ON TWO PROBLEMS CONCERNING TOPOLOGICAL CENTERS

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Abstract. Let Γ be an infinite discrete group and $\beta\Gamma$ its Čech-Stone compactification. Using the well known fact that a free ultrafilter on an infinite set is nonmeasurable, we show that for each element p of the remainder $\beta\Gamma \setminus \Gamma$, left multiplication $L_p : \beta\Gamma \to \beta\Gamma$ is not Borel measurable. Next assume that Γ is abelian. Let $\mathcal{D} \subset \ell^\infty(\Gamma)$ denote the subalgebra of distal functions on Γ and let $D = \Gamma^{\mathcal{D}} = |\mathcal{D}|$ denote the corresponding universal distal (right topological group) compactification of Γ. Our second result is that the topological center of D (i.e. the set of $p \in D$ for which $L_p : D \to D$ is a continuous map) is the same as the algebraic center and that for $\Gamma = \mathbb{Z}$, this center coincides with the canonical image of Γ in D.

1. Introduction

This short note is a direct outcome of the topology conference held in Castellón in the summer of 2007. I was presented during the conference with two problems relating to the topological center of certain right topological semigroups. (A compact semigroup A such that for each $p \in A$ the corresponding right multiplication $R_p : q \mapsto qp$ is continuous is called a *right topological semigroup.* The collection of elements $p \in A$ for which the corresponding left multiplication $L_p : q \mapsto pq$ is continuous is called the *topological center* of A.) The first was a question of Michael Megrelishvili: Given an infinite discrete group Γ, which are the elements of $\beta\Gamma$ for which $L_p : \beta\Gamma \to \beta\Gamma$ is a Baire class 1 map? (It is known that the topological center of $\beta\Gamma$ is exactly Γ itself, considered as a subset of $\beta\Gamma$, see e.g. [3].) The second problem is due to Mahmoud Filali: If $D = D(\Gamma)$ is the universal distal Ellis group of Γ, identify the topological center of D.

I present here a complete answer to Megrelishvili's problem, based on the well known result that a free ultrafilter on an infinite set is nonmeasurable, and an answer to Filali's problem in the case $\Gamma = \mathbb{Z}$, the group of integers.

The interested reader is referred to [1, chapter 1] and the bibliography list thereof, for more information on the abstract theory of topological dynamics, and to [3] for information concerning $\beta\Gamma$.

I thank both Megrelishvili and Filali for addressing to me these nice problems. I also thank the organizers of the Castellón meeting for the formidable effort they put into the details of the conference and for their warm hospitality.

2. On the center of $\beta\Gamma$

Theorem 2.1. *Let Γ be an infinite discrete group and $\beta\Gamma$ its Čech-Stone compactification. For each element p of the remainder $\beta\Gamma \setminus \Gamma$, left multiplication $L_p : \beta\Gamma \to \beta\Gamma$ is not Borel measurable.*

Date: August 6, 2007.

2000 *Mathematics Subject Classification.* Primary 54H20, Secondary 22A15.

Key words and phrases. Topological center, Čech-Stone compactification, Ellis group, distal systems.

Research supported by ISF grant # 1333/07.

2000 Mathematical Subject Classification: Primary 54H20, Secondary 22A15.

Proof. Let $\mathcal{P}(\Gamma)$ denote the collection of all subsets of Γ. Let $\Omega = \{0,1\}^\Gamma$ and let $\chi : \mathcal{P}(\Gamma) \to \Omega$ denote the canonical map $\chi(A) = \mathbf{1}_A$ for $A \subset \Gamma$. We regard Ω as a compact space and let $\mathcal{B}$ denote its Borel σ-algebra. Let $\mu = (\frac{1}{2}(\delta_0 + \delta_1))^\Gamma$ denote the product probability measure on Ω and let $\mathcal{B}_\mu$ denote the completion of $\mathcal{B}$ with respect to μ.

