Competitive Caching with Machine Learned Advice

Seminar on Online Algorithms

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May 25, 2022

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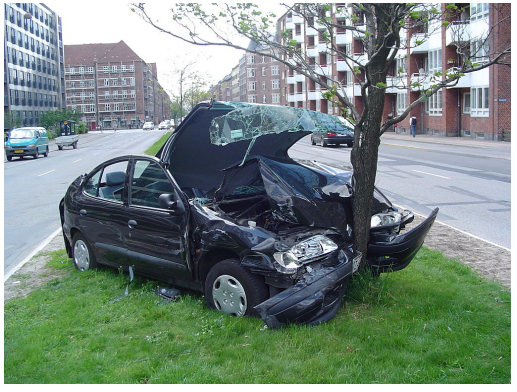

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- Yet, overly cautious

Comparison

ML Algorithms	Online Algorithms
attempt to **predict the unknown**	act without any knowledge
suceptible to **large errors**	**robust againt any input**
exploit **patterns**	overly **cautious**

The main question

What if we could combine the **predictive power** of ML with the **robustness** of online algorithms?

First example

Example: Binary Search

Textbook problem - Sorted array A of size n, and query q.
What is the query cost?

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ML Approach

- train a classifier $h(q)$ to predict $t(q)$.
- How can we use such a classifier?

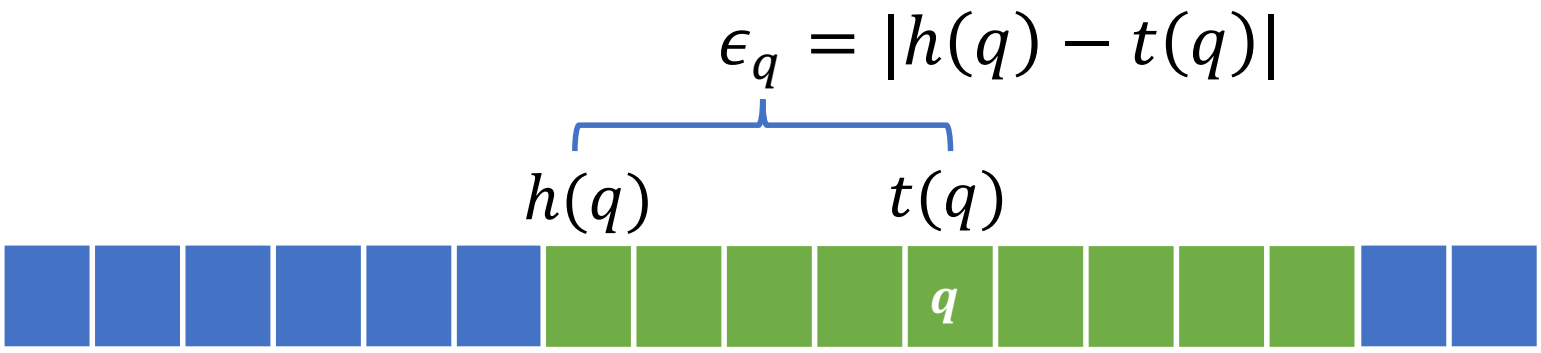

q


$t(q)$
q


$h(q)$
$t(q)$
q


$h(q)$
$t(q)$
q

$$\epsilon_q = |h(q) - t(q)|$$



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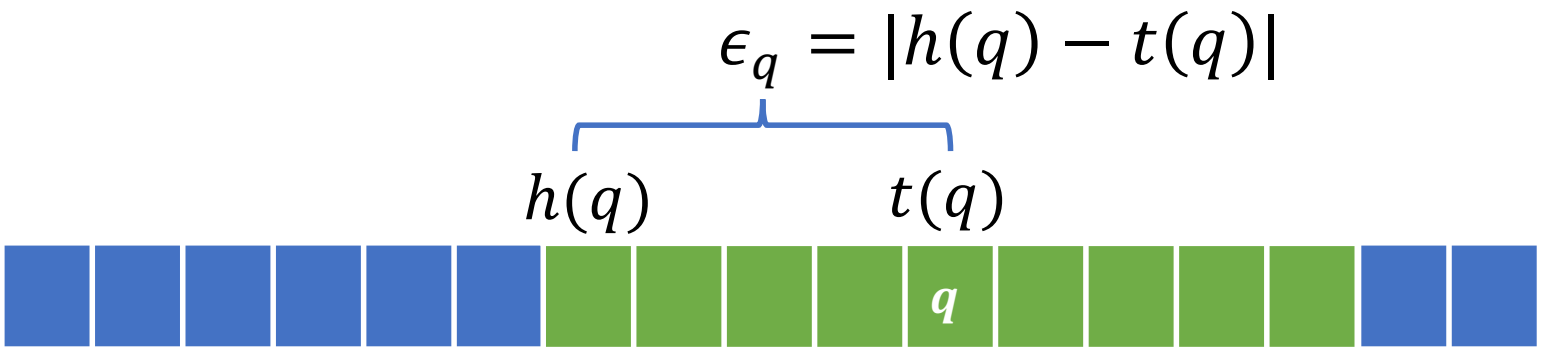

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Is this any good?

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  - In reality, obsereved competitive ratio is much lower.

The main result

The **machine-learning assisted algorithm** reaches a competitive ratio of $2 + O\left(\min\left(\sqrt{\epsilon}, \log k\right)\right)$.

Preliminaries

- To achieve our results we have to define the playing ground for a **new genere of algorithms**: competitive algorithms with machine learning advice.

Preliminaries

- **ML scenarios** consist of:
  - Feature space - $\mathcal{X}$, and labels - $\mathcal{Y}$
  - hypothesis $h : \mathcal{X} \to \mathcal{Y}$
  - loss function: $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$

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  - The loss can be: abosolute ($\ell_1(y, \hat{y}) = |y - \hat{y}|$), squared ($\ell_2(y, \hat{y}) = (y - \hat{y})^2$), or generally: $\ell_c(y, \hat{y}) = \mathbf{1}_{y \neq \hat{y}}$

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- **Online scenarios** consist of a algorithm $\mathcal{A}$ and sequences σ.
- $\mathcal{A}$ has comptitive ratio CR if for every σ:
  $cost_{\mathcal{A}}(\sigma) \leq \mathrm{CR} \cdot \mathrm{OPT}(\sigma)$

The Online ML-Assisted Framework

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- We have a **universe** $\mathcal{Z}$ and **feature** space $\mathcal{X}$
- The input is a sequence of items $\sigma = (\sigma_1, \sigma_2, \dots)$
  Each item σ_i is assosiated with an element $z_i \in \mathcal{Z}$ and with features $x_i \in \mathcal{X}$

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How can we define h to have a general accuracy of ϵ?

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How can we define h to have a general accuracy of ϵ?

Definition

We say that h is $\epsilon - \textit{accurate}$ if for every σ,
$\eta_\ell(h, \sigma) \leq \epsilon \cdot \textsc{Opt}(\sigma)$.

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Define $\mathrm{CR}_{\mathcal{A}}(\epsilon)$ to be the **competitive ratio** of algorithm $\mathcal{A}$ that uses any predictor h that is ϵ-accurate.

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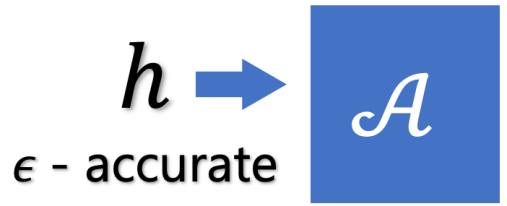

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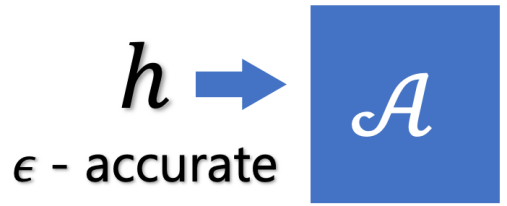

- What would we like $\mathrm{CR}_{\mathcal{A}}(0)$ to be?

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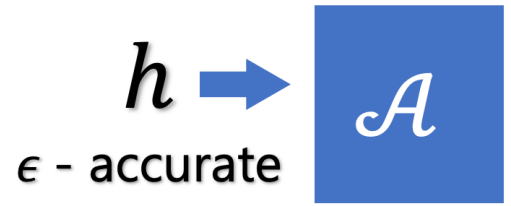

Consistency

$\mathcal{A}$ is β-consistent if $\mathrm{CR}_{\mathcal{A}}(0) = \beta$.

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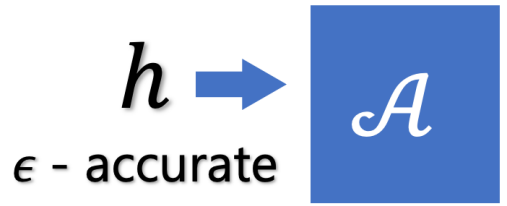

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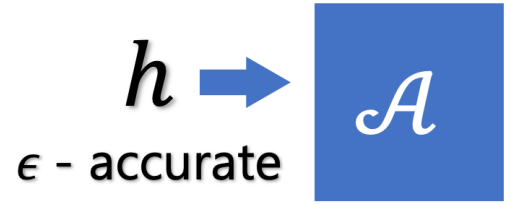

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Competitivness

$\mathcal{A}$ is γ-competitive if $\mathrm{CR}_{\mathcal{A}}(\epsilon) \leq \gamma \cdot \mathrm{OPT}$ for all ϵ.

