kappa_1 = j(\kappa)$.

The GCH and the closure of M under $\kappa^{+3}-$sequences of its elements imply the following:

1. for every $k \leq 4, (\kappa^{+k})^M = \kappa^{+k}$,
2. $|j(\kappa)| = \kappa^{+4}$,
3. $\mathrm{cof}(j(\kappa)) = \kappa^{+4}$,
4. $j(\kappa^{+4}) = \bigcup j''\kappa^{+4}$,
5. $j(\kappa^{+5}) = \kappa^{+5}$,
6. for every $1 \leq k \leq 4, \mathrm{cof}(j(\kappa^{+k})) = \kappa^{+4}$.

Define an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle,$$

as in the gap 2, only replace $Cohen(\beta, \beta^{++})$ by $Cohen(\beta, \beta^{+3})$ and $Col(\beta, \beta^+)$ by $Col(\beta^+, \beta^{+3})$.
Let G be generic subset of $P_{\kappa+1}$. Then, in $V[G]$, $2^\kappa = \kappa^{+3}$. Using the closure of M under $\kappa^{+3}-$sequences of its elements we construct $G^* \subseteq P_{j(\kappa)+1}$ which $M-$generic. Then j extends to $j^* : V[G] \to M[G^*]$ and U to U^*.
Now we proceed as in the gap 2 construction and use $U \times U$. Finally, over κ_1 in $M_{U\times U}[G^*]$, $Col(\kappa_1^+, \kappa_1^{+3})$ is used. Note that we have a generic for it in $V[G]$, by items (1)-(6).
The rest is as in the gap 2 construction.

12 Construction gaps above 2 from optimal assumptions

We would like here to show that the limitations of Section 10. Ideas of [4] which alow to create kind of many Cohen functions from a few will be used.

The following is a typical theorem under this lines:

Theorem 12.1 *Assume GCH. Let κ be a 3-strong cardinal. Then in a generic extension which satisfies $2^\kappa = \kappa^{+3}$ there is a $\kappa-$complete ultrafilter W over κ which includes Cub_κ such that $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$.*

Proof. Let E be a $(\kappa, \kappa^{+3})-$extender. Let $j_1 = j_E : V \to M_1 = M_2$. Consider also the second ultrapower, i.e. $k = j_{j(E)} : M_1 \to M_2$. Also, let $j_2 = j_{E\times E} : V \to M_2$ be the ultrapower by $E \times E$. Denote $j_1(\kappa)$ by κ_1 and $j_2(\kappa)$ by κ_2.

Force with the Cohen forcing $Cohen(\nu, \nu^{+3})$, for every $\nu \leq \kappa$. Let G be a generic.
Then in $V[G]$, let j_1 extends to an embedding $j_1^* : V[G] \to M_1[G^*]$.
Set $U^* = \{X \subseteq \kappa \mid \kappa \in j_1^*(X)\}$. Let $j_2^* : V[G] \to M_2[G^{**}]$ be the extension of j_2 and $k^* : M_1[G^*] \to M_2[G^{**}]$ of k which correspond to $U^* \times U^*$.

Now we would like to extend j_2 and k differently.
Let g_1 denotes $G^* \cap Cohen(\kappa_1, \kappa_1)$.
Use [4], namely—2.18, the end of page 8, and page 9, and reorganize g_1 in order to generate $g^* \subseteq Cohen(\kappa_1, \kappa_1^{+3})^{M_1[G^*\restriction\kappa_1]}$ which is $M_1[G^* \restriction \kappa_1]-$generic. Then $\mathcal{P}(\kappa_1)^{M_1[G^*\restriction\kappa_1, g^*]} \subseteq \mathcal{P}(\kappa_1)^{M_1[G^*\restriction\kappa_1, g_1]}$.
Extend now k to some $k' : M_1[G^* \restriction \kappa_1, g^*] \to M_2[G^{**} \restriction \kappa_2, g^{**}]$.
Then,