A well known and easy fact, which for completeness we will reproduce below (Lemma 2.3), is that a free ultrafilter on an infinite set is nonmeasurable: Viewing an element $p \in \beta\Gamma \setminus \Gamma$ as an ultrafilter on Γ, the collection $\{\chi(A) : A \in p\} \subset \Omega$, is not μ-measurable; i.e. not an element of $\mathcal{B}_\mu$. In particular it is not a Borel subset of Ω. (In fact, it is not even Baire measurable, [4] and [5].)

The compact space Ω becomes a dynamical system when we let Γ act on it by permuting the coordinates:

$$\gamma\omega(\gamma') = \omega(\gamma^{-1}\gamma').$$

Of course the measure μ is Γ-invariant, but we will not need this fact. The action of Γ on Ω extends to an action of $\beta\Gamma$ in the natural way and we write $p\omega$ for the image of $\omega \in \Omega$ under $p \in \beta\Gamma$. (In fact via this "action" $\beta\Gamma$ is identified with the enveloping semigroup of the system (Ω, Γ), see [1, chapter 1].)

For $A \subset \Gamma$ and $p \in \beta\Gamma$ set

$$p \star A = \{\gamma \in \Gamma : \gamma A^{-1} \in p\}$$

and check that $\gamma\mathbf{1}_A = \mathbf{1}_{\gamma A}$ and $p\chi(A) = p\mathbf{1}_A = \mathbf{1}_{p\star A} = \chi(p \star A)$. Moreover if $q \in \beta\Gamma$ then

$$pq \star A = p \star (q \star A).$$

For convenience I sometimes identify a subset $A \subset \Gamma$ with the corresponding element $\mathbf{1}_A = \chi(A)$ in Ω.

Let $\pi_e : \Omega \to \{0,1\}$ denote the projection on the e-component of Ω. Here e is the neutral element of Γ. Let $\omega_0 = \mathbf{1}_D$ be some fixed element of Ω whose Γ orbit is dense in Ω. Let $\psi : \gamma \mapsto \gamma\omega_0$ be the orbit map and let $\hat{\psi}$ denote its unique extension to $\beta\Gamma$. Thus $\hat{\psi}(q) = q\omega_0$ for $q \in \beta\Gamma$. Finally recall that the semigroup product on $\beta\Gamma$ is defined by

$$A \in pq \iff \{\gamma \in \Gamma : \gamma^{-1}A \in q\} \in p.$$

Consider the map $L_p : \beta\Gamma \to \beta\Gamma$ and write $\phi : \beta\Gamma \to \{0,1\}$ for the map $\phi = \pi_0 \circ \hat{\psi} \circ L_p$. (Thus $\phi(q) = (pq\omega_0)(e)$.) We define $J : \Omega \to \Omega$ by $(J(\omega))(\gamma) = \omega(\gamma^{-1})$

We have

$$\mathcal{Q} := \phi^{-1}(1) = \{q \in \beta\Gamma : pq\omega_0(e) = 1\}.$$

Now $pq\omega_0(e) = 1$ iff $e \in \chi^{-1}(pq\omega_0)$ hence

$$\begin{aligned}
\mathcal{Q} &= \{q \in \beta\Gamma : pq\omega_0(e) = 1\} \\
&= \{q \in \beta\Gamma : e \in p \star q\omega_0\} \\
&= \{q \in \beta\Gamma : e \in p \star q \star D\} \\
&= \{q \in \beta\Gamma : e \in \{\gamma \in \Gamma : \gamma(q \star D)^{-1} \in p\}\} \\
&= \{q \in \beta\Gamma : (q \star D)^{-1} \in p\} \\
&= \{q \in \beta\Gamma : J(q\omega_0) \in p\}.
\end{aligned}$$

Thus $(J \circ \hat{\psi})(\mathcal{Q}) = p$ and since also $(\hat{\psi}^{-1} \circ J^{-1})(p) = \mathcal{Q}$ we conclude that $\mathcal{Q} = \phi^{-1}(1)$ is not Borel measurable in $\beta\Gamma$. Finally, since also $\mathcal{Q} = L_p^{-1}(\{q \in \beta\Gamma : (q\omega_0)(e) = 1\})$ we see that L_p is not Borel measurable. ∎

In the next two lemmas let $\Omega = \{0,1\}^{\mathbb{N}}$. As above, we identify subsets A of $\mathbb{N}$ with their characteristic functions $\mathbf{1}_A \in \Omega = \{0,1\}^{\mathbb{N}}$ and, accordingly, filters on $\mathbb{N}$ with subsets of Ω.