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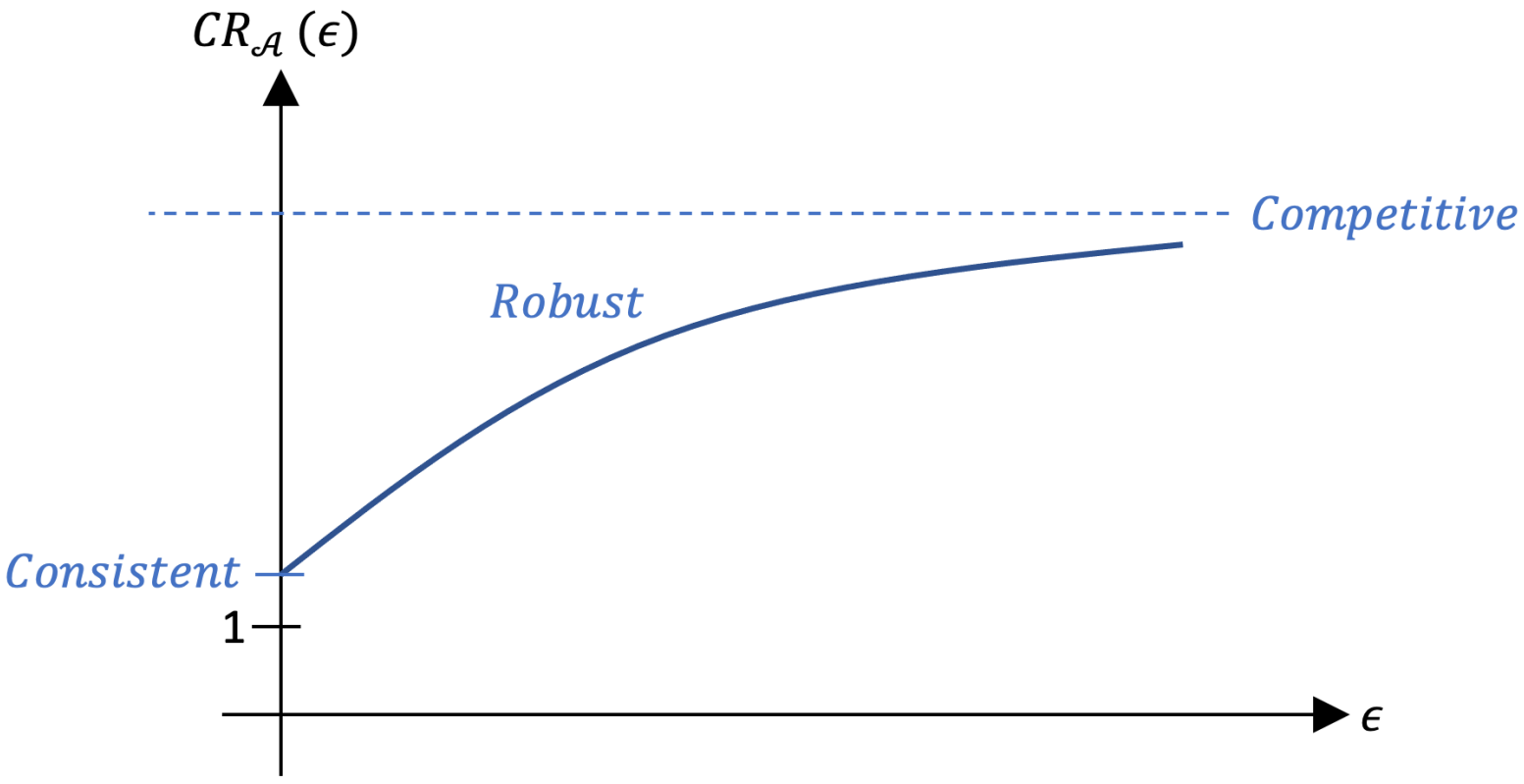

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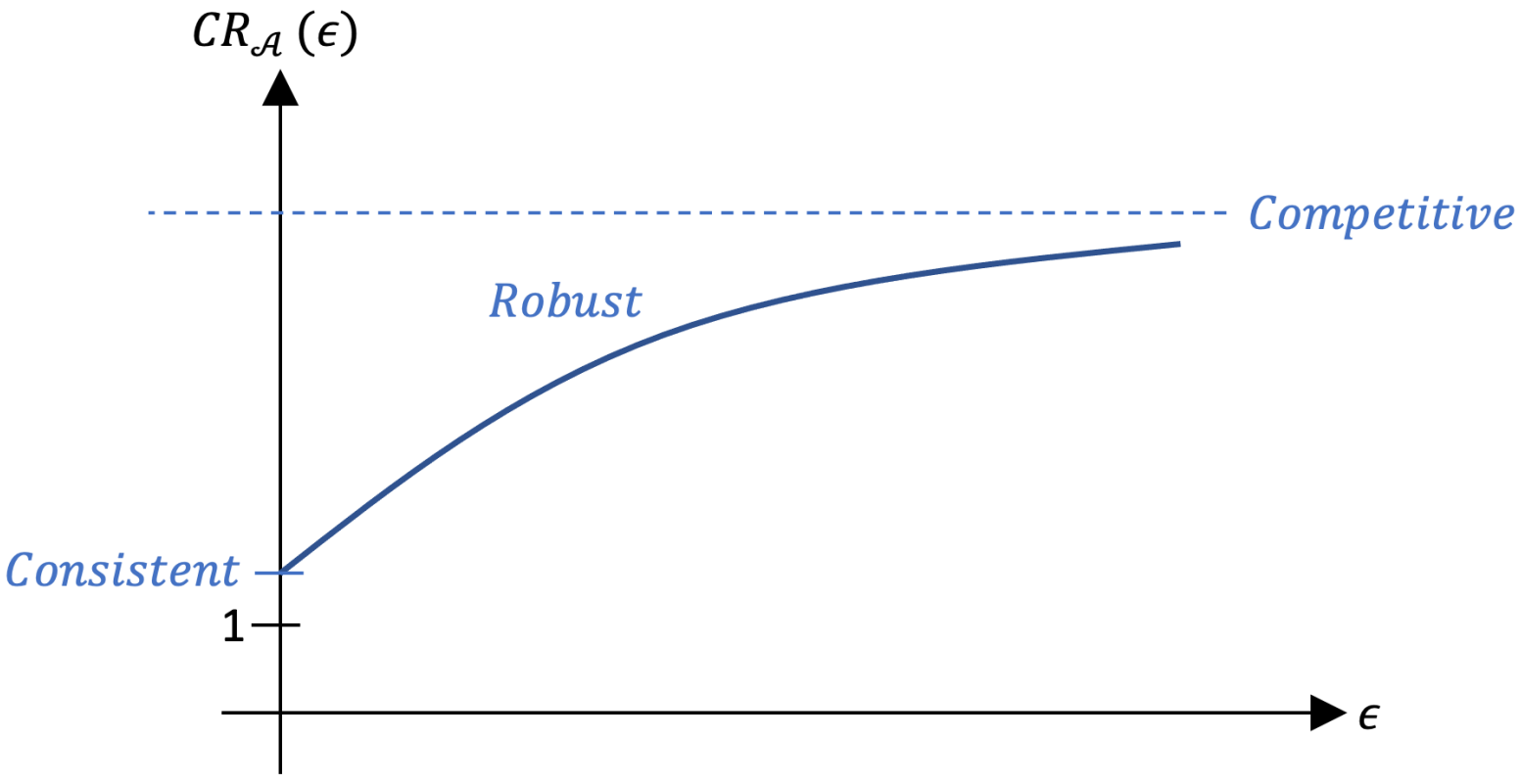

- The holy grail is an algorithm $\mathcal{A}$ which simultaneously optimizes all three properties.

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- $\mathcal{Y} = \mathbb{N}^+$

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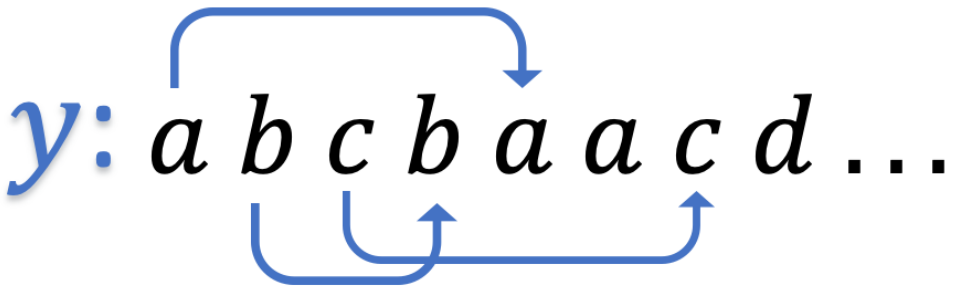

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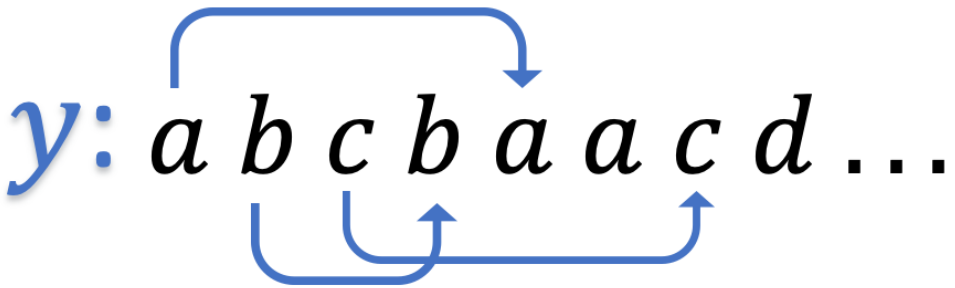

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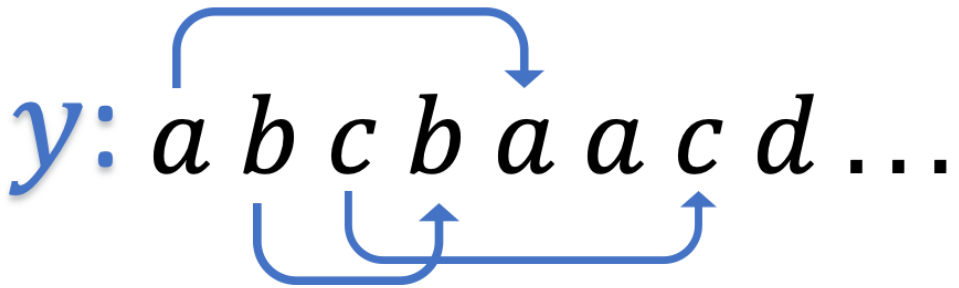

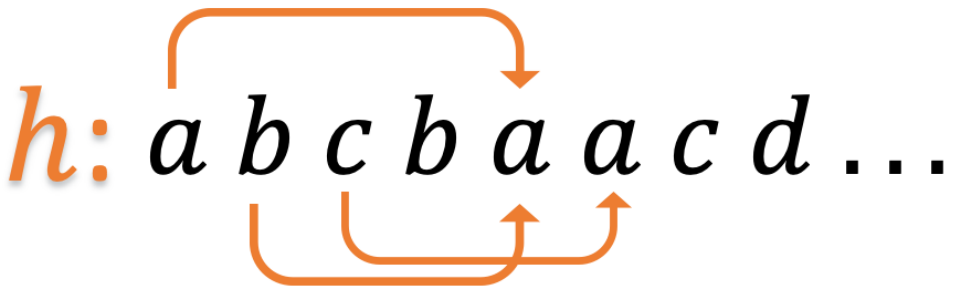

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- Example follows.