$$\{A^2_\alpha \cap \kappa_1 \mid \alpha < (\kappa_2^{+3})^{M_2}\} \subseteq \{B_\beta \mid \beta < (\kappa_1^+)^{M_2}\},$$

where $\langle A^2_\alpha \mid \alpha < (\kappa_2^{+3})^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of subsets of κ_2 and $\langle B_\beta \mid \beta < (\kappa_1^+)^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of subsets of κ_1.
Now, since $\mathrm{cof}(\kappa_1) = \mathrm{cof}((\kappa_1^+)^{M_2} = \kappa^+$ and $\mathrm{cof}((\kappa_2^{++})^{M_2} = \mathrm{cof}((\kappa_1^{++})^{M_1} = \kappa^{++}$,
there will be $S \subseteq \kappa^{++}, |S| = \kappa^{++}$, $I \in M_2$ and $\eta < \kappa_1$, such that
$M_2 \models |I| = \eta$ and $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S\} \subseteq \{B_\beta \mid \beta \in I\}$.
We are exactly in a situation that was excluded in Section 10.
□

A similar construction, based on $(\kappa, \kappa^{++})-$extender, shows that the assumption about preservation of $(\kappa_1^+)^M$ made in Section 10 is possible to realize.

Theorem 12.2 *Assume GCH. Let κ be a 2-strong cardinal. Then in a generic extension which satisfies $2^\kappa = \kappa^{++}$ there is a $\kappa-$complete ultrafilter W over κ which includes Cub_κ such that $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$ and $(j_U(\kappa)^+)^{M_U}$ is preserved, where as usual, $U = \{X \subseteq \kappa \mid \kappa \in j_W(X)\}$.*

Let us deal now with the strength of the principle $\diamondsuit^-_{par}(W)$ of T. Benhamou and G. Goldberg ([3]), which was already considered in Section 10. The results above show that it consistency strength fall down to 2-strong, but let us push it further down to a measurable. The point is that there is no need here to blow up the power of κ.

Theorem 12.3 *Assume GCH. Let κ be a measurable cardinal. Then in a generic extension $\diamondsuit^-_{par}(W)$ holds for a $\kappa-$complete ultrafilter W over κ which includes Cub_κ.*

Proof. Fix a normal ultrafilter U over κ. Let $j_1 = j_U : V \to M_1 = M_2$. Consider also the second ultrapower, i.e. $k = j_{j(U)} : M_1 \to M_2$. Also, let $j_2 = j_{U\times U} : V \to M_2$ be the ultrapower by $U \times U$. Denote $j_1(\kappa)$ by κ_1 and $j_2(\kappa)$ by κ_2.

Force with the Cohen forcing $Cohen(\nu, \nu^+)$, for every $\nu \leq \kappa$. Let G be a generic.
Then in $V[G]$, let j_1 extends to an embedding $j_1^* : V[G] \to M_1[G^*]$.
Set $U^* = \{X \subseteq \kappa \mid \kappa \in j_1^*(X)\}$. Let $j_2^* : V[G] \to M_2[G^{**}]$ be the extension of j_2 and $k^* : M_1[G^*] \to M_2[G^{**}]$ of k which correspond to $U^* \times U^*$.

Now we would like to extend j_2 and k differently.
Let g_1 denotes $G^* \cap Cohen(\kappa_1, \kappa_1^+)$. Use [4], namely - 2.18, the end of page 8, and page 9, and reorganize $g_1 \restriction \kappa_1$ in order to generate $g^* \subseteq Cohen(\kappa_1, \kappa_1^{++})^{M_1[G^* \restriction \kappa_1]}$ which is $M_1[G^* \restriction \kappa_1]-$generic. Then $\mathcal{P}(\kappa_1)^{M_1[G^* \restriction \kappa_1, g^*]} \subseteq \mathcal{P}(\kappa_1)^{M_1[G^* \restriction \kappa_1, g_1]}$.
Extend now k to some $k' : M_1[G^* \restriction \kappa_1, g^*] \to M_2[G^{**} \restriction \kappa_2, g^{**}]$, say by forcing g^{**}.