Let $\phi : \Omega \to \Omega$ denote the "flip" function defined by $\phi(\omega)_n = 1 - \omega_n$. We consider the measure space $(\Omega, \Sigma_\lambda, \lambda)$, where $\Omega = \{0,1\}^{\mathbb{N}}$, λ is the Bernoulli measure $\lambda = (\frac{1}{2}(\delta_0 + \delta_1))^{\mathbb{N}}$, and Σ_λ denotes the completion of the Borel σ-algebra with respect to λ. As usual we use the notation λ_* and λ^* for the induced inner and outer measures.

The assertions of the next lemma are easily verified.

Lemma 2.2. 1. *The involution ϕ is measurable and it preserves λ.*
2. *For $A \subset \mathbb{N}$ we have $\phi(\mathbf{1}_A) = \mathbf{1}_{A^c}$.*
3. *If $\mathcal{F}$ is a filter on $\mathbb{N}$ then $\phi\mathcal{F} \cap \mathcal{F} = \emptyset$.*
4. *If $\mathcal{F}$ is a free filter on $\mathbb{N}$ (i.e. $\bigcap \mathcal{F} = \emptyset$) then, considered as a collection of subsets of $\{0,1\}^{\mathbb{N}}$ it is a "tail event", that is, for every $m \in \mathbb{N}$, $\mathcal{F} = \{0,1\}^m \times \mathcal{F}'$, with $\mathcal{F}' \subset \{0,1\}^{\mathbb{N}}$.*
5. *A filter $\mathcal{F}$ on $\mathbb{N}$ is an ultrafilter iff $\phi(\mathcal{F}) \cup \mathcal{F} = \Omega$.*

Lemma 2.3. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Then*
1. *$\lambda_*(\mathcal{F}) = 0$.*
2. *$\lambda^*(\mathcal{F}) \in \{0,1\}$.*
3. *$\lambda^*(\mathcal{F}) = 1$ if $\mathcal{F}$ is an ultrafilter.*
4. *A free filter $\mathcal{F}$ is measurable iff $\lambda^*(\mathcal{F}) = 0$, and nonmeasurable iff $\lambda^*(\mathcal{F}) = 1$. In particular, every free ultrafilter is nonmeasurable.*

Proof. If $\mathcal{F}$ is a free filter on $\mathbb{N}$ and $\mathcal{G} \subset \mathcal{F}$ is a measurable tail event then it has measure either 0 or 1. Thus $\lambda^*(\mathcal{F}) \in \{0,1\}$. This proves part 2. We also have $\lambda_*(\mathcal{F}) \in \{0,1\}$ and since $\phi(\mathcal{F}) \cap \mathcal{F} = \emptyset$ it follows that

$$1 = \lambda(\Omega) \geq \lambda_*(\phi\mathcal{F}) + \lambda_*(\mathcal{F}) = 2\lambda_*(\mathcal{F}).$$

We conclude that $\lambda_*(\mathcal{F}) = 0$, proving part 1. If $\mathcal{F}$ is an ultrafilter then $\mathcal{F} \cup \phi\mathcal{F} = \{0,1\}^{\mathbb{N}}$ and we conclude that

$$1 = \lambda(\Omega) \leq \lambda^*(\phi\mathcal{F}) + \lambda^*(\mathcal{F}) = 2\lambda^*(\mathcal{F}),$$

whence $\lambda^*(\mathcal{F}) = 1$. This proves part 3. Part 4 is now clear. □

3. On the center of $\Gamma^{\mathcal{D}}$, the universal distal Ellis group of Γ

Let Γ be a discrete abelian group. Let $\mathcal{D}$ denote the closed Γ-invariant subalgebra of (complex valued) distal functions in $\ell^\infty(\Gamma)$. Let $D = \Gamma^{\mathcal{D}} = |\mathcal{D}|$ denote the corresponding Gelfand space. It is well known that D is the largest right topological group compactification of Γ.