$a\ b\ c\ b\ c\ \ldots\ b\ c$

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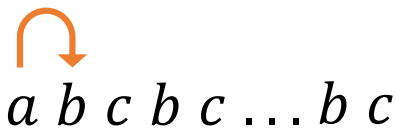

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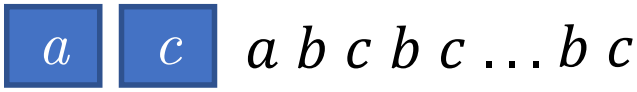

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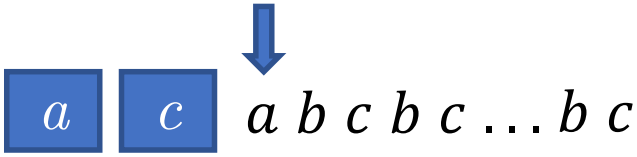

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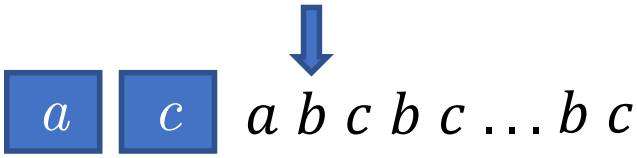

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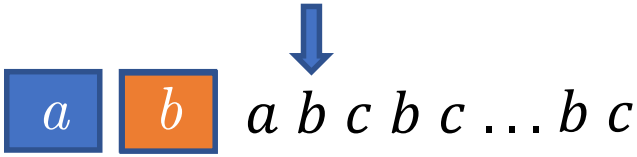

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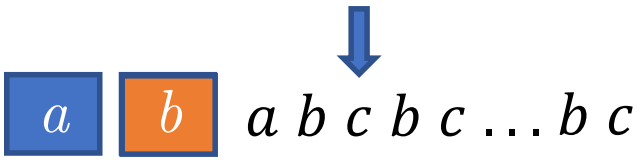

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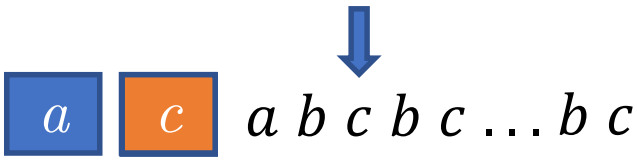

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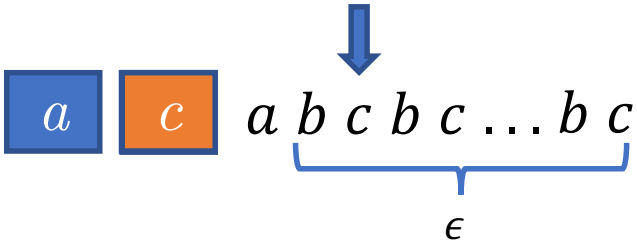

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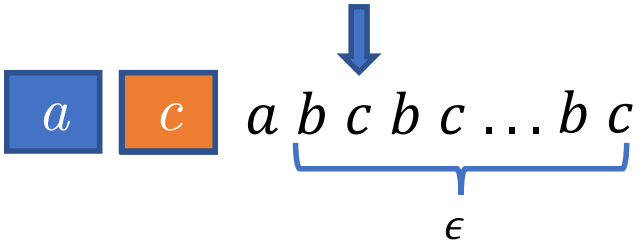

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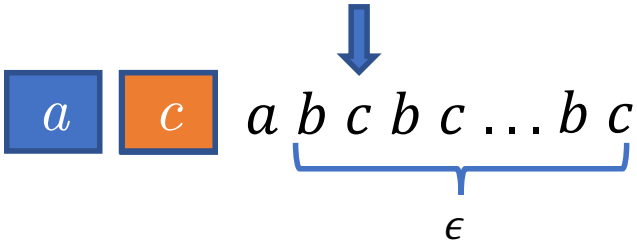

Offline Optimum: $OPT = 1$

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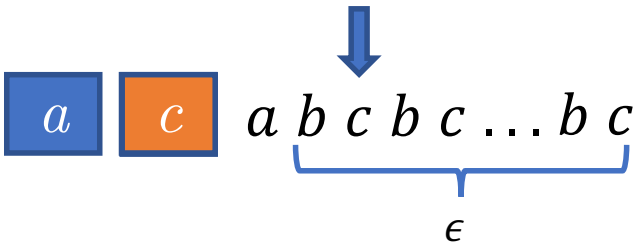

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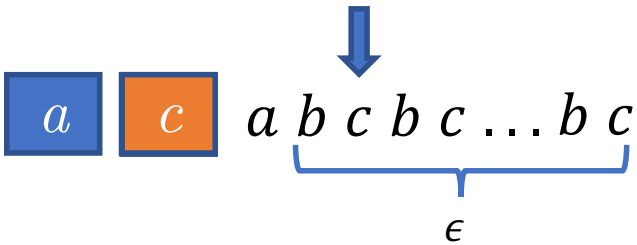

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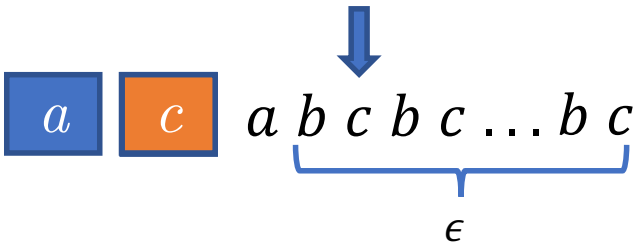

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Absolute Loss: $\eta(h, \sigma) = \varepsilon$

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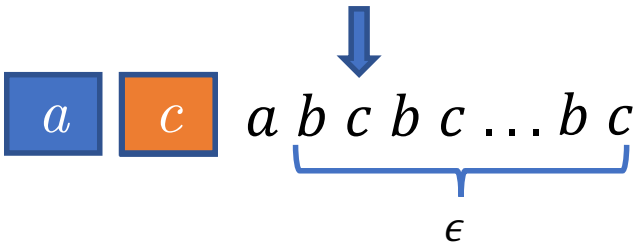

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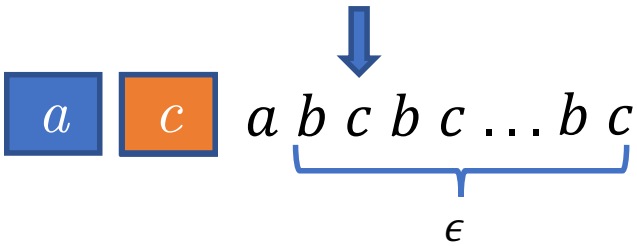
Absolute Loss: $\eta(h, \sigma) = \varepsilon \leq \epsilon \cdot OPT$

Predictor h: $true, except\ h(\sigma_1) = 2$



Offline Optimum: $OPT = 1$

Performance of $\mathcal{B}$: ϵ

Absolute Loss: $\eta(h, \sigma) = \varepsilon \leq \epsilon \cdot OPT$

Competitive ratio: $CR_{\mathcal{B}}(\epsilon) = \frac{\epsilon}{1} = \Omega(\epsilon)$

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- Example follows.

$d\ a\ b\ c\ a\ b\ c \ldots a\ b\ c$

Predictor h: $predict\ d\ correctly$
$but\ predict\ a, b, c\ two\ steps\ too\ early$

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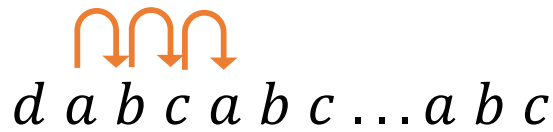

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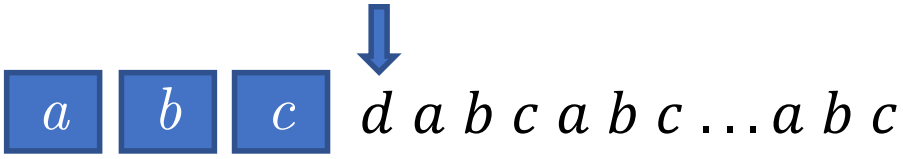

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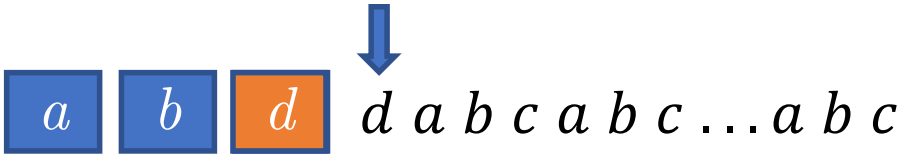