Then,

$$\{A^2_\alpha \cap \kappa_1 \mid \alpha < (\kappa_2^{++})^{M_2}\} \subseteq \{B_\beta \mid \beta < (\kappa_1^{+})^{M_2}\},$$

where $\langle A^2_\alpha \mid \alpha < (\kappa_2^{++})^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of the Cohen subsets of κ_2 produced by g^{**} and $\langle B_\beta \mid \beta < (\kappa_1^{+})^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of subsets of κ_1.

Now, since $\mathrm{cof}(\kappa_1) = \mathrm{cof}((\kappa_1^+)^{M_2}) = \kappa^+$ and $\mathrm{cof}((\kappa_2^{++})^{M_2}) = \mathrm{cof}((\kappa_1^{++})^{M_1}) = \kappa^{++}$, there will be $S \subseteq \kappa^{++}, |S| = \kappa^{++}$, $I \in M_2$ and $\eta < \kappa_1$, such that $M_2 \models |I| = \eta$ and $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S\} \subseteq \{B_\beta \mid \beta \in I\}$.

We may assume without loss of generality that η is an inaccessible cardinal in M_2. Just enlarging I inside M_2 if necessary. Then, by further enlargement inside M_2, we can assume that for every $X \subseteq \kappa$, for every $i \in I$, the set $(B_i \setminus \kappa) \cup X$ appears in $\{B_\beta \mid \beta \in I\}$.

Now pick some $S' \subseteq S, |S'| = \kappa^+$. Still, clearly, $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S'\} \subseteq \{B_\beta \mid \beta \in I\}$. We have that $j_2{}''\kappa^{++}$ is unbounded in $(\kappa_2^{++})^{M_2}$, so there is some $\delta, \kappa^+ \leq \delta < \kappa^{++}$ such that $\sup(j_2{}''S') < j_2(\delta)$.

So, instead of adding $(\kappa_1^{++})^{M_1}-$ many Cohen functions over M_1 we can add and use only $j_1(\delta)-$many.

Clearly, the forcing $Cohen(\kappa, \kappa^+)$ is equivalent to $Cohen(\kappa, \delta)$. Hence, all the embeddings j_1, j_2, k extend now, but the property $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S'\} \subseteq \{B_\beta \mid \beta \in I\}$ remained valid. Note that extending of the embeddings may require changes below κ in A^2_α's in order to have a master condition. However, the sequence $\{B_\beta \mid \beta \in I\}$ was altered to absorb such changes. Hence we will have $\diamondsuit^-_{par}(W)$ in $V[G]$.

□

T. Benhamou and G. Goldberg showed in [3] that if $\diamondsuit^-_{par}(W)$ holds, then W does not have the Galvin property. They asked whether the opposite direction holds as well. Let us point out that this is not true. Namely, in the model of [2], with a failure of the Galvin property, also $\diamondsuit^-_{par}(W)$ fails.

Theorem 12.4 *It is consistent (from a measurable cardinal) to a have a non-Galvin ultrafilter W such that $\diamondsuit^-_{par}(W)$ fails.*

Proof. Let us briefly sketch the construction used in 2.6 of [2].

We start a measurable cardinal κ and assume GCH. Then $Cohen(\nu, \nu^+)$ is iterated with the Easton support through all inaccessibles $\nu \leq \kappa$. Denote by $P_\kappa * Cohen(\kappa, \kappa^+)$ such iteration. Let $G_\kappa \subseteq P_\kappa$ be a generic and let $\langle f_{\kappa\alpha} \mid \alpha < \kappa^+\rangle$ be the Cohen functions added at κ over $V[G_\kappa]$.