Theorem 3.1. *Let Γ be an infinite discrete abelian group. The topological center of $D = \Gamma^{\mathcal{D}}$ is the same as the algebraic center and, when $\Gamma = \mathbb{Z}$, it also coincides with the canonical image of Γ in D.*

Proof. In order to simplify our notation we will identify elements of Γ with their images in D. The coincidence of the topological and algebraic centers of D is easy: Suppose first that $p \in D$ is in the algebraic center of this group. Then, as right multiplication is always continuous, we have for any convergent net $q_\alpha \to q$ in D

$$pq = qp = \lim q_\alpha p = \lim pq_\alpha,$$

i.e. $L_p : D \to D$ is continuous.

Conversely, assume that p is in the topological center; i.e. $L_p : D \to D$ is continuous. We note that if q is an element of D then $\gamma q = q\gamma$ for every $\gamma \in G$. In fact choosing a convergent net $\Gamma \ni \gamma_\alpha \to q$, by the commutativity of Γ,

$$\gamma q = \gamma \lim \gamma_\alpha = \lim \gamma\gamma_\alpha = \lim \gamma_\alpha \gamma = q\gamma.$$

Now, with this in mind, we have

$$pq = p \lim \gamma_\alpha = \lim p\gamma_\alpha = \lim \gamma_\alpha p = qp,$$

so that p is indeed an element of the center.

Now to the more delicate task of showing that this center coincides with Γ. Let $p \in D$ be a central element. If $p \notin \Gamma$ then there exists a *metric* minimal distal dynamical system (Y, Γ) and a point $y_0 \in Y$ such that

$$py_0 \notin \Gamma y_0. \tag{3.1}$$

By assumption the map $L_p : D \to D$ is continuous (in fact a homeomorphism) and as we have seen it also commutes with every element of Γ. In other words, L_p is an automorphism of the system (D, Γ). Now the dynamical system (D, Γ) is the universal distal system and therefore, it admits a unique homomorphism of dynamical systems $\hat{\phi} : (D, \Gamma) \to (E(Y, \Gamma), \Gamma)$ onto the enveloping semigroup $E = E(Y, \Gamma)$ (which by a theorem of Ellis is in fact a group) such that $\hat{\phi}(e_D) = e_E$. Now the map $\phi : p \mapsto \hat{\phi}(p)y_0$ (which we write simply as $p \mapsto py_0$) is a homomorphism $\phi : (D, \Gamma) \to (Y, \Gamma)$ with $\phi(e) = y_0$. If $y_\alpha \to y$ is a convergent net in Y then there are $q_\alpha \in D$ with $y_\alpha = q_\alpha y_0$. With no loss of generality we have $q_\alpha \to q$ in D, so that in particular $y = \lim y_\alpha = \lim q_\alpha y_0 = qy_0$. Now we see that

$$\begin{aligned} py &= pqy_0 = (p \lim q_\alpha)y_0 \\ &= (\lim pq_\alpha)y_0 = \lim p(q_\alpha y_0) \\ &= \lim py_\alpha. \end{aligned}$$

Thus p acts continuously on Y. Since also $p\gamma = \gamma p$ for every $\gamma \in \Gamma$ we conclude that p is an automorphism of the system (Y, Γ).

Note that this argument shows that *p acts as an automorphism of every factor of* (D, Γ). Therefore, our proof will be complete when we find a minimal distal dynamical system (X, Γ) extending (Y, Γ), say $\pi : (X, \Gamma) \to (Y, \Gamma)$, where p is not an automorphism.

At this stage, in order to be able to use a method of construction developed by Glasner and Weiss in [2], we specialize to the case $\Gamma = \mathbb{Z}$. In particular the system (Y, Γ) which was singled out in the above discussion has now the form (Y, T) where $T : Y \to Y$ is a self homeomorphism of Y determined by the element $1 \in \mathbb{Z}$. Of course we can assume that Y is non-periodic (i.e. infinite).