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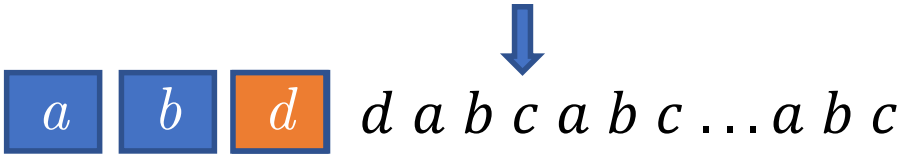

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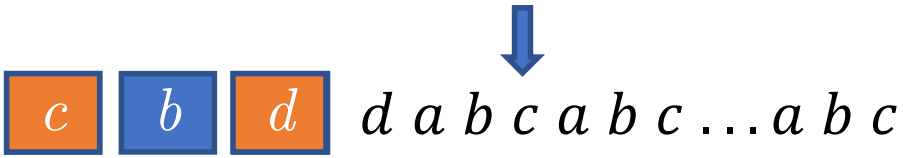

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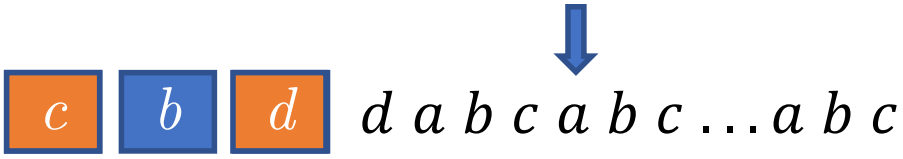

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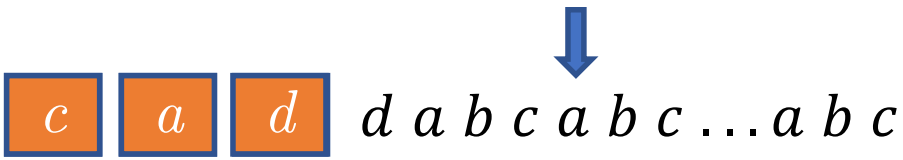

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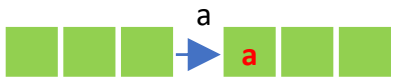
  - However, their **strict** (determisitic) policy leads to weak gaurantees.

Reminder - Classic Marking Algorithm

- Recall the classic Marking Algorithm.