Let U be a normal ultrafilter over κ in V. Let $j_1 : V \to M_1$ and $j_2 : V \to M_2$ be its first and second ultrapowers.
Then, j_1 extends in the usual fashion to $j_1^* : V[G_\kappa, \langle f_{\kappa\alpha} \mid \alpha < \kappa^+\rangle] \to M_1[G_{\kappa_1}, \langle f_{\kappa_1\alpha} \mid \alpha < j_1(\kappa^+)\rangle]$. Let

$$U_1 = \{X \in \mathcal{P}(\kappa)^{V[G_\kappa, \langle f_{\kappa\alpha} | \alpha < \kappa^+\rangle]} \mid \kappa \in j_1^*(X)\}.$$

Further, j_2 extends to $j_2^* : V[G_\kappa, \langle f_{\kappa\alpha} \mid \alpha < \kappa^+\rangle] \to M_2[G_{\kappa_2}, \langle f_{\kappa_2\alpha} \mid \alpha < j_2(\kappa^+)\rangle]$ in a special way, in order to insure that

$$W = \{X \in \mathcal{P}(\kappa)^{V[G_\kappa, \langle f_{\kappa\alpha} | \alpha < \kappa^+\rangle]} \mid \kappa_1 \in j_2^*(X)\}$$

will be non-Galvin. However still, for every $\alpha < \kappa^+$,

$$f_{\kappa_2 j_2(\alpha)} \restriction \kappa_1 = f_{\kappa_1 j_1(\alpha)}$$

and

$$(V_{\kappa_1^{++}})^{M_2[G_{\kappa_2}, \langle f_{\kappa_2\alpha} | \alpha < j_2(\kappa^+)\rangle]} = (V_{\kappa_1^{++}})^{M_1[G_{\kappa_1}, \langle f_{\kappa_1\alpha} | \alpha < j_2(\kappa^+)\rangle]},$$

since the divergence from the second ultrapower by U_1 occurs only by the choice of the Cohen functions $\langle f_{\kappa_2\alpha} \mid \alpha < j_2(\kappa^+)\rangle$, which values are changed only at κ_1.
Let $\langle Y_\xi \mid \xi < \kappa^+\rangle$ be a one to one enumeration of $\mathcal{P}(\kappa)$ in $V[G_\kappa, \langle f_{\kappa\alpha} \mid \alpha < \kappa^+\rangle]$. Then, $j_1^*(\langle Y_\xi \mid \xi < \kappa^+\rangle) = \langle Y_\xi^1 \mid \xi < j_1(\kappa^+)\rangle$ will be a one to one enumeration of $\mathcal{P}(\kappa_1)$ in $M_1[G_{\kappa_1}, \langle f_{\kappa_1\alpha} \mid \alpha < j_(\kappa^+)\rangle]$.
The crucial point is that $j_1^*(\langle Y_\xi \mid \xi < \kappa^+\rangle) = \langle Y_\xi^1 \mid \xi < j_1(\kappa^+)\rangle$ will be also in $M_2[G_{\kappa_2}, \langle f_{\kappa_2\alpha} \mid \alpha < j_2(\kappa^+)\rangle]$, by the agreement of ranks.

Let us argue that $\diamondsuit^-_{par}(W)$ fails. Suppose otherwise. Let $A \in M_W = M_2[G_{\kappa_2}, \langle f_{\kappa_2\alpha} \mid \alpha < j_2(\kappa^+)\rangle]$, λ and $\{X_i \mid i < \kappa^+\}$ a family of κ^+ different subsets of κ be such that

1. $\{j_W(X_i) \cap \lambda \mid i < 2^\kappa\} \subseteq A$,
2. there is no function $f : \kappa \to \kappa$ such that $j_W(f)(|A|^{M_U}) \geq \lambda$.