The following construction is a special case of a general setup designed in [2] for providing minimal extensions of a given non-periodic minimal $\mathbb{Z}$-system (Y, T). We refer the reader to [2] for more details.

Set $X = Y \times K$ where K denotes the circle group $K = S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let Θ be the family of continuous maps $\theta : Y \to K$. For each $\theta \in \Theta$ let $G_\theta : X \to X$ be the map $G_\theta(y, z) = (y, z\theta(y))$ and $S_\theta = G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta$. Thus

$$S_\theta : X \to X, \qquad S_\theta(y, z) = (Ty, z\theta(y)\theta(Ty)^{-1}). \tag{3.2}$$

Form the collection

$$\mathcal{S}(T) = \{G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta : \theta \in \Theta\}.$$

Theorem 1 of [2] ensures that in the set $\overline{\mathcal{S}(T)}$ (closure with respect to the uniform convergence topology in Homeo (X)) there is a dense G_δ subset $\mathcal{R}$ such that for every $R \in \mathcal{R}$

the system (X, R) is minimal, distal, and the projection map $\pi : X \to Y$ is a homomorphism of dynamical systems ($\pi R(y,z) = T\pi(y,z) = Ty$). Note that every $R \in \overline{\mathcal{S}(T)}$ has the form

$$R = T_\phi : X \to X, \quad \text{where} \quad T_\phi(y,z) = (Ty, z\phi(y)),$$

for some continuous map $\phi : Y \to K$. We will often use the fact that for $n \in \mathbb{N}$ the n-th iteration of T_ϕ has the form

$$T_\phi^n(y,z) = (T^n y, z\phi_n(y)), \quad \text{where} \quad \phi_n(y) = \phi(T^{n-1}y)\cdots\phi(Ty)\phi(y). \tag{3.3}$$

Note that when ϕ has the very special form $\phi(y) = \theta(Ty)^{-1}\theta(y)$ for some continuous $\theta : Y \to K$, the equation (3.3) collapses:

$$\phi_n(y) = \theta(T^n y)^{-1}\theta(y), \quad \text{hence} \quad S_\theta^n(y,z) = (T^n y, z\theta(T^n y)^{-1}\theta(y)). \tag{3.4}$$

We temporarily fix an element $R = T_\phi \in \mathcal{R}$. As observed above, the element $p \in D$ defines an automorphism of the system (X, T_ϕ); moreover we have for every $x = (y,z) \in X$:

$$\pi(px) = p\pi(x) = py.$$

This last observation implies that $p : X \to X$ has the form $p(y,z) = (py, \omega(y,z))$ for some continuous map $\omega : Y \times K \to K$.

Lemma 3.2. *The function ω has the form $\omega(y,z) = z\psi(y)$ for some continuous map $\psi : Y \to K$, whence*

$$p(y,z) = (py, z\psi(y))$$

Proof. There exists a net $\{n_\nu\}_{\nu \in I}$ in $\mathbb{Z}$ such that $p = \lim n_\nu$ in D. Thus, for every $(y,z) \in X$

$$p(y,z) = \lim T_\phi^{n_\nu}(y,z) = \lim(T_\phi^{n_\nu} y, z\phi_{n_\nu}(y)) = (py, z\psi(y)),$$

where the *point-wise* limit $\psi(y) := \lim \phi_{n_\nu}(y)$ is necessarily a continuous function. □

The commutation relation $pT_\phi = T_\phi p$ now reads:

$$\begin{aligned} pT_\phi(y,z) &= p(Ty, z\phi(y)) = (pTy, z\phi(y)\psi(Ty)) \\ &= T_\phi p(y,z) = T_\phi(py, z\psi(y)) \\ &= (Tpy, z\psi(y)\phi(py)). \end{aligned}$$

In turn this implies:

$$\phi(y)\psi(Ty) = \psi(y)\phi(py). \tag{3.5}$$

Similarly the commutation relations $pT_\phi^n = T_\phi^n p$ yield:

$$\phi_n(y)\psi(T^n y) = \psi(y)\phi_n(py). \tag{3.6}$$

Next consider any sequence $n_i \nearrow \infty$ such that

- $\lim T^{n_i} y_0 = y_0$,
- $\lim \phi_{n_i}(y_0) = z'$, and
- $\lim \phi_{n_i}(py_0) = z''$.