a b c c d c d a b b e a b


a b c c d c d a b b e a b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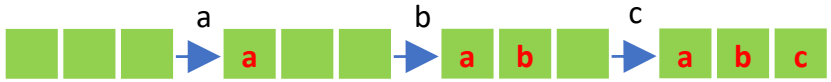
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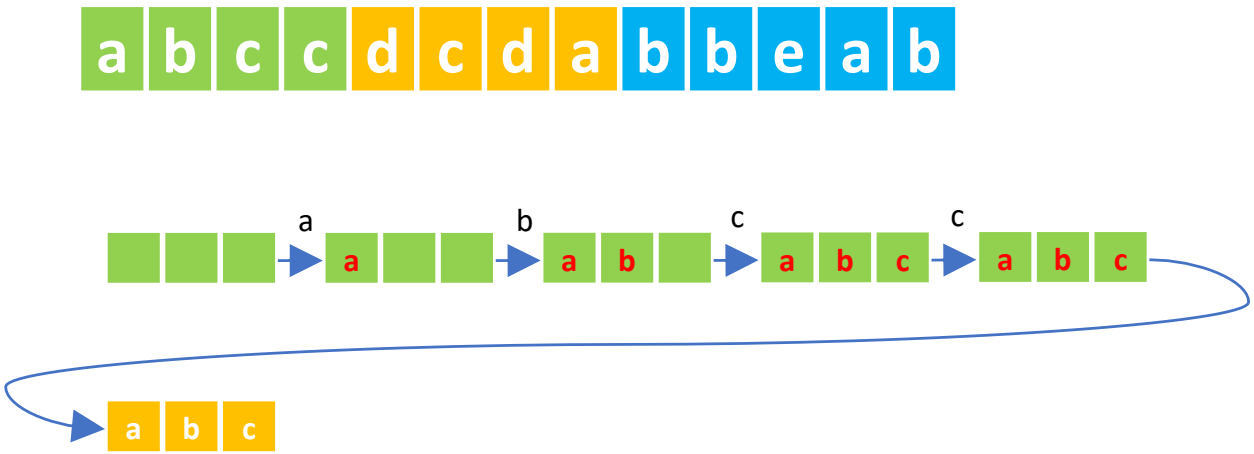
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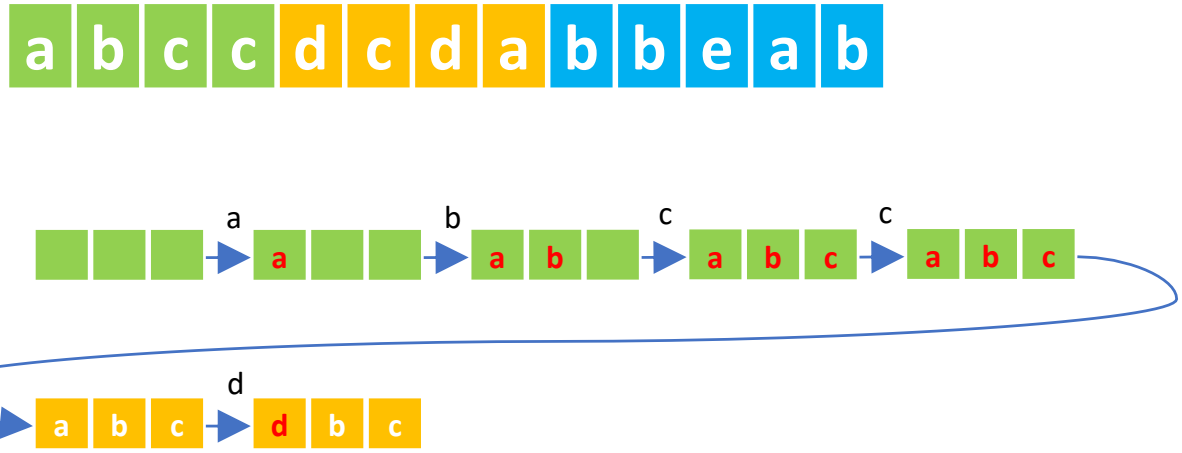
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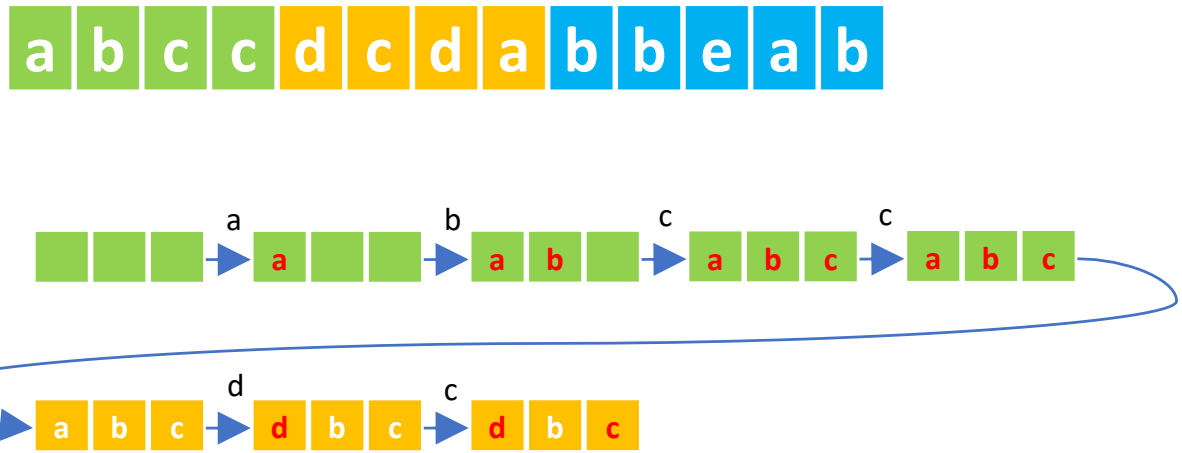
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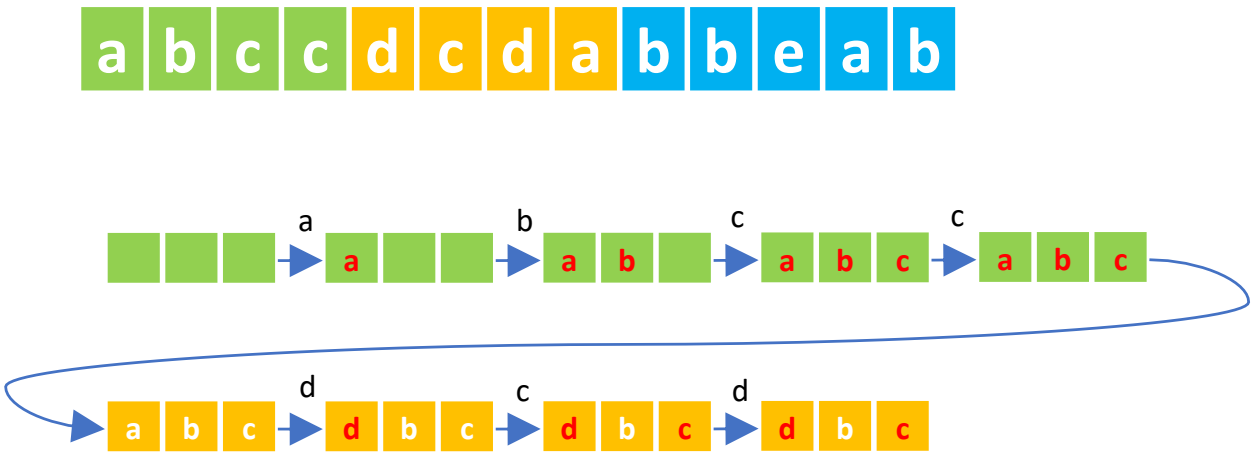
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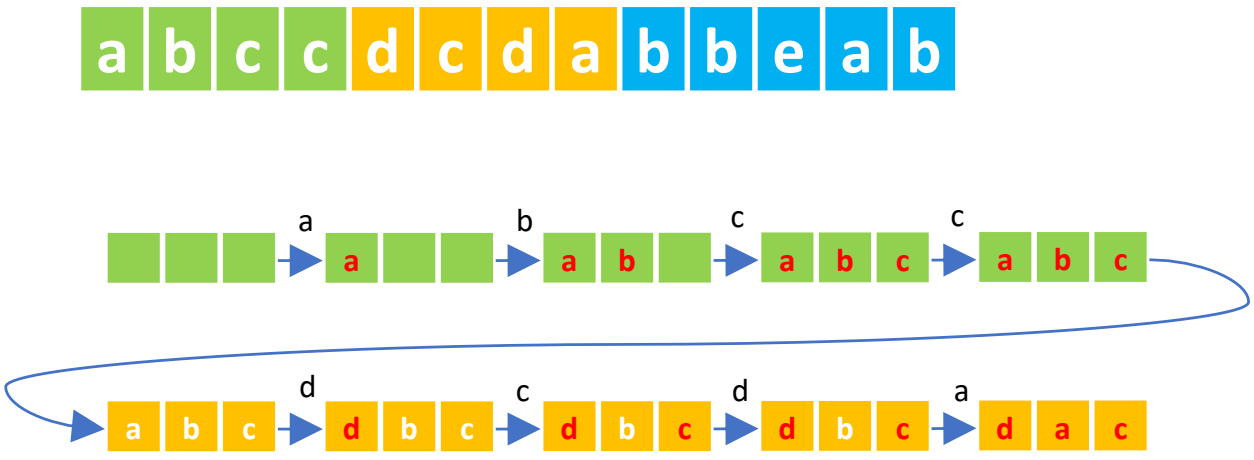
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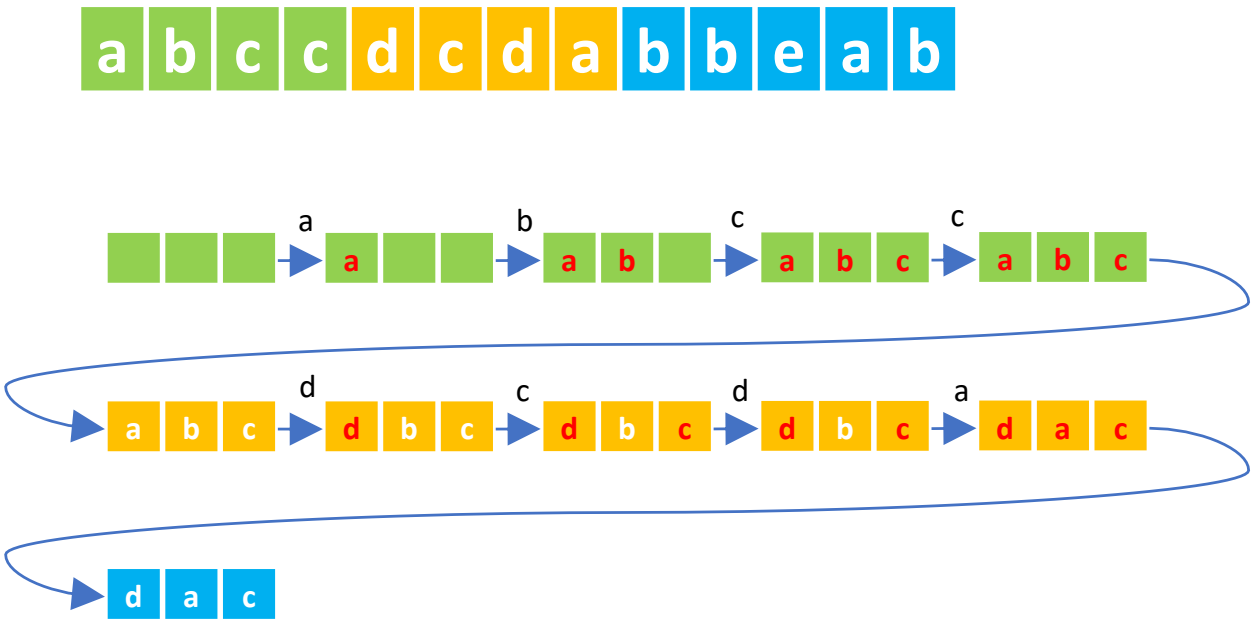
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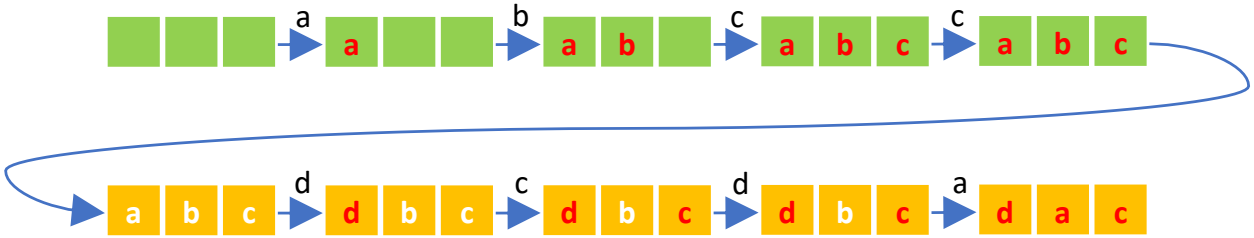
a b c c d c d a b b e a b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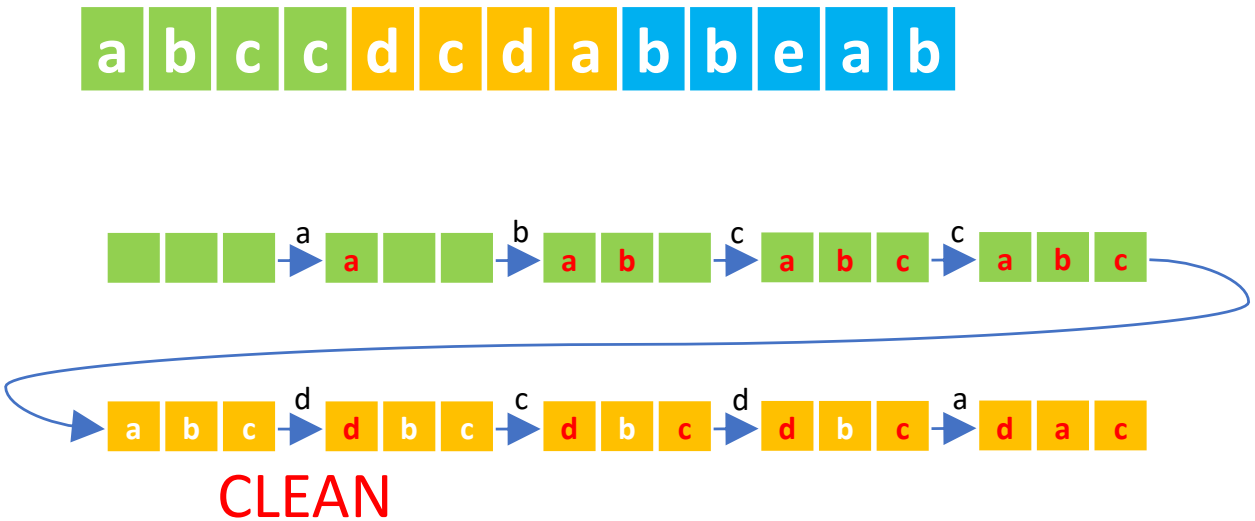
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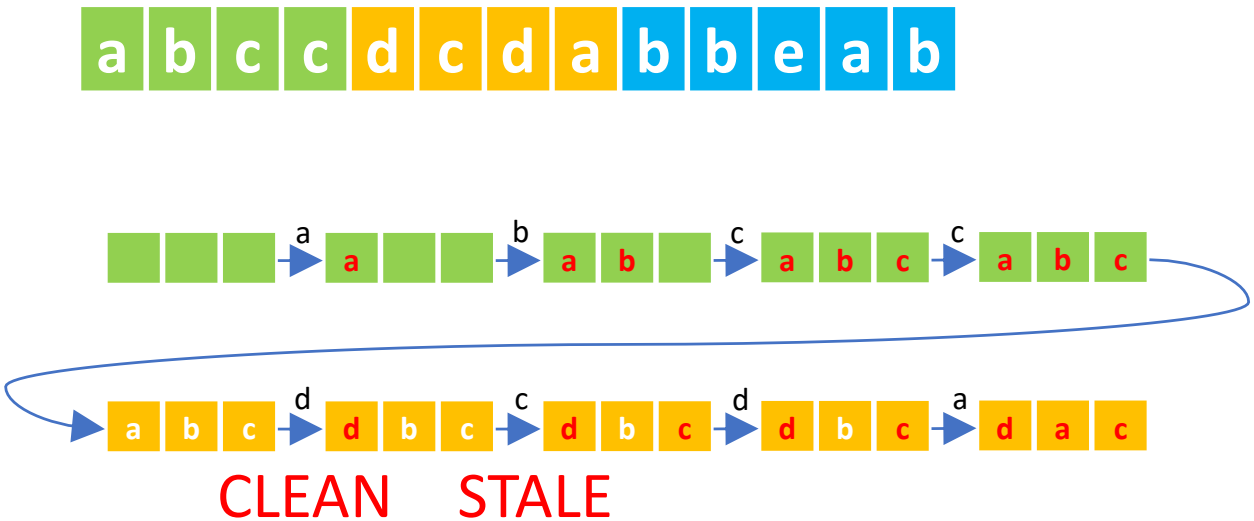
CLEAN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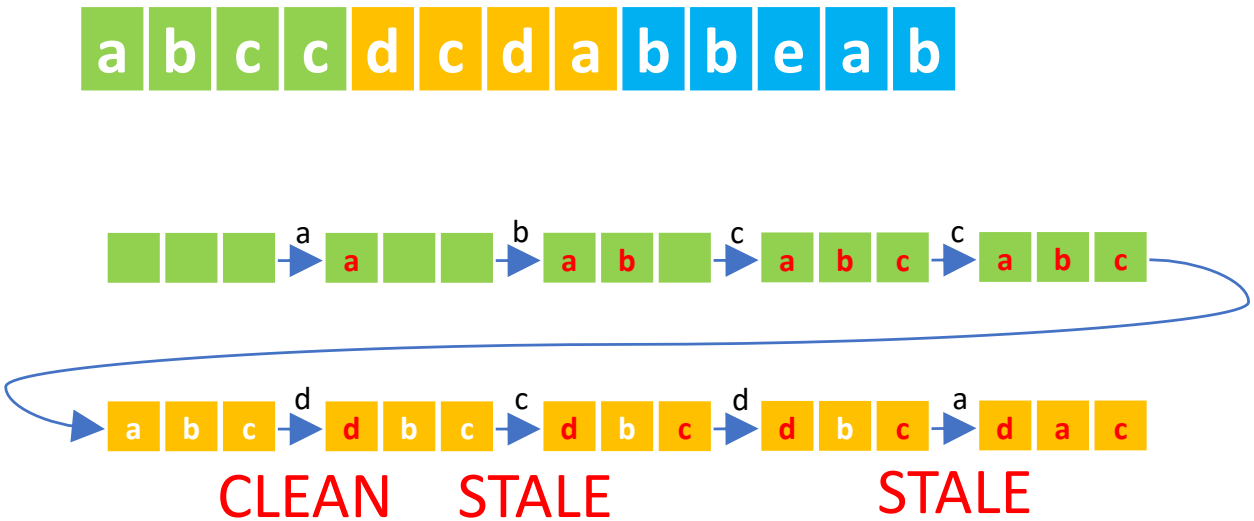
CLEAN
STALE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STALE