Then, the only possible value for λ is κ_1, since the only generators of j_W are κ and κ_1. Let I be the set of indexes of $\{X_i \mid i < \kappa^+\}$ in the enumeration $\langle Y_\xi \mid \xi < \kappa^+\rangle$. Then I must be an unbounded subset of κ^+. So, $j_1''I$ is unbounded in $j_1(\kappa^+)$.

For every $i < \kappa^+$ there $b_i \subseteq \kappa^+$, $|b_i| \leq \kappa$ and a term $t_i \in V$ such that $X_i = t_i(\langle f_{\kappa\xi} \mid \xi \in b_i\rangle)$. We will have, for every $i < \kappa^+$,

$$j_W(X_i) \cap \kappa_1 = j_W(t_i(\langle f_{\kappa\xi} \mid \xi \in b_i\rangle)) \cap \kappa_1 = j_2(t_i)(\langle f_{\kappa_2\xi} \mid \xi \in j_W(b_i)\rangle) \cap \kappa_1 =$$

$$j_1(t_i)(\langle f_{\kappa_1\xi} \mid \xi \in j_1^*(b_i)\rangle).$$

However, we have here $|A|^{M_W} < \kappa_1$, by the second requirement of $\diamondsuit^-_{par}(W)$. So, there is $\eta < \kappa^+$ such that

$$A \subseteq \{Y^1_\xi \mid \xi < j_1(\eta)\}.$$

This is impossible, since $j_1{}''I$ is unbounded in $j_1(\kappa^+)$ as was shown above.

$\square$

Let us deal with two other similar principles which were introduced by T. Benhamou and G. Goldberg in [3].

Definition 12.5 (T. Benhamou and G. Goldberg) κ is called *non-Galvin cardinal* if there are elementary embeddings $j : V \to M, i : V \to N, k : N \to M$ such that

1. $k \circ i = j$,
2. $\text{crit}(j) = \kappa, \text{crit}(k) = i(\kappa)$,
3. ${}^\kappa N \subseteq N, {}^\kappa M \subseteq M$,
4. there is $A \in M$ such that $i''\kappa^+ \subseteq A$ and $M \models |A| < i(\kappa)$.

Proposition 12.6 *Suppose that there exists a non-Galvin cardinal. Then there exists an inner model with a strong cardinal.*

Proof. Suppose that there is no inner model with a strong cardinal. Then the core model $\mathcal{K}$ exists. By Mitchell [16], Schindler [17], $i \restriction \mathcal{K} : \mathcal{K} \to \mathcal{K}^N, j \restriction \mathcal{K} : \mathcal{K} \to \mathcal{K}^M$ are iterated ultrapowers of $\mathcal{K}$. The iterations $i \restriction \mathcal{K}, j \restriction \mathcal{K}$ agree up to $i(\kappa)$, since $\text{crit}(k) = i(\kappa)$. Then, $i(\kappa^+) = (i(\kappa)^+)^{\mathcal{K}_N} = (i(\kappa)^+)^{\mathcal{K}_M}$.

By the definition of a non-Galvin cardinal, there is $A \subseteq i(\kappa^+), A \in M$ which covers $i''\kappa^+$ and $|A|^M < i(\kappa)$.
Note that $i(\kappa)$ is a measurable cardinal in N, since it is the image of κ. Hence it is at least a limit cardinal in M.
Apply the Mitchell Covering Lemma to A over $\mathcal{K}^M$. It will imply existence of measurable cardinals in the interval $(i(\kappa), (i(\kappa)^+)^{\mathcal{K}_M}]$ which is impossible.

$\square$

Another principle $\diamondsuit^-_{thin}(W)$, similar to $\diamondsuit^-_{par}(W)$, was defined in [3], Definition 3.4. Set $U = \{X \mid \kappa \in j_W(X)\}$. If λ of [3], Definition 3.4 is $j_U(\kappa)$, then κ is a non-Galvin cardinal. Hence, in the view of 12.6, this principle is rather strong as well.

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