Applying (3.6) and taking the limit as $i \to \infty$ we get $\psi(y_0)z'' = z'\psi(y_0)$, whence necessarily also $z' = z''$.

The proof of Theorem 3.1 will be complete when we next show that for a residual subset $\mathcal{R}_1$ of $\overline{\mathcal{S}(T)}$, we have $\lim \phi_{n_i}(y_0) = z' \neq z'' = \lim \phi_{n_i}(py_0)$, whenever $R = T_\phi \in \mathcal{R}_1$. Then for any element $T_\phi \in \mathcal{R} \cap \mathcal{R}_1$, (X, T_ϕ) will serve as a minimal distal system where p is not an automorphism.

Proposition 3.3. *For a given sequence $n_i \nearrow \infty$ with $\lim T^{n_i} y_0 = y_0$, the set*

$$\mathcal{R}_1 = \{T_\phi \in \overline{\mathcal{S}(T)} : \ \forall i \ \exists j > i, \ |\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1\}$$

is a residual subset of $\overline{\mathcal{S}(T)}$.

Proof. For $i \in \mathbb{N}$ and $\eta > 0$ set

$$E_{i,\eta} = \{T_\phi \in \overline{\mathcal{S}(T)} : \ \exists j > i, \ |\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1 + \eta\}.$$

Clearly $E_{i,\eta}$ is an open subset of $\overline{\mathcal{S}(T)}$ and for $i < k$ we have $E_{k,\eta} \subset E_{i,\eta}$.

Lemma 3.4. *Given i and $\eta > 0$, for every $\theta_0 \in \Theta$ there exists an $i_0 > i$ such that*

$$G_{\theta_0}^{-1} E_{i_0,\eta} G_{\theta_0} \subset E_{i_0,\eta/2}.$$

Proof. Fix $\theta_0 \in \Theta$. For sufficiently large i_0, for all $j > i_0$ the distances $d(T^{n_j} y_0, y_0)$ and $d(T^{n_j} py_0, py_0)$ are so small that

$$|\theta(T^{n_j} y_0)^{-1} \theta(y_0) \phi_{n_j}(y_0) - \theta(T^{n_j} py_0)^{-1} \theta(y_0) \phi_{n_j}(py_0)| > 1 + \eta/2$$

holds whenever

$$|\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1 + \eta.$$

□

We will show that $E_{i,\eta}$ is also dense in $\overline{\mathcal{S}(T)}$. For this it suffices to show that $G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta \in \overline{E_{i,\eta}}$ for every $\theta \in \Theta$, i.e. $T \times \mathrm{id} \in G_\theta \overline{E_{i,\eta}} G_\theta^{-1}$.

Now for a fixed θ_0 there is by Lemma 3.4, an $i_0 > i$ with $G_{\theta_0}^{-1} E_{i_0,2\eta} G_{\theta_0} \subset E_{i_0,\eta}$, hence it suffices to show that $T \times \mathrm{id} \in \overline{E_{i_0,2\eta}}$, since then

$$T \times \mathrm{id} \in \overline{E_{i_0,2\eta}} \subset G_{\theta_0} \overline{E_{i_0,\eta}} G_{\theta_0}^{-1} \subset G_{\theta_0} \overline{E_{i,\eta}} G_{\theta_0}^{-1}.$$

Finally the next lemma will prove this last assertion and therefore also the density of $E_{i,\eta}$ for every i and $0 < \eta < 1$.

Lemma 3.5. *Given $i \in \mathbb{N}, 0 < \eta < 1$ and $\varepsilon > 0$ there exists $\theta \in \Theta$ such that*

1. $d(T \times \mathrm{id}, G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta) < \varepsilon$.
2. $G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta \in E_{i,\eta}$.