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- Combined, we got:
  $\mathbb{E}[MARK(\sigma)] \leq L \cdot H_k \leq 2 \log k \cdot \textsc{Opt}(\sigma)$.

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- If we'll use the marking algorithm, we'll gain $O(\log k)$ competitive ratio. How can we improve that?
- The natural thing to do then is to **break ties in eviction** using the predictor.

Chains

Chains, Explained

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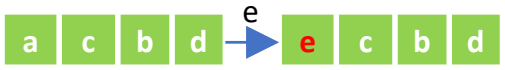

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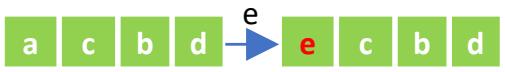

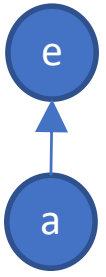

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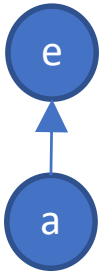

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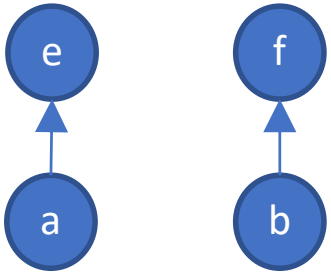

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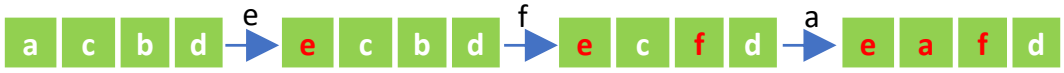

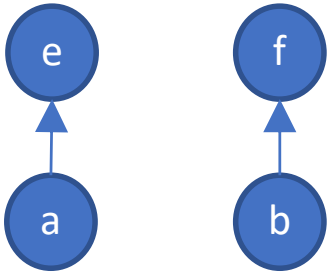

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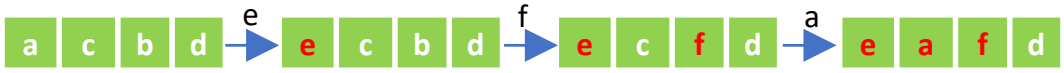

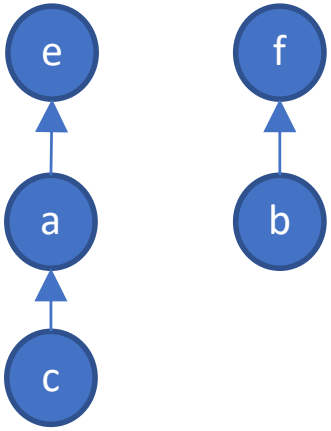

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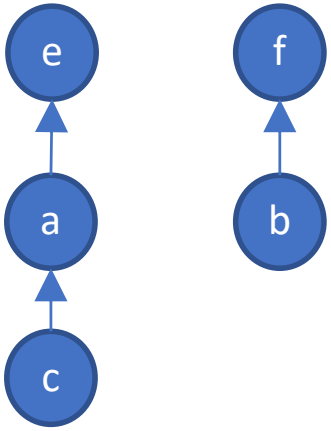

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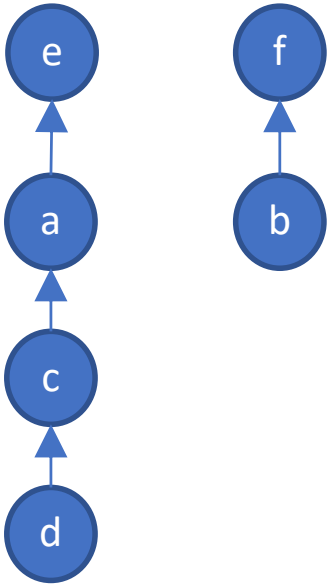
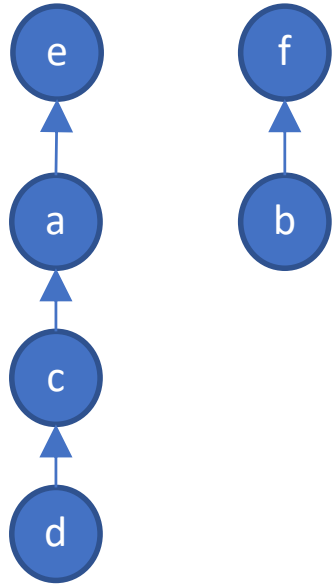
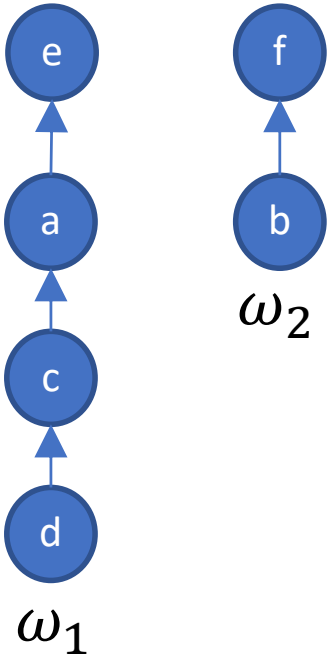


Chains, Explained

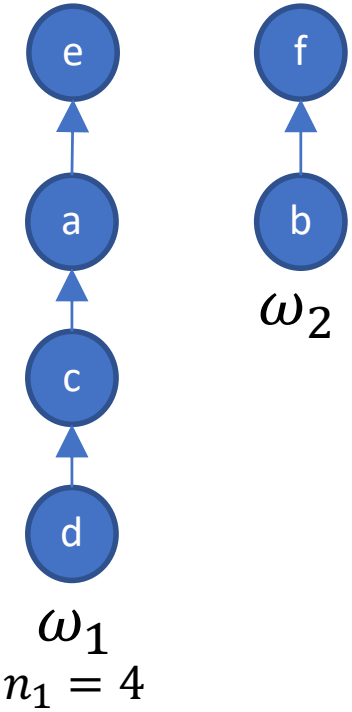


ω_1

Chains, Explained



Chains, Explained

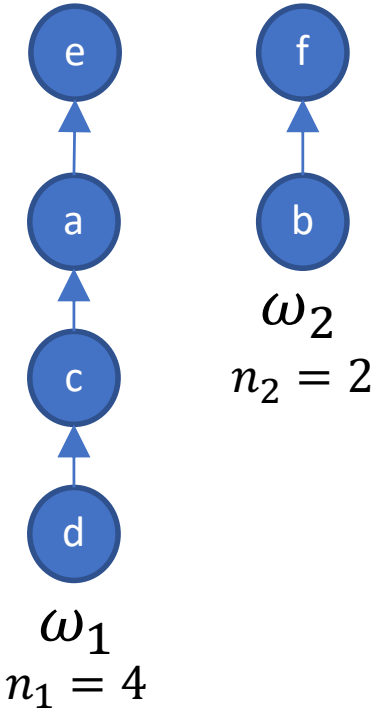


ω_1

$n_1 = 4$

ω_2

Chains, Explained



The Predictive Marker Algorithm

Now to the algorithm:

$\mathcal{PM}$ – *Predictive Marker*

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      evict random unmarked element e.
    - Increase the chain: $n_c \leftarrow n_c + 1$, $\omega_c \leftarrow e$.

Main Theorem

Theorem

Consider the caching scenario, with the prediction model $\mathcal{H}$ and the loss function ℓ_1.
The competitive ratio is of the ϵ-assisted Predictive Marker Algorithm $\mathcal{PM}$ is bounded by:

$$\mathrm{CR}_{\mathcal{PM}}(\epsilon) \leq 2 \cdot \min\left(1 + \sqrt{5\epsilon}, 2H_k\right)$$

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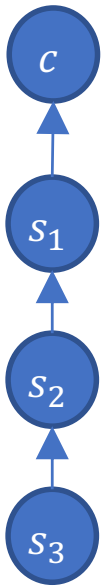
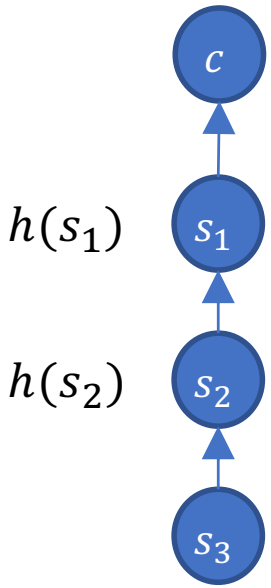
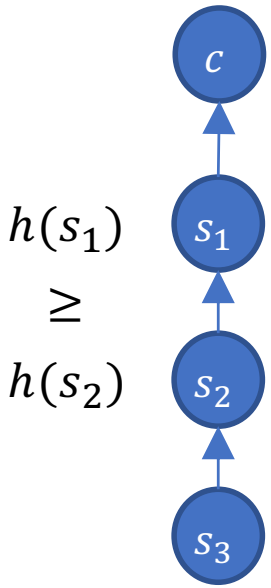
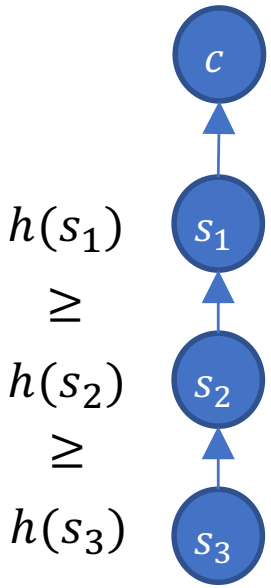
Lemma

If h has error $\leq \eta$ on chain ω_c, then the chain's length is bounded by $n_c \leq 1 + \sqrt{5\eta}$.


c
s_1
s_2
s_3


c
$h(s_1)$
s_1
$h(s_2)$
s_2
s_3


c
$h(s_1)$
s_1
$\geq$
$h(s_2)$
s_2
s_3


c
$h(s_1)$
s_1
$\geq$
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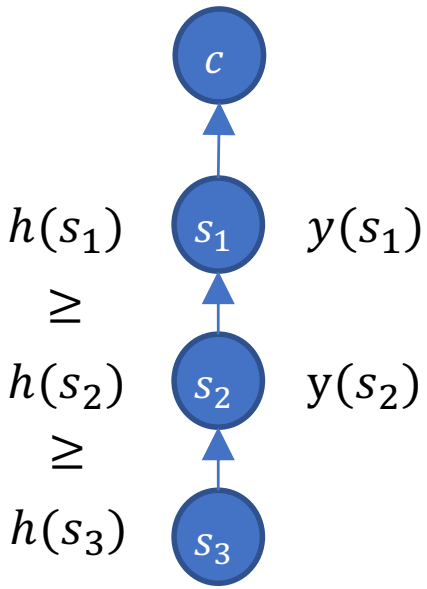
s_2
$\geq$
$h(s_3)$
s_3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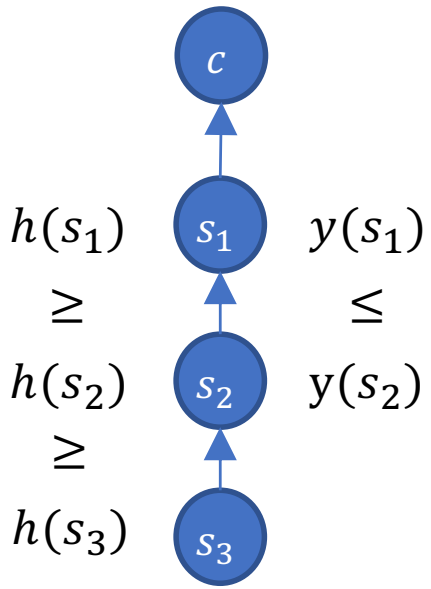
s_3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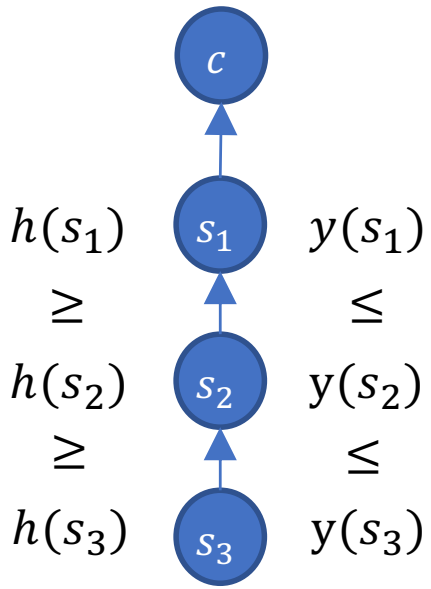
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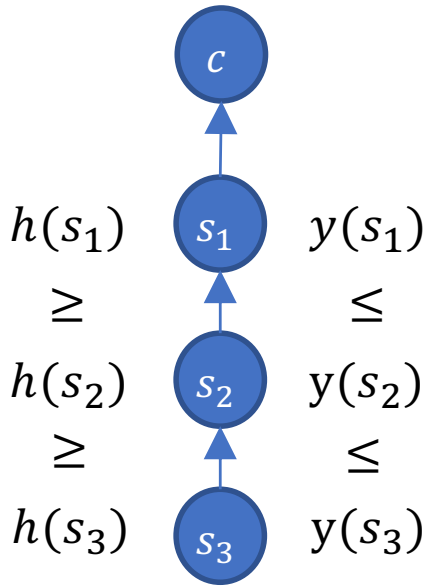

$$h(s_1) \geq h(s_2) \geq h(s_3) \geq \cdots$$

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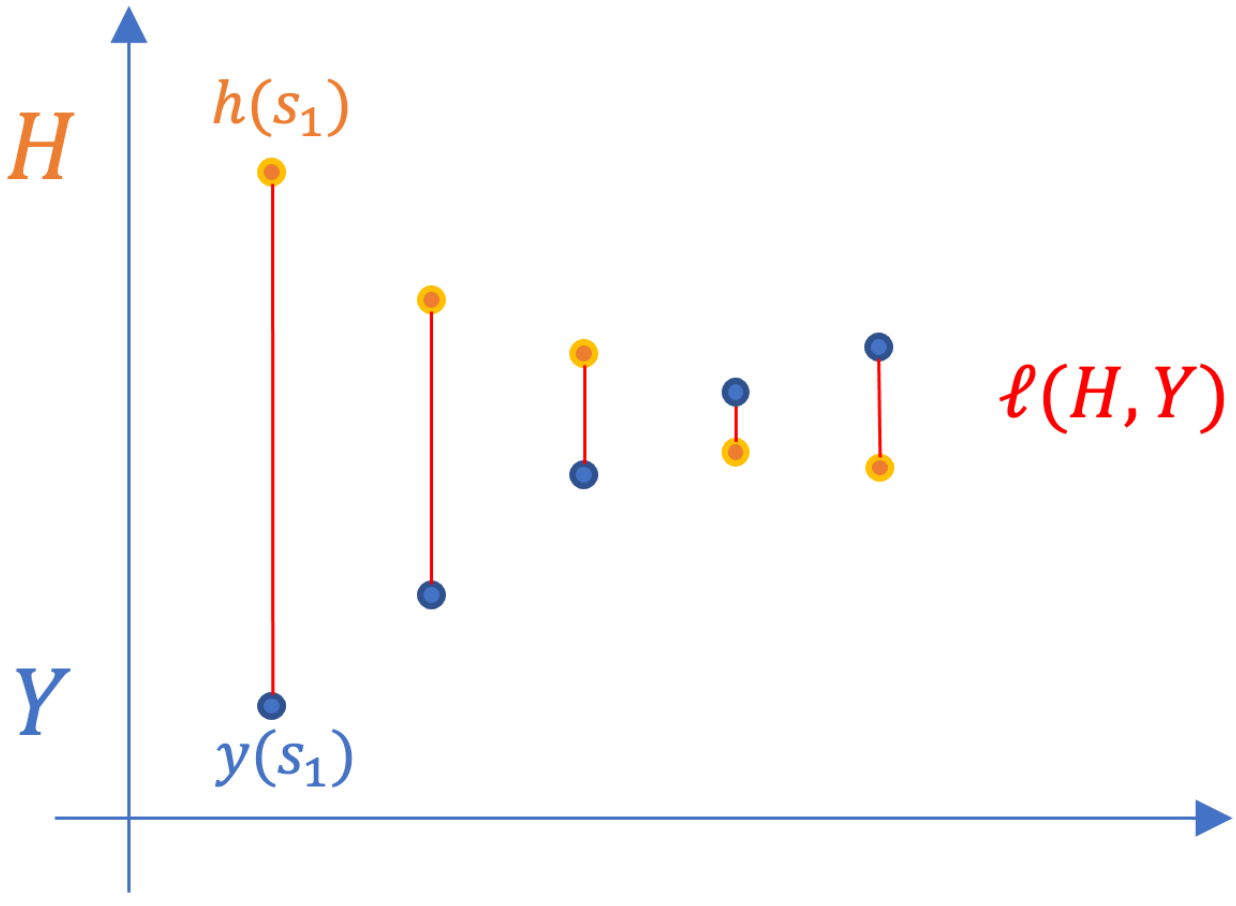
Proof - Chain Lengths