Proof. Let $I = [0, 1]$ and set $h(0) = h(1/3) = h(2/3) = 1, \ h(1) = -1$ and extend this function in an arbitrary way to a continuous $h : I \to S^1$. Choose $\delta > 0$ such that $|t - s| < \delta$ implies $|h(t)^{-1} h(s) - 1| < \varepsilon$. Let $m \in \mathbb{N}$ be such that $2/m < \delta$. Let U_1 and U_2 be open neighborhoods of y_0 and py_0, respectively, in Y such that for $s = 1, 2$, the sets $U_s, TU_s, \dots, T^{m-1} U_s$ are mutually disjoint. (Here we use the facts that Y is infinite and that $py_0 \notin \{T^j y_0 : j \in \mathbb{Z}\}$ (3.1).) Choose $k > i$ so that $T^{n_k} y_0 \in U_1$ and $T^{n_k} py_0 \in U_2$. Let $K_s \subset U_s, \ s = 1, 2$, be Cantor sets such that $y_0, T^{n_k} y_0 \in K_1$ and $py_0, T^{n_k} py_0 \in K_2$.

Next define:

$$g(y_0) = 0, \ g(T^{n_k} y_0) = 1/3, \ g(py_0) = 2/3, \ g(T^{n_k} py_0) = 1$$

and extend this function in an arbitrary way to a continuous function $g : K_1 \cup K_2 \to S^1$. We now extend g to the set $\cup_{j=0}^{m-1} T^j (K_1 \cup K_2)$ by setting $g(y) = g(T^{-j} y)$ for $y \in T^j (K_1 \cup K_2)$. Extend g continuously over all of Y in an arbitrary way.

Set

$$\tilde{g}(y) = \frac{1}{m} \sum_{j=0}^{m-1} g(T^j y).$$

Clearly $\tilde{g} \restriction (K_1 \cup K_2) = g \restriction (K_1 \cup K_2)$, so that

$$\tilde{g}(y_0) = 0, \ \tilde{g}(T^{n_k} y_0) = 1/3, \ \tilde{g}(py_0) = 2/3, \ \tilde{g}(T^{n_k} py_0) = 1.$$

Finally define $\theta : Y \to S^1$ by $\theta(y) = h(\tilde{g}(y))$. Note that

$$\theta(y_0) = \theta(T^{n_k} y_0) = \theta(py_0) = 1, \quad \text{and} \quad \theta(T^{n_k} py_0) = -1. \tag{3.7}$$

Now

$$G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta(y, z) = (Ty, z\theta(Ty)^{-1}\theta(y)) = (Ty, zh(\tilde{g}(Ty))^{-1}h(\tilde{g}(y))).$$

But

$$|\tilde{g}(Ty) - \tilde{g}(y)| < 2/m < \delta,$$

hence $|h(\tilde{g}(Ty))^{-1}h(\tilde{g}(y)) - 1| < \varepsilon$ and therefore also

$$d(T \times \mathrm{id}, G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta) < \varepsilon.$$

This proves part (1) of the lemma and we now proceed to prove part (2). We have to show that $G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta \in E_i$. But this map has the form

$$S_\theta : X \to X, \qquad S_\theta(y, z) = G_\theta^{-1} \circ (T \times \mathrm{id}) \circ G_\theta = (Ty, z\theta(Ty)^{-1}\theta(y)),$$

so that, by (3.4), we have to show that there exists $j > i$ with

$$|\theta(T^{n_j} y_0)^{-1}\theta(y_0) - \theta(T^{n_j} py_0)^{-1}\theta(py_0)| > 1 + \eta.$$

Since, by the choice of θ (3.7), we have

$$|\theta(T^{n_k} y_0)^{-1}\theta(y_0) - \theta(T^{n_k} py_0)^{-1}\theta(py_0)| = |1 - (-1)| = 2 > 1 + \eta,$$

this completes the proof of the lemma. □

To conclude the proof of Proposition 3.3 observe that, for instance, the dense G_δ set $\bigcap_{i=1}^\infty E_{i,1/2}$ is contained in $\mathcal{R}_1$. □

This also concludes the proof of Theorem 3.1.

□

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