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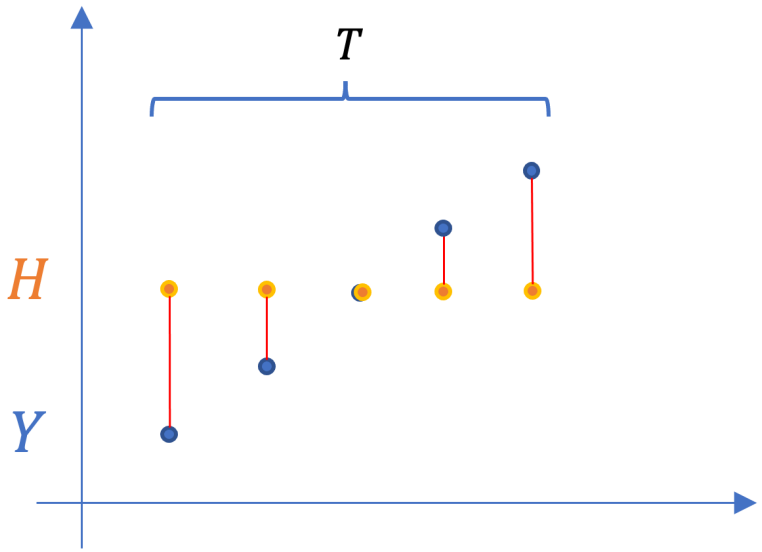

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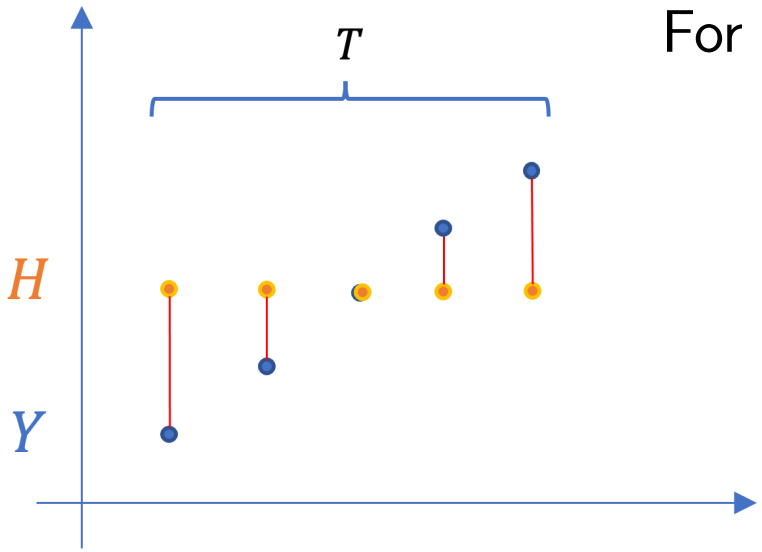

For such chain of length T we get the error:

$$\ell(H, Y) = 2\sum_{i=1}^{\frac{T-1}{2}} i = 2\frac{\frac{T-1}{2}\cdot\frac{T+1}{2}}{2}$$

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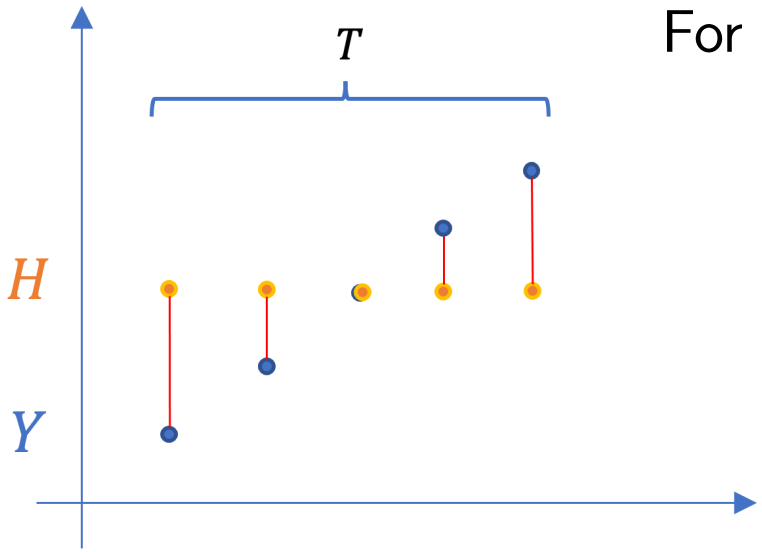

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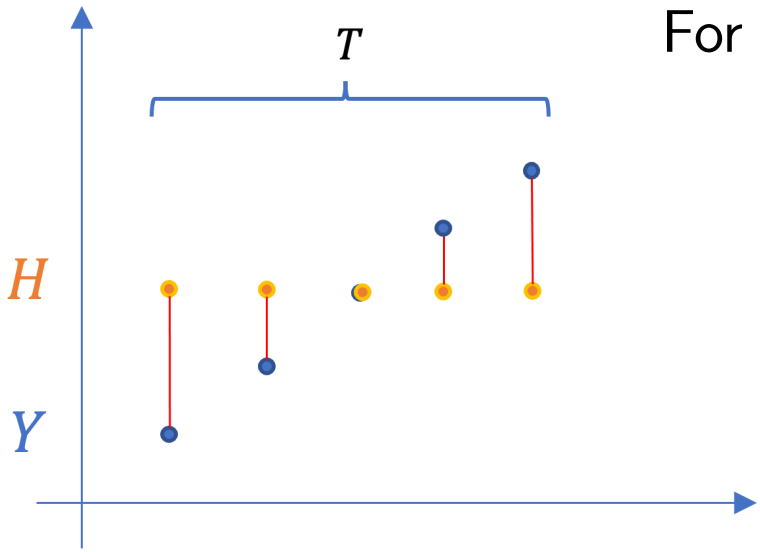

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Corollary

If a chain has error $\leq \eta_c$, its length is at most $\sqrt{4\eta_c+1} \leq \sqrt{5\eta_c}$.

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- So the total cost is bounded by $cost_{\mathcal{PM}(\epsilon)} \leq \sum_{r,c} \min\left(1 + \sqrt{5\eta_{r,c}}, 2H_k\right) \leq ?$

Proof - Continued

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Which means $\mathrm{CR}_{\mathcal{PM}}(\epsilon) \leq 2 \cdot \min\left(1 + \sqrt{5\epsilon}, 2\log k\right)$. □

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Any **deterministic** ϵ-assisted marking algorithm $\mathcal{A}$, that only uses the predictor in tie-breaking among unmarked elements in a deterministic fashion, has a competitive ratio of

$$\mathrm{CR}_{\mathcal{A}}(\epsilon) = \Omega\left(\min\left(\sqrt{\epsilon}, \mathbf{k}\right)\right)$$

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Then, the competitive ratio of ϵ-assisted ϵ-assisted $\mathcal{PM}(\gamma)$ is bounded by:

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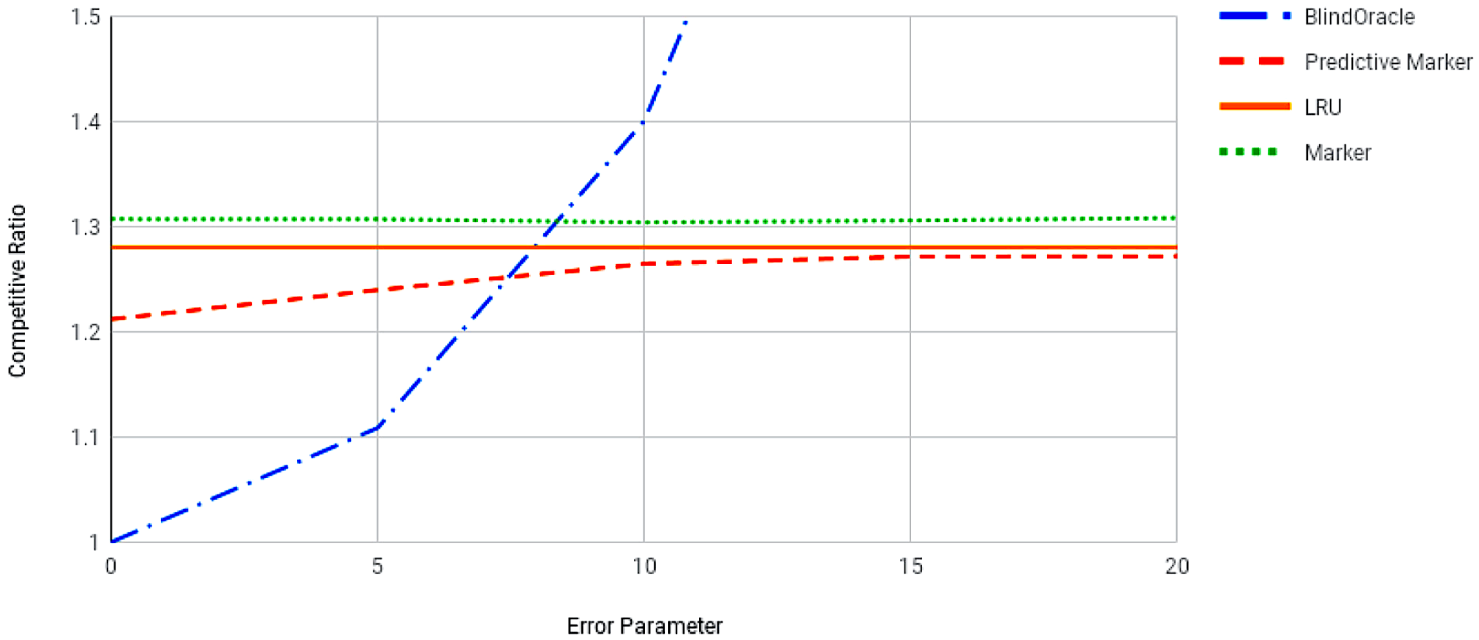
- This new setting reduces the analysis of LRU from $\Theta(k)$ to $O(\log k)$, while still exploiting its predictive power.

Comparing results

- Lets take a look at some real world datasets:

Dataset	Num Sequences	Sequence Length	Unique Elements
BK	100	2,101	67– 800
Citi	24	25,000	593 – 719

Comparing results



Algorithm	Competitive Ratio on BK	Competitive Ratio on Citi
Blind Oracle	2.049	2.023
LRU	1.280	1.859
Marker	1.310	1.869
Predictive Marker	1.266	1.810

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- Thank you!

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J. ACM 68, 4, Article 24 (July 2021), 25 pages.